PHILIPPE BIANE

Intertwining of Markov semi-groups, some examples

Séminaire de probabilités (Strasbourg), tome 29 (1995), p. 30-36

<http://www.numdam.org/item?id=SPS_1995__29__30_0>

Intertwining of Markov semi-groups, some examples.

Philippe Biane

Abstract: We give a general group theoretic scheme for obtaining intertwining relations between Markov semi-groups. This is applied to the Heisenberg group and yields an intertwining relation between the Ornstein-Uhlenbeck and the Yule semi-groups.

1. Introduction.

In this paper we would like to point out the relevance of group theory in the problem of finding intertwining relations between Markov semi-groups, which has been considered recently in [C-P-Y].

Let $(P_t)_{t\in\mathbb{R}_+}$ and $(Q_t)_{t\in\mathbb{R}_+}$ be two Markov semi-groups on the spaces $(E,\mathcal{E})$ and $(F,\mathcal{F})$ respectively. We say that a Markov kernel $\Lambda : (E,\mathcal{E}) \to (F,\mathcal{F})$ intertwines Q and P if one has the relation

$$Q_t\Lambda = \Lambda P_t$$

for all $t > 0$.

The intertwining of two Markov semi-groups is not a symmetric relation; indeed, if Λ intertwines Q and P, there is in general no Markov kernel $\Gamma : (F,\mathcal{F}) \to (E,\mathcal{E})$ such that

$$\Gamma Q_t = P_t\Gamma$$

for all $t > 0$. We refer to [C-P-Y] for a further discussion of the probabilistic properties of the intertwining relation between semi-groups.

There is a general framework which will allow us to build many examples of such intertwining relations, and which we will describe below. The important point in this approach is that we will have to enlarge the notion of Markov semi-group to a non-commutative context, in order to obtain in a natural way intertwinings between purely "commutative" Markov semi-groups.

Let A be a C^*-algebra, and let $(T_t)_{t\in\mathbb{R}_+}$ be a semi-group of completely positive linear contractions of A (recall that a linear T map between two C^*-algebras A and B is said to be completely positive if for any $n \geq 1$ the map $T\otimes Id : A\otimes M_n(\mathbb{C}) \to B\otimes M_n(\mathbb{C})$ is positive). Such a semi-group is a non-commutative analogue of a (sub)-markovian semi-group, indeed if A is a commutative C^*-algebra then it is isomorphic to the algebra $C_0(spec\ A)$ and the maps T_t come from a semi-group of sub-markovian kernels on the topological space $spec\ A$. In the quantum probabilist litterature, semi-groups of completely positive contractions are often called quantum dynamical semi-groups (see e.g. [P]).

Let now B and C be abelian C^*-subalgebras of A which are invariant under the semi-group $(T_t)_{t\in\mathbb{R}_+}$. The restrictions of this semi-group to these subalgebras define thus semi-groups of kernels $(P_t)_{t\in\mathbb{R}_+}$ and $(Q_t)_{t\in\mathbb{R}_+}$ on the spectra of B and C respectively. Denoting by i_B and i_C the inclusions of B and C in A respectively, we have the relations $T_t \circ i_B = i_B \circ P_t$ and $T_t \circ i_C = i_C \circ Q_t$.

In many instances it turns out that there exists a completely positive projection $\pi : A \to C$ such that $\pi \circ T_t = Q_t \circ \pi$ for all $t > 0$. Taking the composition $\pi \circ i_B$

we get a completely positive map $B \longrightarrow C$ which is given by a kernel $\Lambda : spec\, B \longrightarrow spec\, C$ verifying the intertwining relation

$$Q_t\Lambda = \Lambda P_t$$

between the semi-groups $(P_t)_{t\in\mathbf{R}_+}$ and $(Q_t)_{t\in\mathbf{R}_+}$.

In the following section we will give some constructions from group theory which will provide us with many examples of such intertwining relations. In section 3 we will treat explicitly an interesting case, related to the Heisenberg group. In this study we will obtain an intertwining relation between the Ornstein-Uhlenbeck semi-group and the Yule semi-group (the Yule process is a special type of Galton Watson process cf [A-N]).

2. Some groupe theoretic constructions.

2.1 Let G be a locally compact group, there are several algebras commonly associated to G. The convolution algebra $L^1(G)$ is a Banach $*$-algebra. Any weakly continuous unitary representation π of G gives rise to a $*$-representation of $L^1(G)$, and if we let $\|f\| = \sup_\pi |\pi(f)|$, where the supremum is over all weakly continuous unitary representations of G, we obtain a C^*-norm on $L^1(G)$. The completion of $L^1(G)$ for this norm is the C^*-algebra of G, called $C^*(G)$. Of some use is also the enveloping von Neumann algebra of G denoted by $U(G)$. As a Banach space, it is the bidual of $C^*(G)$.

Let φ be a continuous positive definite function on G, with $\varphi(e) \leq 1$. Then the map $\varphi \mapsto \varphi f$ (this is the pointwise product of functions on G) is a completely positive contraction of $L^1(G)$, which extends to a completely positive contraction of $C^*(G)$. Taking the bidual, we get also a normal completely positive contraction on $U(G)$.

If ψ is a continuous, conditionnally positive definite, function with $\psi(e) \leq 0$, then $(e^{t\psi})_{t\in\mathbf{R}_+}$ is a multiplicative semi-group of continuous, positive definite, functions on G and hence gives rise to a semi-group of completely positive contractions of $C^*(G)$. Some properties of such semi-groups have been discussed in [B].

2.2 Let G and ψ be as in 2.1, and denote by $(T_t)_{t\in\mathbf{R}_+}$ the semi-group of completely positive contractions on $C^*(G)$ (or on $U(G)$) generated by ψ.

Let $H \subset G$ be a closed abelian subgroup then there are natural inclusions $C^*(U) \subset U(H) \subset U(G)$, and the semi-group $(T_t)_{t\in\mathbf{R}_+}$ leaves these subalgebras invariant. In fact, its restrition to $C^*(H)$ (or $U(H)$) is the semi-group associated to the conditionnally positive definite function ψ restricted to H.

Let K be a compact group of automorphisms of G, it gives rise to a group of automorphisms of $C^*(G)$. Suppose that the subalgebra of elements of $C^*(G)$ fixed by K is commutative, and let us call $C^*_K(G)$ this subalgebra. If the function ψ is invariant under the automorphisms of K then this subalgebra is invariant by $(T_t)_{t\in\mathbf{R}_+}$. Furthermore the map $\Pi : x \mapsto \int_K \theta(x)d\theta$ where $d\theta$ is the normalized Haar measure on K, is a completely positive projection of $C^*(G)$ onto $C^*_K(G)$, and it verifies $\Pi \circ T_t = Q_t \circ \Pi$ where Q_t is the restriction of T_t to $C^*_K(G)$.

More generally, let (G', K') be a Gelfand pair, so that K' is a compact subgroup of G', such that the algebra $L^1(K'\backslash G'/K')$ of K'-biinvariant functions on G' is abelian. Let again $C^*_K(G')$ be the C^*-algebra generated by $L^1(K'\backslash G'/K')$, and suppose also that ψ is bi-invariant by K', then the semi-group $(T_t)_{t\in\mathbf{R}_+}$ leaves $C^*_K(G')$ invariant, and there is a completely positive projection of $C^*(G')$ onto $C^*_K(G')$ given by $\Pi : x \mapsto \int_{K'}\int_{K'} kxk'dkdk'$, verifiying again $\Pi\circ T_t = Q_t\circ\Pi$. This projection extends to a projection of the corresponding von Neumann algebras $\Pi : U(G') \to U_K(G')$, verifying the same identity.

The preceding example is a special case of this one since one can form the semi-direct product $K \times G$ and $(K \times G, K)$ is then a Gelfand pair.

2.3 Summarizing the preceding section, we see that we obtain an intertwining relation between Markov semi-groups as soon as we have the following data: a Gelfand pair (G, K), and a K-bi-invariant, continuous, conditionnally positive definite, function ψ, on G, with $\psi(e) \leq 0$. In this case, the restrictions of the semi-group T_t, generated by ψ, to the algebras $C^*(H)$, where H is an abelian subgroup of G, and $C^*_K(G)$ are given by submarkovian semi-groups of kernels on the spectra of these C^*-algebras, and the composition of the inclusion $i_H : C^*(H) \to U(G)$ and of the projection onto $U_K(G)$ is given by a kernel which intertwins the two semi-groups.

A simple example is obtained (in the notation of sect. 2.2) by taking $G = \mathbb{R}^d$, $H = \mathbb{R}^k$, K is the orthogonal group $O(d)$, and $\psi(x) = -\frac{1}{2}|x|^2$. In this case, the semi-group T_t is the brownian semi-group on $\mathbb{R}^d$ (considered here as the dual group of G), the semi-group obtained by restriction to $C^*_K(G)$ is the Bessel semi-group of dimension d, the restriction to $C^*(H)$ is the k-dimensional brownian semi-group and, we thus obtain an intertwining between the k-dimensional brownian motion and the d-dimensional Bessel process. We leave the detailed computation to the reader.

3. An example.

We will use the general framework developped in the preceding section to work out an explicit intertwining relation between Markov semi-groups.

Recall that the Heisenberg group is $H_1 = \mathbb{C} \times \mathbb{R}$ with the group law

$$(z, w) \star (z', w') = (z + z', w + w' + \Im m\, z\bar{z}')$$

The function $\psi(z, w) = iw - \frac{1}{2}|z|^2$ is a conditionnally positive definite function on H_1 and the semi-group of completely positive contractions of $C^*(H_1)$ that it induces is a non-commutative analogue of the heat semi-group. It has been studied in [B]. We denote it by $(T_t)_{t\in\mathbb{R}_+}$ in the rest of this section.

Let $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$ which is a maximal abelian subgroup of H_1. The restriction of the semi-group $(T_t)_{t\in\mathbb{R}_+}$ to the corresponding group C^*-algebra (as defined in 2.2) was seen in [B] to be the heat semi-group on $\mathbb{R} \times \mathbb{R}$.

There is a group of automorphisms of H_1 of the form $(z, w) \mapsto (\xi z, w)$ for $\xi \in \mathbb{C}$, $|\xi| = 1$. The subalgebra of $C^*(H_1)$ consisting of elements fixed by this group is abelian and the semi-group obtained by restriction of $(T_t)_{t\in\mathbb{R}_+}$ to this subalgebra was called "non-commutative Bessel semi-group" in [B], where the corresponding transition kernels were also computed. In view of sections 2.2, we can deduce an intertwining relation between this semi-group and the heat semi-group on $\mathbb{R} \times \mathbb{R}$. Instead of carrying out the computations for these semi-groups, we shall do it for related semi-groups obtained by a transformation analogous to the usual transformation yielding Ornstein-Uhlenbeck process from brownian motion. Recall that if $(B_t)_{t\in\mathbb{R}_+}$ is a brownian motion then the processes $(X_s = e^{\pm\frac{s}{2}} B_{e^{\mp s}})_{s\in\mathbb{R}_+}$ are Ornstein-Uhlenbeck processes. The corresponding construction for the Heisenberg group was effected in [B], using the scaling automorphisms of H_1 defined as $\alpha_c(z, w) = (cz, c^2 w)$ for all $c > 0$. Let us recall this construction.

Let $\pi_\pm$ be the two unitary representations of H_1 on $L^2(\mathbf{R})$ defined by

$$\pi_\pm(z,w)f(x) = f(x+p)e^{\pm i(w+pq+2qx)} \tag{3.1}$$

where $z = p + iq$. These representations extend to representations $\pi_\pm : C^*(H_1) \to \mathcal{B}(L^2(\mathbf{R}))$. In [B] we proved the following

3.1 Proposition. *There exists two semi-groups, $(R_s^+)_{s\in\mathbf{R}_+}$ and $(R_s^-)_{s\in\mathbf{R}_+}$, of completely positive contractions on $\mathcal{B}(L^2(\mathbf{R}))$, such that, for all $s \in \mathbf{R}_+$,*

$$\pi_+ \circ \alpha_{e^{-\frac{s}{2}}} \circ T_{1-e^{-s}} = R_s^+ \circ \pi_+ \tag{3.2}$$

$$\pi_- \circ \alpha_{e^{\frac{s}{2}}} \circ T_{e^s-1} = R_s^- \circ \pi_- \tag{3.3}$$

Let us consider the subalgebra of $\mathcal{B}(L^2(\mathbf{R}))$ consisting of multiplication operators by functions in $C_0(\mathbf{R})$. This algebra is invariant by the semi-groups $R^\pm$, indeed one has

3.2 Proposition. *For any $v \in C_0(\mathbf{R})$*

$$R_s^+v(x) = \sqrt{\frac{2}{\pi}}\int_{\mathbf{R}} v(y)\exp\left(-\frac{2(y-e^{-\frac{s}{2}}x)^2}{1-e^{-s}}\right)\frac{dy}{\sqrt{1-e^{-s}}}$$

$$R_s^-v(x) = \sqrt{\frac{2}{\pi}}\int_{\mathbf{R}} v(y)\exp\left(-\frac{2(y-e^{\frac{s}{2}}x)^2}{e^{s}-1}\right)\frac{dy}{\sqrt{e^{s}-1}}$$

Proof Let v be of the form $v(x) = \int_{\mathbf{R}} \eta(q)e^{2iqx}dq$ for some integrable function η. By (3.1) and (3.2) we have, for $f \in L^2(\mathbf{R})$

$$\left(\int_{\mathbf{R}} \eta(q)\pi_+(iq,0)f\,dq\right)(x) = v(x)f(x)$$

and so

$$R_s^+(v)f(x) = \int_{\mathbf{R}} e^{2iqe^{-\frac{s}{2}}x}e^{-\frac{1}{2}(1-e^{-s})|q|^2}\eta(q)f(x)dq$$

Using the formula

$$e^{-\frac{1}{2}(1-e^{-s})|q|^2} = \sqrt{\frac{2}{\pi}}\int_{\mathbf{R}} e^{2iqy}e^{-\frac{2y^2}{1-e^{-s}}}\frac{dy}{\sqrt{1-e^{-s}}}$$

and using Fubini's theorem we obtain that R_s^+v is the multiplication operator by the function

$$\sqrt{\frac{2}{\pi}}\int_{\mathbf{R}} v(e^{-\frac{s}{2}}x+y)e^{-\frac{2y^2}{1-e^{-s}}}\frac{dy}{\sqrt{1-e^{-s}}}$$

and a change of variable gives the required conclusion for R^+. The formula for general $v \in C_0(\mathbf{R})$ is obtained by taking uniform limits, and the case of R^- is treated similarly.

We recognize the restrictions of the two semi-groups $(R_s^\pm)_{s\in\mathbf{R}_+}$ to $C_0(\mathbf{R})$ as the usual Ornstein-Uhlenbeck semi-groups on $\mathbf{R}$, given by the following kernels with respect to

Lebesgue measure

$$p_s^+(x,y) = \frac{\sqrt{2}}{\sqrt{\pi(1-e^{-s})}} \exp(-(\frac{2(y-e^{-\frac{s}{2}}x)^2}{1-e^{-s}}))$$

and

$$p_s^-(x,y) = \frac{\sqrt{2}}{\sqrt{\pi(e^{s}-1)}} \exp(-(\frac{2(y-e^{\frac{s}{2}}x)^2}{e^{s}-1}))$$

We now turn to the Yule semi-group, which can be obtained from $(R_s^\pm)_{s\in\mathbf{R}_+}$ by restriction to a suitable algebra.

Let us introduce the functions

$$\mathcal{E}(\alpha)(x) = (\frac{2}{\pi})^{\frac{1}{4}} e^{-2\alpha x - \frac{\alpha^2}{2} - x^2}$$

for any complex α. One has $\mathcal{E}(\alpha) \in L^2(\mathbf{R})$, and

$$\mathcal{E}(\alpha) = \sum_{n=0}^{\infty} (-\alpha)^n \frac{u_n}{n!}$$

where $u_n(x) = h_n(x)(\frac{2}{\pi})^{\frac{1}{4}} e^{-x^2}$ and the h_n are the Hermite polynomials. One has $|u_n|_{L^2}^2 = n!$. The representations $\pi_\pm$ are given by

$$\pi_+(z,w)\mathcal{E}(\alpha) = \mathcal{E}(\alpha+\bar{z})e^{iw-\alpha z-\frac{|z|^2}{2}} \tag{3.4}$$

$$\pi_-(z,w)\mathcal{E}(\alpha) = \mathcal{E}(\alpha+z)e^{-iw-\alpha\bar{z}-\frac{|z|^2}{2}} \tag{3.5}$$

The number operator, N, of $\pi_\pm$ is defined by the formula

$$Nu_n = nu_n$$

or alternatively, for all $\omega \in \mathbf{R}$,

$$e^{i\omega N}\mathcal{E}(\alpha) = \mathcal{E}(e^{i\omega}\alpha)$$

It turns out that the C^*-algebra generated by the spectral projections of the number operator is the image by $\pi_\pm$ of the algebra of elements of $C^*(H_1)$ invariant under the automorphisms $(z,w) \mapsto (\xi z, w)$, for $\xi \in U(1)$ and so it is stable under the semi-groups $R^\pm$. In [B] we computed the corresponding semi-groups of kernels on the spectrum of this algebra which is exactly the spectrum of N, the set $\mathbf{N}$. We obtained in [B] (corollaire 3.3.3) the following result.

3.3 Proposition. *The semi-groups of kernels on the spectrum of N induced by $R^\pm$ are given by the formulas*

$$q_s^+(k,l) = \frac{k!}{l!(k-l)!} e^{-ls}(1-e^{-s})^{k-l} \quad \text{for } 0 \le l \le k$$

$$q_s^-(k,l) = \frac{l!}{(l-k)!k!} e^{-ks}(1-e^{-s})^{l-k} \text{for } 0 \le k \le l$$

(beware that in [B] the formula for q^- is shifted by 1).

The semi-group of kernels q^- is the Yule semi-group (cf [A-N]).

Let us now show that there is an an intertwining between the kernels p^+ and q^+ on one hand and between p^- and q^- on the other, and compute explicitly this kernel. This is done in the following way.

3.4 Theorem. *There exists a kernel $\Lambda : \mathbb{R} \to \mathbb{N}$ which satisfies the interwining relations*

$$q_s^+ \circ \Lambda = \Lambda \circ p_s^+$$

and

$$q_s^- \circ \Lambda = \Lambda \circ p_s^-$$

This kernel is given by the following formula

$$\Lambda(n, dx) = \frac{1}{n!}\sqrt{\frac{2}{\pi}} h_n^2(x) e^{-2x^2} dx$$

Proof To any bounded operator A on $L^2(\mathbb{R})$ let us associate the diagonal operator $D(A)$ such that $D(A)(u_n) = \frac{<Au_n, u_n>}{n!} u_n$, alternatively, one has $D(A) = \sum_n \Pi_n A \Pi_n$ where the Π_n are the orthogonal projections on the lines $\mathbb{C}.u_n$. This defines a completely positive projection D from $\mathcal{B}(L^2(\mathbb{R}))$ onto the von Neumann algebra generated by the spectral projections of N and hence a completely positive map $D : \mathcal{B}(L^2(\mathbb{R})) \to l^\infty(spec\, N)$. We have the following

3.5 Lemma. *For any $s \in \mathbb{R}_+$ one has*

$$D \circ R_s^\pm = R_s^\pm \circ D$$

Proof. The map D is also given by the formula

$$D(A) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega N} A e^{-i\omega N} d\omega$$

From formulas (3.4) and (3.5) we deduce that, for all $(z, w) \in H_1$,

$$e^{i\omega N} \pi_\pm(z, w) e^{-i\omega N} = \pi_\pm(e^{\pm i\omega} z, w)$$

This implies that

$$\begin{aligned} D \circ R_s^+(\pi_+(z, w)) &= \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega N} \pi_+(ze^{-\frac{s}{2}}, we^{-s}) e^{-i\omega N} e^{(1-e^{-s})(iw - \frac{1}{2}|z|^2)} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \pi_+(e^{i\omega} ze^{-\frac{s}{2}}, we^{-s}) e^{(1-e^{-s})(iw - \frac{1}{2}|z|^2)} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \pi_+(e^{i\omega} ze^{-\frac{s}{2}}, we^{-s}) e^{(1-e^{-s})(iw - \frac{1}{2}|e^{i\omega} z|^2)} d\omega \\ &= R_s^+ \circ D(\pi_+(z, w)) \end{aligned}$$

The result follows then easily for R^+ and a similar computation works for R^-.

Let $i : C_0(\mathbf{R}) \to \mathcal{B}(L^2(\mathbf{R}))$ be the embedding by multiplication operators. It follows from lemma 3.5 that the composition $D \circ i$ is the required map from $C_0(\mathbf{R})$ to $l^\infty(\mathbf{N})$, given by the kernel Λ. So we obtain immediately for $v \in C_0(\mathbf{R})$.

$$\int_{\mathbf{R}} v(x)\Lambda(n,dx) = \frac{< i(v)u_n, u_n >}{n!}$$
$$= \frac{1}{n!}\sqrt{\frac{2}{\pi}}\int_{\mathbf{R}} v(x)h_n^2(x)e^{-2x^2}dx$$

Finally we get the right formula for $\Lambda(n, dx)$.

References.

[A-N] K. B. Athreya and P. E. Ney, *Branching processes, Springer verlag, Berlin, Heidelberg, New york, 1972.*

[B] P. Biane, *Quelques propriétés du mouvement brownien non-commutatif, preprint, Laboratoire de Probabilités, Université Paris VI, 1994.*

[C-P-Y] P. Carmona, F. Petit, M. Yor, *Beta-gamma random variables and intertwining relations between certain Markov processes, preprint, Laboratoire de Probabilités, Université Paris VI, 1994.*

[P] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus, Monographs in Mathematics, Vol 85, Birkhäuser, Basel, 1992.*

C.N.R.S. Laboratoire de Probabilités
Tour 56, 3[e] étage, Université Paris VI
4, place Jussieu, 75252, PARIS cedex 05
FRANCE

The author acknowledges partial support from the HUMAN CAPITAL AND MOBILITY programme, CONTRACT NUMBER: ERBCHRXCT930094