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Notes on the Wiener Semigroup and Renormalization*

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Abstract. In this paper, by using white noise analysis (e.g. Wick product, scaling trasformation) we obtain some results about the ∞-dim. Wiener semigroup. A precise definition of renormalization in white noise analysis is also proposed. The main results are Theorems 2.2, 2.4, 2.5, and 3.2.

1. Introduction and Preliminaries

In this paper we consider the following Gel'fand triple

$$(\mathcal{S})^* \supset (\mathcal{L}^2) = \mathcal{L}^2(\mathcal{S}'(I\!R), \mu) \supset (\mathcal{S})$$

where μ is the white noise measure on $\mathcal{S}'(I\!R)$, the Schwartz space of tempered distributions. Let A denote the self-adjoint operator $-\frac{d^2}{dt^2} + 1 + t^2$ in $\mathcal{L}^2(I\!R)$. For each $p \geq 0$ we put $\mathcal{S}_p(I\!R) = Dom(A^p)$ and $(\mathcal{S})_p = Dom(\Gamma(A^p))$, where $\Gamma(A^p)$ stands for the second quantization of A^p. We denote by $\mathcal{S}_{-p}(I\!R)$ (resp. $(\mathcal{S}_{-p})$) the dual of $\mathcal{S}_p(I\!R)$ (resp. $(\mathcal{S}_p)$). Let $\hat{\mathcal{S}}_p(I\!R^n)$ denote the subspace of all symmetric functions (or distributions) in $\mathcal{S}_p(I\!R^n)$. The norm $|\cdot|_{2,p}$ of $\mathcal{S}_p(I\!R^n)$ is defined by

$$|f^{(n)}|_{2,p} = |(A^p)^{\otimes n} f^{(n)}|_2$$

where $|\cdot|_2$ is the norm of $L^2(I\!R^n)$. Each element ϕ of $(\mathcal{S})_p$ corresponds uniquely to a sequence $(f^{(n)}), f^{(n)} \in \hat{\mathcal{S}}_p(I\!R^n)$, verifying

$$||\phi||_{2,p}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{2,p}^2 < \infty$$

where $||\cdot||_{2,p}$ denotes the norm of $(\mathcal{S})_p$. We write $\phi \sim (f^{(n)}$ for this correspondance. We have

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$$\mathcal{S}(\mathbb{R}^n) = \cap_{p\geq 0}\mathcal{S}_p(\mathbb{R}^n),\ \mathcal{S}'(\mathbb{R}^n) = \cup_{p\geq 0}\mathcal{S}_{-p}(\mathbb{R}^n)$$
$$(\mathcal{S}) = \cap_{p\geq 0}(\mathcal{S})_p,\ (\mathcal{S})^* = \cup_{p\geq 0}(\mathcal{S})_{-p}.$$

The elements of $(\mathcal{S})$ (resp. $(\mathcal{S}^*)$) are called Hida test functionals (resp. Hida distributions).

Now we recall some basic notions and facts in white noise analysis, we denote by $< \cdot, \cdot >$ (resp. $\ll \cdot, \cdot \gg$) the dual pairing between $\mathcal{S}_{-p}(\mathcal{R}^n)$ and $\mathcal{S}_p(\mathbb{R}^n)$ (resp. between $(\mathcal{S})_{-p}$ and $(\mathcal{S})_p$), p running over $\mathbb{R}_+$. Let $\phi \in (\mathcal{S})_p$, $\phi \in (\mathcal{S})_{-p}$ with $\phi \sim (F^{(n)}), \psi \sim (G^{(n)})$. Then

$$\ll \phi, \psi \gg = \sum_{n=0}^{\infty} n! < F^{(n)}, G^{(n)} > . \tag{1.1}$$

Let $\xi \in \mathcal{S}(\mathbb{R})$. Put

$$\mathcal{E}(\xi) = \exp\{< \cdot, \xi > -\frac{1}{2}|\xi|_2^2\}. \tag{1.2}$$

Then $\mathcal{E}(\xi) \in (\mathcal{S})$. Thus for each $\phi \in (\mathcal{S})^*$ we can put

$$S\phi(\xi) = \ll \phi, \mathcal{E}(\xi) \gg,\ \xi \in \mathcal{S}(\mathbb{R}) \tag{1.3}$$

We call $S\phi$ the S-transform of ϕ. Let $\phi, \psi \in (\mathcal{S})^*$. Assume that $\phi \sim (F^{(n)})$ and $\psi \sim (G^{(n)})$. Put

$$H^{(n)} = \sum_{k+j=n} F^{(k)} \hat{\otimes} G^{(j)}.$$

Then $(H^{(n)})$ corresponds to an element of $(\mathcal{S})^*$, which is denoted by $\phi : \psi$ and called the Wick product of ϕ and ψ. We have

$$S(\phi : \psi) = S\phi \cdot S\psi \tag{1.4}$$

It is shown in Meyer-Yan [5] that we have

$$||\phi : \psi||_{2,p} \leq ||\phi||_{2,p+\frac{1}{2}} ||\psi||_{2,p+\frac{1}{2}} \tag{1.5}$$

This inequality will play an important role in the sequel.

Let $\phi \in (\mathcal{S})$. It is shown in Kubo-Yokoi [1] that ϕ admets a continuous version $\tilde{\phi}$ of ϕ (see also Lee [4] and Yan [8]).

Let $\lambda \in \mathbb{R}$ and $y \in \mathcal{S}'(\mathbb{R})$. It is proved in Potthoff-Yan[6] that the following mappings are continuous from $(\mathcal{S})$ into itself:

$$\phi^{(\lambda)}(\cdot) = \tilde{\phi}(\lambda\cdot),\ \tau_y\phi(\cdot) = \tilde{\phi}(\cdot + y), \phi_{(\lambda)} = \Gamma(\lambda)\phi \tag{1.6}$$

where $\Gamma(\lambda)$ is the second quantization of the multiplication by λ. Namely, if $\phi \sim (F^{(n)})$

then $\Gamma(\lambda)\phi \sim (\lambda^n F^{(n)})$. Moreover, $\Gamma(\lambda)$ is a continuous mapping from $(S)^*$ into itself and we have

$$||\phi_{(\lambda)}||_{2,p} \leq ||\phi||_{2,p+\log_2(|\lambda|\vee 1)} \tag{1.7}$$

because for any $\alpha > 0$ we have

$$|F^{(n)}|_{2,p} \leq 2^{-\alpha n}|F^{(n)}|_{2,p+\alpha} \tag{1.8}$$

Let $x \in S'(I\!R)$. The sequence $(\frac{1}{n!}x^{\otimes n})$ corresponds to a Hida distribution, whose S-transform is $\exp < x, \xi >$, $\xi \in S(I\!R)$. We denote it by $\mathcal{E}(x)$. It is easy to see that

$$||\mathcal{E}(x)||_{2,p} = \exp\frac{1}{2}|x|^2_{2,p} \tag{1.9}$$

It is shown in Potthoff-Yan [6] that for $\phi \in (S)$, $F \in (S)^*$ and $x \in S'(I\!R)$ we have

$$\ll \tau_x\phi, F \gg = \ll \phi, \mathcal{E}(x) : F \gg \tag{1.10}$$

Let $x \in S'(I\!R)$. The evaluation mapping at x is a Hida distribution, denoted by δ_x, whose S-transform is

$$S\delta_x(\xi) = \exp\{< x, \xi > -\frac{1}{2}|\xi|^2_2\},\ \xi \in S(I\!R) \tag{1.11}$$

It is shown in Yan [7] that if $p > \frac{1}{2}$ and $x \in S_{-p}(I\!R)$ then $\delta_x \in (S)_{-p}$. By (1.11) we have

$$\delta_x = \mathcal{E}(x) : \delta_0 \tag{1.12}$$

Let $\lambda \in I\!R\backslash\{0\}$. Put

$$\mu^{(\lambda)}(E) = \mu(E/\lambda),\ E \in \mathcal{B}(S'(I\!R)).$$

It is shown in Potthoff-Yan [6] that the " generalized $R-N$ derivative $\frac{d\mu^{(\lambda)}}{d\mu}$ can be regarded as a Hida distribution, whose S-transform is

$$S\frac{d\mu^{(\lambda)}}{d\mu}(\xi) = \exp\{-\frac{1}{2}(1-\lambda^2)|\xi|^2_2\} \tag{1.13}$$

That means $\frac{d\mu^{(\lambda)}}{d\mu}$ corresponds to the following sequence $(F^{(n)})$:

$$F^{(2k)} = \frac{(\lambda^2-1)^k}{2^k k!}T_r^{\otimes k},\ F^{(2k+1)} = 0 \tag{1.14}$$

where T_r is the trace operator which is an element of $\hat{S}_{-p}(\mathbb{R}^2)$ for any $p > \frac{1}{4}$, and we have

$$|T_r|_{2,-p}^2 = \sum_{n=1}^{\infty}(2n)^{-4p} \tag{1.15}$$

If $\lambda^2 \neq 1$ and p_λ be the number such that $|\lambda^2 - 1||T_r|_{2,-p_\lambda} = 1$, then $\frac{d\mu^{(\lambda)}}{d\mu} \in (S_{-p}$ for $p > p_\lambda$ and $\frac{d\mu^{(\lambda)}}{d\mu} \notin S_{-p_\lambda}$ (see Yan [6]).

Let $\mathcal{X}$ be a vector space. We denote by $C\mathcal{X}$ the complexification of $\mathcal{X}$. If $\mathcal{X}$ is a Hilbert space with the norm $||\cdot||$, then the norm of $C\mathcal{X}$ is defined by

$$||x + iy||^2 = ||x||^2 + ||y||^2. \tag{1.16}$$

Let $p > \frac{1}{2}$. It is shown in Lee [4] that each $\phi \in (S)_p$ admets an analytic extension $\tilde{\phi}$ on $CS_{-p}(\mathbb{R})$ and we have

$$\ll \phi, \delta_z \gg = \tilde{\phi}(z), \; z \in CS_{-p}(\mathbb{R}), \tag{1.17}$$

where δ_z is a complex Hida distribution whose S-transform is

$$S\delta_z(\xi) = \exp\{< z, \xi > -\frac{1}{2}|\xi|_2^2\}, \; \xi \in S(\mathbb{R})$$

(see also Yan [8]). Recall that $||A^{-p}||_{H.S.}^2 = \sum_{n=1}^{\infty}(2n)^{-2p} < \infty$ for $p > \frac{1}{2}$, so we have $\mu(S_{-p}(\mathbb{R})) = 1$. The restriction of $\tilde{\phi}$ to $S_{-p}(\mathbb{R})$ is a continuous version of ϕ.

The main purpose of this paper is to study the ∞-dim. Wiener semigroup by using white noise analysis and give a precise definition of renormalizations in white noise analysis.

2. The ∞-Dimensional Wiener Semigroup and White Noise Analysis

In this section we shall study the ∞-dim. Wiener semigroup by using white noise analysis. This investigation was initiated in a joint work with H.H.Kuo and J.Potthoff (see [3]).

We begin with introducing some operators acting on (S).

Definition 2.1 Let $\lambda \in \mathbb{R}\backslash\{0\}$. For each $\phi \in (S)$ we put

$$R_\lambda \phi = (\phi_{(\lambda)})^{(\frac{1}{\lambda})}, \; R_\lambda^{-1}\phi = (\phi^{(\lambda)})_{(\frac{1}{\lambda})} \tag{2.1}$$

Then R_λ and R_λ^{-1} are continuous mappings from (S) into itself.

Lemma 2.1 Let $\phi \in (S)$ and $F \in (S)^*$. Then for any $\lambda \in \mathbb{R}\backslash\{0\}$ we have

$$\ll \phi^{(\lambda)}, F \gg = \ll \phi, F_{(\lambda)} : \frac{d\mu^{(\lambda)}}{d\mu} \gg \tag{2.2}$$

Proof. If $F \in (S)$ then by using S-transform we can obtain

$$F^{(\frac{1}{\lambda})}\frac{d\mu^{(\lambda)}}{d\mu} = F_{(\lambda)} : \frac{d\mu^{(\lambda)}}{d\mu} \tag{2.3}$$

from which it follows (2.2) for $F \in (S)$. If $F \in (S)^*$, by taking a sequence (F_n) of elements of (S) such that $F_n \to F$ in $(S)^*$, we get (2.2) by using (1.5).

Theorem 2.1 Let $p > \frac{1}{4}$ and $\lambda \neq 0$ be such that $|1-\lambda^2||T_r|_{2,-p} < 1$. R_λ and R_λ^{-1} can be extended to a continuous mapping from $(S)_{p+\frac{1}{2}}$ to $(S)_p$. Moreover, we have the following estimates and equalities:

$$||R_\lambda\phi||_{2,p} \leq C(p,\lambda)||\phi||_{2,p+\frac{1}{2}}, \ ||R_\lambda^{-1}\phi||_{2,p} \leq C(p,\lambda)||\phi||_{2,p+\frac{1}{2}} \tag{2.4}$$

$$\ll R_\lambda\phi, F \gg = \ll \phi, F : (\frac{d\mu^{(\frac{1}{\lambda})}}{d\mu})_{(\lambda)} \gg, \tag{2.5}$$

$$\ll R_\lambda^{-1}\phi, F \gg = < \phi, F : \frac{d\mu^{(\lambda)}}{d\mu} \gg \tag{2.6}$$

where $\phi \in (S)_{p+\frac{1}{2}}$, $F \in (S)_{-p}$ and $C(p,\lambda) = ||\frac{d\mu^{(\lambda)}}{d\mu}||_{2,-p}$.

Proof. If $\phi \in (S)$ and $F \in (S)^*$ we obtain (2.5) and (2.6) from (2.2). By using (1.5) we get (2.5) and (2.6) for $\phi \in (S)_{p+\frac{1}{2}}$ and $F \in (S)_{-p}$. Since

$$\frac{d\mu^{(\lambda)}}{d\mu} \sim (\frac{(\lambda^2-1)^k}{k!2^k}T_r^{\otimes k}), (d\mu^{(\frac{1}{\lambda})}d\mu)_{(\lambda)} \sim (\frac{(1-\lambda^2)^k}{k!2^k}T_r^{\otimes k})$$

we have

$$||\frac{d\mu^{(\lambda)}}{d\mu}||^2_{2,-p} = ||(\frac{d\mu^{(\frac{1}{\lambda})}}{d\mu})_{(\lambda)}||^2_{2,-p} = \sum_{k=0}^{\infty} \frac{(2k)!(|1-\lambda^2||T_r|_{2,-p})^{2k}}{(k!2^k)^2} < \infty \tag{2.7}$$

By (2.5), (2.6), (1.5) and (2.7) we obtain

$$|\ll R_\lambda\phi, F \gg| \leq ||\phi||_{2,p+\frac{1}{2}}||F||_{2,-p}||\frac{d\mu^{(\lambda)}}{d\mu}||_{2,-p}$$

$$|\ll R_\lambda^{-1}\phi, F \gg| \leq ||\phi||_{2,p+\frac{1}{2}}||F||_{2,-p}||\frac{d\mu^{(\lambda)}}{d\mu}||_{2,-p}$$

from which it follows (2.4).

Let $\phi \in (S)$. We put

$$P_t\phi(x) = \int_{S'(\mathbb{R})} \tilde{\phi}(x+\sqrt{t}y)\mu(dy) \tag{2.8}$$

and call $(P_t, t \geq 0)$ the Wiener semigroup. Let $\mu_{x,t}$ denote the gaussian measure on $S'(\mathbb{R})$ with mean value x and variance parameter t. Then we have

$$P_t\phi(x) = \int_{S'(\mathbb{R})} \tilde{\phi}\mu_{x,t}(dy) \tag{2.9}$$

Thus the generalized derivative $\frac{d\mu_{x,t}}{d\mu}$ can be regarded as a Hida distribution and its S-transform is given by

$$S(\frac{d\mu_{x,t}}{d\mu})(\xi) = P_t\mathcal{E}(\xi)(x)$$
$$= \exp\{< x, \xi > -\frac{1}{2}(1-t)|\xi|_2^2\}$$

That means

$$\frac{d\mu_{x,t}}{d\mu} = \mathcal{E}(x) : \frac{d\mu^{(\sqrt{t})}}{d\mu} \tag{2.10}$$

Theorem 2.2 Let $\phi \in (\mathcal{S})$. We have

$$P_t\phi = R^{-1}_{\frac{1}{\sqrt{1+t}}}\phi \tag{2.11}$$

$$P_t\phi = R_{\sqrt{1-t}}\phi, \ 0 \le t < 1 \tag{2.12}$$

In particular, P_t is a continuous mapping from $(\mathcal{S})$ into itself.

Proof. By (2.9) and (2.10) we have

$$\begin{aligned} P_t\phi(x) &= \ll \phi, \mathcal{E}(x): \frac{d\mu^{(\sqrt{t})}}{d\mu} \gg \\ &= \ll \phi, \delta_x : \frac{d\mu^{(\sqrt{2})}}{d\mu} : \frac{d\mu^{(\sqrt{t})}}{d\mu} \gg \\ &= \ll \phi, \delta_x : \frac{d\mu^{(\sqrt{1+t})}}{d\mu} \gg \end{aligned} \tag{2.13}$$

Thus, from (2.6) and (2.13) we get

$$P_t\phi(x) = \ll R^{-1}_{\frac{1}{\sqrt{1+t}}}\phi, \delta_x \gg = \widetilde{R^{-1}_{\frac{1}{\sqrt{1+t}}}\phi}(x).$$

If $0 \le t < 1$, then by (2.13) and (2.5) we obtain

$$P_t\phi(x) = \ll R_{\sqrt{1-t}}\phi, \delta_x \gg = \widetilde{R_{\sqrt{1-t}}\phi}(x),$$

because we have

$$(\frac{d\mu^{(\frac{1}{\sqrt{1-t}})}}{d\mu})_{(\sqrt{1-t})} = \frac{d\mu^{(\sqrt{1+t})}}{d\mu}.$$

The theorem is proved.

As an application of (2.12) we obtain the following well known result.

Corollary. Let $\phi \in (\mathcal{S})$. Put

$$Q_t\phi(x) = \int \tilde{\phi}(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(dy) \tag{2.14}$$

Then we have

$$Q_t\phi = e^{-tN}\phi \tag{2.15}$$

where N is the number operator. (Q_t) is called the Ornstein - Uhlenbeck semigroup.

Proof. By (2.14) and (2.12) we have

$$\begin{aligned} Q_t\phi &= (P_{1-e^{-2t}}\phi)^{(e^{-t})} = (R_{e^{-t}}\phi)^{(e^{-t})} = (\phi)_{(e^{-t})} \\ &= \Gamma(e^{-t})\phi = e^{-tN}\phi \end{aligned}$$

Theorem 2.3 Let $\alpha > \frac{1}{4}$ be such that $|T_r|_{2,-\alpha} < \frac{1}{t}$. Then P_t can be extended to a continuous mapping from $(S)_{\alpha+\frac{1}{2}}$ to $(S)_\alpha$ and we have

$$||P_t\phi||_{2,\alpha} \le ||\frac{d\mu^{(\sqrt{1+t})}}{d\mu}||_{2,-\alpha}||\phi||_{2,\alpha+\frac{1}{2}} \tag{2.16}$$

Moreover, for $\phi \in (S)_{\alpha+\frac{1}{2}}$ and $F \in (S)_{-\alpha}$, we have

$$\ll P_t\phi, F \gg = \ll \phi, F : \frac{d\mu^{(\sqrt{1+t})}}{d\mu} \gg \tag{2.17}$$

Proof. (2.17) follows from (2.11) and (2.6). From (2.17) we get (2.16).

Theorem 2.4 Let $\alpha > \frac{1}{4}$ and $\phi \in (S)_{\alpha+\frac{1}{2}}$. Then the following limit exists in $(S)_\alpha$:

$$\triangle\phi = \lim_{t\downarrow 0} \frac{P_t\phi - \phi}{t} \tag{2.18}$$

and for $F \in (S)_{-\alpha}$ we have

$$\ll \triangle\phi, F \gg = \frac{1}{2} \ll \phi, F : I_2(T_r) \gg \tag{2.19}$$

where $I_2(T_r)$ is a Hida distribution whose S-transform is $SI_2(T_r)(\xi) = |\xi|_2^2$.

If $\alpha > \frac{1}{2}$ then we have

$$\widetilde{\triangle\phi}(x) = \lim_{t\downarrow 0} \frac{\widetilde{P_t\phi}(x) - \tilde{\phi}(x)}{t}, \; x \in S_{-\alpha}(I\!R) \tag{2.20}$$

$$\widetilde{\triangle\phi}(x) = -\widetilde{N\tau_x\phi}(0), \; x \in S_{-\alpha}(I\!R) \tag{2.21}$$

Proof. We have

$$\lim_{t\downarrow 0} ||\frac{1}{t}(\frac{d\mu^{(\sqrt{1+t})}}{d\mu} - 1) - \frac{1}{2}I_2(T_r)||_{2,-\alpha}^2$$

$$= \lim_{t\downarrow 0} \sum_{k=2}^{\infty} (2k)! \frac{t^{2(k-1)}}{(k!2^k)^2} |T_r|_{2,-\alpha}^{2k} = 0,$$

from which and (1.5) we see that the limit in (2.18) exists in $(\mathcal{S})_\alpha$ and (2.19) holds. Moreover, for $x \in \mathcal{S}_{-\alpha}(I\!R)$, by (2.18) and (1.17) we have

$$\widetilde{\Delta\phi}(x) = \ll \Delta\phi, \delta_x \gg = \lim_{t\downarrow 0} \frac{1}{t} \ll P_t\phi - \phi, \delta_x \gg$$

from which we get (2.20). Finally, by using (1.5) we can extend (1.10) to the case where $\phi \in (\mathcal{S})_{\alpha+\frac{1}{2}}$ and $x \in \mathcal{S}_{-\alpha}(I\!R), F \in \mathcal{S}_{-\alpha}$. Namely, there exists a unique element of $(\mathcal{S})_\alpha$, denoted by $\tau_x\phi$, such that (1.10) holds for any $F \in (\mathcal{S})_{-\alpha}$. Consequently, for $x \in \mathcal{S}_{-\alpha}(I\!R)$, we have

$$\begin{aligned}
\widetilde{\Delta\phi}(x) &= \ll \Delta\phi, \delta_x \gg = \frac{1}{2} \ll \phi, \delta_x : I_2(T_r) \gg \\
&= \frac{1}{2} \ll \phi, \mathcal{E}(x) : \delta_0 : I_2(T_r) \gg \\
&= - \ll \phi, \mathcal{E}(x) : N\delta_0 \gg \\
&= - \ll \tau_x\phi, N\delta_0 \gg \\
&= - \ll N\tau_x\phi, \delta_0 \gg = -\widetilde{N\tau_x}\phi(0)
\end{aligned}$$

Here we have used the fact that if $\psi \in (\mathcal{S})_p$ then for any $\varepsilon > 0$ we have $N\psi \in (\mathcal{S})_{p-\varepsilon}$. The theorem is proved.

Example. Let $\xi \in \mathcal{S}(I\!R)$. We have

$$\Delta\mathcal{E}(\xi) = \frac{1}{2}|\xi|_2^2\mathcal{E}(\xi)$$

In the literature, the operator 2Δ is often called the Gross Laplacian. The following theorem gives us a good domain of Δ.

Theorem 2.5. Let $\mathcal{D} = \bigcup_{p>\frac{1}{4}} (\mathcal{S})_p$. We define the inductive limit topology on $\mathcal{D}$. Then Δ can be extended to a continuous mapping from $\mathcal{D}$ into itself.

Proof. Let $p > \frac{1}{4}$ and $F \in (\mathcal{S})_{-p}$. Assume that $F \sim (f^{(n)}$. Then we have $F : I_2(T_r) \sim (g^{(n)})$, where

$$g^{(0)} = g^{(1)} = 0, \ g^{(n)} = f^{(n-2)}\hat{\otimes}T_r, \ n \geq 2$$

Therefore, for any $\varepsilon > 0$ if we put

$$C(p,\varepsilon)=\sup_n (n+2)(n+1)2^{-2\varepsilon n}|T_r|^2_{2,-(p+\varepsilon)}$$

then we have (noting that $|f^{(n)}|_{2,-(p+\varepsilon)} \leq 2^{-\varepsilon n}|f^{(n)}|_{2,-p}$)

$$\begin{aligned}||F:I_2(T_r)||^2_{2,-(p+\varepsilon)} &= \sum_{n=0}^{\infty} n!|g^{(n)}|^2_{2,-(p+\varepsilon)} \\ &\leq \sum_{n=0}^{\infty}(n+2)!|f^{(n)}|^2_{2,-(p+\varepsilon)}|T_r|^2_{2,-(p+\varepsilon)} \\ &\leq C(p,\varepsilon)\sum_{n=0}^{\infty} n!|f^{(n)}|^2_{2,-p} = C(p,\varepsilon)||F||^2_{2,-p} \end{aligned} \tag{2.22}$$

We conclude the theorem by (2.19) and (2.22).

Remark 1. Let $\mathcal{D}=\bigcup_{p>\frac{1}{4}}(\mathcal{S})_p$. We denote by ∂_t the Hida derivative (i.e. $\partial_t = D_{\delta_t}$, see Potthoff-Yan [6]). It is shown in Yan [7] that ∂_t is a continuous mapping form $\mathcal{D}$ into itself and we have for $\phi \in \mathcal{D}$ and $\psi \in \mathcal{S}$

$$\ll \partial_t\phi,\psi \gg = \ll \phi,\ \psi : I_1(\delta_t) \gg . \tag{2.23}$$

Since $T_r=\int_{-\infty}^{\infty}\delta_t\otimes\delta_t dt$, it follows from (2.23) and (2.19) that for $\phi\in\mathcal{D}$ we have

$$\Delta\phi = \frac{1}{2}\int_{-\infty}^{\infty}\partial_t^2\phi dt.$$

This formula is due to Kuo [2].

Remark 2. Let $p>\frac{1}{4}$ and $\phi\in(\mathcal{S})_p$ with $\phi\sim(f^{(n)})$. It is easy to prove that $\Delta\phi\sim(h^{(n)})$ with

$$h^{(n)}=\frac{(n+2)(n+1)}{2}f^{(n+2)}\hat{\otimes}_2 T_r,$$

where $f^{(n+2)}\hat{\otimes}_2 T_r$ is an element of $\hat{\mathcal{S}}_p(\mathbb{R}^n)$ verifying

$$< f^{(n+2)}\hat{\otimes}_2 T_r, g^{(n)} > = < f^{(n+2)}, g^{(n)}\hat{\otimes} T_r >, \forall g^{(n)}\in\hat{\mathcal{S}}_{-p}(\mathbb{R}^n).$$

Let z be a complex number. We denote formally by $\frac{d\mu^{(z)}}{d\mu}$ a complex Hida distribution whose S-transform is

$$S\frac{d\mu^{(z)}}{d\mu}(\xi)=\exp\{-\frac{1-z^2}{2}|\xi|_2^2\}.$$

If $p>\frac{1}{4}$ is such that $|T_r|_{2,-p}<\frac{1}{|1-z^2|}$, then $\frac{d\mu^{(z)}}{d\mu}\in\mathcal{C}(\mathcal{S})_{-p}$.

The following theorem extends the Wiener semigroup (P_t) to a group $\{P_z, z\in\mathbb{C}\}$.

Theorem 2.6 Let $\phi \in \mathcal{C}(\mathcal{S})$ and $z \in \mathbb{C}$ We denote by $P_z\phi$ the unique element of $\mathcal{C}(\mathcal{S})$ such that for each $F \in \mathcal{C}(\mathcal{S})^*$

$$\ll P_z\phi, F \gg = \ll \phi, F : \frac{d\mu^{(\sqrt{1+z})}}{d\mu} \gg \tag{2.24}$$

Then $(P_z, x \in \mathbb{C})$ is a group acting on $\mathcal{C}(\mathcal{S})$ which extends the Wiener semigroup $(P_t, t \in \mathbb{R}_+)$. Moreover, for each $x \in \mathcal{S}'(\mathbb{R})$ we have

$$\widetilde{P_z\phi}(x) = \ll \phi, \mathcal{E}(x) : \frac{d\mu^{(\sqrt{z})}}{d\mu} \gg \tag{2.25}$$

If $p > \frac{1}{4}$ is such that $|T_r|_{2,-p} < \frac{1}{|1-z^2|}$, then P_z can be extanded to a continuous mapping from $\mathcal{C}(\mathcal{S})_{p+\frac{1}{2}}$ to $\mathcal{C}(\mathcal{S})_p$.

Proof. By (1.5) we can prove the existence of $P_z\phi$ verifying (2.24). The group property of (P_z) follows from the following trivial fact:

$$\frac{d\mu^{(\sqrt{1+z_1})}}{d\mu} : \frac{d\mu^{(\sqrt{1+z_2})}}{d\mu} = \frac{d\mu^{(\sqrt{1+z_1+z_2})}}{d\mu} \tag{2.26}$$

By (2.24) and (1.17) we have

$$\begin{aligned}
\widetilde{P_z\phi}(x) &= \ll P_z\phi, \delta_x \gg = \ll \phi, \delta_x : \frac{d\mu^{(\sqrt{1+z})}}{d\mu} \gg \\
&= \ll \phi, \mathcal{E}(x) : \delta_0 : \frac{d\mu^{(\sqrt{1+z})}}{d\mu} \gg \\
&= \ll \phi, \mathcal{E}(x) : \frac{d\mu^{(\sqrt{z})}}{d\mu} \gg
\end{aligned}$$

(2.25) is proved. The last conclusion of the theorem is obvious.

Remark. If $\phi \in (\mathcal{S})$, we can prove that $\widetilde{P_z\phi}(x) = \int_{\mathcal{S}'(\mathbb{R})} \tilde{\phi}(x + \sqrt{z}y)\mu(dy)$. But for a general $\phi \in (\mathcal{S})_p$ the integral may not exist.

3. Renormalization in White Noise Analysis

Let $x \in \mathcal{CS}'(\mathbb{R})$. The Wick-transform $: x^{\otimes n} :$ of the tensor product $x^{\otimes n}$ is given by

$$: x^{\otimes n} := \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n!}{k!(n-2k)!2^k} x^{\otimes n-2k} \hat{\otimes} T_r^{\otimes k} \tag{3.1}$$

where $\hat{\otimes}$ stands for the symmetric tensor product. We have

$$x^{\otimes n} = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!2^k} : x^{\otimes n-2k} : \hat{\otimes} T_r^{\otimes k} \tag{3.2}$$

It is shown in Yan [8] that we have also the following formulas

$$: x^{\otimes n} := \int_{S'(\mathbb{R})} (x+iy)^{\otimes n} \mu(dy) \tag{3.3}$$

$$x^{\otimes n} = \int_{S'(\mathbb{R})} : (x+y)^{\otimes n} : \mu(dy) \tag{3.4}$$

If $p > \frac{1}{2}$ and $\phi \in (S)_p$ with $\phi \sim (F^{(n)})$, then we have

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} <: z^{\otimes n} :, F^{(n)} >, \ z \in CS_{-p}(\mathbb{R}) \tag{3.5}$$

where the series is convergent absolutely and uniformly on bounded subsets of $S_{-p}(\mathbb{R})$ (see Lee [4] and Yan [8]).

Let $\lambda \in \mathbb{R}$. The following formula was established in Potthoff-Yan [6]

$$: (\lambda x)^{\otimes n} := \lambda^n \sum_{k=0}^{[\frac{n}{2}]} (1-\lambda^{-2})^k \frac{n!}{k!(n-2k)!2^k} : x^{\otimes n-2k} : \hat{\otimes} T_r^{\otimes k} \tag{3.6}$$

Thus, by (3.6) we obtain

$$: (\sqrt{2}x)^{\otimes n} := \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!2^k} (\sqrt{2})^{n-2k} : x^{\otimes n-2k} : \hat{\otimes} T_r^{\otimes k} \tag{3.7}$$

Let $f \in \hat{S}(\mathbb{R}^n)$. Put

$$\phi(x) = <: x^{\otimes n} :, f >, \ \psi(x) = < x^{\otimes n}, f >$$

Then by (3.7) and (3.2) we have

$$\phi^{(\sqrt{2})} = \psi_{(\sqrt{2})}$$

or equivalently,

$$\phi = (\psi_{(\sqrt{2})})^{(\frac{1}{\sqrt{2}})} = R_{\sqrt{2}}\psi$$

Thus, we can call $R_{\sqrt{2}}$ the renormalization operator, because it transforms a Stratonovich multiple integral into a Wiener multiple integral. In the sequel we denote simply by R (resp. R^{-1}) the operator $R_{\sqrt{2}}$ (resp. $R_{\frac{1}{\sqrt{2}}}$).

As a particular case of Theorem 2.1 we have the following result.

Theorem 3.1 Let $p > \frac{1}{4}$ be such tht $|T_r|_{2,-p} < 1$. R and R^{-1} can be extended to a continuous mapping from $(S)_{p+\frac{1}{2}}$ to $(S)_p$ and we have

$$||R\phi||_{2,p} \leq C(p,\sqrt{2})||\phi||_{2,p+\frac{1}{2}},\ ||R^{-1}\phi||_{2,p} \leq C(p,\sqrt{2})||\phi||_{2,p+\frac{1}{2}} \tag{3.8}$$

$$\ll R\phi, F \gg = \ll \phi, F : \delta_0 \gg \tag{3.9}$$

$$\ll R^{-1}\phi, F \gg = \ll \phi, F : \frac{d\mu^{(\sqrt{2})}}{d\mu} \gg \tag{3.10}$$

where $\phi \in (S)_{p+\frac{1}{2}}$ and $F \in (S)_{-p}$.

Corollary. Let $p > \frac{1}{2}$ and $\phi \in (S)_{p+\frac{1}{2}}$. We have

$$\tilde{\phi}(x) = \ll R\phi, \mathcal{E}(x) \gg,\ x \in S_{-p}(\mathbb{R}) \tag{3.11}$$

In particular, the restriction of $\tilde{\phi}$ to $S(\mathbb{R})$ is the S-transform of $R\phi$.

The following theorem gives us integral representations of $R\phi$ and $R^{-1}\phi$.

Theorem 3.2 Let $p > \frac{1}{2}$ and $\phi \in (S)_{p+\frac{1}{2}}$. We have

$$\widetilde{R\phi}(z) = \int_{S'(\mathbb{R})} \tilde{\phi}(z+iy)\mu(dy),\ z \in \mathcal{C}S_{-p}(\mathbb{R}) \tag{3.12}$$

$$\widetilde{R^{-1}\phi}(z) = \int_{S'(\mathbb{R})} \tilde{\phi}(z+y)\mu(dy),\ z \in \mathcal{C}S_{-p}(\mathbb{R}) \tag{3.13}$$

Proof. Assume $\phi \sim (F^{(n)})$ and $R\phi \sim (G^{(n)})$. By (1.17) we have

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} <: z^{\otimes n} :, F^{(n)} >,\ z \in \mathcal{C}S_{-p}(\mathbb{R}) \tag{3.14}$$

$$\widetilde{R\phi}(z) = \sum_{n=0}^{\infty} <: z^{\otimes n} :, G^{(n)} >,\ z \in \mathcal{C}S_{-p}(\mathbb{R}) \tag{3.15}$$

On the other hand, for $z \in \mathcal{C}S_{-p}(\mathbb{R})$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} |< z^{\otimes n}, G^{(n)} >| &\leq \sum_{n=0}^{\infty} |z|_{2,-p}^{n} |G^{(n)}|_{2,p} \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} |z|_{2,-p}^{n} (\sqrt{n!}|G^{(n)}|_{2,p}) \\ &\leq ||R\phi||_{2,p} \exp \frac{1}{2}|z|_{2,-p}^{2}\end{aligned}$$

Thus, if we put

$$F(z) = \sum_{n=0}^{\infty} < z^{\otimes n}, G^{(n)} >, \ z \in CS_{-p}(I\!R) \tag{3.16}$$

then F is analytic on $CS_{-p}(I\!R)$ and by (3.16) and (3.11) we have

$$F(\xi) = SR\phi(\xi) = \tilde{\phi}(\xi), \ \xi \in S(I\!R)$$

from which it follows

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} < z^{\otimes n}, G^{(n)} >, \ z \in CS_{-p}(I\!R) \tag{3.17}$$

Now by (3.15), (3.17) and (3.3) we get (3.12). Similary, we can prove that

$$\widetilde{R^{-1}}\phi(z) = \sum_{n=0}^{\infty} < z^{\otimes n}, F^{(n)} >, \ z \in CS_{-p}(I\!R) \tag{3.18}$$

Therefore, we can get (3.13) from (3.17), (3.18) and (3.4).

Remark. Let $p > \frac{1}{2}$ and $\phi \in (S)_{p+\frac{1}{2}}$. Assume that $\phi \sim (F^{(n)})$ and $R\phi \sim (G^{(n)})$. By (3.17) ϕ has the following " Stratonovich" decomposition

$$\phi(x) = \sum_{n=0}^{\infty} < x^{\otimes n}, G^{(n)} >, \ x \in S_{-p}(I\!R).$$

Renormalizing ϕ consists in transforming chaos by chaos Stratonovich multiple integrals into Wiener multiple integrals. We obtain the Ito-Wiener decomposition of $R\phi$:

$$R\phi(x) = \sum_{n=0}^{\infty} <: x^{\otimes n} :, G^{(n)} >, \ x \in S_{-p}(I\!R).$$

The following theorem improves Theorem 3.1.

Theorem 3.3 Let p_0 be teh number such that $|T_r|_{2,-p_0} = 1$. Let $p > p_0$ and $\beta > 0$ be such that $2^{-2\beta} + 2^{-2(p-p_0)} < 1$. The operators R and R^{-1} can be extended to continuous mappings from $(S)_p$ to $(S)_{p-\beta}$. Moreover, for $\phi \in (S)_p$ and $F \in (S)_{-p+\beta}$ we have

$$\ll R\phi, F \gg = \ll \phi, F : \delta_0 \gg, \ \ll R^{-1}\phi, F \gg = \ll \phi, F : \frac{d\mu^{(\sqrt{2})}}{d\mu} \gg \tag{3.19}$$

Proof. Let $\alpha > 0$, be such that $2^{-2\beta} + 2^{-2\alpha} = 1$. Then $p - \alpha > p_0$, so we have $c_\alpha = ||\delta_0||_{2,-p+\alpha} = ||\frac{d\mu^{(\sqrt{2})}}{d\mu}||_{2,-p+\alpha} < \infty$ (see Yan [7]). Let $F, G \in (S)^*$. By Yan [7] we have

$$||F : G||_{2,-p} \leq ||F||_{2,-p+\beta}||G||_{2,-p+\alpha}, \tag{3.20}$$

Let $\phi \in (S)$. By (3.20), (3.9) and (3.10) we obtain

$$| \ll R\phi, F \gg | \leq ||\phi||_{2,p}||F||_{2,-p+\beta}||\delta_0||_{2,-p+\alpha}$$
$$| \ll R^{-1}\phi, F \gg | \leq ||\phi|_{2,p}||F||_{2,-p+\beta}||\frac{d\mu^{(\sqrt{2})}}{d\mu}||_{2,-p+\alpha}.$$

Thus we conclude the theorem and we have

$$||R\phi||_{2,p-\beta} \leq c_\alpha||\phi||_{2,p}, \; ||R^{-1}\phi||_{2,p-\beta} \leq c_\alpha||\phi||_{2,p}.$$

Remark. Let p_0 be as above and $\phi \in (S)_p$, where $p > p_0$. Since for each $\xi \in S(\mathbb{R})$ we have $\delta_\xi = \varepsilon(\xi) : \delta_0 \in (S)_{-p}$(by (3.19)), we can put

$$\tilde{\phi}(\xi) = \ll \varphi, \delta_\xi \gg, \; \xi \in S(\mathbb{R}).$$

$\tilde{\phi}$ is a continuous function on $S(\mathbb{R})$. We call $\tilde{\phi}$ the restriction of ϕ on $S(\mathbb{R})$. By (3.9), we have

$$\tilde{\phi}(\xi) = S(R\varphi)(\xi), \; \xi \in S(\mathbb{R})$$

Thus, ϕ is completely determined by its restriction $\tilde{\phi}$.

Recall that if a Hida distribution ϕ corresponds to a sequence $(F^{(n)})$, we can write formally

$$\phi = \sum_{n=0}^{\infty} <: x^{\otimes n} :, F^{(n)} > .$$

Suggested by the above remark, we propose the following general definition of the renormalization.

Definition 3.1 Let $\phi \in (S)^*$ with $\phi \sim (F^{(n)})$. If ψ is a formally defined functional on $S'(\mathbb{R})$ and if ψ admets the following formal expansion:

$$\psi(x) = \sum_{n=0}^{\infty} < x^{\otimes n}, F^{(n)} >$$

then we say that ψ is renormalizable and ϕ is its renormalization. We denote ϕ also by $R\psi$.

We give below some examples.

Example 1. Let $\psi(x) = \exp < x, y >$, where $y \in \mathcal{S}'(I\!R)$. We have formally

$$\psi = \sum_{n=0}^{\infty} \frac{< x, y >^n}{n!} = \sum_{n=0}^{\infty} < x^{\otimes n}, \frac{y^{\otimes n}}{n!} > .$$

Therefore, we get

$$R\psi = \sum_{n=0}^{\infty} <: x^{\otimes n} : \frac{y^{\otimes n}}{n!} >= \mathcal{E}(y).$$

Example 2. Let $\psi(x) = \exp c \int_{-\infty}^{\infty} x(s)^2 ds$, where $c \neq 0$ is a constant. Then we have

$$\begin{aligned}\psi(x) = \exp\{c < x^{\otimes 2}, T_r >\} &= \sum_{n=0}^{\infty} \frac{c^n < x^{\otimes 2}, T_r >^n}{n!} \\ = \sum_{n=0}^{\infty} \frac{< x^{\otimes 2n}, c^n T_r^{\otimes n} >}{n!}\end{aligned}$$

Therefore, we obtain

$$R\psi = \sum_{n=0}^{\infty} <: x^{\otimes 2n} :, \frac{c^n T_r^{\otimes n}}{n!} > .$$

If $c > 0$, then

$$R\psi = \sum_{n=0}^{\infty} <: x^{\otimes 2n} :, \frac{(\sqrt{2c})^{2n} T_r^{\otimes n}}{2^n n!} >= \Gamma(\sqrt{2c}) \frac{d\mu^{(\sqrt{2})}}{d\mu}$$

If $c < 0$, then

$$R\psi = \sum_{n=0}^{\infty} <: x^{\otimes 2n} :, \frac{(\sqrt{-2c})^{2n} (-1)^n T_r^{\otimes n}}{2^n n!} >= \Gamma(\sqrt{-2c}) \delta_0$$

In each case, we have

$$S(R\psi)(\xi) = \exp c |\xi|_2^2, \;\; \xi \in \mathcal{S}(I\!R).$$

Example 3 Let $\psi(x) = \exp c \int_0^t x(s) ds$. Then we have

$$\psi(x) = \sum_{n=0}^{\infty} \frac{c^n}{n!} < x, I_{[o,t]} >^n = \sum_{n=0}^{\infty} < x^{\otimes n}, \frac{c^n I_{[0,t]}^{\otimes n}}{n!} >$$

Thus we get

$$R\psi = \sum_{n=0}^{\infty} <: x^{\otimes n} :, \frac{c^n I_{[0,t]}^{\otimes n}}{n!} >$$

whose S-transform is

$$S(R\psi)(\xi) = \sum_{n=0}^{\infty} < \xi^{\otimes n}, c^n I_{[0,t]}^{\otimes n} >= \exp c \int_0^t \xi(s) ds.$$

Finally, we leave the reader to verify the following identities:

$$R(\phi\psi) = R\phi : R\psi, \ R\phi^{(\lambda)} = (R\phi)_{(\lambda)}.$$

where ϕ and ψ are supposed to be renormalizable.

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