FRANK B. KNIGHT

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Séminaire de probabilités (Strasbourg), tome 27 (1993), p. 36-43

<http://www.numdam.org/item?id=SPS_1993__27__36_0>

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ABSTRACT. Some further results, doubtless evident to the specialists on windings, are obtained concerning the asymptotics of the mutual windings of n independent planar Brownian motions, or n planar random walks, about each other

One of the most intriguing results on the asymptotics of planar Brownian motions, in our opinion, is that of M. Yor (1991), [5], concerning the mutual windings of n independent planar Brownian motions, with distinct starting points, about each other. This result is very simply stated, as follows. Let $Z^1, \ldots, Z^n$ be mutually independent, planar (or complex) Brownian motions with distinct starting points z^k, $1 \le k \le n$. Let $\theta^{i,j}(t)$, $0 \le t$, $1 \le i < j \le n$, be a continuous determination of the argument of $B_t^{i,j} := \frac{1}{\sqrt{2}}(Z_t^i - Z_t^j)$ about 0 (one can take $-\pi < \theta^{i,j}(0) \le \pi$ for convenience—we note that the partially dependent Brownian motions $B_t^{i,j}$ do not reach $0(:= (0,0))$, except on the P-nullset, which we may discard). Then, as $t \to \infty$, the $\frac{1}{2}n(n-1)$ normalized mutual winding angles $(2/\log t)\theta^{i,j}(t)$ converge in law to independent standard Cauchy random variables.

This remarkable result seems a bit overshadowed in the treatment of [5], which incorporates it in a much more general setting. Possibly for this reason, there are several simple corollaries of the result which seem to have gone unstated. Our object here is to call attention to the result itself by presenting a few of these corollaries. For these, we rest largely on the existant literature, and especially on [2], [3], [4], [5], and [6] for the basis of the proofs.

We cannot resist mentioning, from [6, §7.3], that the problem was originally suggested by the study of solar flares which travel randomly on the surface of the sun. For a sphere, however, it seems impossible to define the mutual windings. A better analogy might be to a colony of ants whose ant-hill has been removed. Note that, unlike the theory of windings about fixed points, the mutual windings do not raise the question of dependence on choice of external points (other than the starting points).

The first corollary consists of extending the result from convergence of random variables to convergence of processes.

Corollary 1. *As $t \to \infty$, $(2/\log t)\theta^{i,j}(t^\alpha)$, $1 \le i < j \le n$, converge to $\frac{1}{2}n(n-1)$ independent Cauchy processes with parameter $\alpha \ge 0$, in the sense of convergence in law of the finite dimensional joint distributions (it is noted in [4, p. 765] that it is impossible to strengthen this to the usual convergence in function space).*

Proof. The fact that for each (i,j) the process converges to a Cauchy process limit follows from the general discussion of log scaling limits in [4, Section 8; see especially (8.n) and (8.o)]. However, as that argument is buried rather deeply into [4], we call attention to it by presenting a direct argument which also yields the independence of the limit processes. For any process U_t and $c > 0$ we set $U_t^{(c)} := c^{-1}U_{c^2t}$, the Brownian rescaling of U by c. Transcribing a result of [4, Lemma 3.1] into the notation of [5], we obtain the existence of $\frac{n(n-1)}{2}$ (dependent) pairs of independent Brownian motions $(\rho_t^{i,j}, w_t^{i,j})$, $1 \le i < j \le n$, such that, for $0 < \alpha_1 < \cdots < \alpha_k$, $t > 0$ and $c_m = (\alpha_m/2)\log t$, $1 \le m \le k$ (suppressing a t-dependence)

$$(1.1)\qquad (2/\log t)\theta^{i,j}(t^{\alpha_m}) - \alpha_m w^{i,j,(c_m)}(T_1(\rho^{i,j,(c_m)}) \xrightarrow{P} 0$$

as $t \to \infty$ where $T_\alpha(w) := \inf\{t : w(t) = \alpha\}$; $w \in C(\mathbf{R}_+, \mathbf{R})$. Indeed, this is simply (2.a) of [5], somewhat specialized and with t^{α_m} in place of t. On the other hand, from (2.b) of [5] we have, for each $m \le k$

$$(1.2)\qquad (\rho^{i,j,(c_m)}, w^{i,j,(c_m)})) \xrightarrow{d} (\tilde{\rho}^{i,j,m}, \tilde{w}^{i,j,m})$$

where $(\tilde{\rho}^{i,j,m}, \tilde{w}^{i,j,m})$ are independent pairs of independent Brownian motions starting at 0, and the convergence is that of law on $C(\mathbf{R}_+, \mathbf{R}^{n(n-1)/2})$ with the topology of uniform convergence on compact sets. Ostensibly, the right side of (1.2) depends on m. However, the dependence is transparent, because if we define $(\tilde{\rho}^{i,j}, \tilde{w}^{i,j})$ to satisfy (1.2) with $\alpha_m = 1$, then jointly in $1 \le m \le k$, we have

$$\rho^{i,j,(c_m)}(s) = c_m^{-1}\rho^{i,j}(c_m^2 s) \xrightarrow{d} \alpha_m^{-1}\tilde{\rho}^{i,j}(\alpha_m^2 s), \quad \text{as} \quad t \to \infty, \tag{1.3}$$

with the anaogous fact for $\tilde{w}^{i,j}$. Then combining (1.1) and (1.2) we obtain

$$(2/\log t)\theta^{i,j}(t^{\alpha_m}) \xrightarrow{d} \tilde{w}^{i,j}(\alpha_m^2 T_1(\tilde{\rho}^{i,j,(\alpha_m)})), \quad 1 \le m \le k. \tag{1.4}$$

In fact, the passage time T_1 not being continuous in the topology of uniform convergence on compact sets, we need to appeal here to the (sufficiently remarkable) Lemma B.3 of [4], which says, in essence, that if the law of $(\rho^{i,j,(c_m)}, w^{i,j,(c_m)})$ is fixed (free of (c_m)) for each (i,j), (1.2) implies the joint convergence in law of *any* finite set of measurable functionals $\phi^{i,j}$ of these pairs. To obtain this condition, we can simply replace $(\rho^{i,j}, w^{i,j})$ by $((\rho^{i,j} - \rho_0^{i,j}), (w^{i,j} - w_0^{i,j}))$, which preserves (1.1) since the adjustment is uniformly small as $c_m \to \infty$. We will have further recourse to this lemma below, in treating the "big" windings. Finally, since

$$\begin{aligned}(T_1(\tilde{\rho}^{i,j,(\alpha_m)}) &= T_1(\alpha_m^{-1}\tilde{\rho}^{i,j}(\alpha_m^2 t)) \\ &= \alpha_m^{-2} T_{\alpha_m}(\tilde{\rho}^{i,j}(t)),\end{aligned} \tag{1.5}$$

the right side of (1.4) becomes simply $\tilde{w}^{i,j}(T_{\alpha_m}(\tilde{\rho}^{i,j}))$. Now using the well-known characterization of the Cauchy process as subordinate to the Brownian motion $\tilde{w}^{i,j}$ at the passage times to α of $\tilde{\rho}^{i,j}$, the proof of Corollary 1 is complete.

This result is also an easy consequence of a more general result concerning jointly the large windings, the small windings, and the local times on the unit circle. Indeed, following the pattern for Corollary 1, we have only to transcribe Theorems 4.1 and 4.2 of [4] to the present setting. Still further extensions are, of course, possible. However, with a view to obtaining the analog of Corollary 1 for random walks, we confine our presentation to these three functionals. Let us recall first the necessary definitions. For each $i < j \le n$, we can write $\theta^{i,j}(t) = w^{i,j}(H_t^{i,j})$ where, if $R_t^{i,j} := |B_t^{i,j}|$, then $H_t^{i,j} = \int_0^t (R_s^{i,j})^{-2} ds$, and $\log R_t^{i,j} := \rho^{i,j}(H_t^{i,j})$. This defines the pairs $(\rho^{i,j}, w^{i,j})$ of (1.1). Now we define

the *small windings*

$$\theta_-^{i,j}(t) := \int_0^{H^{i,j}(t)} 1(\rho_u^{i,j} < 0) dw_u^{i,j},$$

the *large windings*

$$\theta_+^{i,j}(t) := \int_0^{H^{i,j}(t)} 1(\rho_u^{i,j} > 0) dw_u^{i,j},$$

and the *local time of $B^{i,j}$ on the unit circle $L^{i,j}(t) := L(\rho^{i,j}, 0, H_t)$*,

where $L(w, x, t)$ is the local time of path w at point x and time t. Here the $\theta_\pm^{i,j}(t)$ measure the increment of $\theta^{i,j}$ during the time when $R_t^{i,j}$ is > 1 (resp. < 1), but as far as the asymptotics as $t \to \infty$ are concerned, it is known that we could replace 1 by any other positive constant. Note that we are following the notation of [5], but the result we need to invoke is given in [4] under entirely different notation. The connection is, that (ρ, w) of [5], with or without ornaments, is (β, θ) of [4], whereas θ of [5] represents, as here, an actual winding angle. Now the proof of Theorem 4.1 of [4] shows immediately that (1.1) may be extended to (from here on, we drop all ornaments in ρ when writing $T_\alpha(\rho)$)

(1.6)

$$(2/\log t)\theta_\pm^{i,j}(t^{\alpha_m}) - \alpha_m \int_0^{T_1(\rho)} 1\left(\rho_u^{i,j,(c_m)} \begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix}\right) dw_u^{i,j,(c_m)} \xrightarrow{P} 0,$$

respectively, as $t \to \infty$, and

$$(2/\log t)L^{i,j}(t^{\alpha_m}) - \alpha_m L(\rho^{i,j,(c_m)}, 0, T_1(\rho)) \xrightarrow{P} 0.$$

Actually, since we must replace ρ by $\rho - \rho_0$ as before, we need to invoke here the continuity of $L(w, x, t)$ in (x, t) at $x = 0$. Now it is only a matter of making a linear change of variables in the stochastic integrals to see that, respectively,

$$\begin{aligned}(1.7)\qquad &\alpha_m \int_0^{T_1(\rho)} 1\left(\rho_u^{i,j,(c_m)}\left\{\begin{matrix}>0\\<0\end{matrix}\right.\right) dw_u^{i,j,(c_m)} \\ &= \alpha_m c_m^{-1} \int_0^{T_{c_m}(\rho)} 1\left(\rho_u^{i,j}\left\{\begin{matrix}>0\\<0\end{matrix}\right.\right) dw_u^{i,j},\end{aligned}$$

which connects to $(2/\log t)\theta_{\pm}^{i,j}(t^{\alpha_m})$ through the tightness argument of Williams (i.e. (3.f) of [4]). The two stochastic integrals and $\alpha_m L(\rho^{i,j,(c_m)}, 0, T_1(\rho))$, $1 \le i < j \le n$, constitute altogether $n(n-1)/2$ measurable functionals of the paths of $(\rho^{i,j,(c_m)}, w^{i,j,(c_m)})$ into R^{3k}, hence by (1.2), (1.3), and Lemma B.3 of [4], they converge as $t \to \infty$ jointly in law to the same functionals of $(\tilde{\rho}^{i,j}, \tilde{w}^{i,j})$. It follows that they are mutually independent in the limit for distinct (i,j), whereas for fixed (i,j) and each m, the limit law of the triple

$$((2\alpha_m^{-1}/\log t)\theta_{\pm}^{i,j}(t^{\alpha_m}), (2\alpha_m^{-1}/\log t)L^{i,j}(t^{\alpha_m}))$$

is given by Theorem 4.2 of [4]. In particular, that of the big windings $(2\alpha_m^{-1}/\log t)\theta_{+}^{i,j}(t^{\alpha_m})$ is the distribution with characteristic function $(\cosh\ \cdot)^{-1}$, called in [1] the "standard hyperbolic secant" because it has density $\frac{1}{2}(\cosh \frac{\pi}{2}\cdot)^{-1}$ over R, while that of the local time is the exponential distribution with mean 2. The corresponding limit processes of (1.7) and $\alpha_m L(\rho^{i,j(c_m)}, 0, T_1(\rho))$ in parameter α are inhomogeneous (unlike the Cauchy process, obtained by adding the two cases in (1.7)). These processes are most easily presented in probabilistic form, as in Table 1 of [4], and we have the following

Corollary 2. *As $t \to \infty$, $\frac{2}{\log t}(\theta_{-}^{i,j}(t^{\alpha}), \theta_{+}^{i,j}(t^{\alpha}), L^{i,j}(t^{\alpha}))$ converges in finite dimensional joint distribution to the $\frac{1}{2}n(n-1)$ independent processes on R^3,*

$$(1.8)\qquad \left(\int_0^{T_\alpha(\tilde{\rho}^{i,j})} I\left(\tilde{\rho}_u^{i,j}\left\{\begin{matrix}>0\\<0\end{matrix}\right.\right) d\tilde{w}_u^{i,j},\quad L(\tilde{\rho}^{i,j}, 0, T_\alpha(\tilde{\rho}^{i,j}))\right),\quad 0 \le \alpha.$$

Proof. Immediate from the preceding remarks, in view of (1.3), (1.5),and a linear change of variables.

We consider now the mutual windings of n random walks on R^2. Let X^j_m, $1 \le m$, $1 \le j \le n$, be mutually independent, mean 0, unit variance, uncorrelated pairs of real random variables, identically distributed in m for each j, and let $S^j = (S^j_m, m \ge 0)$, be the corresponding (independent) random walks on R^2. The winding sequence of S^j about 0, say $\phi^j_m = \sum_{i=1}^m \lambda^j_i$, $-\pi < \lambda^j_i < \pi$, was defined by Bélisle [1] in the evident way (in the treatment below, the random walks eventually do not reach 0, so this case may be discounted). We are concerned here with the sequences $\phi^{i,j}(m)$ giving the windings about 0 of the random walks $S^i_m - S^j_m$, $1 \le i < j \le n$. Under a boundedness plus mild regularity condition, it follows from [1] that, for each (i,j), $2\phi^{i,j}(n)/\log n$ converges in distribution, as $n \to \infty$, to the same standard hyperbolic secant as does the large winding of a plane Brownian motion. Of course this is no coincidence, and a strong Brownian motion approximation is used in the proof, although the details are complicated.

It is natural to suppose that under the same regularity conditions the joint distributions converge to those of independent hyperbolic secant variables. This is probably true, but the obvious method—that of strong approximation by Brownian motion—seems to be technically too complicated even for the case of classical Bernoulli random walk. There is, however, a fairly general hypothesis, and one which has been frequently made in the literature, under which the argument is not difficult, and most of it is already in Bélisle [2]. Namely, we need to assume *circular symmetry*. (It is hardly surprising, in retrospect, that this simplifies treatment of windings about 0). Following [2], we introduce the

Hypothesis. For each $j \le m$, X^j_1 has a distribution which is circularly symmetric, with radial distribution $\mu^j(dr)$ such that $\mu^j\{0\} \ne 1$ and
$\int_1^\infty r^2 \log^2 r \mu^j(dr) < \infty$.

Now we have

Corollary 3. *Under this hypothesis, let* $\phi^{i,j}(t) := \phi^{i,j}([t])$, $0 \le t$. *Then as* $t \to \infty$, *the* *processes $\frac{2}{\log t}\phi^{i,j}(t^\alpha)$ converge in finite dimensional joint distribution to $n(n-1)/2$ independent processes distributed as $\int_0^{T_\alpha(\rho)} 1(\rho_u > 0)dw_u$, $0 \le \alpha$, where (ρ, w) is a Brownian motion on R^2 starting at 0.*

Proof. We first show that $X_1^i - X_1^j$ satisfies the same hypothesis, $1 \le i < j \le n$. Indeed, since X_1^j and $-X_1^j$ have the same distribution, independently of X_1^i, and $|X_1^i - X_1^j| \le |X_1^i| + |X_1^j|$, we have

$$E(|X_1^i - X_1^j|^2 \log^2 |X_1^i - X_1^j|; |X_1^i - X_1^j| > 2)$$

$$\le E((|X_1^i| + |X_1^j|^2 \log^2(|X_1^i| + |X_1^j|; |X_1^i + X_1^j| > 2)$$

$$\le 4E((|X_1^i| \vee |X_1^j|^2 \log^2 2(|X_1^i| \vee |X_1^j|); |X_1^i| \vee |X_1^j| > 1)$$

$$\le 4[E(|X_1^i|^2 \log^2 2(|X_1^i|); |X_1^i| > 1) + E(|X_1^j|^2 \log^2 2|X_1^j|; |X_1^j| > 1)] < \infty.$$

Besides, since $\mathcal{O}(X_1^i + X_1^j) = \mathcal{O}(X_1^i) + \mathcal{O}(X_1^j)$ for any rotation $\mathcal{O}$ of R^2, it is obvious that $X_1^i - X_1^j$ has a spherically symmetric distribution. Hence it satisfies the hypothesis.

Now let W_t^i, $1 \le i \le n$, be independent planar Brownian motions starting at 0, and let $W_t^{i,j} = \frac{1}{\sqrt{2}}(W_t^i - W_t^j)$, $1 \le i < j \le n$. Further, let $R_m^{i,j}$, $1 \le m$, $1 \le i < j \le n$, be independent random varibles on R^+ such that $R_m^{i,j}$ has the distribution of $c^{i,j}\|X_1^i - X_1^j\|$, with $c^{i,j} = \sqrt{2}(E\|X_1^i - X_1^j\|^2)^{-\frac{1}{2}}$. We define $\tau_1^{i,j} = \inf\{t : |W_t^{i,j}| = R_1^{i,j}\}$, and inductively $\tau_{m+1}^{i,j} = \inf\{t > \tau_m^{i,j} : |W_t^{i,j} - W_{\tau_m^{i,j}}^{i,j}| = R_{m+1}^{i,j}\}$. Then clearly the family $W_{\tau_m}^{i,j}$ has the same joint distribution as $c^{i,j}(S_m^i - S_m^j)$, $1 \le m$, $1 \le i < j \le n$. Let us assume, for convenience only, that $P\{X_1^j = 0\} = 0$, so that also $P\{X_1^i - X_1^j = 0\} = 0$ (in any case unless $P\{X_1^j = 0\} = 1$, we would have $P\{S_m^i - S_m^j \ne 0$ for sufficiently large $m\} = 1$, and we could carry out the asymptotics conditionally on $S_m^i - S_m^j \ne 0$). For $\epsilon > 0$, we have $W_\epsilon^{i,j} \ne 0$, $1 \le i < j \le n$, and we can apply Corollary 2 to the large windings $\theta_+^{i,j}(t)$ about 0 of the family $W_{\epsilon+t}^{i,j}$. It follows that $\frac{2}{\log t}\theta_+^{i,j}(t^\alpha)$ converges in finite dimensional joint distribution to $\int_0^{T_\alpha(\rho)} 1(\tilde{\rho}_u^{i,j} > 0)d\tilde{w}_u^{i,j}$, where $(\tilde{\rho}^{i,j}, \tilde{w}^{i,j})$ are independent planar Brownian motions starting at 0. Of course, as far as the large windings is concerned, starting at time

$\epsilon > 0$ is just a technicality to apply Corollary 2, and the same asymptotics hold starting at $t = 0$.

Finally, as shown in Bélisle [2], for each (i,j), $(\theta_+^{i,j}(\tau_m^{i,j}) - \phi^{i,j}(m))/\log m \xrightarrow{P} 0$ as $m \to \infty$, and at the same time $\theta_+^{i,j}(\tau_m^{i,j}) - \theta_+^{i,j}(m) \xrightarrow{P} 0$. It follows that, for each $\alpha > 0$, $\frac{2}{\log t}(\phi^{i,j}(t^\alpha) - \theta_+^{i,j}(t^\alpha)) \xrightarrow{P} 0$, and hence Corollary 3 is a consequence of the Brownian case.

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Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, IL 61801