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ON THE LÉVY TRANSFORMATION

OF BROWNIAN MOTIONS AND CONTINUOUS MARTINGALES

L. E. Dubins¹, M. Émery², M. Yor³

[...] je vous confie aujourd'hui mes
espérances, qui ne reposent encore
que sur des calculs de probabilité.

É. Zola

Introduction

If $(B_t)_{t \geq 0}$ is a Brownian motion started at 0 and $(L_t)_{t \geq 0}$ its local time at 0, Lévy's characterization (see for instance [6] p. 141) implies that

$$\widehat{B} = |B| - L = \int \operatorname{sgn}(B) dB$$

is also a Brownian motion. In other words, the *Lévy transformation* $T : B \rightarrow \widehat{B}$, defined almost everywhere on the Wiener space $W = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, preserves the Wiener measure μ .

Dubins & Smorodinsky [3] have established that an analogue of T for coin-tossing is ergodic. This increases the plausibility of the following conjecture:

(\mathcal{L}) *The Lévy transformation is ergodic,*

that is, the σ -field \mathcal{J} on W of all events a.s. invariant by T is trivial.

Known since the late 70's, the problem of the ergodicity of T is mentioned as an open question in Revuz & Yor [6], page 257.

We shall see that (\mathcal{L}) is closely related to the question of knowing which continuous martingales $M = (M_t)_{t \geq 0}$ with $M_0 = 0$ have the same law as their Lévy transform $\widehat{M} = \int \operatorname{sgn}(M) dM$. (A discussion of this subject is begun in Exercise (2.32) page 231 of Revuz & Yor [6].) Recall that to each continuous martingale M is associated its quadratic variation $\langle M \rangle$; $\langle M \rangle$ is the continuous, non-decreasing, adapted process such that $\langle M \rangle_t = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} (M_{k2^{-n}t} - M_{(k-1)2^{-n}t})^2$. For simplicity, we shall deal only with continuous martingales verifying $M_0 = 0$ and $\langle M \rangle_\infty = \infty$; such processes will be called *divergent martingales*. As is well known

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(see [1] and [2]), each divergent martingale M is obtained by time-changing a Brownian motion, the time-change being given by $\langle M \rangle$. More precisely, to M is associated a (unique) Brownian motion β^M such that $M_t = (\beta^M)_{\langle M \rangle_t}$ for all t ; this defines a map $\beta_M : \Omega \rightarrow W$ transforming \mathbb{P} into the Wiener measure μ . The law of M is characterized by the joint law of β^M and $\langle M \rangle$.

Changing time in the integral $\int \text{sgn}(M) dM$ gives $\beta^{\widehat{M}} = \widehat{\beta^M}$; as $\langle \widehat{M} \rangle = \langle M \rangle$, \widehat{M} has the same law as M if and only if

$$(\beta^M)_{\langle M \rangle} \stackrel{(\text{law})}{=} T(\beta^M)_{\langle M \rangle}.$$

A sufficient condition is the independence of β^M and $\langle M \rangle$. We conjecture that this sufficient condition is also necessary.

Let us give a name to this conjecture:

(\mathcal{M}) *A divergent martingale M has the same law as its Lévy transform \widehat{M} (if and) only if the processes β^M and $\langle M \rangle$ are independent.*

A reason to believe in this conjecture is Ocone's Theorem A of [5]: The independence of β^M and $\langle M \rangle$ is a necessary and sufficient condition for M to have the same law as all integrals $\int H dM$, where H ranges either over all deterministic processes of the form $H = \mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, or over all $\{-1,1\}$ -valued processes that are predictable for the natural filtration of M .

The next section gives some preliminary observations about (\mathcal{L}). Then comes our main result, the equivalence of (\mathcal{L}) and (\mathcal{M}), established, with some further precisions, in the third section. In the last section, we try to understand (\mathcal{L}) better, in particular by constructing examples of martingales which are not identical in law with their Lévy transform. The appendix borrows from [5] Ocone's theorem and its proof, with a few remarks.

Preliminary remarks

The following lemma from ergodic theory is well known.

LEMMA 1. — *Let (W, \mathcal{G}, μ) be a probability space and T a measurable transformation of W which preserves μ . A random variable $Z \in L^2(W, \mu)$ is a. s. invariant by T if and only if*

$$\langle Y \circ T, Z \rangle_{L^2} = \langle Y, Z \rangle_{L^2}$$

for all $Y \in L^2(W, \mu)$.

PROOF. — If Z is invariant, $Z = Z \circ T$ a. s. and $\langle Y \circ T, Z \rangle = \langle Y \circ T, Z \circ T \rangle = \langle Y, Z \rangle$ by the invariance of μ .

Conversely, if $\langle Y \circ T, Z \rangle = \langle Y, Z \rangle$ for every Y , $\langle Y \circ T, Z \rangle = \langle Y \circ T, Z \circ T \rangle$ and $Z \circ T$ is the conditional expectation of Z given $T^{-1}\mathcal{G}$; as Z and its projection $Z \circ T$ on $L^2(T^{-1}\mathcal{G})$ have the same L^2 -norm (invariance of μ), they must be equal. ■

LEMMA 2. — *With the notations of the introduction, let $S : W \rightarrow [0, \infty]$ be a stopping time for B . The stopped processes B^S and \widehat{B}^S have the same law if and only if S is a. s. invariant.*

PROOF. — If S is invariant, $\widehat{B}^S = (B \circ T)^{S \circ T} = (B^S) \circ T$ has the same law as B^S .

Conversely, if B^S and \widehat{B}^S have the same law, the pairs (B^S, S) and (\widehat{B}^S, S) also have the same law, because S is a function of the path of B^S (for instance $S = \sup \{t \in \mathbb{Q} : B_t^S \neq B_\infty^S\}$). Now the Markov property at time S makes it possible to deduce the law of (B, S) from that of (B^S, S) and similarly for \widehat{B} ; hence (B, S) and (\widehat{B}, S) have the same law. This gives $\langle Y, e^{-S} \rangle = \langle Y \circ T, e^{-S} \rangle$ for every $Y \in L^2(W)$ and S is invariant by Lemma 1. ■

REMARK. — The Markov property in the above proof cannot be dispensed of: if the random variable S is not a stopping time, it may happen that B^S and \widehat{B}^S have the same law but S is not invariant. Take for instance any $[0, 1]$ -valued random variable S independent of \mathcal{F}_1 ; B^S and \widehat{B}^S have the same law, namely that of a Brownian motion stopped at some independent time distributed as S .

Equivalence of (\mathcal{L}) and (\mathcal{M})

THEOREM 1. — *Let M be a divergent martingale. The following three properties are equivalent:*

- (i) M and \widehat{M} have the same law;
- (ii) $(\beta^M, \langle M \rangle)$ and $(T(\beta^M), \langle M \rangle)$ have the same law;
- (iii) β^M and $\langle M \rangle$ are conditionally independent given the σ -field $\beta_M^{-1}(\mathcal{J})$.

Examples of this situation are obtained by taking $\langle M \rangle$ independent of β^M ; if (\mathcal{L}) is true, $\beta_M^{-1}(\mathcal{J})$ is trivial and there are no other examples.

PROOF. — Since $\langle M \rangle = \langle \widehat{M} \rangle$ and the law of $(\beta^M, \langle M \rangle)$ depends only on that of M , (i) implies (ii). Conversely, (ii) \Rightarrow (i) follows from $M = (\beta^M)_{\langle M \rangle}$ and $\widehat{M} = T(\beta^M)_{\langle M \rangle}$.

(ii) \Rightarrow (iii). Let F be a bounded random variable measurable with respect to $\sigma\{\langle M \rangle_t, t \geq 0\}$; there exists a bounded measurable function f on W such that $\mathbb{E}[F|\beta^M] = f(\beta^M)$. For every $g \in L^2(W)$, using the definition of f , hypothesis (ii) and again the definition of f , one can write

$$\mathbb{E}[f(\beta^M)g(\beta^M)] = \mathbb{E}[Fg(\beta^M)] = \mathbb{E}[Fg \circ T(\beta^M)] = \mathbb{E}[f(\beta^M)g \circ T(\beta^M)]$$

and Lemma 1 gives $f(\beta^M) = f \circ T(\beta^M)$ a.s. So f is \mathcal{J} -measurable and coming back to the definition of f one gets $\mathbb{E}[F|\beta^M] = \mathbb{E}[F|\beta_M^{-1}(\mathcal{J})]$, the desired result.

(iii) \Rightarrow (ii). Keep the same notations and call \mathcal{J}' the σ -field $\beta_M^{-1}(\mathcal{J})$. Hypothesis (iii) gives on the one hand

$$\mathbb{E}[Fg(\beta^M)] = \mathbb{E}[\mathbb{E}[F|\mathcal{J}'] \mathbb{E}[g(\beta^M)|\mathcal{J}']]$$

and on the other hand

$$\mathbb{E}[Fg \circ T(\beta^M)] = \mathbb{E}[\mathbb{E}[F|\mathcal{J}'] \mathbb{E}[g \circ T(\beta^M)|\mathcal{J}']] .$$

Since every \mathcal{J}' -measurable random variable has the form $h(\beta^M)$ where h is \mathcal{J} -measurable, $\mathbb{E}[g(\beta^M)|\mathcal{J}'] = \mathbb{E}[g \circ T(\beta^M)|\mathcal{J}']$ and we obtain

$$\mathbb{E}[Fg(\beta^M)] = \mathbb{E}[Fg \circ T(\beta^M)] ,$$

which means precisely that (ii) holds. ■

Recall that a martingale M is called *pure* if it is divergent and if for each $t \geq 0$ its quadratic variation $\langle M \rangle_t$ is measurable for the σ -field $\sigma\{\beta_s^M, s \geq 0\}$.

A weaker form of Conjecture (\mathcal{M}) is obtained by restricting to pure martingales the demand that $M \stackrel{\text{law}}{=} \widehat{M}$ if and only if β^M and $\langle M \rangle$ are independent; since $\langle M \rangle$ is measurable with respect to β^M , we get the statement:

(\mathcal{M}') *A pure martingale M has the same law as its Lévy transform \widehat{M} (if and only if $\langle M \rangle$ is deterministic.*

As will be shown below, (\mathcal{M}') is in fact not weaker than but equivalent to (\mathcal{M}) .

Similarly, a weaker form of (\mathcal{L}) is obtained by restricting to stopping times the statement that all random variables on W invariant by T are constant:

(\mathcal{L}') *On the canonical space W , every stopping time invariant by T is constant.*

THEOREM 2. — *The four conjectures (\mathcal{L}) , (\mathcal{L}') , (\mathcal{M}) and (\mathcal{M}') are equivalent.*

PROOF. — If (\mathcal{L}) is true, \mathcal{J} is trivial and (i) \Rightarrow (iii) in Theorem 1 gives (\mathcal{M}) . In turn, (\mathcal{M}) trivially implies (\mathcal{M}') . The theorem will be proved by showing that

$$(\mathcal{L}) \text{ is false } \implies (\mathcal{L}') \text{ is false } \implies (\mathcal{M}') \text{ is false.}$$

Assume (\mathcal{L}) is false. On W endowed with Wiener measure, there exists a non-trivial invariant bounded random variable F . Call B the canonical Brownian motion on W , \widehat{B} its Lévy transform, $(\mathcal{F}_t)_{t \geq 0}$ and $(\widehat{\mathcal{F}}_t)_{t \geq 0}$ their natural filtrations. Since F is a functional of \widehat{B} , the $(\widehat{\mathcal{F}}_t)$ -martingale $M_t = \mathbb{E}[F|\widehat{\mathcal{F}}_t]$ has the form $\mathbb{E}[F] + \int_0^t \widehat{H}_s d\widehat{B}_s$ with \widehat{H} predictable in the filtration $(\widehat{\mathcal{F}}_t)_{t \geq 0}$; so M is also a martingale for $(\mathcal{F}_t)_{t \geq 0}$ and

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F|\widehat{\mathcal{F}}_t] = \mathbb{E}[F \circ T|\widehat{\mathcal{F}}_t] = \mathbb{E}[F|\mathcal{F}_t] \circ T :$$

the process M is invariant by T .

Using M , it is easy to construct a finite, non-constant stopping time S invariant by T , for instance $S = t + \mathbb{1}_\Gamma(M_t)$ where t is large enough for M_t to be non-constant and Γ is a suitable Borel set. So (\mathcal{L}') is false too.

If (\mathcal{L}') is false, let S be a finite, non-constant, invariant stopping time on Wiener space. For $\alpha > 0$, the increasing process

$$A_t = \int_0^t [\mathbb{1}_{[0,S]}(s) + \alpha \mathbb{1}_{]S,\infty[}(s)] ds$$

is not deterministic if $\alpha \neq 1$; it is also invariant, and $(B, A) \stackrel{(\text{law})}{=} (\widehat{B}, A)$; the inverse of A , obtained by replacing α with α^{-1} , is adapted and each A_t is a stopping time. Consequently, $M_t = B_{A_t}$ is a martingale (for the filtration $\mathcal{G}_t = \mathcal{F}_{A_t}$), satisfying condition (ii) of theorem 1, hence $M \stackrel{(\text{law})}{=} \widehat{M}$. As $\langle M \rangle = A$ is measurable with respect to $\beta^M = B$ but not deterministic, M is a counterexample to (\mathcal{M}') and (\mathcal{M}') does not hold. ■

Some Remarks

a) It can be observed that the stopping time S constructed in the first part of the proof takes only two values. This leads to another variant of (\mathcal{L}) , namely, there are no invariant stopping times taking exactly two values. Of course, this just means that each invariant event belonging to some \mathcal{F}_t is trivial.

b) (\mathcal{L}') amounts to stating that every non constant Brownian stopping time S is not invariant. According to Lemma 2, this means that the stopped processes B^S and $\widehat{B}^S = |B^S| - L^S$ do not have the same law. A sufficient condition is that the random variables B_S and $\widehat{B}_S = |B_S| - L_S$ have different laws. Many stopping times have this property, for instance the first hitting time of a given level by B , or by $|B|$ or by $L \dots$

However, there also exist many stopping times (for the filtration of B) such that B_S and \widehat{B}_S are not only identical in law, but a.s. equal; for instance $\inf \{t \geq 1 : B_t = \widehat{B}_t\}$. This stopping time is a.s. finite since the martingale $B - \widehat{B}$ is divergent (for its bracket is $4 \int \mathbb{1}_{\{B_t \leq 0\}} dt$).

But a finite stopping time S such that $B_S = \widehat{B}_S$ cannot be invariant unless it vanishes identically. For in that case $B_S = \widehat{B}_S = \widehat{B}_{S \circ T} = B_S \circ T$ is a function of \widehat{B} only; and, since B and $-B$ have the same conditional law given \widehat{B} , B_S must be its own opposite and $B_S = 0$, giving $L_S = |B_S| - \widehat{B}_S = 0$ and $S = 0$. (The same argument shows more generally that a finite, non-negative random variable S measurable for \widehat{B} and verifying $B_S = \phi(\widehat{B}_S)$ must vanish if ϕ is a function such that $\phi(x) \neq 0$ for every $x < 0$.)

More precisely, it can be proved that if S is an invariant, finite, positive stopping time, $\mathbb{P}[B_S = \widehat{B}_S] \leq 1/2$. Let indeed A denote the event $\{B_S = \widehat{B}_S\}$ and suppose $\mathbb{P}[A] > 1/2$. Since $\widehat{B}_S = |B_S| - L_S$ and $L_S > 0$, on A one has $\widehat{B}_S = -B_S - L_S$ and $2B_S = -L_S$; this can be rewritten $2\widehat{B}_S = \inf_{t \leq S} \widehat{B}_t$. But $\mathbb{P}[T^{-1}A]$ is also larger than one half, so the event $A \cap T^{-1}A$ is not negligible. On this event, one has on the one hand $2\widehat{B}_S = \widehat{I}_S$ and on the other hand $2\widehat{B}_S = -\widehat{L}_S$

(where $\widehat{I}_t = \inf_{s \leq t} \widehat{B}_s$ and \widehat{L} is the local time at zero of \widehat{B}). To establish the claim, we shall prove that, if \widehat{B} is a Brownian motion started at 0, the event $\{\exists t > 0 : 2\widehat{B}_t = \widehat{I}_t = -\widehat{L}_t\}$ is negligible; replacing \widehat{B} with $-\widehat{B}$, dropping the hats for typographical simplicity and letting $S_t = \sup_{s \leq t} B_s$, this amounts to showing that

$$\mathbb{P}[\exists t > 0 : 2B_t = S_t = L_t] = 0.$$

Since both processes S and L are locally constant in the random open set $\{t : B_t \neq S_t \text{ and } B_t \neq 0\}$, if equality $S = L$ holds at some time $t > 0$ such that $B_t = \frac{1}{2}S_t$, it also holds identically in some neighborhood of t and hence at some rational t ; so we just have to show that $\mathbb{P}[S_t = L_t] = 0$ for each $t > 0$.

From the scaling property of Brownian motion, this probability does not depend on t ; hence it is also equal to $\mathbb{P}[S_T = L_T]$ where T is an exponential random variable independent of B . Now, the joint law of (S_T, L_T) is easily computed from excursion theory arguments; in particular it is absolutely continuous with respect to Lebesgue measure on \mathbb{R}_+^2 , with identical exponential marginals. This proves the claim.

c) Still working in the σ -field generated by B , notice that (\mathcal{L}) is true if and only if, for H ranging over a total subset of L^2 , $\mathbb{E}[H|\mathcal{J}] = \mathbb{E}[H]$, or equivalently by the ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H \circ T^k = \mathbb{E}[H].$$

We shall use the subset consisting of the constant 1 (for which the above equality is trivial) and of all multiple Wiener integrals

$$H = \int_0^\infty dB_{u_1} \int_0^{u_1} dB_{u_2} \dots \int_0^{u_{p-1}} dB_{u_p} f(u_1, \dots, u_p)$$

where $p \geq 1$ and f satisfies

$$\int_0^\infty du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{p-1}} du_p f^2(u_1, \dots, u_p) < \infty.$$

This programme can be carried out successfully in the case $p = 1$. Indeed, writing $B^{(k)} = T^k(B)$ and $\varepsilon_t^{(k)} = \text{sgn } B_t^{(k)}$, Tanaka's formula gives

$$B_t^{(k)} = \int_0^t dB_s^{(k-1)} \varepsilon_s^{(k-1)} = \int_0^t dB_s^{(k-2)} \varepsilon_s^{(k-2)} \varepsilon_s^{(k-1)} = \int_0^t dB_s \varepsilon_s \varepsilon_s^{(1)} \dots \varepsilon_s^{(k-1)}.$$

Now, if $H = \int_0^\infty dB_u f(u)$, where $f \in L^2$, we have

$$\frac{1}{n} \sum_{k=1}^n H \circ T^k = \int_0^\infty dB_u f(u) \left(\frac{1}{n} \sum_{k=1}^n \varepsilon_u \varepsilon_u^{(1)} \dots \varepsilon_u^{(k-1)} \right)$$

so that

$$\mathbb{E} \left[\left(\frac{1}{n} \sum_{k=1}^n H \circ T^k \right)^2 \right] = \int_0^\infty du f^2(u) \mathbb{E} \left[\left(\frac{1}{n} \sum_{k=1}^n \varepsilon_u \varepsilon_u^{(1)} \dots \varepsilon_u^{(k-1)} \right)^2 \right].$$

For fixed u , $\varepsilon_u^{(\ell)}$ is independent of $B^{(\ell+1)}$ and hence also of all the $\varepsilon_u^{(m)}$ for $m > \ell$; so the sequence $\varepsilon_u, \varepsilon_u^{(1)}, \varepsilon_u^{(2)}, \dots$ of Bernoulli variables is independent, whence

$$\mathbb{E} \left[\left(\frac{1}{n} \sum_{k=1}^n \varepsilon_u \varepsilon_u^{(1)} \dots \varepsilon_u^{(k-1)} \right)^2 \right] = \frac{1}{n},$$

and $\frac{1}{n} \sum_{k=1}^n H \circ T^k$ does tend to zero.

For random variables H belonging to chaoses of higher order, the same method, and the well-known isometry between the p^{th} chaos and the space of square-integrable symmetric functions of p variables, reduce (\mathcal{L}) to an equivalent property.

PROPOSITION. — (\mathcal{L}) is true if and only if for each $p > 1$

$$\lim_{n \rightarrow \infty} \int_0^1 du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{p-1}} du_p \mathbb{E} \left[\left(\frac{1}{n} \sum_{k=1}^n \prod_{\substack{0 \leq \ell < k \\ 1 \leq m \leq p}} \varepsilon_{u_m}^{(\ell)} \right)^2 \right] = 0.$$

Hence, the first step in that direction would be to take $p = 2$ and to get a good estimate of

$$\mathbb{E} \left[\left(\frac{1}{n} \sum_{k=1}^n \varepsilon_u \varepsilon_v \varepsilon_u^{(1)} \varepsilon_v^{(1)} \dots \varepsilon_u^{(k-1)} \varepsilon_v^{(k-1)} \right)^2 \right].$$

APPENDIX: Ocone’s Theorem

Ocone’s Theorem A of [5] consists of the equivalence between (ii), (iii) and (iv) below. His setting is more general than the following rephrasing: he deals with local martingales (and further extends his results to the càdlàg case).

THEOREM. — Let M be a continuous, divergent martingale with natural filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. The following five statements are equivalent:

- (i) the processes β^M and $\langle M \rangle$ are independent;
- (ii) for every \mathcal{F} -predictable process H taking values in $\{-1, 1\}$, the pairs of processes $(\int H dM, \langle M \rangle)$ and $(M, \langle M \rangle)$ have the same law (in particular the martingales $\int H dM$ and M have the same law);
- (iii) for every deterministic function h of the form $\mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, the martingale $\int h dM$ has the same law as M ;
- (iv) for every \mathcal{F} -predictable process H measurable for the product σ -field $\mathcal{B}(\mathbb{R}_+) \otimes \sigma(\langle M \rangle)$ and such that $\int_0^\infty H_s^2 d\langle M \rangle_s < \infty$ a. s.,

$$\mathbb{E} \left[\exp \left(i \int_0^\infty H_s dM_s \right) \mid \langle M \rangle \right] = \exp \left(-\frac{1}{2} \int_0^\infty H_s^2 d\langle M \rangle_s \right);$$

- (v) for every deterministic function h of the form $\sum_{j=1}^n \lambda_j \mathbb{1}_{[0,a_j]}$,

$$\mathbb{E} \left[\exp \left(i \int_0^\infty h(s) dM_s \right) \right] = \mathbb{E} \left[\exp \left(-\frac{1}{2} \int_0^\infty h^2(s) d\langle M \rangle_s \right) \right].$$

Notice that (iii) is a symmetry assumption: given the past of M , the conditional law of the future increments is symmetric; (iv) says that, conditionally given $\langle M \rangle$, M is a Gaussian martingale with variance $\langle M \rangle$.

LEMMA 1. — *If a right-continuous process X is a martingale for some (non necessarily right-continuous) filtration $(\mathcal{G}_t)_{t \geq 0}$, it is also a martingale for its right-continuous enlargement $(\mathcal{G}_{t+})_{t \geq 0}$.*

PROOF. — When $\varepsilon > 0$ tends to 0, $X_{s+\varepsilon}$ tends to X_s in L^1 by uniform integrability; so, for $s < t$, $\mathbb{E}[X_t - X_s | \mathcal{G}_{s+}] = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \mathbb{E}[X_t - X_{s+\varepsilon} | \mathcal{G}_{s+}] = 0$. ■

LEMMA 2. — *Let β be a Brownian motion with natural filtration \mathcal{B} and \mathcal{G} a σ -field independent of β . If a process H taking values in $\{-1, 1\}$ is predictable for the filtration \mathcal{E} defined by $\mathcal{E}_t = \bigcap_{\varepsilon > 0} (\mathcal{B}_{t+\varepsilon} \vee \mathcal{G})$, the process $\int H d\beta$ is a Brownian motion independent of \mathcal{G} .*

PROOF. — Lemma 1 and the independence of β and \mathcal{G} imply that β is a \mathcal{E} -Brownian motion. By Lévy's characterization, $\int H d\beta$ is also a \mathcal{E} -Brownian motion; so it is independent of \mathcal{E}_0 , hence of \mathcal{G} . ■

PROOF OF OCONE'S THEOREM. — We shall show that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) and (i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i). Implications (ii) \Rightarrow (iii) and (iv) \Rightarrow (v) are trivial.

(i) \Rightarrow (ii) and (iv). We suppose that $\beta = \beta^M$ and $\langle M \rangle$ are independent. Let $A_t = \inf \{s : \langle M \rangle_s > t\}$, so that, with the convention $A_{0-} = 0$, its left-limit is $A_{t-} = \inf \{s : \langle M \rangle_s \geq t\}$; denote by \mathcal{B} the natural filtration of β and by \mathcal{E} the right-continuous enlargement of the filtration $\mathcal{B}_t \vee \sigma(\langle M \rangle)$. If T is a \mathcal{F} -stopping time, $\langle M \rangle_T$ is a \mathcal{E} -stopping time since

$$\{\langle M \rangle_T \leq t\} = \{T \leq A_t\} \in \mathcal{F}_{A_t} \subset \bigcap_{\varepsilon > 0} \sigma(M^{A_t+\varepsilon}) \subset \bigcap_{\varepsilon > 0} [\mathcal{B}_{t+\varepsilon} \vee \sigma(\langle M \rangle)] = \mathcal{E}_t.$$

If H is a bounded, \mathcal{F} -predictable process, $K_t = H_{A_{t-}}$ is bounded and \mathcal{E} -predictable and $\int_0^t H_s dM_s = \int_0^{(M)_t} K_u d\beta_u$ (when $H = \mathbb{1}_{[0, T]}$ with T a \mathcal{F} -stopping time, $K = \mathbb{1}_{[0, (M)_T]}$ and both integrals agree since $M_{T \wedge t} = \beta_{\langle M \rangle_{T \wedge t}}$; the general case follows by a monotone class argument).

If furthermore H takes values in $\{-1, 1\}$, Lemma 2 with $\mathcal{G} = \sigma(\langle M \rangle)$ says that $\gamma = \int K d\beta$ is a Brownian motion independent of $\langle M \rangle$ (as is β). Consequently, both processes $M = \beta_{\langle M \rangle}$ and $\int H dM = \gamma_{\langle M \rangle}$ have the same conditional law given $\langle M \rangle$; this proves (ii).

Taking now H bounded, \mathcal{F} -predictable, $[\mathcal{B}(\mathbb{R}_+) \otimes \sigma(\langle M \rangle)]$ -measurable and such that $\int_0^\infty H_s^2 d\langle M \rangle_s < \infty$, K is also $[\mathcal{B}(\mathbb{R}_+) \otimes \sigma(\langle M \rangle)]$ -measurable,

$\int_0^\infty K_s^2 ds < \infty$ and

$$\begin{aligned} \mathbb{E}\left[\exp\left(i\int_0^\infty H_s dM_s\right)\middle|\langle M\rangle\right] &= \mathbb{E}\left[\exp\left(i\int_0^\infty K_s d\beta_s\right)\middle|\langle M\rangle\right] \\ &= \exp\left(-\frac{1}{2}\int_0^\infty K_s^2 ds\right) = \exp\left(-\frac{1}{2}\int_0^\infty H_{A_s}^2 ds\right) \\ &= \exp\left(-\frac{1}{2}\int_0^\infty H_s^2 d\langle M\rangle_s\right). \end{aligned}$$

This proves (iv) when H is bounded; the general case follows by taking limits.

(iii) \Rightarrow (i). Let $s < t$. Since $M' = \int (\mathbb{1}_{[0,s]} - \mathbb{1}_{(s,\infty)}) dM$ has the same law as M , the triple $(M^s, \langle M\rangle, M_t - M_s)$ has the same law as $(M'^s, \langle M'\rangle, M'_t - M'_s)$. But the stopped processes M^s and M'^s are the same, the quadratic variations $\langle M\rangle$ and $\langle M'\rangle$ are equal and $M'_t - M'_s = -(M_t - M_s)$, yielding

$$(M^s, \langle M\rangle, M_t - M_s) \stackrel{\text{law}}{=} (M^s, \langle M\rangle, -(M_t - M_s)).$$

Denoting by \mathcal{G}_s the σ -field generated by the processes M^s and $\langle M\rangle$ and the null events, this implies that M is a martingale for the filtration \mathcal{G} , whence also for its right-continuous enlargement \mathcal{H} (Lemma 1).

The random variables $A_t = \inf\{s : \langle M\rangle_s > t\}$ are stopping times for the filtration \mathcal{H} (they are \mathcal{H}_0 -measurable!); the stopped martingales M^{A_t} are square-integrable and one has $\langle M\rangle_{A_t} = t$. Introducing the filtration $\mathcal{K}_t = \mathcal{H}_{A_t}$ and the Brownian motion $\beta = \beta^M$, one can write

$$\mathbb{E}[\beta_t - \beta_s | \mathcal{K}_s] = \mathbb{E}[\beta_{\langle M\rangle_{A_t}} - \beta_{\langle M\rangle_{A_s}} | \mathcal{H}_{A_s}] = \mathbb{E}[M_{A_t} - M_{A_s} | \mathcal{H}_{A_s}] = 0.$$

Consequently β is a \mathcal{K} -martingale, hence (Lévy's characterization) a \mathcal{K} -Brownian motion; so it is independent of \mathcal{K}_0 , and a fortiori of $\mathcal{G}_0 = \sigma(\langle M\rangle)$.

(v) \Rightarrow (i). Let B be a Brownian motion independent of $\langle M\rangle$. Applying (i) \Rightarrow (v) to the martingale $N = B_{\langle M\rangle}$ and remarking that, since $\langle M\rangle = \langle N\rangle$, the right-hand side of (v) is the same for M and N , we see that M and N have the same law. Consequently $(\beta^M, \langle M\rangle)$ and $(\beta^N, \langle N\rangle) = (B, \langle M\rangle)$ also have the same law and β^M is independent of $\langle M\rangle$. ■

REMARKS. — a) The hypotheses that \mathcal{B} is the natural filtration of β in Lemma 2 and \mathcal{F} that of M in (ii) cannot be dropped.

If one supposes only that \mathcal{B} is a filtration such that β is a \mathcal{B} -Brownian motion, Lemma 2 becomes false: Take a Brownian motion B with natural filtration \mathcal{B} , call $\beta = \int \text{sgn}(B) dB$ the Lévy transform of B and \mathcal{G} the σ -field generated by $\text{sgn}(B_1)$. Now $H = \text{sgn}(B)$ is \mathcal{B} -predictable and a fortiori \mathcal{E} -predictable, where $\mathcal{E}_t = \bigcap_{\varepsilon > 0} (\mathcal{B}_{t+\varepsilon} \vee \mathcal{G})$. But $\int H d\beta = B$ is certainly not independent of $\text{sgn}(B_1)$. (What makes this example work is that for $t < 1$ both random variables $\text{sgn}(B_t)$ and $\text{sgn}(B_1)$ are independent of β , but the pair $(\text{sgn}(B_t), \text{sgn}(B_1))$ is not.)

Similarly, the theorem becomes false if \mathcal{F} is no longer the natural filtration of M , but only some filtration for which M is a martingale. In that case, (i), (iii), (iv) and (v) are still equivalent, but (ii) may become stronger, as shown by the following example. Take as above B with natural filtration \mathcal{B} and Lévy transform β ; define an increasing process A independent of β by $A_t = t$ for $t \leq 1$ and $A_t = 1 + [u\mathbb{1}_{\{B_1 > 0\}} + v\mathbb{1}_{\{B_1 \leq 0\}}](t-1)$ for $t > 1$, where u and v are strictly positive real numbers. Our martingale verifying (i) will be the Lévy transform $M_t = \beta_{A_t}$ of B_{A_t} ; as the latter is a martingale for the filtration $\mathcal{F}_t = \mathcal{B}_{A_t}$, so is also M . The process

$$H_t = \begin{cases} \operatorname{sgn}(B_t) & \text{if } t \leq 1 \\ \operatorname{sgn}(B_1) \operatorname{sgn}(B_{A_t}) & \text{if } t > 1 \end{cases}$$

is \mathcal{F} -predictable, but the random variables

$$M_2 = \begin{cases} \beta_{1+u} & \text{if } B_1 > 0 \\ \beta_{1+v} & \text{if } B_1 \leq 0 \end{cases} \quad \text{and} \quad \int_0^2 H_s dM_s = \begin{cases} B_{1+u} & \text{if } B_1 > 0 \\ 2B_1 - B_{1+v} & \text{if } B_1 \leq 0 \end{cases}$$

do not have the same law in general. Indeed, on the one hand the law of M_2 is symmetric (β is independent of $\operatorname{sgn}(B_1)$) and on the other hand, if u is chosen large and v small, $\mathbb{P}[B_{1+u} > 0 \text{ and } B_1 > 0]$ is close to $1/4$ and $\mathbb{P}[2B_1 - B_{1+v} > 0 \text{ and } B_1 \leq 0]$ to 0, yielding by addition

$$\mathbb{P}\left[\int_0^2 H_s dM_s > 0\right] \approx \frac{1}{4} \neq \frac{1}{2} = \mathbb{P}[M_2 > 0].$$

This shows that the filtration \mathcal{F} is too large for (ii) to hold.

b) If M is a martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and if \mathcal{C} is a sub- σ -field of \mathcal{A} , then M is still a martingale for the enlarged filtration $(\mathcal{F}_t \vee \mathcal{C})$ if and only if $\mathbb{E}[\int_0^t H_s dM_s | \mathcal{C}] = 0$ for each t and each simple, \mathcal{F} -predictable process H verifying $|H| = 1$. Indeed, if M is a martingale for $(\mathcal{F}_t \vee \mathcal{C})$, so is also $\int H dM$, yielding $\mathbb{E}[\int_0^t H_s dM_s | \mathcal{F}_0 \vee \mathcal{C}] = 0$. Conversely, if the condition holds, $\mathbb{E}[(M_t - M_s)U | \mathcal{C}] = 0$ for each $C \in \mathcal{C}$ and each \mathcal{F}_s -measurable U with values in $\{-1, 1\}$; but for $A \in \mathcal{F}_s$, $2\mathbb{1}_A = (\mathbb{1}_A - \mathbb{1}_{A^c}) + 1$ is the sum of two such U 's, whence $\mathbb{E}[(M_t - M_s)\mathbb{1}_A | \mathcal{C}] = 0$, and M is a martingale for the large filtration.

c) If $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of some Brownian motion B and if \mathcal{C} is a non-trivial sub- σ -field of $\bigvee_t \mathcal{F}_t$, no \mathcal{F} -Brownian motion can be a martingale for $(\mathcal{F}_t \vee \mathcal{C})$. For such a Brownian motion β would have the form $\int H d\beta$ for an \mathcal{F} -predictable H with $|H| = 1$; so $B = \int H d\beta$ would also be a $(\mathcal{F}_t \vee \mathcal{C})$ -martingale, hence a $(\mathcal{F}_t \vee \mathcal{C})$ -Brownian motion and would be independent of $\mathcal{F}_0 \vee \mathcal{C} = \mathcal{C}$.

d) Yet, keeping the notations of c), there exist a non-trivial sub- σ -field \mathcal{C} of $\bigvee_t \mathcal{F}_t$ and a process that is both a (\mathcal{F}_t) - and a $(\mathcal{F}_t \vee \mathcal{C})$ -martingale, for instance the σ -field $\mathcal{C} = \mathcal{F}_1$ and the process $\int h dB$, with $h = \mathbb{1}_{[1, \infty)}$. This example generalizes as follows: Let A be a (\mathcal{F}_t) -predictable set and assume, for simplicity, that (almost) all sections $A^c(\omega)$ of its complementary have infinite Lebesgue

measure. Let $M = \int \mathbf{1}_A dB$, $N = \int \mathbf{1}_{A^c} dB$ and denote by \mathcal{C} the σ -field generated by the Brownian motion β^N . Time-changing by $\langle N \rangle = \int \mathbf{1}_{A^c} dt$ the predictable representation property with respect to β^N shows that every square-integrable, \mathcal{C} -measurable random variable assumes the form

$$U = \mathbb{E}[U] + \int_0^\infty K_s \mathbf{1}_{A^c}(s) dB_s,$$

where K is predictable and such that $\mathbb{E}[\int_0^\infty K_s^2 \mathbf{1}_{A^c}(s) ds] < \infty$. This easily implies that $M = \int \mathbf{1}_A dB$ satisfies the condition in b) above, showing that M is a $(\mathcal{F}_t \vee \mathcal{C})$ -martingale.

It seems worthwhile to present such examples here as they play an important rôle in some martingale proofs of the Ray-Knight theorems for Brownian local times (see, for instance, Exercises (2.8) and (2.9) pages 426–427 of Revuz & Yor [6] and Jeulin [4]).

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