

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

JIA-AN YAN

## **An asymptotic evaluation of heat kernel for short time**

*Séminaire de probabilités (Strasbourg)*, tome 30 (1996), p. 104-107

[http://www.numdam.org/item?id=SPS\\_1996\\_\\_30\\_\\_104\\_0](http://www.numdam.org/item?id=SPS_1996__30__104_0)

© Springer-Verlag, Berlin Heidelberg New York, 1996, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

# An Asymptotic Evaluation of Heat Kernel for Short Time<sup>1</sup>

In Honor of P.A. Meyer and J. Neveu

J.A. Yan

Consider the following heat equation

$$\frac{\partial u}{\partial t} = \left(\frac{\Delta}{2} + V\right)u, \quad (1)$$

where  $\Delta$  is the Laplacian operator on  $\mathbb{R}^d$  and  $V$  is a continuous function on  $\mathbb{R}^d$ . Under mild assumptions on  $V$  the fundamental solution of equation (1) exists and can be expressed by the Feynman-Kac formula (cf.[2]). This fundamental solution is called the heat kernel.

The purpose of this paper is to prove the following theorem, which gives an asymptotic evaluation of the heat kernel for short time.

**Theorem.** Let  $V$  be a continuous function on  $\mathbb{R}^d$ . Assume there exist positive constants  $C$ ,  $C_1$  and  $C_2$  such that

$$V(x)^+ \leq C(1 + |x|^2), \quad (2)$$

$$V(x)^- \leq C_1 e^{C_2|x|^2}. \quad (3)$$

Let  $q(t, x, y)$  be the fundamental solution of the heat equation (1). Then we have

$$\lim_{t \downarrow 0} \frac{1}{t} \log \frac{q(t, x, y)}{p(t, x, y)} = \int_0^1 V((1-s)x + sy) ds, \quad (4)$$

where  $p(t, x, y)$  is the transition density of a standard Brownian motion.

The main tool for proving this theorem is the Feynman-Kac formula. We recall it for the reader's convenience.

Let  $\Omega = C([0, \infty), \mathbb{R}^d)$  be the collection of all continuous functions from  $[0, \infty)$  to  $\mathbb{R}^d$ . For  $\omega \in \Omega$ , let  $X_t(\omega) = \omega(t)$ . Let  $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$ ,  $\mathcal{F} = \sigma\{X_s, s < \infty\}$ . We denote by  $(\mathbb{P}_x, x \in \mathbb{R}^d)$  the unique family of probability measures on  $(\Omega, \mathcal{F})$  such that  $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x)$  is a standard Brownian motion. Let  $y \in \mathbb{R}^d$  and  $t > 0$ . Put

---

<sup>1</sup>Work supported by the National Natural Science Foundation of China.

$$Y_s(\omega) = X_s(\omega) - \frac{s \wedge t}{t} (X_t(\omega) - y), \quad s \geq 0.$$

Then under  $\mathbb{P}_x$  the process  $(Y_s, 0 \leq s \leq t)$  is a *Brownian bridge from  $x$  to  $y$  on  $[0, t]$*

and  $(Y_s, t \leq s < \infty)$  is a Brownian motion with  $Y_t = y$ . Moreover, under  $\mathbb{P}_x$  these two processes are independent. We denote by  $\mathbb{P}_{x,y,t}$  the distribution of the process  $(Y_s, s \geq 0)$  on  $(\Omega, \mathcal{F})$  under  $\mathbb{P}_x$ . We call  $\mathbb{P}_{x,y,t}$  the  $(0, x; t, y)$ -*Brownian bridge measure*. Under mild assumptions on  $V$  it was shown that the heat kernel for (1) can be expressed by the following Feynman-Kac formula( cf.[2,Theorem 3.2]) :

$$\begin{aligned} q(t, x, y) &= p(t, x, y) \mathbb{E}_{x,y,t} [e^{\int_0^t V(X_s) ds}] \\ &= p(t, x, y) \mathbb{E}_0 [e^{\int_0^t V(x + \frac{s}{t}(y-x) + X_s - \frac{s}{t} X_t) ds}] \\ &= p(t, x, y) \mathbb{E}_0 [e^{\int_0^t V(x + s(y-x) + \sqrt{t}(X_s - sX_1)) ds}], \end{aligned} \quad (5)$$

where

$$p(t, x, y) = (2\pi t)^{-\frac{d}{2}} \epsilon x p\left\{-\frac{|x-y|^2}{2t}\right\}.$$

We are going to prove the theorem. To begin with we prepare a lemma.

**Lemma.** Let  $\{\xi(\varepsilon), \varepsilon > 0\}$  be a family of integrable random variables such that  $\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]$  exists and is finite. If

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)(e^{\varepsilon \xi(\varepsilon)} - 1)] = 0, \quad (6)$$

then we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \log \mathbb{E}[e^{\varepsilon \xi(\varepsilon)}] = \lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]. \quad (7)$$

*Proof.* Since  $1 + x \leq e^x \leq 1 + x + x(e^x - 1)$ , we have

$$\xi(\varepsilon) \leq \frac{1}{\varepsilon} [e^{\varepsilon \xi(\varepsilon)} - 1] \leq \xi(\varepsilon) + \xi(\varepsilon)[e^{\varepsilon \xi(\varepsilon)} - 1].$$

This together with (6) imply

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathbb{E}[e^{\varepsilon \xi(\varepsilon)}] - 1) = \lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)], \quad (8)$$

which is equivalent to (7). ■

**Corollary 1.** If instead of (6) we assume

$$\lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{E}[\xi(\varepsilon)^2 (e^{\varepsilon \xi(\varepsilon)} + 1)] = 0, \quad (9)$$

then (7) holds.

*Proof.* Immediate from the fact that  $x(e^x - 1) \leq x^2(e^x + 1)$ . We leave the proof of this fact to the reader. ■

**Corollary 2.** Let  $\{\xi(\varepsilon), \varepsilon > 0\}$  be a family of integrable random variables such that  $\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]$  exists and is finite. If there exists an  $\varepsilon_0 > 0$  such that  $\{\xi(\varepsilon), 0 < \varepsilon \leq \varepsilon_0\}$  is uniformly integrable (u.i. for short) and  $\mathbb{E}[e^{\delta \sup_{0 < \varepsilon \leq \varepsilon_0} \xi(\varepsilon)^+}] < \infty$  for some  $\delta > 0$ , then (6) is satisfied. In particular, we have (7).

*Proof.* For any  $c > 0$ , we have

$$\xi(\varepsilon)^+ e^{\varepsilon \xi(\varepsilon)^+} \leq \frac{1}{c} e^{(c+\varepsilon)\xi(\varepsilon)^+}.$$

Thus, by assumption we can find an  $\varepsilon_1 > 0$  with  $\varepsilon_1 \leq \varepsilon_0$  such that  $\{\xi(\varepsilon)^+ e^{\varepsilon \xi(\varepsilon)^+}, 0 < \varepsilon \leq \varepsilon_1\}$  is u.i.. On the other hand, we have

$$|\xi(\varepsilon)| e^{\varepsilon \xi(\varepsilon)} \leq \xi(\varepsilon)^+ e^{\varepsilon \xi(\varepsilon)^+} + |\xi(\varepsilon)|.$$

Consequently,  $\{\xi(\varepsilon)[e^{\varepsilon \xi(\varepsilon)} - 1], \varepsilon_1 \geq \varepsilon > 0\}$  is u.i.. Therefore, (6) holds, because  $\varepsilon \xi(\varepsilon)$  tends to 0 in probability as  $\varepsilon$  tends to 0. ■

Now we are in a position to prove our theorem. Put

$$\xi(\varepsilon) = \int_0^1 V(x + s(y-x) + \sqrt{\varepsilon}(X_s - sX_1)) ds. \quad (10)$$

We may assume  $C_1 \geq C, C_2 \geq 1$ . Then by (2) and (3) we get

$$\begin{aligned} \sup_{0 < \varepsilon \leq \varepsilon_0} |\xi(\varepsilon)| &\leq C_1 \int_0^1 e^{C_2 \sup_{0 \leq s \leq \varepsilon_0} |x+s(y-x) + \sqrt{\varepsilon}(X_s - sX_1)|^2} ds \\ &\leq C_1 \int_0^1 e^{2C_2 |x+s(y-x)|^2} ds e^{4C_2 \varepsilon_0 \sup_{0 \leq s \leq 1} |X_s|^2}. \end{aligned}$$

Thus by Fernique's theorem ([1]) for sufficiently small  $\varepsilon_0 > 0$ ,  $\{\xi(\varepsilon), 0 < \varepsilon \leq \varepsilon_0\}$  is u.i. and we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)] = \int_0^1 V(x + s(y-x)) ds.$$

On the other hand, by (10) and (2) we have

$$\begin{aligned} &e^{\sup_{0 \leq \varepsilon \leq \varepsilon_0} \xi(\varepsilon)^+} \\ &\leq e^{\int_0^1 \sup_{0 \leq \varepsilon \leq \varepsilon_0} C(1+|x+s(y-x) + \sqrt{\varepsilon}(X_s - sX_1)|^2) ds} \\ &\leq e^{\int_0^1 C(1+2|x+s(y-x)|^2) ds} e^{4C\varepsilon_0 \sup_{0 \leq s \leq 1} |X_s|^2}. \end{aligned}$$

Thus, once again by Fernique's theorem, for sufficiently small  $\varepsilon_0$ ,  $\mathbb{E}[e^{\sup_{0 \leq \varepsilon \leq \varepsilon_0} \xi(\varepsilon)^+}] < \infty$ . Consequently, in view of (5) and (10) we can apply Corollary 2 to conclude the theorem.

**References**

- [1 ] X. Fernique, Intégrabilité des vecteurs Gaussiens, C.R.A.S. Paris 270, Series A(1970), 1698-1699.
- [2 ] J.A. Yan, From Feynman-Kac formula to Feynman integrals via analytic continuation, *Stoch. Processes and their Applications*, 54 (1994), 215 - 232.

J.A. Yan

Institute of Applied Mathematics

Academia Sinica

P.O. Box 2734

Beijing 100080, P.C. China

Email: jayan@bepc2.ihep.ac.cn