JEAN-CLAUDE GRUET
ZHAN SHI

On the Spitzer and Chung laws of the iterated logarithm for brownian motion

Séminaire de probabilités (Strasbourg), tome 29 (1995), p. 237-247
<http://www.numdam.org/item?id=SPS_1995__29__237_0>

On the Spitzer and Chung laws of the iterated logarithm for Brownian motion

by

J.-C. Gruet[(1)] and Z. Shi[(2)]

(1) *Laboratoire de Probabilités, Université Paris VI, 4 Place Jussieu, 75252 Paris Cedex 05, France & Université de Reims, UFR Sciences, 51062 Reims Cedex, France*

(2) *L.S.T.A., Université Paris VI, 4 Place Jussieu, 75252 Paris Cedex 05, France*

1. Introduction.

Let $\{ W(t);\, t \geq 0 \}$ be a two-dimensional Brownian motion. It is well-known that the Brownian path is almost surely dense in the plane, but never hits a given point at positive time. A natural question is thus to study the rate with which the small values of $\|W - x\|$ (the symbol "$\|\cdot\|$" denoting the usual Euclidean modulus) approach 0 for any $x \in \mathbb{R}^2$. Without loss of generality, we assume $W(0) = (1,0)$ and $x = (0,0)$. Let

$$X(t) = \inf_{0 \leq s \leq t} \|W(s)\|, \quad t > 0.$$

The following celebrated Spitzer (1958) integral test characterizes the lower functions of X:

Theorem A (Spitzer 1958). *For any non-decreasing function $f > 1$,*

$$\mathbb{P}\Big[\, X(t) < \frac{t^{1/2}}{f(t)},\ \text{i.o.} \,\Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t \log f(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Here and in the sequel, "i.o." stands for "infinitely often" as t tends to infinity.

Spitzer's Theorem A answers the how-small-are-the-small-values-of-$\|W\|$ question. We propose to study the corresponding "how big" problem for the small values. Our Theorem 1, stated as follows, provides a characterization of the upper functions of X.

Theorem 1. *If $g > 1$ is non-decreasing, then*

$$\mathbb{P}\Big[\, X(t) > \exp\Big(-\frac{\log t}{g(t)}\Big),\ \text{i.o.}\Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t\, g(t) \log t} dt \begin{cases} < \infty \\ = \infty \end{cases}.$$

Theorems A and 1 together give an accurate description of the almost sure asymptotic behaviours of X. For example, it is immediately seen from the aboves theorems that

$$\mathbb{P}\Big[\, X(t) > \exp\Big(-\frac{\log t}{(\log\log t)^a}\Big),\ \text{i.o.}\Big] = \begin{cases} 0 & \text{if } a > 1; \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathbb{P}\Big[\, X(t) < \exp\Big(-(\log t)(\log\log t)^a\Big),\ \text{i.o.}\Big] = \begin{cases} 0 & \text{if } a > 1; \\ 1 & \text{otherwise.} \end{cases}$$

What about the lower functions of the big values of Brownian motion? Let us recall the classical Chung (1948) integral test for linear Brownian motion.

Theorem B (Chung 1948). *Let B be a real-valued Brownian motion. For every non-decreasing function $h > 0$ such that $t^{-1/2}h(t)$ is non-increasing, we have*

$$\mathbb{P}\Big[\sup_{0\le s\le t} |B(s)| < \frac{t^{1/2}}{h(t)},\ \text{i.o.}\Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t} h^2(t) \exp\Big(-\frac{\pi^2}{8} h^2(t)\Big) \begin{cases} < \infty \\ = \infty \end{cases}.$$

Chung's Theorem B was obtained for linear Brownian motion. The following natural question was raised by Révész (1990, p.195): for the planar Brownian motion W, what can be said on the liminf behaviour of $\sup_{0\le s\le t} \|W(s)\|$?

The same question can be asked for a Brownian motion of any dimension. Let $\{V(t);\, t \geq 0\}$ denote a d-dimensional Brownian motion, and let

$$Y(t) = \sup_{0 \leq s \leq t} \|V(s)\|, \quad t > 0.$$

Our answer to the problem is the following

Theorem 2. *Let $d \geq 1$ and let $h > 0$ be a non-decreasing function. Then*

$$\mathbb{P}\Big[Y(t) < \frac{t^{1/2}}{h(t)}, \text{ i.o.} \Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} dt\, \frac{h^2(t)}{t} \exp\Big(-\frac{j_\nu^2}{2} h^2(t)\Big) \begin{cases} < \infty \\ = \infty \end{cases},$$

where j_ν denotes the smallest positive zero of the Bessel function $J_\nu(x)$ of index $\nu \equiv (d-2)/2$.

Remarks. (i) Since $j_{-1/2} = \pi/2$, Theorem B is a special case of the above result.

(ii) An interesting feature in Theorem 2 is that we do not suppose $t^2/h(t)$ to be non-decreasing. Thus the latter condition can be removed from Chung's Theorem B.

(iii) As usual, Theorem 2 has a "local" version for t tending to 0, of which the statement and proof are omitted.

Corollary 1. *We have, for $d \geq 1$,*

$$\liminf_{t \to \infty} \Big(\frac{\log\log t}{t}\Big)^{1/2} Y(t) = \frac{j_\nu}{2^{1/2}} \quad \text{a.s.}, \tag{1.1}$$

with rate of convergence

$$\liminf_{t \to \infty} \frac{(\log\log t)^{3/2}}{t^{1/2} \log\log\log t} \Big(Y(t) - \frac{j_\nu}{2^{1/2}} \frac{t^{1/2}}{(\log\log t)^{1/2}} \Big) = -\frac{j_\nu}{2^{1/2}}, \quad \text{a.s.}$$

Remark. The LIL (1.1) was previously obtained by Lévy (1953) for $d = 2$ and by Ciesielski & Taylor (1962) for any dimension d.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

2. The proof of Theorem 1.

Let W be as before a Brownian motion in the plane, starting from $(1,0)$. Define

$$H(x) = \inf\{\, t > 0 : \, \|W(t)\| = x \,\}, \quad 0 < x \le 1,$$

the first hitting time of $\|W\|$ at level x. Obviously the process $y \mapsto H(1/y)$ (for $y \ge 1$) is increasing, and it has independent increments by using the strong Markov property of $\|W\|$. Since

$$H(2^{-n}) = \sum_{k=1}^{n} \Big(H(2^{-k}) - H(2^{-(k-1)}) \Big),$$

using Brownian scaling gives

$$H(2^{-n}) = \sum_{k=1}^{n} 2^{-2(k-1)} \xi_k,$$

where $(\xi_k)_{k \ge 1}$ is an i.i.d. sequence of random variables having the same law as $H(1/2)$. Consequently,

$$2^{-2(n-1)} \sum_{k=1}^{n} \xi_k \le H(2^{-n}) \le \sum_{k=1}^{n} \xi_k. \tag{2.1}$$

Let us first establish a preliminary result for the partial sum of (ξ_k):

Lemma 1. *Let $\{\, \Lambda(t); t \ge 0 \,\}$ be a subordinator, and assume that $\Lambda(1)$ has the same law as $H(1/2)$. Then for any function $f > 1$ such that $f(t)/t$ is non-decreasing, we have*

$$\limsup_{t \to \infty} \frac{\Lambda(t)}{f(t)} = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \quad \Longleftrightarrow \quad \int^{\infty} \frac{dt}{\log f(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Proof of Lemma 1. The Laplace transform of $H(1/2)$ is well-known (see Kent (1978 Theorem 3.1)):

$$\mathbb{E} \exp\big(-\lambda H(1/2)\big) = \frac{K_0(\sqrt{2\lambda})}{K_0(\sqrt{\lambda/2})}, \quad \forall \lambda > 0,$$

where K_0 is the modified Bessel function. Recall that $\Lambda(1)$ has the same law as $H(1/2)$. Write

$$\mathbb{E} \exp\big(-\lambda \Lambda(1)\big) = \exp\big(-\Psi(\lambda)\big).$$

Thus

$$\Psi(\lambda) = \log K_0(\sqrt{\lambda/2}) - \log K_0(\sqrt{2\lambda}).$$

Using elementary asymptotics of K_0, we immediately arrive at the following estimate

$$\Psi(\lambda) - \lambda\Psi'(\lambda) \sim \frac{2\log 2}{\log(1/\lambda)}, \quad \lambda \to 0,$$

(the usual symbol "$a(x) \sim b(x)$" ($x \to x_0$) means $\lim_{x\to x_0} a(x)/b(x) = 1$). Now the statement of Lemma 1 follows by applying a general result for subordinators (see for example Fristedt (1974 Theorem 6.1)) which tells that $\limsup_{t\to\infty} \Lambda(t)/f(t) = 0$ or ∞ (almost surely) according as

$$\int^{\infty} \left(\Psi(\frac{1}{f(t)}) - \frac{1}{f(t)}\Psi'(\frac{1}{f(t)}) \right) dt$$

converges or diverges. □

Lemma 2. *If $h > 1$ is a non-decreasing function with $\int^{\infty} dt/h(t) = \infty$, then*

$$\int^{\infty} \frac{dt}{t + h(t)} = \infty.$$

Proof of Lemma 2. The proof is briefly sketched, since it involves only elementary computations. Set $\mathcal{A} = \{t : h(t) \le t\}$ and $\mathcal{B} = \{t : h(t) > t\}$. Obviously, we have

$$\frac{1}{t + h(t)} \ge \frac{1}{2}\left(\frac{1}{t}\mathbb{1}_{\mathcal{A}}(t) + \frac{1}{h(t)}\mathbb{1}_{\mathcal{B}}(t) \right).$$

Assume $\int^{\infty} \mathbb{1}_{\mathcal{A}}(t)(dt/t) < \infty$. We only have to show $\int^{\infty} \mathbb{1}_{\mathcal{B}}(t)(dt/h(t)) = \infty$. Write $F_{\mathcal{A}}(t) \equiv \int_1^t \mathbb{1}_{\mathcal{A}}(s)ds$. Using integration by parts for $\int \mathbb{1}_{\mathcal{A}}(s)(ds/s)$, it is seen that $t \mapsto F_{\mathcal{A}}(t)/t$ is a Cauchy family for $t > 1$. Thus $\lim_{t\to\infty} F_{\mathcal{A}}(t)/t$ exists. If $\lim_{t\to\infty} F_{\mathcal{A}}(t)/t > 0$, then $\int_1^t F_{\mathcal{A}}(s)(ds/s^2)$ would diverge, which contredicts the convergence of $\int_1^{\infty} \mathbb{1}_{\mathcal{A}}(s)(ds/s)$ (the latter is obviously greater than $\int_1^t F_{\mathcal{A}}(s)(ds/s^2)$ by integration by parts). Consequently, $\lim_{t\to\infty} F_{\mathcal{A}}(t)/t = 0$. Thus $\int_1^t \mathbb{1}_{\mathcal{B}}(s)ds \ge t/2$ for sufficiently large t. Again using integration by parts, we obtain

$$\int_1^t \mathbb{1}_{\mathcal{B}}(t)\frac{ds}{h(s)} \ge \frac{1}{2}\int_1^t \frac{ds}{h(s)} + \text{a finite term},$$

which diverges as t tends to infinity. Lemma 2 is proved. □

Proof of Theorem 1. Pick $0 < x < 1$, and let us write $2^{-n} \le x \le 2^{-(n-1)}$ (which means $(n-1)\log 2 \le \log(1/x) \le n \log 2$). Then $H(2^{-(n-1)}) \le H(x) \le H(2^{-n})$. Using (2.1), we have

$$x^2 \Lambda\Big(\log(1/x)/\log 2 - 1\Big) \le H(x) \le \Lambda\Big(\log(1/x)/\log 2 + 1\Big). \tag{2.2}$$

First, we show the following integral test for H:

$$\limsup_{x\to 0+} \frac{H(x)}{f(x)} = \begin{cases} 0 \\ \infty \end{cases}, \text{ a.s.} \iff \int_{0+} \frac{dx}{x \log f(x)} \begin{cases} < \infty \\ = \infty \end{cases}, \tag{2.3}$$

for any function $f > 1$ such that $\log f(x)/\log(1/x)$ is non-increasing. Indeed, assume that $\int_{0+} dx/(x \log f(x))$ converges. Define $\hat{f}(t) = f(2^{-t/2})$. Then $\hat{f}(t)/t$ is non-decreasing, with $\int^{\infty} dt/\log \hat{f}(t) < \infty$. By Lemma 1, we have $\limsup_{t\to\infty} \Lambda(t)/\hat{f}(t) = 0$, with probability 1. Thus $\limsup_{t\to\infty} \Lambda(t)/f(2^{-(t-1)}) = 0$. Using the second part of (2.2), this implies $\limsup_{x\to 0+} H(x)/f(x) = 0$. It remains to verify the divergent half of (2.3). Suppose $\int_{0+} dx/(x \log f(x)) = \infty$. Then $\int^{\infty} dt/\log f(2^{-2(t+1)})$ diverges as well. According to Lemma 2, this implies

$$\int^{\infty} \frac{dt}{\log \tilde{f}(t)} = \infty,$$

for $\tilde{f}(t) \equiv 2^{2(t+1)} f(2^{-2(t+1)})$. Applying Lemma 1 gives $\limsup_{t\to\infty} \Lambda(t)/\tilde{f}(t) = \infty$, which, by means of the first part of (2.2), yields $\limsup_{x\to 0+} H(x)/f(x) = \infty$. Hence (2.3) is proved. By noting $[H(x) > t] = [X(t) > x]$ (for any $0 < x < 1$ and $t > 0$), several lines of standard calculation readily confirm that the integral test (2.3) is equivalent to that in Theorem 1. $\square$

3. The proof of Theorem 2.

In this section, V denotes a d-dimensional Brownian motion, which, without loss of generality, is assumed to start from 0. Let $H(x) = \inf\{ t > 0 : \|V(t)\| = x \}$ (for $x > 0$). The proof of Theorem 2 is essentially based on the following exact density function of $H(1)$ due to Ciesielski & Taylor (1962):

$$\mathbb{P}\big[H(1) \in dt \big]/dt = \frac{1}{2^{\nu}\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \exp\Big(-\frac{j_{\nu,n}^2}{2} t\Big), \quad t > 0,$$

where $\nu = (d-2)/2$, and $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ are the positive zeros of the Bessel function J_ν (and of course $J_{\nu+1}$ denotes the Bessel function of index $\nu+1$). Let Y be as before the supremum process of $\|V\|$. By Brownian scaling, we have, for any $x > 0$,

$$\begin{aligned} \mathbb{P}\left[Y(1) < x\right] &= \mathbb{P}\left[H(x) > 1\right] = \mathbb{P}\left[H(1) > 1/x^2\right] \\ &= \frac{2^{1-\nu}}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{1-\nu} J_{\nu+1}(j_{\nu,n})} \exp\Big(-\frac{j_{\nu,n}^2}{2x^2}\Big), \end{aligned}$$

which implies

$$\mathbb{P}\left[Y(1) < x\right] \sim \frac{2^{1-\nu}}{\Gamma(\nu+1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp\Big(-\frac{j_\nu^2}{2x^2}\Big), \quad \text{as } x \to 0, \tag{3.1}$$

(recall that $j_\nu \equiv j_{\nu,1}$ is the smallest positive zero of J_ν). Write in the sequel $\rho \equiv j_\nu^2/2$.

Let $h > 0$ be a non-decreasing function. In the rest of the note, generic constants will be denoted by K_i $(1 \le i \le 9)$.

We begin with the convergent part of Theorem 2, which is an immediate consequence of the tail estimation (3.1). Indeed, pick a sufficiently large initial value t_0 and define the sequence $(t_k)_{k\ge1}$ by $t_{k+1} = (1 + h^{-2}(t_k))t_k$ for $k = 0, 1, 2, \cdots$, and write $h_k \equiv h(t_k)$ for notational convenience. Obviously t_k increases to infinity (as $k \to \infty$). Assume that $\int^\infty (dt/t) h^2(t) \exp(-\rho h^2(t))$ converges. This implies, by several lines of elementary calculation, the convergence of $\sum_k \exp(-\rho h_k^2)$. From (3.1) and scaling it follows that

$$\begin{aligned} \mathbb{P}\Big[Y(t_k) < \frac{t_k^{1/2}}{h_k - 1/h_k}\Big] &= \mathbb{P}\Big[Y(1) < \frac{1}{h_k - 1/h_k}\Big] \\ &\le K_1 \exp(-\rho(h_k - 1/h_k)^2) \\ &\le K_2 \exp(-\rho h_k^2), \end{aligned}$$

which sums. According to Borel-Cantelli lemma, (almost surely) for large k, $Y(t_k) \ge t_k^{1/2}/(h_k - 1/h_k)$. Let $t \in [t_k, t_{k+1}]$. Then by our construction of (t_k),

$$Y(t) \ge Y(t_k) \ge \frac{t_k^{1/2}}{h_k - 1/h_k} = \frac{t_{k+1}^{1/2}}{(1 + 1/h_k^2)(h_k - 1/h_k)} \ge \frac{t_{k+1}^{1/2}}{h_k} \ge \frac{t^{1/2}}{h(t)},$$

which yields the convergent part of Theorem 2.

It remains to show the divergent part. Let h be such that

$$\int^\infty \frac{dt}{t} h^2(t) \exp(-\rho h^2(t)) = \infty. \tag{3.2}$$

In view of (1.1), we assume without loss of generality that

$$\frac{(\log\log t)^{1/2}}{(2\rho)^{1/2}} \le h(t) \le \frac{(2\log\log t)^{1/2}}{\rho^{1/2}}. \tag{3.3}$$

Define $t_k = \exp(k/\log k)$ (for $k \ge k_0$) and write as before $h_k \equiv h(t_k)$. In what follows, we only deal with the index k tending ultimately to infinity. Therefore our results, sometimes without further mention, are to be understood for sufficiently large k's. Obviously (3.2) is equivalent to the following

$$\sum_k \exp(-\rho h_k^2) = \infty. \tag{3.4}$$

Using (3.3) gives

$$\frac{(\log k)^{1/2}}{(3\rho)^{1/2}} \le h_k \le \log k. \tag{3.5}$$

Fix an $\varepsilon > 0$, then

$$k^{-1/2} h_k^2 \le \frac{\varepsilon}{\rho}, \tag{3.6}$$

(for $k \ge k_0$). Consider the measurable event $A_k = \{ Y(t_k) < t_k^{1/2}/h_k \}$. From (3.1) it follows that for $k \ge k_0$,

$$\mathbb{P}(A_k) = \mathbb{P}\Big[Y(1) < \frac{1}{h_k} \Big] \ge (1-\varepsilon) \frac{2^{1-\nu}}{\Gamma(\nu+1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp\Big(-\rho h_k^2\Big), \tag{3.7}$$

which, by means of (3.4), yields

$$\sum_k \mathbb{P}(A_k) = \infty. \tag{3.8}$$

Let $k < \ell$. Since V has independent and stationary increments, we have

$$\begin{aligned} \mathbb{P}(A_k A_\ell) &= \mathbb{P}\Big[\sup_{0 \le t \le t_k} \|V(t)\| < \frac{t_k^{1/2}}{h_k}, \sup_{0 \le t \le t_\ell} \|V(t)\| < \frac{t_\ell^{1/2}}{h_\ell} \Big] \\ &\le \mathbb{P}(A_k) \sup_{\|x\| \le t_k^{1/2}/h_k} \mathbb{P}\Big[\sup_{0 \le t \le t_\ell - t_k} \|V(t) + x\| < \frac{t_\ell^{1/2}}{h_\ell} \Big]. \end{aligned}$$

Using a general property of Gaussian measures (see for example Ledoux & Talagrand (1991 p.73)), it follows that

$$\begin{aligned} \mathbb{P}(A_k A_\ell) &\le \mathbb{P}(A_k) \mathbb{P}\Big[\sup_{0 \le t \le t_\ell - t_k} \|V(t)\| < \frac{t_\ell^{1/2}}{h_\ell} \Big] \\ &= \mathbb{P}(A_k) \mathbb{P}\Big[Y(1) < \frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell} \Big]. \end{aligned} \tag{3.9}$$

For every $n > k_0$, define

$$\mathcal{E}(n) = \{\, k_0 \le k < \ell \le n : \ell - k \le (\log k)^3 \,\},$$

$$\mathcal{F}(n) = \{\, k_0 \le k < \ell \le n : \ell - k > (\log k)^3 \,\}.$$

It is seen that when $k < \ell < k + (\log k)^3$,

$$\begin{aligned} \frac{\ell}{\log \ell} - \frac{k}{\log k} &= \frac{\ell \log k - k \log \ell}{\log k \, \log \ell} \\ &= \frac{(\ell - k)\log k - k \log\bigl(1 + (\ell - k)/k\bigr)}{\log k \, \log \ell} \\ &\sim \frac{\ell - k}{\log k}, \qquad (k \to \infty), \end{aligned}$$

which implies

$$\frac{t_k}{t_\ell} \le \exp\Bigl(-\frac{\ell - k}{2 \log k} \Bigr).$$

Let $(k, \ell) \in \mathcal{E}(n)$. From the above estimate it follows that

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2}} \le \Bigl(1 - \exp\bigl(-\frac{\ell - k}{2 \log k}\bigr) \Bigr)^{-1/2} \le \Bigl[K_3 \min\Bigl((\ell - k)/\log k, 1 \Bigr) \Bigr]^{-1/2},$$

which, by means of (3.5), yields

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell} \le \frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_k} \le \Bigl[K_4 \min\,(\ell - k, \log k) \Bigr]^{-1/2}.$$

From (3.9) and (3.1) it follows that

$$\mathbb{P}(A_k A_\ell) \le K_5 \mathbb{P}(A_k) \exp\bigl(-K_6(\ell - k)\bigr) + K_5 \mathbb{P}(A_k) k^{-K_6}.$$

Obviously,

$$\sum_{\ell > k} \exp\bigl(-K_6(\ell - k)\bigr) \le K_7,$$

$$\sum_{k < \ell < k + (\log k)^3} k^{-K_6} \le k^{-K_6} (\log k)^3 \le K_8.$$

Therefore,

$$\sum\sum_{(k,\ell) \in \mathcal{E}(n)} \mathbb{P}(A_k A_\ell) \le K_9 \sum_{k = k_0}^{n} \mathbb{P}(A_k). \tag{3.10}$$

Now let $(k, \ell) \in \mathcal{F}(n)$. In this case, $\ell - (\log \ell)^2 \ge k + (\log k)^3 - (\log(k + (\log k)^3))^2 > k$, thus $\ell - k > (\log \ell)^2$. Since

$$\frac{\ell}{\log \ell} - \frac{k}{\log k} = \frac{(\ell - k)\log k - k \log\bigl(1 + (\ell - k)/k\bigr)}{\log k \, \log \ell} \sim \frac{\ell - k}{\log \ell},$$

we have

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell} \le \frac{1}{\left[1 - \exp\left(-(\ell - k)/2\log \ell\right)\right]^{1/2} h_\ell} \le \frac{1}{\left(1 - \ell^{-1/2}\right)^{1/2} h_\ell}.$$

By means of (3.9), (3.1), (3.6) and (3.7), this implies

$$\begin{aligned}\mathbb{P}(A_k A_\ell) &\le \mathbb{P}(A_k)(1+\varepsilon)\exp\left(\rho \ell^{-1/2} h_\ell^2\right) \frac{2^{1-\nu}}{\Gamma(\nu+1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp\left(-\rho h_k^2\right) \\ &\le (1+3\varepsilon)e^{\varepsilon}\mathbb{P}(A_k)\mathbb{P}(A_\ell).\end{aligned}$$

Combining the above estimate together with (3.10) and (3.8) yields

$$\liminf_{n\to\infty} \sum_{k=k_0}^{n} \sum_{\ell=k_0}^{n} \mathbb{P}(A_k A_\ell) \Big/ \Big(\sum_{k=k_0}^{n} \mathbb{P}(A_k)\Big)^2 \le 1.$$

According to a well-known version of Borel-Cantelli lemma (see for example Révész (1990 p.28)), we have $\mathbb{P}(A_k, \text{i.o.}) = 1$. The proof of the divergent part of Theorem 2 is completed. □

Acknowledgement. We are grateful to Davar Khoshnevisan for helpful comments on the first draft of this work.

REFERENCES

Chung, K.L. (1948): On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* 64, 205-233.

Ciesielski, Z. & Taylor, S.J. (1962). First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.* 103 434-450.

Fristedt, B.E. (1974): Sample functions of stochastic processes with stationary independent increments. In: *Advances in Probability 3* (Eds.: P. Ney & S. Port) pp. 241-396. Dekker, New York.

Itô, K. & McKean, H.P. (1965): *Diffusion Processes and their Sample Paths.* Springer, Berlin.

Ledoux, M. & Talagrand, M. (1991): *Probability in Banach Spaces: Isoperimetry and Processes.* Springer, Berlin.

Lévy, P. (1953): La mesure de Hausdorff de la courbe du mouvement brownien. *Giornale dell'Istituto Italiano degli Attuari* 16 1-37.

Révész, P. (1990): *Random Walk in Random and Non-Random Environments.* World Scientific, Singapore.

Revuz, D. & Yor, M. (1994): *Continuous Martingales and Brownian Motion.* 2nd Edition. Springer, Berlin.

Spitzer, F. (1958): Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* 87 187-197.