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Some Markov properties of Stochastic Differential Equations with jumps

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1 Introduction

In previous articles [3] [4] we were interested in SDE's on manifolds driven by non continuous semimartingales; we got an extension of Meyer-Schwartz second order calculus [16], [19]. The main idea to deal with macroscopic jumps was to substitute 2-jets at x (only local behaviour is needed in the continuous case) by functions that are twice differentiable at x . In this setting a typical coefficient of SDE is a function φ from $V \times W \times V$ to W that describes how the solution Y jumps when the driving semimartingale X has a jump, we write :

$$Y_t = \varphi(X_{t^-}, Y_{t^-}, X_t) \quad \text{if } X_{t^-} \neq X_t.$$

In [3] the existence and uniqueness of a strong solution are obtained for such SDE's with locally Lipschitz coefficients, and the stochastic development of a càdlàg semimartingale in the tangent space of a Riemannian manifold was presented as an example of a SDE with jumps in [4]. Here we will be concerned with Markov properties of solutions, when the driving process is a Lévy process living in $\mathbf{R}^v$, more precisely we will compute their infinitesimal generator. This type of processes was already studied by Fujiwara, Applebaum [8] [1] but the techniques used were completely different; they construct Lévy flows on manifolds. Since we have not studied flows for SDE's with jumps we will not go further in this direction. In [18] Rogerson has defined the α-stable process with values in a Riemannian manifold M as a Brownian motion time-changed by a suitable subordinator; he has also constructed another process, called pseudo α-stable, as the stochastic development in M of a vector valued α-stable process in $T_{x_0}M$. These definitions agree when M is a Euclidean vector space but not in general. We will apply results on SDE's with jumps to compare pseudo and α-stable processes. In Section 2 we recall the existence and uniqueness theorem, and its application to stochastic development. Then we exhibit sufficient conditions for the solution to be Markovian, and compute its infinitesimal generator. In section 4 we give a probabilistic proof that pseudo α-stable and α-stable processes do not have the same law if the manifold is Riemannian with a pole and a rotationally invariant metric. We have to assume that $\Delta r < \frac{dim(M)-1}{r}$ (respectively >) where r represents the distance from the pole because we are studying the radial part of a Brownian motion. This evidently includes sphere in all dimensions. On the sphere in dimension 2, it is worth mentioning that $\Delta/2$ and the infinitesimal generator of the pseudo 1-stable process are linked via a formula involving a concave, piecewise affine function. As a consequence of this remark the pseudo 1-stable process on the 2-sphere is not a Brownian motion time-changed by a subordinator.

2 Summary on SDE's with jumps.

All filtered spaces $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$ will verify the so called "usual conditions". If H is a real valued process, a new process starting at zero is defined by :

$$\mathbb{S}(H)_t = \sum_{0<s\leq t} H_s$$

if the sum is absolutely convergent, otherwise $\mathbb{S}(H)_t = \infty$.

All manifolds will possess a countable atlas and we will use Einstein's convention for summation.

To introduce SDE's with jumps we define first constrained coefficients of SDE's with jumps.

Definition 1 *Suppose that C is a closed submanifold of $V \times W$, such that the projection p_1 from C to V be onto and submersive. A measurable application φ from $C \times V$ to W will be called a constrained coefficient of SDE's with jumps if*

- *for each $z = (x, y)$ in C, $\varphi(z, x) = y$*
- *φ is C^3 in a neighborhood of $\{(z, p_1(z)) / z \in C\}$*
- *$\forall x \in V, \forall z \in C;\ (x, \varphi(z, x)) \in C$.*

Remark 1 *The particular case when $C = V \times W$ can be interpreted as the unconstrained case .*

The following existence and uniqueness result was shown in [3].

Theorem 1 *Take a constrained coefficient φ from C on to V, a semimartingale X living on V. We will note a SDE with jumps*

$$\begin{cases} \overset{\Delta}{d}Y &= \Phi(Y, \overset{\Delta}{d}X) \\ Y_0 &= y_0, \end{cases} \tag{1}$$

Equation (1) admits a unique solution (Y_{sol}, η), where η is the previsible stopping time of explosion of Y_{sol}, with the meaning that if $(x^i)_{i=1 \text{ to } v}$, $(y^\alpha)_{\alpha=1 \text{ to } w}$ are two C^2

imbeddings from V to $\mathbf{R}^v$, respectively from W to $\mathbf{R}^w$

$$\forall\alpha = 1 \text{ to } w \quad Y^\alpha = y_0^\alpha + \int \frac{\partial\varphi^\alpha}{\partial x^i}((X_-,Y_-),X_-)dX^i + 1/2\int \frac{\partial^2\varphi^\alpha}{\partial x^i\partial x^j}((X_-,Y_-),X_-)d\langle (X^i)^c,(X^j)^c\rangle + \mathcal{S}(\varphi^\alpha((X_-,Y_-),X) - Y_-^\alpha - \frac{\partial\varphi^\alpha}{\partial x^i}((X_-,Y_-),X_-)\Delta X^i). \tag{2}$$

Remark 2 *Because of equation (2) it is clear that the solution jumps only when X has a jump, and $Y_t = \varphi(X_{t^-}, Y_{t^-}, X_t)$ if $X_{t^-} \neq X_t$, which gives an intuitive interpretation of the coefficient φ.*

As an example of such SDE's stochastic development of semimartingales with jumps has been presented in [4]. For continuous processes this method had been used to obtain a Brownian motion on a Riemannian manifold from a flat Brownian motion. Stochastic development has been extended to discontinuous driving process in [18] [6]. We suppose that M is a complete connected Riemannian manifold with a C^3 atlas and dimension m, $O(M)$ will be the orthonormal frame bundle, and x_0 a reference point of M. With these notations $R_0 \in O_{x_0}(M)$ will be an isomorphism from $T_{x_0}M$ to $\mathbf{R}^m$. Two steps are necessary to develop a deterministic or random curve : solve a SDE between $V = \mathbf{R}^m$ and $W = O(M)$, then project the solution onto M. Let us describe the coefficient we need for the first step when the driving process may have jumps. The process $R_0(X_t)$ is the driving semimartingale in $\mathbf{R}^m$, and the coefficient of this SDE is constructed with the Riemannian exponential. If x, x' are two points in $\mathbf{R}^m$ and if R belongs to $O_\xi(M)$ we define $\varphi(x,(\xi,R),x')$ as the result of parallel transporting R along the geodesic $exp_\xi(tR^{-1}(x-x'))$ until time $t = 1$. In this example no constraint is needed. The equation

$$\begin{cases} \overset{\Delta}{d}Y &= \Phi(Y,\overset{\Delta}{d}X) \\ Y_0 &= (x_0,R_0) \end{cases}$$

has a solution Y until the explosion time η. We call this process Y the horizontal lift of X and $\pi(Y) = y$ is the developed curve associated to X where $\pi : O(M) \longmapsto M$ is the trivial bundle projection. If X is smooth enough y is nothing else than the usual developed curve of textbooks in Mechanics. This setting includes continuous stochastic development of Brownian motions that leads to horizontal Brownian motion as in [5]. In Section 4 we will study with some care what happens when X is the α-stable process, extending results of [18].

3 Markov solutions to SDE's with Jumps

In a vector space, the pair of the driving process and the solution is Markovian, when the driving process is Markovian. But if you want the solution alone to be Markovian we "practically" have to suppose that the driving process is a Lévy process. A precise formulation of this is given by Theorem 32 in [17], and by [14] for the converse. Therefore in this geometric setting we will suppose that the driving Lévy process lives in a vector space. Actually only the group structure is needed. Furthermore if X jumps from X_{t^-} to X_t, solution will not explicitly depend on X_{t^-} but on jumps $X_t - X_{t^-}$.

Let us introduce classical notations for Markov processes. Consider an homogeneous Markov V-valued process X_t and its filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Suppose the existence of a family of transition probabilities $P(t, x, A)$ defined for all x, $0 \leq s \leq t$, and Borel sets A such that

$$E(X_t \in A|\mathcal{F}_s) = E(X_t \in A|X_s) = P(t-s, X_s, A) \;\; P \;\; ps$$

for all s, t $0 \leq s \leq t$ and Borel sets A. Usual hypothesis is assumed for transition probabilities. Because of the possible explosion of solutions of SDE's we have to consider submarkov transition probabilities. Adding to V a cemetery point is the usual trick to deal with such problem. Last we state the classical representation for Lévy processes see [15].

Theorem 2 *Let X be a càdlàg Lévy process living in $\mathbf{R}^v$, then X has a decomposition*

$$X_t = at + \sigma B_t + \int_{\{|x|<1\}\times[0,t]} \{\nu(dx, ds) - \pi(dx)ds\} + S_t(\Delta X 1_{\{|\Delta X| \geq 1\}}) \qquad (3)$$

where a is a vector in $\mathbf{R}^v$, σ belongs to $\mathcal{L}(\mathbf{R}^{\subseteq}, \mathbf{R}^{\subseteq})$, B is a $\mathbf{R}^v$ Brownian motion; $\nu(dx, ds)$ a vector Poisson measure and π a measure on $\mathbf{R}^v$ such that

$$E(\nu(A \times [0,t])) = t\pi(A) \;\; avec \;\; A \in \mathcal{B}(\mathbf{R}^{\subseteq}).$$

In the next proposition we exhibit sufficient conditions for the solution of a SDE to be a Markov process. The emphasis is put on the non-linear treatment of jumps in the infinitesimal generator.

Proposition 1 *Consider a unconstrained coefficient φ. Suppose that the driving process lives in $\mathbf{R}^v$, y_0 is a random variable on W, φ is a function of the increment $\varphi(x, y, z) = \psi(y, z - x)$, and the driving semimartingale is a Lévy process with a decomposition as in Theorem 2. The solution of*

$$\begin{cases} \overset{\triangle}{d}Y &= \Phi(Y, \overset{\triangle}{d}X) \\ Y_0 &= y_0 \end{cases}$$

is an homogeneous Markov process with transition probability

$$P(t, x, C) = P(Y_t^y \in C)$$

where $y \in W$ and Y^y verifies

$$\begin{cases} \overset{\triangle}{d}Y^y &= \Phi(Y^y, \overset{\triangle}{d}X) \\ Y_0^y &= y. \end{cases}$$

Furthermore if $f \in C^2(W), f(Y) - f(Y_0) - \int_0 Af(Y_s)ds$ is a local martingale and A is defined by

$$\begin{aligned} Af(y) = \langle df(y), \frac{\partial \psi}{\partial x}(y,0)a\rangle + 1/2\, Tr(Hess(f).(\frac{\partial \psi}{\partial x}(y,0)\sigma)(\frac{\partial \psi}{\partial x}(y,0)\sigma)^*) \\ + \int_{\mathbf{R}^v} \{f(\psi(y,x)) - f(y) - df(y)\frac{\partial \psi}{\partial x}(y,0)1\{|x|<1\}\}\pi(dx) \end{aligned} \qquad (4)$$

where $\frac{\partial \psi}{\partial x}(y,0)$ stands for $\frac{\partial \psi}{\partial x}(y,x)|_{x=0}$ and maps $\mathbf{R}^v$ to V.

Proof : Choose an imbedding $(y^\alpha)_{\alpha=1 \text{ to } w}$, and recall the system (2). For all $\alpha =$ 1 to w on $[[0,\eta[[$ we get

$$
\begin{aligned}
Y^\alpha &= y_0^\alpha + \int_0 \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)dX_s^i \\
&+1/2 \int_0 \frac{\partial^2 \psi^\alpha}{\partial x^i \partial x^j}(Y_{s^-},0)d\langle (X^i)^c,(X^j)^c\rangle_s \\
&+\mathsf{S}(\psi^\alpha(Y_-,0) - Y_-^\alpha - \frac{\partial \psi^\alpha}{\partial x^i}(Y_-,0)\Delta X^i).
\end{aligned} \tag{5}
$$

Replace X by its decomposition to obtain

$$
\begin{aligned}
&\forall \alpha = 1 \text{ to } w \quad \text{on } [[0,\eta[[\\
&Y^\alpha = y_0^\alpha + \int_0 \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)a^i ds + \int_0 \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)(\sigma dB)_s^i \\
&+1/2 \int_0 \frac{\partial^2 \psi^\alpha}{\partial x^i \partial x^j}(Y_{s^-},0)(\sigma\sigma^*)_{i,j} ds \\
&+ \int_{[0,.]\times\{|x|<1\}} \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)x^i\{\nu(dx,ds) - \pi(dx)ds\} \\
&+\mathsf{S}(\frac{\partial \psi^\alpha}{\partial x^i}(Y_-,0)\Delta X^i 1_{|\Delta X|\geq 1}) \\
&+\mathsf{S}(\psi^\alpha(Y_-,0) - Y_-^\alpha - \frac{\partial \psi^\alpha}{\partial x^i}(Y_-,0)\Delta X^i).
\end{aligned} \tag{6}
$$

We can write this expression with Poisson measures

$$
\begin{aligned}
Y^\alpha &= y_0^\alpha + \int_0 \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)a^i ds + \int_0 \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)(\sigma dB)_s^i \\
&+1/2 \int_0 \frac{\partial^2 \psi^\alpha}{\partial x^i \partial x^j}(Y_{s^-},0)(\sigma\sigma^*)_{i,j} ds \\
&+ \int_{[0,.]\times\{|x|<1\}} \frac{\partial \psi^\alpha}{\partial x^i}(Y_{s^-},0)x^i\{\nu(dx,ds) - \pi(dx)ds\} \\
&+ \int_{[0,.]\times\{|x|<1\}} \{\psi^\alpha(Y_-,0) - Y_-^\alpha - \frac{\partial \psi^\alpha}{\partial x^i}(Y_-,0)\Delta X^i)\}\nu(dx,ds).
\end{aligned} \tag{7}
$$

SDE's with jumps are translated in Rogerson's language, they become SDE's with Poisson jumps, and Rogerson [18] proves in theorem 3 p 4.4 that the solution is a homogeneous Markov process. He proceeds as in the classical Gihman Skorokhod's demonstration [10], the main steps are similar to those for diffusions : first verify the measurability of Y_t^y with respect to (y,t), and conclude using the flow property of the solution. However there is an additional technical problem with the explosion. The computation of the infinitesimal generator is an immediate consequence of Itô's formula. ◇

4 Pseudo α-stable and α-stable processes

Stochastic development of jump processes was presented in Section 2 . Rogerson in [18] tried to develop symmetric α-stable processes in order to find a Markov process with infinitesimal generator $-1/2\ (-\Delta)^{\alpha/2}$ on the Riemannian manifold. But what works for continuous processes like Brownian motion fails when a jump occurs. Rogerson has depicted the situation with a non-commutative diagram : the vertical arrows correspond to the stochastic development, the horizontal one to subordination. A classical construction of a Markov process with the infinitesimal generator $-1/2\ (-\Delta)^{\alpha/2}\ \ \alpha \in]0,2[$ is achieved in [7] with a pair of independent processes (B, T^{α}), where B is a Brownian motion, and T^{α} a one sided $\alpha/2$- stable $\mathbf{R}^{+}$-valued process. The subordination is simply the time-change consisting in taking B at random time T_t^{α}, and it works on any state space whatsoever as soon as a Laplacian is defined on it.

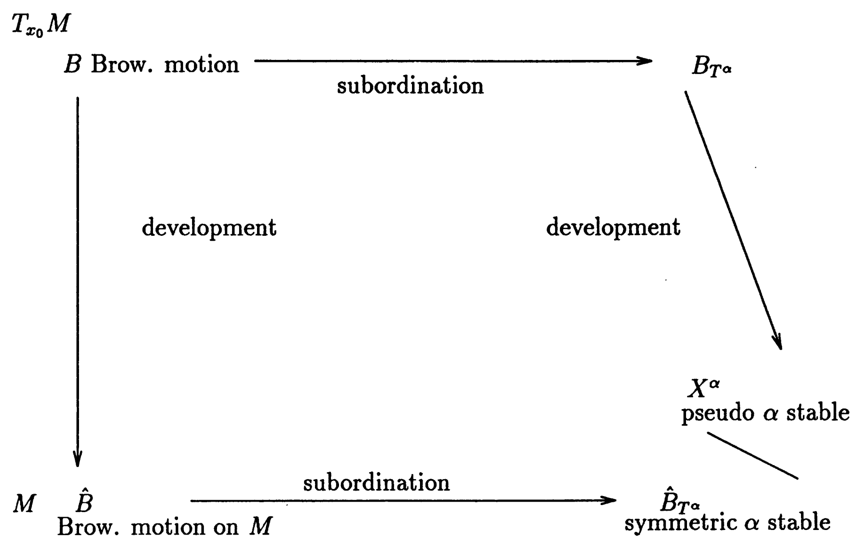

Figure 1: non commutative diagram

The framework of SDE's with jumps explains that the laws of pseudo α-stables and α-stables processes differ because of the curvature effect. When the driving semimartingale jumps, the stochastic development uses the interpolation between $B_{T_{t-}^{\alpha}}$ and $B_{T_t^{\alpha}}$ instead of the Brownian curve from time T_{t-}^{α} until T_t^{α}. But one knows that two curves that possess the same ending points are not necessarily mapped to curves with the same ending points. We can state this claim precisely as soon as we can compare how fast the Brownian motion goes to infinity in a vector space and on the particular Riemannian manifold. The radial part of the Brownian motion will be the basic tool of the next result.

Theorem 3 *Take a Riemannian connected manifold V with dimension $v > 1$. Suppose that V has a pole x_0 : in polar geodesic coordinates centered at x_0, the metric has the form $ds^2 = dr^2 + g^2(r)d\theta$, suppose also that $\Delta_y d(x,y) < \frac{v-1}{d(x,y)}$ for all x, y in a normal neighborhood of x_0 (respectively $>$). The laws of pseudo α-stable and α-stable processes are distinct.*

Proof : Assume that exp_{x_0} is a diffeomorphism from the ball $B_\rho(T_{x_0}\mathbf{R}^v)$ with center x_0 and diameter $\rho < \infty$ onto $\mathcal{V}$, we will choose ρ small enough such that there exists no (x,y) belonging to $\mathcal{V}$ and y conjugate to x. A Brownian motion B^V living on V with a starting point x_0 is constructed with a stochastic development, and we call τ the first exit time of B^V out of $\mathcal{V}$. Taking a one-sided α-stable process T^α independent of B^V, we define a sequence of stopping times

$$T_n = \inf\{t \geq 0 / d^V(B^V_{T^\alpha_{t-}}, B^V_{T^\alpha_t}) > 1/n\},$$

where d^V is the geodesic distance on V. The limit of T_n is almost surely 0, the sequence of events $A_n = \{\tau > T^\alpha_{T_n}\}$ is non decreasing, and $P(\bigcup_n A_n) = 1$. Applying the Itô formula to the function $d^V(B^V_{(T^\alpha_{T_n})^-}, x)$ when $s > (T^\alpha_{T_n})^-$, leads to

$$dr^V_s = dW_s + 1/2\frac{(v-1)g'(r^V_s)}{g(r^V_s)}ds \tag{8}$$

if $(T^\alpha_{T_n})^- < s < \tau$ and $r^V_s = d^V(B^V_{(T^\alpha_{T_n})^-}, B^V_s)$. In equation (8) W is a real valued Brownian motion, and one can construct a $T_{x_0}V$ Brownian motion, with $r_s = d(B_{(T^\alpha_{T_n})^-}, B_s)$ satisfying

$$dr_s = dW_s + 1/2\frac{(v-1)}{r_s}ds. \tag{9}$$

Therefore thanks to the comparison Theorem of solutions of SDE's, we know that

$$d^V(B^V_{(T^\alpha_{T_n})^-}, B^V_.) < d(B_{(T^\alpha_{T_n})^-}, B_.)$$

on $]](T^\alpha_{T_n})^-, \tau[[$ and

$$d^V(B^V_{(T^\alpha_{T_n})^-}, B^V_{T^\alpha_{T_n}}) < d(B_{(T^\alpha_{T_n})^-}, B_{T^\alpha_{T_n}}) \text{ on } A_n.$$

But when $B_{(T^\alpha)}$ jumps, the pseudo α-stable process X^α jumps too and remark 2 implies $d^V(X^\alpha_{T_n^-}, X^\alpha_{T_n}) = d(B_{(T^\alpha_{T_n})^-}, B_{T^\alpha_{T_n}})$. Hence almost surely for n big enough

$$d^V(B^V_{(T^\alpha_{T_n})^-}, B^V_{T^\alpha_{T_n}}) < d^V(X^\alpha_{T_n^-}, X^\alpha_{T_n}).$$

Consequently if $\tilde{T}_n = \inf\{t \geq 0 / d^V(X^\alpha_{t-}, X^\alpha_t) > 1/n\}$, we know that $\tilde{T}_n \leq T_n$ and when $\tilde{T}_n = T_n$ then $d^V(B^V_{(T^\alpha_{\tilde{T}_n})^-}, B^V_{T^\alpha_{\tilde{T}_n}}) < d^V(X^\alpha_{\tilde{T}_n^-}, X^\alpha_{\tilde{T}_n})$. If $P(\exists n \in \mathbf{N}, \tilde{T}_n < T_n) > 0$ we take the conditional probability $P(\ |\exists n \in \mathbf{N}, \tilde{T}_n < T_n)$, $B^V_{T^\alpha}$ and X^α do not have the same conditional law, since $\tilde{T}_n$ is obtained with the same construction from X^α as T_n from $B^V_{T^\alpha}$. But if we consider $P_2 = P(\ |\forall n \in \mathbf{N}, \tilde{T}_n = T_n)$ assuming $P(\forall n \in \mathbf{N}, \tilde{T}_n = T_n) > 0$, then $B^V_{T^\alpha}$ and X^α do not have the same law on $(\Omega, P_2, \mathcal{F})$ because $d^V(B^V_{(T^\alpha_{T_n})^-}, B^V_{T^\alpha_{T_n}}) < d^V(X^\alpha_{\tilde{T}_n^-}, X^\alpha_{\tilde{T}_n})$. So the laws of pseudo α-stable and α-stable processes are distinct. The same proof works when inequalities are reversed.
◇

5 Spectral comparison between pseudo α-stable and α-stable processes

Rogerson proved the same theorem in the special case of sphere in dimension 2, but he used different techniques : he compared the spectra of the infinitesimal generator of both processes. In this particular case the spherical harmonic functions are the eigenfunctions for the pseudo α-stable infinitesimal generator A_α. This phenomenon is quite general and it is a consequence of a geometrical remark. It is clear that both processes have their law invariant under a rotation preserving the starting point. We should consider this remark as a hint to study those processes on symmetric spaces. As a conclusion we will exhibit a fairly strange geometrical property of the spectrum of the pseudo α-stable process that prevents it to be a subordinated process of a Brownian motion.

The next proposition computes the infinitesimal generator of the pseudo α-stable process, it is a mere consequence of Proposition 1 applied to the first step in stochastic development. For the second step we use the rotational invariance of the Lévy measure of an α-stable process to get the Markov property of the projected process from $O(V)$ onto V.

Proposition 2 *If* $f \in \mathcal{D}(\mathcal{A}_\alpha) \bigcap \mathcal{C}^\epsilon_{\mathfrak{l}}(\mathcal{V})$

$$A_\alpha f(x) = \int_{T_x V} \{f(exp_x(u)) - f(x) - df(x).u1_{\|u\|<1}\}\pi(du) \tag{10}$$

where $\pi(du) = \frac{C(\alpha,v)}{\|u\|^{\alpha+v}}du$ *is the Lévy measure associated to* α*-stable symmetric process in* $\mathbf{R}^v$.

Geometers working on symmetric spaces usually introduce the spherical functions as the eigenfunctions of all differential operators that are invariant by the action of the isometry group. We can show that the spherical functions are also eigenfunctions of A_α, although A_α is not a differential operator. We first have to recall basic facts and notation for globally symmetric spaces.

Definition 2 *Let* V *be an analytic Riemannian manifold;* V *is called a symmetric space if each* $x \in V$ *is an isolated fixed point of an involutive isometry* s_x.

Symmetric spaces are analytically diffeomorph to the quotient of the connected component $G = I_0(V)$ of the isometry group which contains the identity by the subgroup K of G for which x_0 is a fixed point of V

$$\begin{array}{rcl} G/K & \longmapsto & V \\ gK & \longmapsto & g(x_0). \end{array}$$

On the other hand G acts on G/K by $(g, hK) \longmapsto \tau(g)(hK)$ where $\tau(g) : hK \longmapsto ghK$, and $D(G/K)$ represents the space of differential operators on G/K invariant under the action of G. We get an easy example with the Laplace Beltrami operator of V when it is identified to G/K with its Riemannian structure.

If *proj* is the projection of G onto G/K recall the definition of spherical functions in [13].

Definition 3 *Let φ be a complex-valued function on G/K of class C^∞ which satisfies $\varphi(proj(e)) = 1$, φ is called a spherical function if*

(i) $\varphi^{\tau(k)} = \varphi \quad \forall k \in K$.

(ii) $D\varphi = \lambda_D \varphi \quad \forall D \in D(G/K)$.

Although A_α is not a differential operator, its law is invariant under the action of K, and the spherical functions are still eigenfunctions.

Proposition 3 *Let φ be a spherical function which belongs to the domain of A_α. We get*

$$A_\alpha(\varphi) = \lambda_{A_\alpha}\varphi.$$

Proof : Take $x_0 = proj(e)$

$$A_\alpha(\varphi)(x_0) = \int_{T_{x_0}V} \{\varphi(exp_{x_0}(u)) - 1 - d\varphi(x_0).u1_{\|u\|<1}\}\pi(du)$$

thanks to symmetry of π we can write

$$A_\alpha(\varphi)(x_0) = 1/2 \int_{T_{x_0}V} \{\varphi(exp_{x_0}(u)) + \varphi(exp_{x_0}(-u)) - 2\}\pi(du).$$

We have then to solve the problem in G. Theorem 3.3 in [12] explains that Riemannian exponential of a symmetric space may be expressed with Exponential of the Lie group G. Take the notations of [12], $\mathfrak{G}$ is the Lie algebra associated to G, $\mathfrak{T}$ with K, and $\mathfrak{P}$ satisfies $\mathfrak{G} = \mathfrak{T} \oplus \mathfrak{P}$, moreover $d(proj)_e$ is an isomorphism from $\mathfrak{P}$ to $T_{x_0}V$. If we note $\tilde{\varphi} = \varphi \circ proj$, we express $A_\alpha(\varphi)(x_0)$ as an integral on $\mathfrak{P}$ with

$$A_\alpha(\varphi)(x_0) = 1/2 \int_{\mathfrak{P}} \{\tilde{\varphi}(Exp(u)) + \tilde{\varphi}(Exp(-u)) - 2\tilde{\varphi}(e)\}\pi(du).$$

With a technical transformation it appears as

$$A_\alpha(\varphi)(x_0) = 1/2 \int_K \int_{\mathfrak{P}} \{\tilde{\varphi}(kExp(u)) + \tilde{\varphi}(kExp(-u)) - 2\tilde{\varphi}(e)\}\pi(du)dk$$

where dk stands for the Haar measure on K. If we want to compute $A_\alpha(\varphi)(x)$ where $x = proj(h)$, we apply an isometry h which maps x_0 onto x and we get

$$A_\alpha(\varphi)(x) = 1/2 \int_K \int_{\mathfrak{P}} \{\tilde{\varphi}(hkExp(u)) + \tilde{\varphi}(hkExp(-u)) - 2\tilde{\varphi}(h)\}\pi(du)dk.$$

One knows that the spherical functions are solutions of the following functional equation (Proposition 2.2 [13])

$$\forall g, h \in G \quad \int_K \tilde{\varphi}(gkh)dk = \tilde{\varphi}(g)\tilde{\varphi}(h). \tag{11}$$

So we get

$$A_\alpha(\varphi)(x) = (A_\alpha(\varphi)(x_0))\tilde{\varphi}(h)$$

as we claimed. ◇

A probabilistic approach of this problem is related to the study of semi groups that are invariant under action of K as presented in [9][11]. Let (X_t^x) be a Markov process starting from x the law of X^x is assumed to be invariant under the action of the subgroup of isometries fixing x. The commutation property of the semi groups (P_t) and (Q_s) corresponding to another process Y^y of same type can be expressed as follows :

$$P_t(Q_s(f))(x) = E_\omega(Q_s(f)(X_t^x(\omega))) = E_\omega(E_{\omega'}(f(Y_s^{X_t^x(\omega)}(\omega')))).$$

But if G/K is put for V, the law of $X_t^{Y_s^x}$ is nothing but the law of $\tau(Y_s^x).(X_t^x)$ on G/K, this law is called in [9] the convolution of Y_s^x by X_t^x. The convolution commutes on the symmetric spaces (we can find the proof in [9][11]), and it implies the commutation of P_t , Q_s. It is the same phenomenon than in Proposition 3 expressed on the law of random variables and not on infinitesimal generators.

Nevertheless more can be deduced from the spectral study of those processes. We will show that there is no subordinator T such that the law of the pseudo 1-stable process on the sphere in dimension 2 is the same as that of B_T where B is a S^2-Brownian motion and (B,T) are independent. We will introduce the Bernstein functions by considering the Laplace transform of subordinators.

Definition 4 *If T_t is a real non decreasing càdlàg Lévy process*

$$E(e^{-pT_t}) = e^{-\psi(p)t}$$

where ψ is a Bernstein function.

We get in [2] two other characterizations of the Bernstein functions.

Definition 5 *A function $\psi \in C^\infty(]0,\infty[,\mathbf{R})$ is called a Bernstein function if*

$$\psi \geq 0,\ (-1)^p\psi^{(p)} \leq 0 \quad \forall p \geq 1\ ;$$

but there is also an integral representation of these functions built on the same pattern as the Lévy Khintchine formula.

Definition 6 *A function ψ is called a Bernstein function if and only if there exist two positive constants a, b, and a positive measure μ on $]0,\infty[$ with $\int_0^\infty \frac{s}{1+s}d\mu(s) < \infty$ such that*

$$\psi(x) = a + bx + \int_0^\infty (1 - e^{xs})d\mu(s) \quad \forall x > 0\ ;$$

the triple (a,b,μ) is uniquely determined by ψ.

The Bernstein functions give the correspondence between the eigenvalues of $(-\Delta/2)$ and those of the infinitesimal generator of subordinated processes. Take a process B_T subordinated to a Brownian motion with a subordinator T, which corresponds to the Bernstein function ψ. If φ is a bounded spherical function for the eigenvalue λ with respect to $(-\Delta/2)$, it will be associated to the eigenvalue $-\psi(\lambda)$ as eigenfunction of the infinitesimal generator of B_T. Actually if P_t is the semi group of B_T

$$\begin{aligned} P_t(\varphi)(x) &= E(\varphi(B_{T_t}^x)) \\ &= E(E(\varphi(B_{T_t}^x)|T_t = a)) \\ &= \int_0^\infty e^{-a\lambda}\varphi_{\lambda(x)}dP_{T_t}(a) \\ &= exp(-t\psi(\lambda))\varphi_\lambda(x). \end{aligned}$$

Let us go back to the special case of the sphere in dimension 2. We can fix as reference point the north pole with Cartesian coordinates $PN = (0,0,1)$, we write (r,θ) for the polar geodesic coordinates from the north pole. The spherical functions can be expressed with Legendre polynomials of first kind

$$f_n(x) = P_n(cos(r(x))) \text{ and } (-\Delta/2))f_n(PN) = \frac{n(n+1)}{2} = \lambda_n.$$

On the other hand if A is the infinitesimal generator associated to the pseudo 1-stable process, formula (10) leads to

$$Af_n(PN) = 2\pi \int_0^\infty \frac{P_n(cos(r)) - 1}{r^2} C(1,2)dr = \mu_n.$$

But on the graph of $(\lambda_n, -\mu_n)$ for $n = 1$ to 10 we remark that those points are aligned three by three. Since Bernstein functions are either affine or strictly concave thanks to definition 6. we know that there is no Bernstein function such that $\psi(\lambda_n) = -\mu_n$. Hence

Proposition 4 *The pseudo* 1*-stable on the* 2*-sphere is not a Brownian motion time-changed by a subordinator : more precisely it is not equal in law to* B_T *where* B *is a Brownian motion and* T *a subordinator independent of* B*.*

6 Exact computation of eigenvalues

In this part we would like to present the computations of eigenvalues which allow us to prove the last result. They have been obtained using MAPLE symbolic software. It is the reason why we can claim that points $(\lambda_n, -\mu_n)$ are aligned and not nearly aligned as the answer given by classical numerical program. The same result could have been obtained without a computer, but we are not sure that we would have tried it because it is a lot of work, and you do not know at the beginning if it will be useful. We first recall the expression of μ_n as

$$\mu_n = 2\pi \int_0^\infty \frac{P_n(cos(r)) - 1}{r^2} C(1,2)dr.$$

In the next table we present on the first row the eigenvalue corresponding to $-(\Delta/2)$ $\lambda_n = \frac{n(n+1)}{2}$, on second line $-\mu_n$, and we can read that points $(\lambda_n, -\mu_n)$ are aligned three by three since $\delta_n = \frac{\mu_n - \mu_{n+1}}{\lambda_{n+1} - \lambda_n}$ is printed on the third line.

n	1	2	3	4	5
λ_n	1	3	6	10	15
μ_n	$(-1/4)\pi$	$(-3/8)\pi$	$(-9/16)\pi$	$(-45/64)\pi$	$(-225/256)\pi$
δ_n	$(-1/16)\pi$	$(-1/16)\pi$	$(-9/256)\pi$	$(-9/256)\pi$	$(-25/1024)\pi$

n	6	7	8	9
λ_n	21	28	36	45
μ_n	$(-525/512)\pi$	$(-1225/1024)\pi$	$(-11025/8192)\pi$	$(-99225/65536)\pi$
δ_n	$(-25/1024)\pi$	$(-1225/65536)\pi$	$(-1225/65536)\pi$	$(-3969/262144)\pi$

This phenomenon was first suggested by the graph where you put λ_n on the X axis and $-\mu_n$ on the Y axis. It can also be seen on the graph that the points are close to the parabola

$$y = 1/2(x^{1/2}),$$

which is a graphic representation of the idea that the pseudo α-stable processes are perturbated α-stable processes in a spectral sense.

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