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A MARTINGALE PROOF OF THE KHINCHIN ITERATED LOGARITHM LAW FOR WIENER PROCESSES

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Let w_t be a d-dimensional Wiener process. Our goal here is to give a martingale proof of the following celebrated Khinchin log log law:

Theorem 1. *With probability one*

$$\limsup_{t\to\infty} \frac{|w_t|}{\sqrt{2t\log\log t}} = 1.$$

The "standard" proof of this result can be found in many places (cf. for instance [1]). Our martingale proof is based on formula (4), which might be interesting by itself. In particular, this formula allows us to give a rather short proof of the Kolmogorov–Petrovskii criterion allowing one to recognize when a given function $\alpha(t)$ is an "upper" or "lower" function.

Let $\alpha(t)$ be a strictly positive continuously differentiable function on $[0,\infty)$. For $t > 0$, $x \in \mathbb{R}^d$ define

$$h(t,x) = \frac{1}{t^{d/2}} e^{|x|^2/(2t)}, \quad \beta(t) = \frac{1}{t^{d/2}} e^{\alpha^2(t)/(2t)}, \quad \gamma(t) = (\frac{1}{\beta(t)})',$$

$$\tau(t) = \inf\{s \geq t : |w_s| \geq \alpha(s)\}.$$

Lemma 2. *For any $\varepsilon > 0$ the process*

$$m_\varepsilon(t) := h(t+\varepsilon, w_t)\beta^{-1}(t+\varepsilon) - \int_0^t h(r+\varepsilon, w_r)\gamma(r+\varepsilon)\, dr$$

is a martingale.

To prove the lemma it suffices to observe that for $0 \leq s \leq t$ we have

$$\beta^{-1}(t+\varepsilon) - \beta^{-1}(s+\varepsilon) = \int_s^t \gamma(r+\varepsilon)\, dr,$$

then take a normal $(0,(t+\varepsilon)I)$ variable ξ independent of w and notice that

$$E\{h(t+\varepsilon, w_t)|\mathcal{F}_s^w\} = Eh(t+\varepsilon, x + \xi\left(\frac{t-s}{t+\varepsilon}\right)^{1/2})|_{x=w_s} =$$

$$\frac{1}{(2\pi)^{d/2}(t+\varepsilon)^d}\int_{\mathbb{R}^d} \exp\frac{1}{2(t+\varepsilon)}[|w_s + y\left(\frac{t-s}{t+\varepsilon}\right)^{1/2}|^2 - |y|^2]\, dy,$$

$$\frac{1}{2(t+\varepsilon)}[|w_s + y\left(\frac{t-s}{t+\varepsilon}\right)^{1/2}|^2 - |y|^2] =$$

$$\frac{1}{2(s+\varepsilon)}|w|_s^2 - \frac{1}{2(t+\varepsilon)}|y\left(\frac{s+\varepsilon}{t+\varepsilon}\right)^{1/2} + w_t\left(\frac{t-s}{s+\varepsilon}\right)^{1/2}|^2,$$

$$E\{h(t+\varepsilon, w_t)|\mathcal{F}_s^w\} =$$

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$$\frac{1}{(2\pi)^{d/2}(t+\varepsilon)^d}\int_{\mathbf{R}^d}\exp[\frac{1}{2(s+\varepsilon)}|w|_s^2-\frac{s+\varepsilon}{2(t+\varepsilon)^2}|y|^2]\,dy=h(s+\varepsilon,w_s)\ \ (a.s.).$$

Corollary 3. *If $0<s<T<\infty$, then*

$$P\{s<\tau(s)\le T\}+E\frac{1}{\beta(T)}h(T,w_T)I_{\tau(s)>T}=$$

$$\frac{\alpha^d(s)}{\beta(s)s^d}\kappa_d+E\int_s^T h(r,w_r)\gamma(r)I_{\tau(s)>r}\,dr, \tag{1}$$

where κ_d is $(2\pi)^{-d/2}$ times the volume of the unit sphere.

Indeed, $Em_\varepsilon(s)I_{\tau(s)>s}=Em_\varepsilon(\tau(s)\wedge T)I_{\tau(s)>s}$, which means that

$$E\frac{h(s+\varepsilon,w_s)}{\beta(s+\varepsilon)}I_{\tau(s)>s}+E\int_s^{\tau(s)\wedge T}h(r+\varepsilon,w_r)\gamma(r+\varepsilon)\,dr=$$

$$E\frac{h(\tau(s)\wedge T+\varepsilon,w_{\tau(s)\wedge T})}{\beta(\tau(s)\wedge T+\varepsilon)}I_{\tau(s)>s}.$$

It remains only to let here $\varepsilon\downarrow 0$ and apply the dominated convergence theorem along with the observations that $|w_r|\le\sup_{r\le T}\alpha(r)$ for $s<r\le\tau(s)\wedge T$,

$$E\frac{h(s,w_s)}{\beta(s)}I_{\tau(s)>s}=E\frac{h(s,w_s)}{\beta(s)}I_{|w_s|<\alpha(s)}=$$

$$\frac{1}{\beta(s)s^d(2\pi)^{d/2}}\int_{|x|<\alpha(s)}dx=\frac{\alpha^d(s)}{\beta(s)s^d}\kappa_d=\kappa_d[\frac{\alpha^2(s)}{s}]^{d/2}\exp[-\frac{\alpha^2(s)}{2s}], \tag{2}$$

$$\frac{h(\tau(s)\wedge T,w_{\tau(s)\wedge T})}{\beta(\tau(s)\wedge T)}I_{\tau(s)>s}=\frac{1}{\beta(T)}h(T,w_T)I_{\tau(s)>T}+$$

$$\frac{1}{\beta(\tau(s))}h(\tau(s),w_{\tau(s)})I_{T\ge\tau(s)>s}=\frac{1}{\beta(T)}h(T,w_T)I_{\tau(s)>T}+I_{T\ge\tau(s)>s}.$$

By letting $T\to\infty$ in (1) and by applying the monotone convergence theorem, relations (2) and the fact that $\{\tau(s)>T\}\subset\{|w_T|<\alpha(T)\}$ we immediately get the first assertion of the following lemma

Lemma 4. *Define $f(t)=\kappa_d[\frac{\alpha^2(t)}{t}]^{d/2}\exp[-\frac{\alpha^2(t)}{2t}]$, let $\gamma(t)\ge 0$ for large t and let*

$$\lim_{t\to\infty}\frac{\alpha(t)}{\sqrt{t}}=\infty\ \ (\lim_{t\to\infty}f(t)=0\,). \tag{3}$$

Then for any $s>0$

$$P\{s<\tau(s)<\infty\}=f(s)+\int_s^\infty Eh(r,w_r)\gamma(r)I_{\tau(s)>r}\,dr; \tag{4}$$

$$\int^\infty r^{-d/2}\gamma(r)\,dr=\infty\Longrightarrow\limsup_{t\to\infty}[|w_t|-\alpha(t)]\ge 0\ \ (\text{a.s.}); \tag{5}$$

$$\int^\infty \alpha^d(r)r^{-d}\gamma(r)\,dr<\infty\Longrightarrow\limsup_{t\to\infty}[|w_t|-\alpha(t)]\le 0\ \ (\text{a.s.}). \tag{6}$$

Proof. To prove (5) notice that $h(r,w_r) \geq r^{-d/2}$, so that

$$\int_s^\infty Eh(r,w_r)\gamma(r)I_{\tau(s)>r}\,dr \geq \int_s^\infty r^{-d/2}\gamma(r)\,dr\,P\{\tau(s)=\infty\},$$

and under the condition in (5) we have $P\{\tau(s)=\infty\}=0$. It remains only to observe that

$$\{\omega : \limsup_{t\to\infty}[|w_t|-\alpha(t)]<0\} \subset \bigcup_{n=1}^{\infty}\{\omega : \tau(n)=\infty\}.$$

To prove (6) we use (2) and that $\{\tau(s)>r\}\subset\{|w_r|<\alpha(r)\}$ if $r>s$. Then

$$\int_s^\infty Eh(r,w_r)\gamma(r)I_{\tau(s)>r}\,dr \leq \kappa_d\int_s^\infty \alpha^d(r)r^{-d}\gamma(r)\,dr,$$

and from (4) we see that under the condition in (6), $P\{s<\tau(s)<\infty\}\to 0$ as $s\to\infty$. Finally, for any $s>0$

$$\{\omega : \limsup_{t\to\infty}[|w_t|-\alpha(t)]>0\}\subset\{\omega : |w_s|\geq\alpha(s)\}\cup\{\omega : s<\tau(s)<\infty\},$$

$$P\{\limsup_{t\to\infty}[|w_t|-\alpha(t)]>0\}\leq\lim_{s\to\infty}P\{|w_s|\geq\alpha(s)\}=\lim_{s\to\infty}P\{|w_1|\geq\frac{\alpha(s)}{\sqrt{s}}\}=0.$$

The lemma is proved.

Proof of Theorem 1. Take $\varepsilon\in[0,1)$ and define $\alpha(t)=((1+\varepsilon)2t\log\log t)^{1/2}$ if $t\geq 10$ and for $t<10$ define $\alpha(t)$ in any way just to get a positive differentiable function on $[0,\infty)$. Then $\beta(t)=t^{-d/2}(\log t)^{1+\varepsilon}$, $\alpha(t)/\sqrt{t}\to\infty$ for any ε, and as easy to check

$$\int^\infty r^{-d/2}\,d\frac{1}{\beta(r)}=\infty \text{ if } \varepsilon=0,\quad \int^\infty \alpha^d(r)r^{-d}\,d\frac{1}{\beta(r)}<\infty \text{ if } \varepsilon>0;$$

from Lemma 4 it follows that

$$\limsup_{t\to\infty}\frac{|w_t|}{(2t\log\log t)^{1/2}}\geq 1,\quad \limsup_{t\to\infty}\frac{|w_t|}{(2t\log\log t)^{1/2}}\leq 1+\varepsilon \text{ (a.s.) if } \varepsilon>0.$$

The theorem is proved.

Next observe that

$$\int^\infty\frac{\alpha^d(r)}{r^d}\gamma(r)\,dr=\int^\infty\frac{\alpha^d(r)}{r^d}\,d(r^{d/2}\exp[-\frac{\alpha^2(r)}{2r}])=$$

$$\frac{d}{2}\int^\infty\frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\frac{dr}{r}-\frac{1}{2}\int^\infty\frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\,d\frac{\alpha^2(r)}{r},$$

and under condition (3) the integral

$$\int^\infty\frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\,d\frac{\alpha^2(r)}{r}=\int^\infty x^{d/2}e^{-x/2}\,dx$$

is finite, so that the integrals

$$\int^\infty\frac{\alpha^d(r)}{r^d}\gamma(r)\,dr,\quad\int^\infty\frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\frac{dr}{r}$$

converge or diverge simultaneously.

Now we see that the statement (6) in Lemma 4 implies the second statement in the following Kolmogorov–Petrovskii criterion.

Theorem 5. *Assume that $t^{d/2}\exp[-\alpha^2(t)/(2t)]$ and $t\alpha^{-1}(t)$ increase for large t, and assume (3). Then*

$$\int^\infty \frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\frac{dr}{r}=\infty \implies \limsup_{t\to\infty}\frac{|w_t|}{\alpha(t)}\geq 1 \text{ (a.s.)}; \qquad (7)$$

$$\int^\infty \frac{\alpha^d(r)}{r^{d/2}}\exp[-\frac{\alpha^2(r)}{2r}]\frac{dr}{r}<\infty \implies \limsup_{t\to\infty}\frac{|w_t|}{\alpha(t)}\leq 1 \text{ (a.s.)}.$$

Proof. We only need to prove (7), and to do this we come back to statement (4) of Lemma 4 but analyze it slightly more carefully. Notice that for any given $r>0$ the process $w_t - w_r t/r$, $t\in[0,r]$, and the random vector w_r are independent. Therefore for $r>s$

$$Eh(r,w_r)I_{\tau(s)>r}=\frac{1}{(2\pi r)^{d/2}}\int_{|x|<\alpha(r)}e^{-|x|^2/(2r)}h(r,x)P\{\tau(s)>r|w_r=x\}\,dx=$$

$$\frac{1}{(2\pi)^{d/2}r^d}\int_{|x|<\alpha(r)}P\{\tau(s)>r|w_r=x\}\,dx=$$

$$\frac{1}{(2\pi)^{d/2}r^d}\int_{|x|<\alpha(r)}P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}w_r+\frac{t}{r}x|\alpha^{-1}(t)<1\}\,dx=$$

$$\frac{\alpha^d(r)}{(2\pi)^{d/2}r^d}\int_{|x|<1}P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}w_r+\frac{t}{r}\alpha(r)x|\alpha^{-1}(t)<1\}\,dx.$$

From (4) it now follows that if $\int^\infty \alpha^d(r)r^{-d}\gamma(r)\,dr=\infty$, then for any $s>0$

$$\liminf_{r\to\infty}\int_{|x|<1}P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}w_r+\frac{t}{r}\alpha(r)x|\alpha^{-1}(t)<1\}\,dx=0. \qquad (8)$$

Now observe that for $\varepsilon\in(0,1)$, $|x|\leq\varepsilon$ and s so large that $t\alpha^{-1}(t)$ increases for $t\geq s$, we have $(t/r)\alpha(r)|x|\alpha^{-1}(t)\leq\varepsilon$,

$$P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}w_r+\frac{t}{r}\alpha(r)x|\alpha^{-1}(t)<1\}\geq$$

$$P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}\alpha(r)[w_r\alpha^{-1}(r)]|\alpha^{-1}(t)<1-\varepsilon\}\geq$$

$$P\{\sup_{t\in[s,r]}|w_t-\frac{t}{r}\alpha(r)[w_r\alpha^{-1}(r)]|\alpha^{-1}(t)<1-\varepsilon,\ |w_r|\alpha^{-1}(r)\leq\varepsilon\}\geq$$

$$P\{\sup_{t\in[s,r]}\frac{|w_t|}{\alpha(t)}<1-2\varepsilon\}-P\{|w_r|>\varepsilon\alpha(r)\}.$$

Since $\alpha(r)/\sqrt{r}\to\infty$, the last probability tends to zero as $r\to\infty$, and from (8) it follows that for any $\varepsilon\in(0,1)$ we have

$$P\{\sup_{t\geq s}\frac{|w_t|}{\alpha(t)}<1-2\varepsilon\}\leq\lim_{r\to\infty}P\{\sup_{t\in[s,r]}\frac{|w_t|}{\alpha(t)}<1-2\varepsilon\}=0.$$

This obviously yields the first assertion of our theorem, which is thus proved.

Remark 6. From the zero-one law it follows easily that for one-dimensional Wiener process B_t

$$\limsup_{t\to\infty}\frac{B_t}{\alpha(t)}\leq(\geq)1 \text{ (a.s.)} \implies \limsup_{t\to\infty}\frac{|w_t|}{\alpha(t)}\leq(\geq)1 \text{ (a.s.)}.$$

Therefore, consideration of arbitrary d does not yield any advantage, though it actually might happen that for $d = 1$ the integral in (7) converges and, say for $d = 2$ diverges. In this case $\limsup$ in (7) simply equals 1 (a. s.).

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