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Chaoticity on a stochastic interval $[0, T]$

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Abstract

The chaotic representation property is given a meaning and established for a class of martingales X defined on some stochastic interval $[0, T]$ and having only finitely many jumps before $T - \varepsilon$.

1.Introduction

Let X be a martingale with predictable bracket $< X, X >_t = t$, $(\mathcal{F}_t)$ be its filtration and $\mathcal{F} = \bigcup_{t>0} \mathcal{F}_t$. We say that the martingale X has the chaotic representation property (C.R.P) or is chaotic, if for all $F \in L^2(\Omega, \mathcal{F})$, there exists a sequence (f_k) with $f_k \in L^2(\mathbb{R}_+^k, dt^{\otimes k})$, such that

$$F = \sum_{k=0}^{\infty} F_k,$$

where $F_0 = \mathbb{E}[F]$ and for $k > 0$

$$F_k = \int_{0<t_1<...<t_k} f_k(t_1, ..., t_k)\, dX_{t_1} ... dX_{t_k}.$$

(For the definition of the latter multiple stochastic integral, see [7].)

The random variables $F_k, k \in N$, are such that

$$\mathbb{E}[F_k F_j] = \delta_j(k) \left(\int_{0<t_1<...<t_k} f_k^2(t_1, ..., t_k)\, dt_1 ... dt_k \right),$$

where $\delta_j(k) = 0$ if $k \neq j$ and $\delta_k(k) = 1$.

It is interesting to express the chaotic representation property as an isomorphism between $L^2(\Omega, \mathcal{F})$ and the symmetric Fock space over $H = L^2(\mathbb{R}_+, dt)$, defined by

$$Fock(H) = \oplus_{k=0}^{\infty} H^{\odot k}.$$

For, $k \in \mathbb{N}*$, the space $H^{\odot k} = L^2_{sym}(\mathbb{R}_+^k, dt_1 ... dt_k)$ is the set of the class of square integrable functions with respect to $dt_1 ... dt_k$, which are symmetric with respect to the k parameters $(t_1, ..., t_k)$. The scalar product over $H^{\odot k}$ is defined by

$$< f, g > = \int_{0<t_1<...<t_k} f(t_1, ..., t_k) g(t_1, ..., t_k)\, dt_1 ... dt_k,$$

and $H^{\odot 0} = \mathbb{R}$.

The well known examples of martingales having the chaotic representation property are the Brownian motion and the standard Poisson process [6].

Moreover, *He* and *Wang* [5] have characterized the Lévy processes which have the predictable representation property but until 1987 we did not know if these processes have the chaotic representation property.

In 1987, the author [2] proved that for the Lévy processes the chaotic representation property and the predictable representation property are equivalent.

In 1988, *Emery* [3] showed that a martingale earlier discovered by Azéma [1] has the chaotic representation property, introducing at the same time other examples which satisfy the "structure equation" of the form

$$d[X,X]_t = dt + \Phi(t)dX_t, \quad X_0 = x.$$

He later proved in [4] that if the predictable process $\Phi(t)$ is such that the integral $A_t = \int_0^t \Phi^{-2}(s)ds$ is a.s. finite for all t, then the predictable representation property implies the chaotic representation property. This applies to structure equations with Φ of the form

$$\Phi(t) = \phi_1(t)1_{]0,T_1]}(t) + \sum_{n\geq 2} \phi_n(t, T_{n-1}, ..., T_1)1_{]T_{n-1},T_n]}(t)$$

where ϕ_n are deterministic and the T_n's are the successive jumps of the solution X to the structure equation

$$d[X,X]_t = dt + \Phi(t)dX_t, \quad X_0 = x.$$

The hypothesis $A_t < \infty$ implies that there are only finitely many jumps on finite intervals since A_t is the predictable compensator of the number of jumps

$$C_t = \sum_{n\geq 1} 1_{[T_n,\infty[}(t).$$

The aim of this work is to study the following problem : Dropping the finiteness assumption for A_t and putting $T_\infty = \sup_n T_n$, we will allow T_∞ to be finite. The above formulas define (in law) the martingale X only on the interval $[0, T_\infty]$. We will prove that X still has the chaotic representation property, in the following sense : If M is a chaotic martingale independent of X(possibly defined on an enlargement of Ω), the martingale

$$Y_t = \begin{cases} X_t & \text{for } t \leq T_\infty \\ X_{T_\infty} + M_{t-T_\infty} - M_0 & \text{for } t \geq T_\infty \end{cases}$$

has the chaotic representation property (we will see in Lemma 2.2. that this does not depend on the choice of M).

2.Chaoticity before a stopping time

This section is devoted to giving a rigorous meaning to the chaotic representation property for a martingale defined only up to some stopping time.

Definition. *Let $(X_t)_{t\geq 0}$ be a martingale such that $< X, X >_t$ is equal to t, $(\mathcal{F}_t)$ be its filtration and T be a stopping time of $(\mathcal{F}_t)$. We say that X is chaotic on $[0,T]$ if $L^2(\mathcal{F}_T)$ is included in the chaotic space of X, i.e. if each $F \in L^2(\mathcal{F}_T)$ has an expansion $F = \sum_{k=0}^{\infty} F_k$ with $F_0 = \mathbb{E}[F]$ and for $k > 0$*

$$F_k = \int_{0<t_1<...<t_k} f_k(t_1, ..., t_k)dX_{t_1}...dX_{t_k}$$

with $(f_k)_{k\geq 0} \in Fock(H)$.

Lemma 2.1. *If the martingale $(X_t)_{t>0}$ is chaotic on $[0,T]$ and if a martingale $(Y_t)_{t>0}$ verifies $< Y,Y >_t = t$ and $X = Y$ on $[0,T]$, then T is a stopping time for the filtration generated by Y and Y is also chaotic on $[0,T]$.*

Proof. By proposition (1, ii) of [4], each element of $L^2(\mathcal{F}_T)$ is a sum of multiple integrals with respect to Y; so it only remains to prove that T is a stopping time for Y. For each $t \geq 0$, the indicator of the event $\{T \leq t\}$ is in both $L^2(\mathcal{F}_T)$ and $L^2(\mathcal{F}_t)$, so it is of the form

$$I\!P(T \leq t) + \sum_{k=1}^{\infty} \int_{0<t_1<...<t_k<t} f_k(t_1, ..., t_k) dX_{t_1} ... dX_{t_k}.$$

By proposition (1, ii) of [4] again, it is also equal to

$$I\!P(T \leq t) + \sum_{k=1}^{\infty} \int_{0<t_1<...<t_k<t} f_k(t_1, ..., t_k) dY_{t_1} ... dY_{t_k}$$

and this shows that T is a stopping time for Y.

Lemma 2.2. *Let T be a stopping time and X be a martingale defined on the interval $[0,T]$ only and verifying $< X,X >_t = t$ on this interval. The following conditions are equivalent.*

1) For some chaotic martingale M independent of X (and possibly defined on an enlargement of Ω), the martingale

$$Y_t = \begin{cases} X_t & \text{for } t \leq T \\ X_T + M_{t-T} - M_0 & \text{for } t \geq T \end{cases}$$

has the chaotic representation property.

2) Same statement as 1), with "for every M" instead of "for some M".

3) There exists a martingale $(X'_t)_{t>0}$ (possibly defined on an enlargement of Ω), verifying $< X',X' >_t = t$, chaotic on $[0,T]$, with restriction X to $[0,T]$.

4) Every martingale $(X'_t)_{t>0}$ (possibly defined on an enlargement of Ω), verifying $< X',X' >_t = t$, with restriction X to $[0,T]$, is chaotic on $[0,T]$.

Proof. The implications 2) $\Rightarrow$ 1) $\Rightarrow$ 3) are trivial and 3) is equivalent to 4) by Lemma 2.1. So it suffices to prove 3) $\Rightarrow$ 2). The proof is completely similar to the proof of Proposition (1, iii) of [4] and Corollary 2 of [4] except for one detail: With the notations of [4], X is no longer supposed to have the $C.R.P$ but only to be chaotic on $[0,T]$. So in the proof of $(1, iii)$, page 14, it is not obvious that there exists an element g in $Fock(H)$ such that

$$U = \int g(A) dX_A := \sum_{k=0}^{\infty} \int_{0<t_1<...<t_k} g_k(t_1, ..., t_k) dX_{t_1} ... dX_{t_k}.$$

But we know that $U = \int_{A \subset [T,\infty[} f(A) dX_A$, so for almost every A, $I\!E[f^2(A) 1_{A \subset [T,\infty[}]$ is finite, and the chaoticity of X on $[0,T]$ implies that there exists $h(B,A)$ such that $\int h(B,A) dX_B$ is equal to $f(A) 1_{A \subset [T,\infty[}$. Since $f(A) 1_{A \subset [T,\infty[} \in L^2(\mathcal{F}_{\inf A})$, then $h(B,A)$ is null if $\sup B > \inf A$ and the existence of g is obtained by putting

$$g(\{t_1, ..., t_k\}) = \sum_{i=1}^{k+1} h(\{t_1, ..., t_{i-1}\}, \{t_i, ..., t_k\})$$

this proves the lemma.

Definition. *Let T be a stopping time and X be a martingale defined only on the interval $[0,T]$ and verifying $< X, X >_t = t$ on this interval. We say that X is chaotic on $[0,T]$ if the four conditions of Lemma 2.2 are met.*

Lemma 2.3. *Let $(T_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of stopping times and T_∞ its limit.*

1)If a martingale $(X_t)_{t \geq 0}$ verifying $< X, X >_t = t$ is chaotic on each interval $[0, T_n]$, it is also chaotic on $[0, T_\infty]$.

2)Let X be a martingale defined only on $[0, T_\infty]$ and verifying $< X, X >_t = t$. If for each n the restriction of X to $[0, T_n]$ is chaotic on $[0, T_n]$, then X is chaotic on $[0, T_\infty]$

Proof. 1) For each n, we know that $L^2(\mathcal{F}_{T_n})$ is included in the chaotic space of X. As this chaotic space is closed and as, by the martingale convergence theorem, $\bigcup_n L^2(\mathcal{F}_{T_n})$ is dense in $L^2(\mathcal{F}_{T_\infty})$, the latter is also included in the chaotic space of X.

2) Using Lemma 2.2, it suffices to apply 1) to the martingale

$$Y_t = \begin{cases} X_t & \text{for } t \leq T_\infty \\ X_{T_\infty} + B_{t - T_\infty} - B_0 & \text{for } t \geq T_\infty \end{cases}$$

where B is a Brownian motion independent of X.

3.Construction of the martingale

This section is devoted to constructing the martingale X announced in the introduction.

The set $\Omega = \mathbb{R}_+^{\mathbb{N}}$ is the set of the sequences $\omega = (S_n, n \in \mathbb{N})$ with S_0 equal to zero and $S_n \in \mathbb{R}_+$ for all $n \in \mathbb{N}$.

The sequence ω defines the following increasing sequence :

$$T_n = \sum_{i=0}^{n} S_i \quad \text{for} \quad n \in \mathbb{N}.$$

Let $T_\infty = \lim_{n \to \infty} T_n$.

For $i \in \mathbb{N}$, let ϕ_{i+1} be a measurable $\mathbb{R}_*$ valued function defined on $\mathbb{R}_+^{i+1}$. We define the point process p_t by

$$p_t = \begin{cases} 0 & \text{for } t \in [0, T_1[\\ \sum_{j=1}^{i} \phi_j(T_j, ..., T_1) & \text{for } t \in [T_i, T_{i+1}[. \end{cases}$$

The process (p_t) generates the increasing family of σ-fields $\mathcal{F}_t^0$ defined by

$$\mathcal{F}_t^0 = \sigma(p_s, s \leq t), \quad \mathcal{F}^0 = \sigma(p_s, s > 0).$$

We use the following notations:

$$\Phi_{i+1}(t) = \phi_{i+1}(t, T_i, ..., T_1) \quad \text{for} \quad i \geq 1,$$

$$\Phi(t) = \Phi_{i+1}(t) \quad \text{if} \quad t \in]T_i, T_{i+1}].$$

We suppose that, for all $i \in \mathbb{N}$, there exists a $\mathcal{F}_{T_i}^0$ measurable positive function $\tau_{i+1} > T_i$, such that

$$\int_{T_i}^{t} \Phi_{i+1}^{-2}(s) ds < +\infty \text{ for } t \in [T_i, \tau_{i+1}[\quad \text{and} \quad \int_{T_i}^{\tau_{i+1}} \Phi_{i+1}^{-2}(s) ds = \infty.$$

The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}^0)$ is defined by the law of T_1, with density

$$\Phi_1^{-2}(t) exp\left\{-\int_0^t \Phi_1^{-2}(s)\, ds\right\} \; 1_{]0,\tau_1[}(t) dt$$

and the conditional law of T_{i+1}, with density

$$\Phi_{i+1}^{-2}(t) exp\left\{-\int_{T_i}^t \Phi_{i+1}^{-2}(s)\, ds\right\} \; 1_{]T_i,\tau_{i+1}[}(t) dt.$$

The σ-fields $\mathcal{F}_t^0$ are augmented with all subsets of $\mathbb{P}$-null sets of $\mathcal{F}^0$ and denoted by $\mathcal{F}_t$. For all $i \in \mathbb{N}$, T_i is a stopping time of $(\mathcal{F}_t)$.

Proposition 3.1. *Let $N(dt, dx)$ be the random measure on $\mathbb{R}_+ \times \mathbb{R}_*$ defined for $t > 0$ and A a measurable set of $\mathbb{R}_*$ by*

$$N(]0,t] \times A) = \sum_{T_n \leq t} 1_A(\, \Phi_n(T_n)\,).$$

The predictable projection of $N(dt, dx)$ with respect to the probability $\mathbb{P}$ is given by

$$\nu(dt, dx) = \Phi^{-2}(t) 1_{[0,T_\infty[}(t)\, dt\, \delta_{\Phi(t)}(dx).$$

Proof. Let $n \in \mathbb{N}_*$, f be a bounded measurable function on $\mathbb{R}_+^n$ and g be a bounded measurable function on $\mathbb{R}$.

Let us consider the predictable process

$$Z(t,x) = 1_{]T_n,T_{n+1}]}(t) f(T_1, ..., T_n) g(x).$$

We have to prove that

$$\mathbb{E}[\int_0^\infty \int_{\mathbb{R}*} Z(t,x) N(dt, dx)\,] = \mathbb{E}[\int_0^\infty Z(\,t, \Phi(t)\,)\Phi^{-2}(t) dt].$$

From the equality

$$\int_0^\infty \int_{\mathbb{R}*} Z(t,x) N(dt, dx) = f(T_1, ..., T_n) g(\, \Phi_{n+1}(T_{n+1})\,),$$

and using the conditional law of T_{n+1}, with respect to $(T_1, ..., T_n)$, we obtain

$$\mathbb{E}[\int_0^\infty \int_{\mathbb{R}*} Z(t,x) N(dt, dx)\,]$$

$$= \mathbb{E}\left[f(T_1, ..., T_n) \int_{T_n}^{\tau_{n+1}} g(\Phi_{n+1}(t_{n+1}))\, \Phi_{n+1}^{-2}(t_{n+1}) \right.$$
$$\left. exp\left(-\int_{T_n}^{t_{n+1}} \Phi_{n+1}^{-2}(s) ds\right)\, dt_{n+1} \right].$$

An integration by parts gives

$$\int_{T_n}^{T_{n+1}} \int_{T_n}^{t_{n+1}} g\left(\Phi_{n+1}(t)\right) \Phi_{n+1}^{-2}(t)\, dt$$
$$\left\{ \Phi_{n+1}^{-2}(t_{n+1})\, exp\left(-\int_{T_n}^{t_{n+1}} \Phi_{n+1}^{-2}(s) ds\right)\right\} dt_{n+1}$$
$$= \int_{T_n}^{T_{n+1}} g\left(\Phi_{n+1}(t_{n+1})\right) \Phi_{n+1}^{-2}(t_{n+1})\, exp\left(-\int_{T_n}^{t_{n+1}} \Phi_{n+1}^{-2}(s) ds\right) dt_{n+1}.$$

Thus,

$$I\!E[\,f(T_1,...,T_n) \int_{T_n}^{T_{n+1}} g\left(\Phi_{n+1}(t)\right)\right) \Phi_{n+1}^{-2}(t)\, dt\,]$$
$$= I\!E[\,f(T_1,...,T_n)\, g\left(\Phi_{n+1}(T_{n+1})\right)],$$

which is exactly what was to be proved.

Proposition 3.2. *Let $m \in I\!R$.*

1) The process (X_t) defined on the predictable interval $[0, T_\infty[$ by

$$X_t = m + p_t - \int_0^t \Phi(s)^{-1} ds$$

is a $(\mathcal{F}_t)$ square integrable martingale with $< X, X >_t = t$, it verifies the structure equation

$$d[X, X]_t = dt + \Phi(t) dX_t, X_0 = m.$$

2) When the definition of X is extended to $[0, T_\infty]$ by $X_{T_\infty} = \lim_{n\to\infty} X_{T_n}$ on $T_\infty < \infty$, the martingale X has the chaotic representation property on $[0, T_\infty]$.

In the case when $T_\infty = \infty$ *a.s.*, the chaotic property 2) is a consequence of the Theorem 5 of [4].

Proof. 1) Since X_t is also equal to

$$X_t = m + \int_0^t \int_{I\!R_*} x p(dx, dt) - \int_0^t \int_{I\!R_*} x \nu(dx, dt)$$

by Proposition 3.1, X is a martingale with predictable bracket $< X, X >_{t\wedge T_\infty} = t \wedge T_\infty$ and satisfies the structure equation

$$d[X, X]_t = 1_{[t<T_\infty]}\, dt + \Phi(t) dX_t, \; X_0 = m.$$

2)By Lemma 2.3, it suffices to verify that, for each finite n, X is chaotic on $[0, T_n]$. Define a martingale X^n by the same construction as X, but with $\phi_i \equiv 1$ for $i > n$. The martingale M^n is identical in law to X on $[0, T_n]$ and is a compensated standard Poisson process after T_n. It has the chaotic representation property by Theorem 5 of [4]; this implies in particular that it is chaotic on $[0, T_n]$. So the restriction of X to $[0, T_n]$ is chaotic by Lemma 2.2, and X is chaotic on $[0, T_\infty]$ by Lemma 2.3.

4.Examples

Let $(\lambda_n, n \in I\!N*)$ be a sequence of strictly positive real numbers and $(T_n, n \in I\!N*)$ be the successive jumps such that the sojourn times $(T_{n+1} - T_n, n \in I\!N)$ being independent exponentially distributed variables. The density of $T_n - T_{n-1}$ is

$$\lambda_n e^{-\lambda_n t}.$$

When

$$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty,$$

T_∞ is finite almost surely; or else, it is infinite almost surely. For $t \in [T_{n-1}, T_n[$ The predictable process Φ is given by
$\Phi(t) = \sqrt{\lambda_n^{-1}}$ and the martingale X by

$$X_t = -\sqrt{\lambda_n}(t - T_{n-1}) + \sum_{i=1}^{n-1} \left(\sqrt{\lambda_i^{-1}} - \sqrt{\lambda_i}(T_i - T_{i-1})\right).$$

It is chaotic on $[0, T_\infty]$ by the preceding proposition.

Another example is given by the structure equation

$$d[X,X]_t = dt + f(X_{t-})dX_t, \quad X_0 = m$$

with m is such that $f(m) \neq 0$ and f is a deterministic continuous function. Let $T_\infty = \inf\{t > 0, X_t = 0\}$, for $t < T_\infty$, X_t can be constructed as follows: let $(T_n, n \in I\!N*)$ be the jump times of X_t, and suppose that the integral equation

$$x_t = f(X_{T_n} - \int_{T_n}^t x_s^{-1} ds),\ t > T_n,$$

has a unique solution $t \longrightarrow \Phi_{n+1}(t, X_{T_n}, \tau_{n+1})$ on the widest interval $[T_n, \tau_{n+1}[$ of $[T_n, \infty[$ where x_t is defined.

If x_t is such that

$$\int_{T_n}^t x_s^{-2} ds < \infty, \quad \text{for } t \in [T_n, \tau_{n+1}[\quad \text{and} \int_{T_n}^{\tau_{n+1}} x_s^{-2} ds = \infty,$$

then we can see that $x_{T_{n+1}} = \Delta X_{T_{n+1}}$ is the jump size at T_{n+1},

$$X_{T_{n+1}} = X_{T_n} + \Phi_{n+1}(T_{n+1}, X_{T_n}, \tau_{n+1}) - \int_{T_n}^{T_{n+1}} \Phi_{n+1}^{-1}(s, X_{T_n}, \tau_{n+1}) ds,$$

and for $t \in [T_n, T_{n+1}[$,

$$X_t = X_{T_n} - \int_{T_n}^t \Phi_{n+1}^{-1}(s, X_{T_n}, \tau_{n+1}) ds.$$

If we put $T_0 = 0$, then for all $n \in I\!N$ the law of T_{n+1}, with respect to $(T_0, ..., T_n)$, is supported by $]T_n, \tau_{n+1}[$ and has the density

$$\Phi_{n+1}^{-2}(t, X_{T_n}, \tau_{n+1}) exp\left\{- \int_{T_n}^t \Phi_{n+1}^{-2}(s, X_{T_n}, \tau_{n+1})\, ds\right\}.$$

By Proposition 3.2, X is chaotic on $[0, T_\infty]$.

If $f(x) = \beta x$ we find again the Azéma martingale with parameter $\beta \notin \{-1, 0\}$ on the interval $[0, T_\infty]$, where T_∞ is the first time when $X = 0$ (T_∞ is also the first accumulation point of jump times of X).

Remark.

The solution of the differential equation $x_t = f(a - \int_0^t x_s^{-1} ds)$ allows us to construct the martingale X on $[0, T_\infty]$; the existence and the uniqueness of the solution of this equation implies the existence and the uniqueness in law of X on $[0, T_\infty]$.

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