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On conditioning random walks in an exponential family to stay nonnegative

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SUMMARY. We show that the probability measures resulting from conditioning different random walks in an exponential family to stay nonnegative coincide with the measures obtained by taking one member of the family and conditioning it both to stay nonnegative and to go to infinity at a prescribed rate. This extends results in [1] where this relation was established for certain special members of an exponential family.

In this note, we present a relation involving conditioning to stay nonnegative for the collection of random walks which arise from an exponential family of step distributions. Let us first introduce some notation concerning the exponential family. Consider $(p(k), k \in \mathbb{Z})$ a probability law on $\mathbb{Z}$ which is not supported by any sub-lattice (the restriction to distributions on integers is only a matter of convenience, the extension to non-lattice distributions is easy). Denote the moment generating function by $M(s) = \sum_k s^k p(k)$, $s > 0$, and define $\alpha = \inf\{s : M(s) < \infty\}$, $\beta = \sup\{s : M(s) < \infty\}$, the end points of the interval where M is finite. As usual, it is convenient to introduce $m(s) = sM'(s)/M(s)$. The mapping $s \to m(s)$ is an increasing bijection from (α, β) to, say, (μ^-, μ^+), and the inverse bijection is denoted by $m \to s(m)$. We will assume throughout the note that $\alpha < \beta$ and $\mu^+ > 0$. The exponential family indexed by $m \in (\mu^-, \mu^+)$ is specified by

$$p^{(m)}(k) = s(m)^k p(k) \tilde{M}(m)^{-1} \ , \ k \in \mathbb{Z}, \tag{1}$$

where $\tilde{M}(m) = M \circ s(m)$. Notice that $m = \sum k p^{(m)}(k)$, so the exponential family is parametrized by the mean.

We consider a probability space Ω, a sequence of random variables $X_1, \cdots, X_i, \cdots$, and the partial sums $S_n = \sum_{i=1}^n X_i$, $n \geq 0$. For every $m \in (\mu^-, \mu^+)$, let P^m be a probability law on Ω under which $X_1, \cdots, X_i, \cdots$ are i.i.d. with common distribution $p^{(m)}$. For simplicity we put $P = P^{m(1)}$, and we write P_x^m for the law of $S + x$ under P^m.

We now review some material on conditioning a random walk to stay nonnegative. Introduce the first passage time below 0,

$$\tau = \min\{n : S_n < 0\} .$$

For $m \in (0 \vee \mu^-, \mu^+)$, the random walk S drifts to $+\infty$ under P^m, that is $P^m(\tau = \infty) > 0$. We denote the conditional law $P^m(\cdot \mid \tau = \infty)$ by $P^{m,+}$. The function

$$h^{(m)}(x) = P^m_x(\tau = \infty) \ , \ x \in \mathbb{N} \ , \tag{2}$$

is harmonic for S killed at time τ under P^m (see also section 2 in [1] for alternative expressions for $h^{(m)}$), and $P^{m,+}$ corresponds to the Doob's $h^{(m)}$–transform. That is

$$D_i = \begin{cases} h^{(m)}(S_i) & \text{for } i < \tau \\ 0 & \text{for } i \geq \tau \end{cases}$$

is a P^m- martingale, and for any event Λ which depends only on the i first steps of S, we have

$$P^{m,+}(\Lambda) = \frac{1}{h^{(m)}(0)} E^m(D_i, \Lambda) .$$

Finally, S is a Markov chain under $P^{m,+}$, with transition function

$$p^{(m,+)}(x,y) = p^{(m)}(y-x) \ h^{(m)}(y)/h^{(m)}(x) \ , \ x,y \in \mathbb{N}. \tag{3}$$

For the limit case $m = 0$, S oscillates under P^0 and we cannot condition P^0 on $\{\tau = \infty\}$ in the usual way. Nonetheless, Spitzer [7] showed that there exists a unique (up to a multiplicative constant) positive harmonic function $h^{(0)}$ for S killed at time τ under P^0. More precisely, $h^{(0)}$ is the renewal function based on the strict ascending ladder heights process of $-S$ under P^0 and can be identified as the limit of the ratio $P^0_\cdot(\tau > n)/P^0(\tau > n)$ as $n \to \infty$. Then we can consider $P^{0,+}$, the law under which S is a Markov chain with transition function $p^{(0,+)}$ given in (3). Moreover $P^{0,+}$ is the limit (in the sense of weak convergence of finite dimensional distributions) of P^0 conditioned on $\{\tau > n\}$. See section 3 in [1] for details.

When $m(1) < 0$, Keener [4] proved that the conditional law $P(\cdot \mid \tau > n)$ converges as $n \to \infty$ to $P^{0,+}$. When $\mu^- < 0 < m(1)$, it follows from a result of Veraverbeke and Teugels [8] that the conditional law $P(\cdot \mid n < \tau < \infty)$ converges to $P^{0,+}$ again. On the other hand, the authors [1] observed that if there exists $t \in (1, \beta)$ with $M(t) = 1$ (this is essentially Cramer's condition), then $P(\cdot \mid$ S exceeds level n before time $\tau)$ also converges as $n \to \infty$, but towards a different limit, namely $P^{m(t),+}$. It is therefore natural to ask whether for any $m \in (0 \vee \mu^-, \mu^+)$, $P^{m,+}$ can be obtained as the limit of a suitably conditioned version of P. This question also has an interpretation in terms of the space-time Martin boundary which we discuss briefly in the remark at the end of this note.

Since $E^m(X_1) = m$, the law of large numbers implies

$$\lim_{n \to \infty} S_n/n = m \ , \ P^{m,+}\text{-a.s.}$$

This suggests the following simple solution to our problem.

Theorem. *Let $f(S) = f(S_1, \cdots, S_i)$ be a bounded Borel functional which depends only on a finite number of steps. We have*

(1) *If $m \in (0 \vee m(1), \mu^+)$, then*

$$\lim_{n\to\infty} E(f(S) \mid \tau > n, S_n \geq mn) = E^{m,+}(f(S)) .$$

(2) *If $m(1) > 0$ and $m \in (0 \vee \mu^-, m(1))$, then*

$$\lim_{n\to\infty} E(f(S) \mid \tau > n, S_n \leq mn) = E^{m,+}(f(S)) .$$

This result should be compared with the following relation between P and P^m which derives readily from a classical theorem of large deviation of Petrov [5]. If $f(S)$ is a bounded functional which depends only on a finite number of steps then, for $m \in (m(1), \mu^+)$

$$\lim_{n\to\infty} E(f(S) \mid S_n \geq mn) = E^m(f(S)) ,$$

and for $m \in (\mu^-, m(1))$

$$\lim_{n\to\infty} E(f(S) \mid S_n \leq mn) = E^m(f(S)) .$$

Proof of the Theorem. The first step consists of establishing the following asymptotic estimate. For every $x \in I\!N$ and $m \in (0 \vee \mu^-, \mu^+)$, we have

$$\lim_{n\to\infty} \sup_k \left| \sqrt{n}\, P_x^m(S_n = mn + k, \tau > n) - h^{(m)}(x) g\left(\frac{k}{\sqrt{n}}\right) \right| = 0 , \qquad (4)$$

where we agree here and thereafter that k varies in the set $\{k \in I\!R : mn + k \in Z\!\!Z\}$. In (4), $g(u) = (2\pi c)^{-1/2} \exp(-u^2/2c)$ is the centered Gaussian density with variance c, where c is the variance of $p^{(m)}$, and $h^{(m)}$ is given by (2). Indeed, we have

$$P_x^m(S_n = mn + k, \tau > n) = I_1(n,k) - I_2(n,k) - I_3(n,k) ,$$

with

$$\begin{aligned} I_1(n,k) &= P_x^m(S_n = mn + k) , \\ I_2(n,k) &= \sum_{1 \leq i \leq \sqrt{n}} P_x^m(S_n = mn + k, \tau = i) , \\ I_3(n,k) &= \sum_{\sqrt{n} < i \leq n} P_x^m(S_n = mn + k, \tau = i) . \end{aligned}$$

Applying the local limit theorem of Gnedenko (see e.g. [2] on p. 351), we have

$$\lim_{n\to\infty} \sup_k \left| \sqrt{n}\, I_1(n,k) - g\left(\frac{k}{\sqrt{n}}\right) \right| = 0 .$$

On the other hand, by the Markov property,

$$I_2(n,k) = \sum_{1 \leq i \leq \sqrt{n}} \sum_{-\infty < y < 0} P_x^m(\tau = i, S_i = y)\, P_y^m(S_{n-i} = mn + k)$$

and again by the local limit theorem of Gnedenko,

$$\lim_{n\to\infty} \sup_k \left| \sqrt{n}\, I_2(n,k) - \sum_{1\le i\le\sqrt{n}} \sum_{-\infty<y<0} g\left(\frac{k-y}{\sqrt{n-i}}\right) P_x^m(\tau=i, S_i=y) \right| = 0.$$

Using the inequality

$$\begin{aligned}&\sup_k \sum_{1\le i\le\sqrt{n}} \sum_{-\infty<y<0} P_x^m(\tau=i,S_i=y) \left| g\left(\frac{k-y}{\sqrt{n-i}}\right) - g\left(\frac{k}{\sqrt{n}}\right)\right| \\ &\le \sum_{1\le i\le\sqrt{n}} \sum_{-\infty<y<0} P_x^m(\tau=i,S_i=y) \sup_k \left| g\left(\frac{k-y}{\sqrt{n-i}}\right) - g\left(\frac{k}{\sqrt{n}}\right)\right|\end{aligned}$$

and the dominated convergence theorem, we deduce

$$\lim_{n\to\infty} \sup_k \left| \sqrt{n}\, I_2(n,k) - g\left(\frac{k}{\sqrt{n}}\right) P_x^m(\tau \le \sqrt{n}) \right| = 0\ .$$

Since $P_x^m(\tau \le \sqrt{n})$ converges to $1 - h^{(m)}(x)$ [by (2)], we have

$$\lim_{n\to\infty} \sup_k \left| \sqrt{n}\, I_2(n,k) - g\left(\frac{k}{\sqrt{n}}\right) (1 - h^{(m)}(x)) \right| = 0\ .$$

Finally, writing j for the integer part of $\sqrt{n}$, we have

$$\begin{aligned} I_3(n,k) &\le P_x^m(\sqrt{n} < \tau < \infty) \\ &\le \sum_{r=0}^{\infty} P_x^m(S_j = r)\ P_r^m(\tau < \infty)\ . \end{aligned}$$

Since $m > 0$, we can pick $t \in (0,1)$ such that $E^m(t^{S_1}) = a < 1$. Then $(t^{S_n}, n \ge 0)$ is a P_r^m-supermartingale and the optional sampling theorem yields $P_r^m(\tau < \infty) \le t^r$. Hence

$$I_3(n,k) \le \sum_{r=0}^{\infty} P_x^m(S_j = r) t^r \le t^{-x}\ a^{\sqrt{n}-1}\ ,$$

in particular $\sqrt{n}\, I_3(n,k)$ goes to 0 as $n \to \infty$. The proof of (4) is now complete.

Next we use (1) to rewrite (4) in terms of P_x as

$$\begin{aligned}\lim_{n\to\infty} \sup_k\ |\sqrt{n}\ s(m)^{mn+k-x}\ \tilde{M}(m)^{-n}\ P_x(S_n = mn+k, \tau > n) \\ -\ h^{(m)}(x)\ g\left(\frac{k}{\sqrt{n}}\right)| \ =\ 0\ . \end{aligned} \qquad (5)$$

For $m \in (0 \vee m(1), \mu^+)$, $s(m) > 1$ and we deduce from (5) that for $i \ge 0$

$$\begin{aligned} &P_x(S_{n-i} \ge mn, \tau > n-i) \\ &\sim K\ h^{(m)}(x)\ n^{-1/2}\ s(m)^{x-mn}\ \tilde{M}(m)^{i-n} \quad (n \to \infty)\ , \end{aligned}$$

for some constant $K > 0$. In turn, this implies

$$\lim_{n\to\infty} \frac{P_x(S_{n-i} \geq mn, \tau > n-i)}{P(S_n \geq mn, \tau > n)} = h^{(m)}(x)\, s(m)^x\, \tilde{M}(m)^{-i}\,. \tag{6}$$

Similarly, if $m(1) > 0$ and $m \in (0 \vee \mu^-, m(1))$, then $s(m) < 1$ and we derive from (5)

$$\lim_{n\to\infty} \frac{P_x(S_{n-i} \leq mn, \tau > n-i)}{P(S_n \leq mn, \tau > n)} = h^{(m)}(x)\, s(m)^x\, \tilde{M}(m)^{-i}\,. \tag{7}$$

Finally, consider $f(S)$, a bounded functional which depends only on the i first steps. With no loss of generality, we may suppose that $0 \leq f(S) \leq 1$. It is plain from (6), the Markov property and Fatou's lemma, that for $m \in (0 \vee m(1), \mu^+)$

$$\begin{aligned}
&\liminf_{n\to\infty} E(f(S) \mid \tau > n, S_n \geq mn) \\
&= \liminf_{n\to\infty} E(f(S) P_{S_i}(\tau > n-i, S_{n-i} \geq mn), \tau > i)/P(\tau > n, S_n \geq mn) \\
&\leq E(f(S)\, s(m)^{S_i}\, \tilde{M}(m)^{-i}\, h^{(m)}(S_i), \tau > i) \\
&= E^m(f(S)\, h^{(m)}(S_i), \tau > i) \\
&= E^{m,+}(f(S)) \quad \text{(by (3))}.
\end{aligned}$$

Replacing f by $1 - f$, we get

$$\begin{aligned}
&\limsup_{n\to\infty} E(f(S) \mid \tau > n, S_n \geq mn) \\
&= 1 - \liminf_{n\to\infty} E((1-f)(S) \mid \tau > n, S_n \geq mn) \\
&\leq 1 - E^{m,+}((1-f)(S)) \\
&= E^{m,+}(f(S))\,.
\end{aligned}$$

(The last equality comes from the fact that $P^{m,+}$ is conservative and would be false otherwise).

The second assertion of the Theorem follows from the Markov property and (7) in the same way.

Remark. The estimate (5) is clearly sharper than the Theorem. It can also be used to derive information on the space-time exit Martin boundary of S killed at time τ under P, see Doob [3] and Revuz [6]. In particular, it entails that for $m \in (0 \vee \mu^-, \mu^+)$, the function

$$k^{(m)}(x,i) = h^{(m)}(x)\, s(m)^x\, \tilde{M}(m)^{-i} \qquad (x, i \in \mathbb{N})$$

can be identified as the limit

$$\lim_{n\to\infty} \frac{P_x(S_{n-i} = mn + k, \tau > n-i)}{P(S_n = mn + k, \tau > n)}$$

where k is any fixed integer, and hence it is a *minimal* point of the boundary. However, the estimate (5) does not seem to yield the complete characterization of the space-time Martin boundary. Technically, our approach via Gnedenko's local limit theorem allows us to determine the asymptotic behaviour of the ratio

$$\frac{P_x(S_{n-i} = a(n), \tau > n - i)}{P(S_n = a(n), \tau > n)}$$

when the sequence $a(n)$ is such that $a(n) = mn + O(\sqrt{n})$ for some $m \in (0 \vee \mu^-, \mu^+)$, but not otherwise.

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