KRZYSZTOF BURDZY

Excursion laws and exceptional points on brownian paths

Séminaire de probabilités (Strasbourg), tome 27 (1993), p. 177-181
<http://www.numdam.org/item?id=SPS_1993__27__177_0>

EXCURSION LAWS AND EXCEPTIONAL POINTS ON BROWNIAN PATHS

KRZYSZTOF BURDZY

University of Washington

The purpose of this note is to present an example of a family of "exceptional points" on Brownian paths which cannot be constructed using an entrance law.

Watanabe (1984, 1987) proved that various families of exceptional points on Brownian paths may be constructed using entrance laws. Special cases include excursions of one-dimensional Brownian motion within square root boundaries (see Watanabe (1984); the original construction was given by Davis (1983) and Greenwood and Perkins (1983)) and "cone-points" on the outer boundary of the 2-dimensional Brownian path (Burdzy (1989) Theorem 2.4 (i)). Watanabe's construction consists of generating an infinite but countable number of excursions (finite pieces of Brownian path) and then splicing them in a suitable way. The excursions are generated by a Poisson Point Process and they come ordered in a natural way corresponding to "local time." The excursions may be spliced together if the lifetimes of excursions corresponding to the local time interval $[0, c]$ are summable for each c. This condition is satisfied when the expected lifetime of an excursion under the excursion law is finite. Hence, one of the main conditions in Watanabe's theorem is that of finiteness of the expected lifetime of the excursion under the excursion law.

One may ask whether there exists a converse to Watanabe's theorem which would say that exceptional points on Brownian paths corresponding to an excursion law exist only when the expected lifetime of an excursion under the excursion law is finite. This could settle an open problem of whether there exist excursions within $\pm c\sqrt{t}$ for the critical case $c = 1$ (they do for $c > 1$ and do not for $c < 1$; see Davis and Perkins (1985)). Our theorem shows that such a general result cannot be proved.

The reader may consult the books of Blumenthal (1992), Burdzy (1987) and Sharpe (1988) regarding excursion theory and further references.

Let X denote the standard Brownian motion starting from 0. Suppose that $f : [0, \infty) \to [0, \infty]$. We will say that $\{X(s), s \in (t, t + \varepsilon)\}$ is an *excursion within f-boundaries* if $\varepsilon > 0$ and $|X(t + u) - X(t)| < f(u)$ for all $u \in (0, \varepsilon)$. The starting point of an excursion within f-boundaries may be called an *exceptional point* if for every fixed $t \geq 0$, the time t is not the starting point of an excursion within f-boundaries a.s. Let $C_*[0, \infty)$ denote the space of functions defined on $[0, \infty)$ which take real values and are continuous on some interval $[0, \zeta)$ and are equal to Δ (a

Supported in part by NSF grant DMS 91-00244 and AMS Centennial Research Fellowship

point outside $\mathbb{R}$) otherwise. The case $\zeta = \infty$ is not excluded. A σ-finite measure H on $C_*[0,\infty)$ will be called an *excursion law within f-boundaries* if

(i) the canonical process under H is strong Markov with the transition probabilities of Brownian motion killed upon hitting the graph of f or $-f$, and

(ii) H is supported by paths starting from 0.

Theorem 1. *There exists a continuous function f such that*

(i) w.p.1 there exist excursions within f-boundaries and their starting points are exceptional points for Brownian paths, and

(ii) we have $H\zeta = \infty$ where H is the Brownian excursion law within f-boundaries.

Remark. It is a part of our assertion that there exists only one (up to a multiplicative constant) excursion law within f-boundaries. This seems to be true for any function f but we will indicate how to prove it just for our special choice of f.

Our proof of Theorem 1 uses in an essential way the fact that our function f is not monotone.

Problem 1. *Does there exist a monotone function f which satisfies Theorem 1?*

Proof of Theorem 1. First we will define a function g and prove the theorem for g in place of f. Choose a sequence of strictly positive numbers $\{p_k\}_{k\geq 1}$ such that $\sum_{k=1}^{\infty} p_k < 1/2$. We will also need $a_k, b_k > 0$ for $k \geq 1$ whose values will be specified later. Let

$$g(t) = \begin{cases} a_k & \text{for } t = b_k,\ k \geq 1, \\ \infty & \text{otherwise.} \end{cases}$$

It will be convenient to work with the time reversed process, i.e., $Y(t) = X(1-t) - X(1)$ for $t \in [0,1]$. The process Y is a Brownian motion.

The construction of g is based on the following observation. For every $p < 1$ and $c_1, c_2 > 0$ there exists $b \in (0, c_1)$ such that with probability greater than p there exists $t \in (b, c_1)$ such that $Y(t) = Y(t-b)$ and $|Y(t)| < c_2$. We can use the continuity of Brownian paths to strengthen this statement as follows. Suppose that p, c_1, c_2 and b are as above. Then for every $a > 0$ there exists $d > 0$ such that with probability greater than p there exists $t \in (b, c_1)$ such that $Y(t) = Y(t-b)$, $|Y(t)| < c_2$ and $|Y(t-b) - Y(s)| < a$ for all $s \in [t-b, t-b+d]$.

We will define inductively sequences of strictly positive numbers $\{a_k\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$, $\{d_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$. The first constraint we impose on these sequences is that $b_{k+1} < d_k < b_k/2 < 2^{-k-1}$ for all k. Let $a_1 = 1$, $d_0 = 1/2$ and let $b_1, d_1 > 0$ be so small that with probability greater than $1 - p_1$ there exists $t \in (b_1, d_0/2)$ such that $Y(t) = Y(t-b_1)$ and $|Y(t-b_1) - Y(s)| < a_1/4$ for all $s \in [t-b_1, t-b_1+d_1]$. Let $q_1 = d_0/4 \wedge d_1/4$. Next choose $b_2 \in (0, q_1)$ so small that with probability greater than $1 - p_2$ there exists $t \in (b_2, q_1)$ such that $Y(t) = Y(t-b_2)$ and $|Y(t)| < a_1/8$. Let P^x denote the distribution of Brownian motion starting from x. Choose $a_2 > 0$ so small that for every real x and every $t \geq b_2/2$

$$P^x(|Y(t)| \leq a_2) < 2^{-1} b_2/b_1.$$

Find $d_2 \in (0, d_1/8)$ so small that with probability greater than $1 - p_2$ there exists $t \in (b_2, q_1)$ such that $Y(t) = Y(t-b_2)$, $|Y(t)| < a_1/8$ and $|Y(t-b_2) - Y(s)| < a_2/4$ for all $s \in [t-b_2, t-b_2+d_2]$.

Now we proceed by induction. Suppose that $a_1, \dots, a_k$ and $b_1, \dots, b_k$ have been chosen. Let $q_k = \min_{j \le k} 2^{-k-2} d_j$. Let $b_{k+1} \in (0, q_k)$ be so small that with probability greater than $1 - p_{k+1}$ there exists $t \in (b_{k+1}, q_k)$ such that $Y(t) = Y(t - b_{k+1})$ and $|Y(t)| < 2^{-k-2} a_j$ for every $j \le k$. Find $a_{k+1} > 0$ so small that for every real x and every $t \ge b_{k+1}/2$

$$P^x(|Y(t)| \le a_{k+1}) < 2^{-k} b_{k+1}/b_k.$$

Choose $d_{k+1} > 0$ so small that with probability greater than $1 - p_{k+1}$ there exists $t \in (b_{k+1}, q_k)$ such that $Y(t) = Y(t - b_{k+1})$, $|Y(t)| < 2^{-k-2} a_j$ for every $j \le k$ and $|Y(t - b_{k+1}) - Y(s)| < a_{k+1}/4$ for all $s \in [t - b_{k+1}, t - b_{k+1} + d_{k+1}]$. This completes the inductive definition of $\{a_k\}_{k \ge 1}$, $\{b_k\}_{k \ge 1}$, $\{d_k\}_{k \ge 1}$ and $\{q_k\}_{k \ge 1}$.

Next we will prove that there exist excursions within g-boundaries. Let T_1 be the smallest $t > b_1$ such that $Y(t) = Y(t - b_1)$ and $|Y(t - b_1) - Y(s)| < a_1/4$ for all $s \in [t - b_1, t - b_1 + d_1]$. By our choice of b_1, a_1, d_0 and d_1, we have $T_1 < 1/4$ with probability greater than $1 - p_1$. Let T_2 be the smallest $t \in (T_1 + b_2, T_1 + q_1)$ such that $Y(t) = Y(t - b_2)$, $|Y(t) - Y(T_1)| < a_1/8$ and $|Y(t - b_2) - Y(s)| < a_2/4$ for all $s \in [t - b_2, t - b_2 + d_2]$. If there is no such t, we let $T_2 = T_3 = \cdots = \infty$. Note that T_1 is a stopping time for Y. Using the strong Markov property for Y at T_1 and the definition of b_2, a_2, q_1 and d_2 we see that T_2 is finite (and, therefore, $T_2 < T_1 + q_1$) with probability exceeding $1 - p_2$. We continue by induction. Suppose that $T_1, \dots, T_k$ have been chosen and are finite. Let T_{k+1} be the smallest $t \in (T_k + b_{k+1}, T_k + q_k)$ such that $Y(t) = Y(t - b_{k+1})$, $|Y(t) - Y(T_k)| < 2^{-k-2} a_j$ for every $j \le k$ and $|Y(t - b_{k+1}) - Y(s)| < a_{k+1}/4$ for all $s \in [t - b_{k+1}, t - b_{k+1} + d_{k+1}]$. If such t does not exist then we let $T_{k+1} = T_{k+2} = \cdots = \infty$. The strong Markov property applied at T_k and the definitions of the constants ensure that T_{k+1} is finite and bounded by $T_k + q_k$ with probability greater than $1 - p_{k+1}$. We see that all T_k's are finite with probability greater than $1 - \sum_{k=1}^{\infty} p_k > 1/2$. Let $T_\infty = \lim_{k \to \infty} T_k$. Note that if T_∞ is finite then

$$T_\infty < 1/4 + \sum_{k \ge 1} q_k \le 1/4 + \sum_{k \ge 1} 2^{-k-2} \le 1/2.$$

Suppose that $T_\infty < 1$ and let $U = 1 - T_\infty$. We will show that U is the starting point for an excursion of X within g-boundaries. Fix an arbitrary $k > 1$. We have

$$Y(T_k) = Y(T_k - b_k) \tag{1}$$

and $|Y(T_k - b_k) - Y(s)| < a_k/4$ for all $s \in [T_k - b_k, T_k - b_k + d_k]$. Since $T_{j+1} - T_j < q_j < 2^{-j-2} d_k$ for all $j \ge k$, we have

$$T_\infty - T_k = \sum_{j \ge k} T_{j+1} - T_j < d_k.$$

Hence $T_\infty - b_k \in [T_k - b_k, T_k - b_k + d_k]$ and, therefore,

$$|Y(T_k - b_k) - Y(T_\infty - b_k)| < a_k/4. \tag{2}$$

It follows from the definition of T_j's that $|Y(T_{j+1})-Y(T_j)| < 2^{-j-2}a_k$ for all $j \geq k$. This and the continuity of Y implies that

$$|Y(T_\infty) - Y(T_k)| \leq \sum_{j\geq k} |Y(T_{j+1}) - Y(T_j)| < a_k/4.$$

This, (1) and (2) imply that $|Y(T_\infty) - Y(T_\infty - b_k)| < a_k/2$ for all k. We may express this in terms of X and U as $|X(U) - X(U + b_k)| < a_k/4$. Now it follows directly from the definition that U is the starting point of an excursion of X within g-boundaries.

We have proved that an excursion within g-boundaries exists with probability greater than 1/2. An easy modification of the argument shows that for each $k > 1$, with probability greater than 1/2 there exists an excursion within g-boundaries which has a starting point in $(0, 1/k)$. A standard application of the 0-1 law then shows that an excursion within g-boundaries exists with probability 1.

In order to prove that the starting points of excursions within g-boundaries are exceptional points it will suffice to show that with P^0-probability 1, $|X(b_k)| > a_k$ for infinitely many k. This can be achieved by choosing each a_k sufficiently small so that $P^0(|X(b_k)| > a_k) > 1 - 2^{-k}$.

Let us prove that there exists only one excursion law within g-boundaries. Note that a_k's may be chosen so small that

$$P^x(X(b_{k-1} - b_k) \in dy)/P^z(X(b_{k-1} - b_k) \in dy) \in (1/2, 2)$$

for all $x, z \in (-a_k, a_k)$ and $|y| < a_{k-1}$. Then an argument similar to that in the proof of Theorem 2.2 (b) of Burdzy (1987) shows that for every j and $\varepsilon > 0$ there exists $k_0 < \infty$ such that for every $k > k_0$

$$P^x(X(b_j - b_k) \in dy)/P^z(X(b_j - b_k) \in dy) \in (1 - \varepsilon, 1 + \varepsilon)$$

for all $x, z \in (-a_k, a_k)$ and $|y| < a_j$. Suppose that H and $\widetilde{H}$ are excursion laws within g-boundaries. An application of the Markov property at time b_k shows that

$$\frac{H(X(b_j) \in dy)}{\widetilde{H}(X(b_j) \in dy)} \cdot \frac{\widetilde{H}(X(b_k) \in (-a_k, a_k))}{H(X(b_k) \in (-a_k, a_k))} \in (1 - \varepsilon, 1 + \varepsilon)$$

for all $|y| < a_j$. Since ε can be made arbitrarily small by choosing large k, we conclude that the distributions of H and $\widetilde{H}$ at time b_j are constant multiples of each other. This is true for every j and clearly implies that H is a constant multiple of $\widetilde{H}$.

We will show that the excursion law H within g-boundaries has infinite expected lifetime. Let $H(\zeta \geq b_1) = c > 0$. Since $P^x(|X(b_3)| \leq a_2) < 2^{-1}b_2/b_1$, an application of the Markov property for H at time b_3 implies that

$$H(\zeta \geq b_2) > 2cb_1/b_2.$$

Similarly $P^x(|X(b_{k+2})| \leq a_{k+1}) < 2^{-k}b_{k+1}/b_k$ for all $k \geq 1$ and induction shows that

$$H(\zeta \geq b_k) > 2^k cb_1/b_k$$

for $k \geq 1$. For every k

$$H\zeta \geq b_k H(\zeta \geq b_k) > b_k 2^k c b_1 / b_k = 2^k c b_1.$$

It follows that $H\zeta = \infty$.

This completes the proof of the theorem with function g playing the role of the function f in the statement of the theorem. The function g is not continuous. We will now sketch an argument explaining how to modify the function g in order to obtain a continuous function f which also satisfies the theorem.

The modulus of continuity for Brownian paths is $\delta(t) = \sqrt{t}$ up to a logarithmic correction, so with probability 1 we have $|X(s+t) - X(s)| < t^{1/4}$ for all s and all $t < c(s)$ where $c(s) > 0$ is random. Let $h(t) = g(t) \wedge t^{1/4}$. Then every starting point of an excursion within g-boundaries is a starting point of an excursion within h-boundaries. The proof that the excursion law within h-boundaries has infinite expected lifetime does not need any essential changes. Note that h is finite and continuous at 0. It is not hard to see that we can smooth h away from 0 (leaving its values at b_k's) to obtain a continuous function f which has all the desired properties. □

REFERENCES

1. R. Blumenthal, *Excursions of Markov Processes*, Birkhäuser, Boston, 1992.
2. K. Burdzy, *Multidimensional Brownian Excursions and Potential Theory*, Longman, Harlow, Essex, 1987.
3. K. Burdzy, *Geometric properties of 2-dimensional Brownian paths*, Probab. Th. Rel. Fields 81 (1989), 485–505.
4. B. Davis, *On Brownian slow points*, Z. Wahrsch. verw. Gebiete 64 (1983), 359-367.
5. B. Davis and E. Perkins, *Brownian slow points: the critical case*, Ann. Probab. 13 (1985), 779–803.
6. P. Greenwood and E.A. Perkins, *Local time on square root boundaries*, Ann. Probab. 11 (1983), 227–261.
7. M. Sharpe, *General Theory of Markov Processes*, Academic Press, New York, 1988.
8. S. Watanabe, *Excursion point processes and diffusions*, Proceedings of the International Congress of Mathematicians, 1983, Warszawa (Z. Ciesielski and Cz. Olech, eds.), PWN-Polish Scientific Publishers, Warsaw, 1984, pp. 1117–1124.
9. S. Watanabe, *Construction of semimartingales from pieces by the method of excursion point processes*, Ann. Inst. Henri Poincaré 23 (1987), 297–320.

DEPARTMENT OF MATHEMATICS, GN-50, SEATTLE, WA 98195