WILHELM VON WALDENFELS

Continuous Maassen kernels and the inverse oscillator

Séminaire de probabilités (Strasbourg), tome 30 (1996), p. 117-161
<http://www.numdam.org/item?id=SPS_1996__30__117_0>

CONTINUOUS MAASSEN KERNELS AND THE INVERSE OSCILLATOR

Wilhelm von Waldenfels
Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294
D-69120 Heidelberg

Dedicated to P.A. Meyer to his 60th birthday

Summary: The quantum stochastic differential equation of the inverse oscillator in a heat bath of oscillators is solved by the means of a calculus of continuous and differentiable Maassen kernels. It is shown that the time development operator does not only map the Hilbert space of the problem into itself, but also vectors with finite moments into vectors with finite moments. The vacuum expectation of the occupancy numbers coincides for pyramidally ordered times with a classical Markovian birth process showing the avalanche character of the quantum process.

§ 0. Introduction

The quantum mechanical oscillator has the Hamiltonian $\omega_0 b^+b$, where b and b^+ are the usual annihilation and creation operators. The inverse oscillator has the Hamiltonian $-\omega_0 b^+b$. Coupled to a heat bath the inverse oscillator has the Hamiltonian

$$-\omega_0 b^+b + \sum_{\lambda\in\Lambda} (\omega_0+\omega_\lambda)\, a_\lambda^+ a_\lambda + \sum_{\lambda\in\Lambda} (g_\lambda a_\lambda b+\overline{g_\lambda}\, a_\lambda^+ b^+).$$

As this Hamiltonian is not bounded below it cannot describe a real physical system; it can be used, however, to approximate the initial behavior of real physical systems, e.g. in the case of superradiance, at it is shown in § II.1 [2], [3], [11].

Using the interaction representation and singular coupling limit we arrive to the quantum stochastic differential equation for the time development operator

(1) $$dU_{t,s} = (-ibda_t - ib^+da_t^+ - \tfrac{1}{2}\, bb^+dt)U_{t,s}.$$

This is a well-known equation, already mentioned in one of the early papers of Hudson and Parthasarathy [5].
The mathematical problem is that the coefficients b and b^+ are unbounded operators. We treat it in considering the matrix elements

(2) $$\langle m|U_{t,s}|n\rangle$$

as Maassen kernels. Here again a problem arises as the kernels are not bounded in the Maassen sense. Due to the simple algebraic structure, however, all convolutions of these kernels are allowed.

In chapter I we reconstruct the theory of Maassen kernels without the exponential bond used by Maassen. We introduce continuity and differentiability in a slightly different way and obtain an elementary theory which uses only calculus and Lebesgue integrations. We regain Maassen's theorem connecting differentiation and integration similar to the fundamental theorem of calculus and Maassen's and Robinson's general Itô-formula [7], [8], [9], [10], [12]. From there one can obtain several Itô tables for adapted processes. We have the usual Itô table for forward adapted processes

(3)

	da	da_t^+
da	0	dt
da_t^+	0	0

For backward adapted processes we obtain

(4)

	da	da_t^+
da	0	$-dt$
da_t^+	0	0

and if one of the processes is forward adapted and the other backward adapted we have

(5)

	da_t	da_t^+
da	0	0
da_t^+	0	0.

In chapter II we investigate the special structure of the inverse oscillator in a bath. Due to the quadratic Hamiltonian the Heisenberg equations are linear and can be solved easily. In II.2 we calculate the Heisenberg equations going back to the finite heat bath and performing the singular coupling limit. We obtain

(6) $$b_{t,s}^+ = U_{t,s}^+ b^+ U_{t,s} = e^{(t-s)/2}\, b^+ + i \int_s^t e^{(t-t')/2}\, da_t.$$

We see that for $t \to \infty$

(7) $$e^{-t/2}\, b_{t,0}^+ \to b^+ + i \int_0^t e^{-t'/2}\, da_{t'} = B^+.$$

As B and B^+ commute we can interpret them as classical quantities which might be understood as the macroscopic quantities after amplification [2]. Assume for $t = 0$ as statistical operator the vacuum for the bath and the density matrix ρ for the b and b^+, then B and B^+ are distributed with respect to the classical probability law given by a smeared out Wigner transform of ρ.

It is easy to solve the stochastic equation (1) by Maassen kernels. We obtain a uniquely determined matrix

$$\left(\langle m|u_{t,s}|n\rangle\,(\sigma,\tau)\right)_{m,n\;=\;0,1,2,\ldots}$$

and are left with the problem to show that this is the matrix of a unitary operator $U_{t,s}$. By assumption

$$t \to \langle m|u_{t,s}|n\rangle$$

is forward adapted. From the explicit formula one concludes that

$$s \to \langle m|u_{t,s}|n\rangle$$

is backward adapted. Using the differentiation calculus and (3), (4) and (5) we conclude that in matrix form

$$u_{t,s} * u_{s,r} = u_{t,r}$$

for $r < s < t$ (Proposition 2 of § II.4). Let $p = p(b, b^+)$ be a polynomial in b and b^+, then by differentiating with respect to t we obtain

(8) $$u^+_{t,s} * p(b,b^+)\,\delta_{\emptyset,\emptyset} * u_{t,s} = p(b_{t,s},b^+_{t,s})$$

where $b_{t,s}$ is given by (6) and by differentiating with respect to s

(9) $$u_{t,s} * p(b,b^+)\,\delta_{\emptyset,\emptyset} * u^+_{t,s} = p(b_{s,t},b^+_{s,t})$$

with

$$b^+_{s,t} = e^{(t-s)/2}\,b^+\delta_{\emptyset,\emptyset} - i\int_s^t e^{(t-t')/2}da_{t'}.$$

The equations (8) and (9) hold for $|t-s| < 1$. Choosing $p = 1$, one can deduce the unitarity of $U_{t,s}$. But there is more. Call Λ the operator of the total number of particles in the Fock space

$$(\Lambda\xi)(\omega) = \#\omega\xi(\omega),$$

then there exist constants C_k, Γ_k such as

$$U^+_{t,s}\,(\Lambda + bb^+)^k U_{t,s}$$

and

$$U_{t,s}\,(\Lambda + bb^+)^k U^+_{t,s}$$

are

(10) $$\leq C_k\, e^{\Gamma_k |t-s|}(\Lambda + bb^+)^k.$$

From there we establish a unitary evolution $U_{t,s}$ for all t and s; furthermore $U_{t,s}$ maps the space

$$\mathcal{D}_k = \{\xi : \|(\Lambda + bb^+)^{k/2}\,\xi\| < \infty\}$$

onto itself (Theorem II.5). The Heisenberg equations can now be established in a rigorous way.

Call $X(t)$ the classical Markov process on $\mathbf{N}$ which is able to make only jumps of $+1$ and has the transition probabilities

$$\mathbf{P}(X(t+dt) = n+1 \mid X(t) = n) = (n+1)dt$$
$$\mathbf{P}(X(t+dt) = n \mid X(t) = n) = 1-(n+1)dt,$$

then $X(t)$ and

$$N(t) = U^+_{t,0}\, b^+b\, U_{t,0}$$

have the same marginal distributions and moments for pyramidally ordered times. For non pyramidally ordered times there are differences. To establish this result was one of the major difficulties of the paper. We had to use (10) heavily.

I. Continuous Maassen kernels

§ I.1. Measurable kernels

We follow Maassen's original notation [9]. Let $I \subset \mathbf{R}$ be an interval. Denote by $\Omega(I)$ the set of all finite subsets of I.

$$\Omega(I) = \bigcup_{n=0}^{\infty} \Omega_n(I),$$

$$\Omega_0(I) = \{\emptyset\};\ \ \Omega_n(I) = \{\omega \in \Omega(I) : \#\omega = n\}$$

where $\#\omega$ denotes the cardinality. $\Omega_n(I)$ can be identified with the subset $\{(t_1, \ldots, t_n) \in I^n : t_1 < \ldots < t_n\}$ and inherits the structure of a measure space from I^n. Let $d\omega$ denote the measure on $\Omega(I)$ which has $\emptyset$ as an atom of measure 1 and which equals the Lebesgue measure on $\Omega_n(I)$ for $n = 1, 2, \ldots$. So

$$\int f(\omega)d\omega = f(\emptyset) + \sum_{n=1}^{\infty} \int_{t_1<\ldots<t_n}\!\!\int dt_1 \ldots dt_n\, f(\{t_1, \ldots, t_n\}).$$

A kernel is a measurable function

$$x : \Omega(I) \times \Omega(I) \to \mathbf{C}.$$

Two kernels x and y are called multipliable if

$$\sum_{\alpha\subset\sigma} \sum_{\beta\subset\tau} \int_{\Omega(I)} d\gamma \, | (x(\alpha, \beta + \gamma) | \, | y((\sigma\backslash\alpha) + \gamma, \tau\backslash\beta)| < \infty$$

for almost all $\sigma, \tau \in \Omega(I)$. The sum $\omega + \omega'$ of two finite subsets of I is equal to $\omega \cup \omega'$ if $\omega \cap \omega' = \emptyset$ and is not defined if $\omega \cap \omega' \neq \emptyset$. So the integrand is defined almost everywhere.

If two kernels are multipliable their product $x * y$ is defined by

$$(x * y)(\sigma, \tau) = \sum_{\alpha\subset\sigma} \sum_{\beta\subset\tau} \int_{\Omega(I)} d\gamma \, (x(\alpha, \beta + \gamma) \, y((\sigma \setminus \alpha) + \gamma, \tau \setminus \beta).$$

For n factors we have the formula

$$(x_1 * \dots * x_n)(\sigma, \tau) = \sum_{\substack{a_1+ \dots + \alpha_n=\sigma \\ \beta_1+ \dots + \beta_n=\tau}} \int \cdots \int d\gamma_{12}\, d\gamma_{13} \dots d\gamma_{1n}\, d\gamma_{23} \dots d\gamma_{2n} \dots d\gamma_{n-1,n}$$

$$x_1(\alpha_1; \beta_1+\gamma_{12}+ \dots + \gamma_{1,n})$$
$$x_2(\alpha_2+\gamma_{1,2}; \beta_2+\gamma_{23}+ \dots + \gamma_{2,n})$$
$$\vdots$$
$$x_{n-1}(\alpha_{n-1}+\gamma_{1,n-1}+ \dots \gamma_{n-2,n-1}; \beta_{n-1}+\gamma_{n-1,n})$$
$$x_n(\alpha_n+\gamma_{1,n}+ \dots + \gamma_{n-1,n}; \beta_n$$

$$= \sum_{\substack{a_1+ \dots + \alpha_n=\sigma \\ \beta_1+ \dots + \beta_n=\tau}} \int \cdots \int \prod_{1\leq i<j\leq n} d\gamma_{ij} \prod_{i=1}^{n} x_i(\alpha_i+\gamma_{1,i}+\dots+\gamma_{i-1,i}; \beta_i+\gamma_{i,i+1}+\dots+\gamma_{i,n}).$$

So the product $x_1 * \dots * x_n$ exists if the product $|x_1| * \dots * |x_n|$ given by the formula above is finite a.e. and if that is the case the product is given by the formula.
It is easy to prove that e.g.

$$(x_1 * \dots * x_n) * x_{n+1} = x_1 * \dots * x_n * x_{n+1}$$

using the $\Sigma\int$-Lemma [6]:

$$\int d\sigma \sum_{\alpha_1+\ldots+\alpha_d=\sigma} f(\alpha_{1,\ldots}\, \alpha_d) = \int \ldots \int d\sigma_1 \ldots d\sigma_d\, f(\sigma_1, \ldots, \sigma_d).$$

A vector ξ is a measurable function $\Omega(I) \to \mathbf{C}$. The application of a kernel to a vector is given by

$$(x * \xi)(\omega) = \sum_{\sigma \subset \omega} \int d\tau\, x(\sigma, \tau)\, \xi((\omega \setminus \sigma) + \tau)$$

if this expression exists.
Define

$$\hat{\xi}(\sigma, \tau) = \xi(\sigma)\delta_{\emptyset}(\tau).$$

Then

$$(x * \hat{\xi})(\omega, \emptyset) = (x * \xi)(\omega).$$

This reduces the multiplication $x * \xi$ to the multiplication of kernels. Denote

$$x^T(\sigma, \tau) = x(\tau, \sigma).$$

Let ξ be a vector, define

$$(\xi^T * x)(\omega) = \sum_{\tau \subset \omega} \int d\sigma\, \xi((\omega \setminus \tau) + \sigma)\, x\,(\sigma, \tau) = (\check{\xi} * x)(\emptyset, \omega)$$

with $\check{\xi}(\sigma, \tau) = \delta_{\emptyset}(\sigma)\xi(\tau)$.

Let η be another vector, define

$$\xi^T * \eta = \int \xi(\omega)\eta(\omega)d\omega = (\check{\xi} * \hat{\eta})(\emptyset, \emptyset).$$

One has the usual rules

$$(x * y)^T = y^T * x^T$$
$$(x * \xi)^T = \xi^T * x^T.$$

Define as usual

$$x^+ = \bar{x}^T, \;\; \xi^+ = \bar{\xi}^T,$$

where $\bar{x}$ is the complex conjugate.

§ 2. Introducing continuity

At first some notations. Let S be a set, $A \subset S$ and $B \subset S^d$, then

$$A(\times)B = \{(a,b) : a \in A, b = (b_1, \ldots, b_d) \in B : a \neq b_1, \ldots, a \neq b_d\}.$$

So

$$A^{(d)} = A(\times) \ldots (\times)A = \{(a_1, \ldots, a_d) \in A^d : a_1 \neq a_j \text{ for } i \neq j\}.$$

If $A \subset \mathbf{P}(S)$, $B \subset \mathbf{P}(S)^d$, where $\mathbf{P}(S)$ is the set of all subsets of S, then

$$A(\times)B = \{(\alpha, \beta) : \alpha \in A, \beta = (\beta_1, \ldots, \beta_d) \in B : \alpha \cap (\beta_1 \cup \ldots \cup \beta_d) = \emptyset\}$$

and

$$A^{(d)} = A(\times) \ldots (\times) A = \{\alpha_1, \ldots, \alpha_d) \in A^d : \alpha_i \cap \alpha_j = \emptyset \text{ for } i \neq j\}.$$

If $A \subset S$ and $B \subset \mathbf{P}(S)^d$, then

$$A(\times)B = \{(a, \beta) : a \in A, \beta = (\beta_1, \ldots, \beta_d) \in B : a \notin \beta_1 \cup \ldots \cup \beta_d\}.$$

We introduce in $\Omega_n(I) = \{(t_1, \ldots, t_n) \in I^n : t_1 < \ldots < t_n\}$ the usual topology and define so a topology on $\Omega(I)$. We denote by $\mathcal{C}(I)$ the set of continuous functions $\Omega(I) \to \mathbf{C}$ such that for $\xi \in \mathcal{C}(I)$ and all p

$$\|\xi\|_p = \sup_{\#\omega=p} |\xi(\omega)| < \infty.$$

$\Omega(I)^{(2)}$ inherits its topology from $\Omega(I)^2$. A continuous kernel x is a continuous function on $\Omega(I)^{(2)}$, such that

$$\|x\|_{p,q} = \sup_{(\sigma,\tau)\in \Omega_p(I)(\times)\Omega_q(I)} |x(\sigma, \tau)| < \infty.$$

Denote by $\mathcal{C}_0(I)$ the subspace of $\mathcal{C}(I)$ of all ξ such that $\xi(\omega) = 0$ for all ω with $\#\omega$ bigger than some bound depending on ξ.

Remark 1: The assumptions $\|\xi\|_p < \infty$ and $\|x\|_{p,q} < \infty$ are essentially integrability conditions and can be replaced by much weaker ones.

Proposition 1: Let I be a finite interval of length L. Let x be a continuous kernel in $\Omega(I)^{(2)}$. Then for $\xi \in \mathcal{C}_0(I)$ the product $x * \xi$ is defined and

$$\xi \to x * \xi$$

is a mapping from $\mathcal{C}_0(I) \to \mathcal{C}(I)$ such that

$$\|x * \xi\|_p \leq \sum_{q=0}^{p} \sum_{r=0}^{\infty} \binom{p}{q} \frac{L^r}{r!} \|x\|_{q,r} \|\xi\|_{p-q+r}.$$

The sum is finite as $\|\xi\|_{p-q+r}$ vanishes for r sufficiently big.

Proof: Recall

$$(x * \xi)(\omega) = \sum_{\alpha+\beta=\omega} d\tau\, x(\alpha,\tau)\xi(\beta+\tau)$$

and put $\omega = \{\omega_1, \ldots, \omega_p\}$. Choose $A, B \subset \{1, \ldots, p\}$ with $A+B = \{1, \ldots, p\}$ and $\alpha = \omega_A = \{\omega_j : j \in A\}$ and $\beta = \omega_B$. Then with $\tau = \{t_1, \ldots, t_r\}$

$$(x * \xi)(\omega) = \sum_{A+B=\{1,\ldots,p\}} \sum_{r=0}^{R} \int_{t_1<\ldots<t_r} \int x(\omega_A;\{t_1, \ldots, t_r\})\xi(\omega_B + \{t_1, \ldots, t_r\})\, dt_1 \ldots dt_r$$

where $R < \infty$ is some integer. Call

$$\eta(\tau; \omega, A, B) = x(\omega_A; \tau)\, \xi(\omega_B + \tau)$$

with $\tau = \{t_1, \ldots, t_r\}$. The function

$$\tau \in \Omega_r(I) \to \eta(\tau; \omega, A, B)$$

is continuous for $\tau \cap \omega = \emptyset$, hence measurable and bounded by
$\|x\|_{\#A,r}\, \|\xi\|_{\#B+r}$.

For $\omega^{(n)} \to \omega$ the sets $\omega_A^{(n)} \to \omega_A$ and $\omega_B^{(n)} \to \omega_B$. So

$$\eta(\tau; \omega^{(n)}, A, B) \to \eta(\tau; \omega, A, B)$$

for $\tau \cap \omega = \emptyset$, that means a.e. By Lebesgue's theorem

$$\int \eta(\tau; \omega^{(n)}, A, B)\, d\tau \to \int \eta(\tau; \omega, A, B).$$

From there one gets the result immediately.
Definition: We say a pair (x,y) of kernels has the finite product property (FP) if

$$x(\alpha_1, \beta_1 + \gamma)\, y\, (\alpha_2 + \gamma, \beta_2)$$

vanishes for $\#\gamma$ sufficiently big for fixed $\#\alpha_1, \#\beta_1, \#\alpha_2, \#\beta_2$.
Proposition 2: Let I be a finite interval of length L. Let x and y be continuous kernels on I with the finite product property (FP). Then $x * y$ is a continuous kernel on I.
Proof: Assume $(\sigma, \tau) \in \Omega_p(\times)\Omega_q$. Then

$$(x * y)(\sigma, \tau) = \sum_{\substack{A_1+A_2=\{1,\ldots,p\} \\ B_1+B_2=\{1,\ldots,q\}}} \sum_{r=0}^{\infty} \int_{\#\omega=r} d\omega\, z_r(\omega; \sigma, \tau, A_1, A_2, B_1, B_2)$$

with

$$z_r(\omega; \sigma, \tau, A_1, A_2, B_1, B_2) = x(\sigma_{A_1}, \tau_{B_1} + \omega) y(\sigma_{A_2} + \omega, \tau_{B_2}).$$

If $\sigma = \{s_1, \ldots, s_p\}$, then $\sigma_{A_1} = \{s_i : i \in A_1\}$ etc. Now $\omega \to z_r(\omega)$ is continuous for $\omega \cap (\sigma \cup \tau) = \emptyset$, hence it is measurable. Moreover it is bounded. Let $\sigma^{(n)} \to \sigma$ and $\tau^{(n)} \to \tau$, then $\sigma_{A_1}^{(n)} \to \sigma_{A_1}$, etc. and

$$z_r(\omega; \sigma^{(n)}, \tau^{(n)}, A_1, A_2, B_1, B_2) \to z_r(\omega; \sigma, \tau, A_1, A_2, B_1, B_2)$$

for $\omega \cap (\sigma \cup \tau) = \emptyset$. As the integrand stays bounded, we have continuity of the integral and hence of the sums.

Remark 2: Assume instead of (FP) that

$$C_{p',p'';q',q''} = \sum_{r=0}^{\infty} \frac{L^r}{r!} \|x\|_{p',q'+r} \|y\|_{p''+r,q''} < \infty$$

for all p', p'', q', q''. Then $x * y$ exists, is continuous and

$$\|x * y\|_{p,q} \leq \sum_{\substack{p'+p''=p \\ q'+q''=q}} \frac{p!q!}{p'!p''!q'!q''!} C_{p',p'';q',q''}.$$

Remark 3: We say that a kernel x has Maassen's property if

$$\|x\|_{p,q} \leq c\, M^{p+q}$$

where c and M are some constants. If x and y are continuous kernels on a finite interval and have Maassen's property, then $x * y$ is a continuous kernel on I and has Maassen's property. For then

$$C_{p',p'';q',q''} \leq c^2 e^{L+M^2} M^{p'+p''+q'+q''}$$

and

$$\|x * y\|_{p,q} \leq c^2 e^{L+M^2} 2^{2(p+q)} M^{p+q}.$$

§ I.3. Continuous processes and their integrals

Definition 1: A continuous kernel process is a continuous mapping

$$x : I(\times)\, \Omega(I)^{(2)} \to \mathbf{C}$$

such that

$$\|x\|_{p,q} = \sup \{|x_t(\sigma, \tau)| : t \in I, \#\sigma = p, \#\tau = q\} < \infty$$

for all p, q.

If $f : I \to \mathbf{C}$ is measurable define the measurable kernels

$$a(f)(\sigma, \tau) = \begin{cases} \bar{f}(t) & \text{if } \sigma = \emptyset, \tau = \{t\} \\ 0 & \text{otherwise} \end{cases}$$

$$a^+(f)(\sigma, \tau) = \begin{cases} f(s) & \text{if } \sigma = \{s\}, \tau = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

Then a_t and a_t^+ are examples of continuous kernel processes where

$$a_t(\sigma, \tau) = a(\mathbf{1}_{I \cap]-\infty, t]})(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \tau = \{t'\}, t'<t \\ 0 & \text{otherwise} \end{cases}$$

the case $t = t'$ is not defined and similar

$$a_t^+(\sigma, \tau) = a^+(\mathbf{1}_{I\cap]-\infty,t]})(\sigma, \tau) = \begin{cases} 1 & \text{if } \tau = \emptyset, \sigma = \{s\}, s < t \\ 0 & \text{otherwise} \end{cases}.$$

The following proposition shows that the stochastic integral is a Riemann integral.

Proposition 1: Let x be a continuous process on I. Let $[t_0, t_1] \subset I$ and $(\sigma, \tau) \in \Omega^{(2)}$. Let $t_0 = t^{(0)} < t^{(1)} < \ldots < t^{(n)} = t_1$ and $\delta = \max_{i=1,\ldots,n} (t^{(i)} - t^{(i-1)})$. Assume $t^{(i-1)} \leq u_i \leq t^{(i)}$ and $u_i \notin \sigma \cup \tau$. Then for $\delta \to 0$

$$\sum (x_{u_i} * (a_{t^{(i-1)}} - a_{t^{(i-1)}}))(\sigma, \tau) \to \sum_{t^+ \in [t_0, t_1] \cap \tau} x_s(\sigma, \tau \setminus \{t\}).$$

and
$$\sum \left(a^+_{t^{(i)}} - a^+_{t^{(i-1)}}\right) * x_{u_i} \to x_s(\sigma \setminus \{s\}, \tau).$$

The right-hand side is well defined as

$$t \notin \sigma \cup (\tau \setminus \{t\}) \text{ and } s \notin (\sigma \setminus \{s\}) \cup \tau.$$

Proof: Call $\Delta_1 = [t^{(0)}, t^{(1)}]$, $\Delta_2 =]t^{(1)}, t^{(2)}]$, ..., $\Delta_n =]t^{(n-1)}, t^{(n)}]$.
Then

$$\sum_i (x_{u_i} * (a_{t^{(i)}} - a_{t^{(i-1)}}))(\sigma, \tau) = \sum_i (x_{u_i} * a(1_{\Delta_i}))(\sigma, \tau)$$
$$= \sum_{i=1}^{k} \sum_{t \in \tau} x_{u_i}(\sigma, \tau \setminus \{t\}) \mathbf{1}_{\Delta_i}(t).$$

If δ is sufficiently small there is at most one element of τ in Δ_i. We continue, the last expression equals

$$\sum_{i: \exists t \in \tau \cap \Delta_i} x_{u_i}(\sigma, \tau \setminus \{t\}) \to \sum_{t \in \tau \cap [t_0, t_1]} x_t(\sigma, \tau \setminus \{t\}).$$

Definition 2 [1], [6], [10]: Assume $A \subset I$

$$\left(\int_A x_t * da_t\right)(\sigma, \tau) = \sum_{t \in \tau \cap A} x_t(\sigma, \tau \setminus \{t\})$$

$$\left(\int_A da_t^+ * x_t\right)(\sigma, \tau) = \sum_{s \in \sigma \cap A} x_s(\sigma \setminus \{s\}, \tau).$$

Definition 3: Let $x : I(\times)\Omega(I)^{(2)}$ be a continuous process. Then x is called continuous differentiable ($\mathbf{C}^1$) if

$$\frac{d}{dt} x_t(\sigma, \tau) \text{ for } (t \notin \sigma \cup \tau)$$
$$(R_+^e)_t (\sigma, \tau) = x_{t+0}(\sigma \cup \{t\}, \tau)$$
$$(R_-^e)_t (\sigma, \tau) = x_{t-0}(\sigma \cup \{t\}, \tau)$$
$$(R_+^r)_t (\sigma, \tau) = x_{t+0}(\sigma, \tau \cup \{t\})$$
$$(R_-^r)_t (\sigma, \tau) = x_{t-0}(\sigma, \tau \cup \{t\})$$

exist and form continuous processes.

Remark 1: Let x be a continuous differentiable process. Then

$$x_{t\pm 0}(\sigma, \tau) \text{ exist for all } t \in I.$$

The following theorem goes back to H. Maassen [9].

Theorem 1: Let x be $\mathcal{C}^1$ then for $t_0 < t$ in I

$$x_{t_1-0} - x_{t_0+0} = \int_{]t_0 t_1[} da_t^+ * f_t + \int_{]t_0 t_1[} g_t * da_t + \int_{t_0}^{t_1} h_t dt.$$

with

$$f_t = R_+^{\ell} x - R_-^{\ell} x$$
$$g_t = R_+^r x - R_-^r x$$
$$h_t = \frac{d}{dt} x_t.$$

We write for short

$$d_t x_t = da_t^+ * f + g_t * da_t + h_t dt.$$

Proof: Call

$$t^{(0)} = t_0 < t^{(1)} < \ldots < t^{(n)} < t^{(n+1)} = t_1$$

and $\{t^{(1)}, \ldots, t^{(n)}\} = (\sigma \cup \tau) \cap]t_0, t_1[$.

Then

$$x_{t_1-0}(\sigma, \tau) - x_{t_0+0}(\sigma, \tau)$$

$$= \sum_{i=1}^{n+1} (x_{t^{(i)}-0}(\sigma, \tau) - x_{t^{(i-1)}+0}(\sigma, \tau)) + \sum_{i=1}^{n} (x_{t^{(i)}+0}(\sigma, \tau) - x_{t^{(i)}-0}(\sigma, \tau))$$
$$= \sum_{i=1}^{n+1} \int_{t^{(i-1)}}^{t^{(i)}} dt\, h_t(\sigma, \tau) + \sum_{i=1}^{n} \begin{cases} f_{t^{(i)}}\left(\sigma \setminus \{t^{(i)}\}, \tau\right), & \text{if } t^{(i)} \in \sigma \\ g_{t^{(i)}}\left(\sigma, \tau \setminus \{t^{(i)}\}\right), & \text{if } t^{(i)} \in \tau \end{cases}.$$

Now h_t is locally integrable w.r.t. This gives the theorem.

Proposition 2: Let $x^{(n)}$ be a sequence of $\mathcal{C}^1$ processes such that $\|x^{(n)}-x\|_{p,q}$ and $\|\dot{x}^{(n)}-\dot{x}\|_{p,q}$ converge to zero for all p, q. Then the $R_\pm^{\ell} x^{(n)}$ and $R_\pm^r x^{(n)}$ converge to $R_\pm^{\ell} x$ and $R_\pm^r x$ and hence the $f_t^{(n)}, g_t^{(n)}, h_t^{(n)}$ of the last theorem converge to f_t, g_t, h_t.

Lemma 1 [9]: Fix $\omega_0 \in \Omega(I)$. Assume a function

$$z: I(\times)\Omega(I) \to \mathbf{C}$$

to be continuous and bounded for $(\{t\} \cup \omega) \cap \omega_0 = \emptyset$ and that

$$\frac{d}{dt} z_t(\omega) = \dot{z}_t(\omega)$$
$$(R_\pm z)_t(\omega) = z_{t\pm 0}(\omega + \{t\})$$

are defined and continuous and bounded for $(\{t\} \cup \omega) \cap \omega_0 = \emptyset$. Assume furthermore that $z(\omega)$ vanishes for $\#\omega$ sufficiently big. Then $t \to \int z_t(\omega)\, d\omega$ is continuous differentiable for $t \notin \omega_0$ and

$$\frac{d}{dt} \int z_t(\omega) d\omega = \int \dot{z}_t(\omega) d\omega + \int d\omega\, (R_+ z - R_- z)(\omega)$$

Proof: Choose $t \notin \omega_0$ and $\varepsilon > 0$ such that $I_\varepsilon = [t-\varepsilon, t+\varepsilon]$ does not meet ω_0.

$$\frac{1}{2\varepsilon}\int (z_{t+\varepsilon}(\omega) - z_{t-\varepsilon}(\omega))d\omega = \int_{\Omega(I\setminus I_\varepsilon)} d\omega \int_{\Omega(I_\varepsilon)} d\gamma \frac{1}{2\varepsilon} (t_{t+\varepsilon}(\omega+\gamma) - z_{t-\varepsilon}(\omega+\gamma))$$

$$= \int_{\Omega(I\setminus I_\varepsilon)} d\omega \frac{1}{2\varepsilon} (z_{t+\varepsilon}(\omega) - z_{t-\varepsilon}(\omega) + \int_{\Omega(I\setminus I_\varepsilon)} d\omega \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} (z_{t+\varepsilon}(\omega + \{s\} - z_{t-\varepsilon}(\omega + \{s\})ds$$

$$+ \int_{\Omega(I\setminus I_\varepsilon)} d\omega \frac{1}{2\varepsilon} \int_{\#\gamma \geq 2} d\gamma\, (z_{t+\varepsilon}(\omega+\gamma) - z_{t-\varepsilon}(\omega+\gamma))$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Now

$$\mathrm{I} = \frac{1}{2} \int_{\Omega(I\setminus I_\varepsilon)} \int_{-1}^{+1} ds\, \dot{z}_{t+\varepsilon s}(\omega)$$

and $\dot{z}_{t+\varepsilon s}(\omega) \rightarrow \dot{z}_t(\omega)$ for $t \notin \omega$, that means a.e. As the integral is bounded

$$\mathrm{I} \rightarrow \int d\omega\, \dot{z}_t(\omega).$$

We have

$$\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} ds\, z_{t+\varepsilon}(\omega+\{s\}) = \frac{1}{2}\int_{-1}^{+1} ds\, z_{t+\varepsilon}(\omega+\{t+\varepsilon s\}) \rightarrow z_{t+0}(\omega+\{t\})$$

as

$$z_{t+\varepsilon}(\omega+\{t+\varepsilon s\}) = \int_{t+\varepsilon s}^{t+\varepsilon} \dot{z}_{t'}(\omega+\{t+\varepsilon s\})dt' + z_{(t+\varepsilon s)+0}(\omega+\{t+\varepsilon s\}) \rightarrow z_{t+0}(\omega+\{t\})$$

by the continuity of R_+z and by the boundedness of $\dot{z}$. So $\mathrm{II} \rightarrow \int d\omega((R_+z)_t(\omega) - (R_-z)_t(\omega))$. It is easy to see that $\mathrm{III} \rightarrow 0$. That $\frac{d}{dt}\int z_t(\omega)d\omega$ is continuous for $t \notin \omega_0$ can be shown in the usual way.

The following theorem is a generalized Itô-product formula and can be found without proof in [12].

Definition 4 (cf. Def. 1 of I.2): We say that the processes x_t and y_t have the finite product property (FP) if for fixed $\#\alpha_1, \#\alpha_2, \#\beta_1, \#\beta_2$ there exists a constant R such that

(PF) $\qquad x_t(\alpha_1, \beta_1 + \gamma)y_t(\alpha_2 + \gamma, \beta_2) = 0$ for $\#\gamma > R$ and all $t \notin I$.

Theorem 2: Assume that the process x_t and y_t are $\mathcal{C}^1$ and that they have the finite product property (FP). Then $x * y$ exists and is $\mathcal{C}^1$ and

$$R_+^{\ell}(x * y) = (R_+^{\ell}x) * y + x * (R_+^{\ell}y)$$

and similar for R^{ℓ}_{-} and $R^{r}_{\pm}$ and

$$\frac{d}{dt}(x * y)_t = \dot{x}_t * y_t + x_t * \dot{y}_t + (R^r_+ x)_t * (R^\ell_+ y)_t - (R_- x)_t * (R^r_- y)_t.$$

Proof: We have

$$(x * y)_s(\sigma+\{t\}, \tau) = \sum_{\substack{\alpha_1+\alpha_2=\sigma \\ \beta_1+\beta_2=\tau}} \int d\gamma(x_s(\alpha_1+\{t\}, \beta_1+\gamma)y_s(\alpha_2+\gamma,\beta_2) + x_s(\alpha_1,\beta_1+\gamma)y_s\{\alpha_2+\{t\}+\gamma,\beta_2)).$$

For $s \downarrow t$ the integrand stays bounded and converges. So

$$R^\ell_+(x * y) = (R^\ell_+ x) * y + x * (R^\ell_+ y);$$

these are continuous processes by proposition 1 of I.2 and its proof.
We have for $t \notin \sigma \cup \tau$

$$(x * y)_t(\sigma,\tau) = \sum_{\substack{\alpha_1+\alpha_2=\sigma \\ \beta_1+\beta_2=\tau}} \int d\gamma x_t(\alpha_1,\beta_1+\gamma)y_t(\alpha_2+\gamma,\beta_2)$$

Apply the previous lemma for

$$z_t(\omega) = x_t(\alpha_1,\beta_1+\omega)y_t(\alpha_2+\omega,\beta_2)$$

and $\omega_0 = \sigma \cup \tau$. Then we obtain the wished result as

$$(R_+ z)_t(\omega) = \dot{z}_{t+0}(\omega+\{t\}) = x_{t+0}(\alpha_1, \beta_1 + \omega + \{t\})y_{t+0}(\alpha_2 + \omega + \{t\}, \beta_2)$$
$$= (R^r_+ x)_t (\alpha_1, \beta_1 + \omega) (R^\ell_+ y)_t(\alpha_2 + \omega, \beta_2).$$

§ I.4. Adapted processes

Definition: Let x: $I(\times)\Omega(I)^{(2)} \to \mathbf{C}$ be a continuous process. x is called forward adapted if

$$x_t(\sigma,\tau) = 0 \text{ for } t < \max(\sigma\cup\tau)$$

and x is called backward adapted if

$$x_t(\sigma,\tau) = 0 \text{ for } t > \min(\sigma\cup\tau).$$

Remark 1: Assume $A \subset I$ measurable and x,y two measurable kernels on I such that

$$x(\sigma,\tau) = 0 \text{ for } \sigma \cup \tau \not\subset A$$
$$y(\sigma,\tau) = 0 \text{ for } \sigma \cup \tau \not\subset A^c.$$

Then $x * y = y * x$ and

$$(x * y)(\sigma,\tau) = x(\sigma \cap A, \tau \cap A)y(\sigma \cap A^c, \sigma \cap A^c).$$

From this remark one deduces

Proposition 1: Let x be a forward adapted continuous process. Then using the terminology of proposition 1 of I.2, the Itô sum

$$\sum x_{t(i-1)} * (a_{t^{(i)}} - a_{t^{(i-1)}})$$
$$= \sum (a_{t^{(i)}} - a_{t^{(i-1)}}) * x_{t(i-1)} \to \int x_t * da_t.$$

Similarly, let x be a backward adapted process, then the "backward-Itô" sum

$$\sum x_{t(i)} * (a_{t^{(i)}} - a_{t^{(i-1)}}) = \sum (a_{t^{(i)}} - a_{t^{(i-1)}}) * x_{t(i)} \to \int x_t * da_t.$$

Similar assertions hold for a^+.
Hence we will use the notations

$$\int da_t * x_t = \int x_t * da_t$$
$$\int da_t^+ * x_t = \int x_t * da_t^+$$

for forward or backward adapted processes.

Proposition 2: Let x be forward adapted, then

$$\int_A x_t * da_t = \begin{cases} x_{\max\tau}(\sigma, \tau \setminus \{\max\tau\}) & \text{if } \max\sigma < \max\tau \text{ and } \max\tau \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\int_A x_t * da_t^+ = \begin{cases} x_{\max\sigma}(\sigma \setminus \{\max\sigma\}, \tau) & \text{if } \max\tau < \max\sigma \text{ and } \max\sigma \in A \\ 0 & \text{otherwise} \end{cases}$$

Similarly, if x is backward adapted

$$\int_A x_t * da_t = \begin{cases} x_{\min\tau}(\sigma, \tau \setminus \{\min\tau\}) & \text{if } \min\sigma > \min\tau \text{ and } \min\tau \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\int_A x_t * da_t^+ = \begin{cases} x_{\min\sigma}(\sigma \setminus \{\min\sigma\}, \tau) & \text{if } \min\tau > \min\sigma \text{ and } \min\sigma \in A \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3: Let x be forward adapted and $\mathcal{C}^1$; then

$$(R_+^{\ell} x)_t(\sigma, \tau) = x_{t+0}(\sigma + \{t\}, \tau)$$

may be different from zero only if $t > \max(\sigma \cup \tau)$, the case $t = \max(\sigma \cup \tau)$ being not defined. R_-^{ℓ} is always $= 0$. One has similar results for $R_\pm^{r} x$.

Let x be a backward adapted process and $\mathcal{C}^1$, then $R_+^{\ell} x$ and $R_+^{r} x$ are zero and R_-^{ℓ} and $R_-^{r} x$ are $\neq 0$ only if $t < \min(\sigma \cup \tau)$.
From these results we draw the corollary used again and again.

Corollary: Let x be a forward adapted $\mathcal{C}^1$-process and assume

$$dx_t = f_t * da_t^+ + g_t * da_t + h_t dt.$$

Then $t \to x_t(\emptyset, \emptyset)$ has no jump and is C^1, so

$$x_t(\emptyset, \emptyset) = x_{t_0}(\emptyset, \emptyset) + \int_{t_0}^{t} h_{t'}(\emptyset, \emptyset) dt'.$$

Assume $\sigma \cup \tau \neq \emptyset$. Then there exists $t_{max} = \max(\sigma \cup \tau)$.
For $t < t_{max}$ we have $x_t(\sigma, \tau) = 0$, for $t > t_{max}$ the function $t \to x_t(\sigma,\tau)$ has no jump and is C^1, and we have that

$$x_{t_{max}+0}(\sigma,\tau) = \begin{cases} f_{t_{max}}(\sigma \setminus \{t_{max}\},\tau) & \text{for } t_{max} \in \sigma \\ g_{t_{max}}(\sigma,\tau \setminus \{t_{max}\}) & \text{for } t_{max} \in \tau \end{cases}.$$

So finally

$$x_t(\sigma,\tau) = \begin{cases} 0 \quad \text{for} \quad t < t_{max} \\ x_{t_{max}+0}(\sigma,\tau) + \displaystyle\int_{t_{max}}^{t} h_{t'}(\sigma,\tau)dt' \quad \text{for } t > t_{max}. \end{cases}$$

Similarly, if x is a backward adapted process we have

$$x_t(\emptyset,\emptyset) = x_{t_o}(\emptyset,\emptyset) + \int_{t_o}^{t} h_{t'(\emptyset,\emptyset)dt'}$$

for all $t \in I$ and for $\sigma \cup \tau \neq \emptyset$ and $t_{min} = \min(\sigma \cup \tau)$,

$$x_t(\sigma,\tau) = \begin{cases} 0 \quad t > t_{min} \\ x_{t_{min}-0}(\sigma,\tau) - \displaystyle\int_{t}^{t_{min}} h_{t'}(\sigma,\tau)dt' \quad \text{for } t < t_{min} \end{cases}$$

and

$$x_{t_{min}-0}(\sigma,\tau) = \begin{cases} -f_{min\ t}(\tau \setminus \{t_{min}\}, \sigma) & \text{if } t_{min} \in \sigma \\ -g_{min\ t}(\tau,\sigma \setminus \{t_{min}\}) & \text{if } t_{min} \in \tau \end{cases}.$$

Proposition 4: Let $x^{(1)}$ and $x^{(2)}$ be forward adapted processes and $\mathcal{C}^1$ such that having the finite product property PF of definition 4 of I.3.
If

$$dx_t^{(i)} = f_t^{(i)} * da_t^t + g_t^{(i)} * da_t + h_t^{(i)}dt \quad (i = 1, 2).$$

Then

$$\begin{aligned} d_t(x^{(1)} * x^{(2)})_t &= \left(f_t^{(1)} * x_t^{(2)} + x_t^{(1)} * f_t^{(2)}\right) * da_t \\ &+ \left(g_t^{(1)} * x_t^{(2)} + x_t^{(1)} * g_t^{(2)}\right) * da_t \\ &+ \left(h_t^{(1)} * x_t^{(2)} + x_t^{(1)} * h_t^{(2)} + g_t^{(1)} * f_t^{(2)}\right)dt. \end{aligned}$$

If $x^{(1)}$ and $x^{(2)}$ are $\mathcal{C}^1$ and backward adapted, the "Itô term" $+ g_t^{(1)} * f_t^{(2)}$ has to be replaced by $-g_t^{(1)} * f_t^{(2)}$. So we have the usual Itô table for forward adapted processes

	da_t	da_t^+
da_t	0	dt
da_t^+	0	0

and the slightly different Itô table for backward adapted processes

	da_t	da_t^+
da_t	0	-dt
da_t^+	0	0

This result is an easy consequence of proposition 3 and theorem 2 of § I.3.

Proposition 5: Let x be forward adapted and y be backward adapted, then

$$(x_t * y_t)(\sigma,\tau) = x_t(\sigma\cap]-\infty,t],\tau\cap]-\infty,t])y_t(\sigma\cap[t,\infty[,\tau\cap[t,\infty[)$$

and

$$x_t * y_t = y_t * x_t.$$

Assume x and y to be $\mathcal{C}^1$, then $x * y$ is $\mathcal{C}^1$ and one has the classical rules of differentiation without any Itô term. So the Itô table is

	da_t	da_t^+
da_t	0	0
da_t^+	0	0 .

Proof: See Remark 1. As the integral term in $x*y$ does not appear, the calculations are much simpler than those of theorem 2, § I.3 and we do not need the property (PF).

§ I.5. The number operator and the splitting of the Fock space

The number process has been introduced by Meyer into the framework of Maassen kernels. We will not follow him, but will consider it only as an operator on $\mathcal{C}(I)$.

Definition: Let $\xi \in \mathcal{C}(I)$, so we define

$$\Lambda\xi \in \mathcal{C}(I) \text{ by } \Lambda\xi(\omega) = (\#\omega)\xi(\omega).$$

The operator Λ is called <u>number</u> operator. We could introduce Λ into the framework of Maassen kernels defining

$$\lambda(\sigma,\tau) = \begin{cases} \delta(s-t) & \text{if } \sigma = \{s\}, \tau = \{t\} \\ 0 & \text{otherwise} \end{cases}.$$

We will not persue this line of ideas. Instead we want to investigate the operator Λ-$a^+(f)a(f)$, where $f \in I \to \mathbf{C}$, $\int |f|^2 dt = 1$.

Let $I \subset \mathbf{R}$ be an interval and $f : I \to \mathbf{C}$ continuous and bounded and such that $\int |f(t)|^2 dt = 1$. Define the operator

$$\Phi_0 : \mathcal{C}_0(I) \to \mathcal{C}_0(I)$$

(1)

$$\Phi_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^+(f)^m a(f)^m$$

where $a^\#(f)$ is the operator $\xi \to a^\#(f) * \xi$. In $\Phi_0\xi$, where $\xi \in \mathbf{C}_0(I)$, there are only

finitely many terms in the sum.
By direct calculation using the commutation relations we obtain

(2) $a(f)\Phi_0 = \Phi_0 a^+(f) = 0.$

From there one obtains immediately $\Phi_0^2 = \Phi_0$.
Call

(3) $\Phi_k = \frac{1}{k!} a^+(f)^k \Phi_0 a(f)^k.$

Then

(4) $\Phi_k \Phi_\ell = \delta_{k\ell} \Phi_k.$

Furthermore

(5) $$\sum_{k=0}^{\infty} \Phi_k = 1.$$

We shall prove as an example the last equation.

$$\sum_{k=0}^{\infty} \Phi_k = \sum \frac{(-1)^\ell}{k!\ell!} a^{+(k+\ell)} a^{(k+\ell)} = \sum \frac{c_p}{p!} a^{+p} a^p$$

with $$c_p = \sum_{\ell=0}^{p} (-1)^\ell \binom{p}{\ell} = \begin{cases} 1 & p = 0 \\ 0 & p \neq 0 \end{cases}.$$

Call $\mathbf{C}^{(n)}(I)$ the set of bounded continuous functions $\Omega_n(I) \to \mathbf{C}$, then $\mathbf{C}_0(I) = \mathbf{C}^{(0)}(I) \oplus \mathbf{C}^{(1)}(I) \oplus \ldots$
Call $\mathcal{K} = \{\xi \in \mathbf{C}_0(I) : a(f)\xi = 0\}$.
Then

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$$

with

$$\mathcal{K}_i = \mathcal{K} \cap \mathbf{C}^{(i)}(I).$$

Proposition 1: Φ_0 maps $\mathbf{C}_0(I)$ onto $\mathcal{K}$.
Proof: Immediate.
Proposition 2: Any $\xi \in \mathbf{C}^{(n)}(I)$ can be expressed in a unique way in the form

$$\xi = \sum_{k=0}^{n} \frac{a(f)^{+k}}{\sqrt{k!}} \xi_k$$

with

$$\xi_k \in \mathcal{K}_{n-k}\,.$$

Proof: We split

$$\xi = \sum_{k=0}^{n} \Phi_k \xi = \sum_{k=0}^{n} \frac{a(f)^{+k}}{\sqrt{k!}} \Phi_0 \frac{a(f)^k}{\sqrt{k!}} \xi.$$

Then $\frac{a(f)^k}{\sqrt{k!}} \xi \in \mathbf{C}^{(n-k)}$, by the properties of $a(f)$, hence $\Phi_0 \frac{a(f)^k}{\sqrt{k!}} \in \mathcal{K}^{(n-k)}$. The uniqueness follows out of eq. (4).
Corollary: Call ϕ_k the k-th vector of the standard basis of $\ell^2(\mathbf{N})$:

$$\sum_{k=0}^{\infty} \frac{a(f)^k}{\sqrt{k!}} \xi_k \to \sum_{k=0}^{\infty} \phi_k \otimes \xi_k$$

is an isomorphism from $\mathcal{C}_0(I)$ onto $\mathcal{L}_0(N) \otimes \mathcal{K}$ which preserves the scalar product. Here $\mathcal{L}_0(N)$ is the set of finite linear combinations of the ϕ_ℓ.
We split the number operator accordingly

$$\Lambda = (\Lambda - a^+(f)a(f)) + a^+(f)a(f) = \Lambda_0 + a^+(f)a(f).$$

We have on $\mathcal{K}$:

$$\xi \in \mathcal{K}: \Lambda f = \Lambda_0 f.$$

If $\xi \in C^{(n)}$, then in the decomposition of proposition 2

$$\Lambda_0 \xi_k = \Lambda \xi_k = (n-k)\xi_k.$$

II. The quantum stochastic differential equation of the inverse oscillator

§ II.1. The physical model

The quantum oscillator has the Hamiltonian $\omega_0 b^+ b$, where ω_0 is the frequency and b and b^+ the usual annihilator and creation operators. The inverse oscillator has the Hamiltonian $-\omega_0 b^+ b^+$. Coupled to a heat bath of oscillators $(a_\lambda^\#)_{\lambda \in \Lambda}$ the total Hamiltonian is given by

$$H_0 = -\omega_0 b^+ b + \Sigma(\omega_0 + \omega_\lambda) a_\lambda^+ a_\lambda + \sum_{\lambda \in \Lambda} (g_\lambda a_\lambda b + \bar{g}_\lambda a_\lambda^+ b^+) \tag{1}$$

where we have used rotating wave approximation. This Hamiltonian, however, is not bounded below, so it cannot describe a real physical system. Nevertheless, it is able to give the initial behavior of superradiance and can be used as a model of an amplifier. We give a sketch of these ideas.
The physical model of superradiance can be found in [2] and more explicitly in [11]. We use the normalization of [4]. Consider a system of N two level atoms contained in a region of space smaller than the wave length c/ω_0, where ω_0 is the transition frequency of the atoms. We assume the atoms coupled to a heat bath of harmonic oscillators and obtain in rotating wave approximation the Dicke Hamiltonian

$$H_{Dicke} = S_3 \omega_0 + \Sigma(\omega_0 + \omega_\lambda) a_\lambda^+ a_\lambda + \sum_{\lambda \in \Lambda} \left(\frac{g_\lambda}{\sqrt{N}} S_+ a_\lambda + \frac{\bar{g}_\lambda}{\sqrt{N}} S_- a_\lambda^+ \right).$$

The Hilbert space of the atoms is $(\mathbf{C}^2)^{\otimes N}$. The operators S_i are defined by

$$S_i = \sigma_i \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \sigma_i$$

with

$$\sigma_i = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The operators S_i obey the spin commutation relations. So $(\mathbb{C}^2)^{\otimes N}$ can be considered as a spin-representation space. Any irreducible representation is invariant under the application of H_{Dicke}.

In the case of superradiance at $t = 0$ all atoms are in the upper state. Then, due to spontaneous emission, one atom emits a photon, the radiation increases the probability of the emission of a second photon, so an avalanche is created which is dying out when the majority of atoms is in the lower state.

At $t = 0$ the state of the atomic system is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes N} = \psi_{N/2}$. This vector is the vector of highest weight of the representation space and application of the S_i generates an invariant irreducible subspace spanned by ψ_m, $m = -N/2, -N/2+1, \ldots, +N/2$. One has

$$S_3\psi_m = m\psi_m$$
$$S_\pm\psi_m = \left(\frac{N}{2}\left(\frac{N}{2}+1\right)-m(m\pm1)\right)^{1/2}\psi_{m\pm1}.$$

Put

$$\phi_k = \psi_{N/2-k},$$

then

$$\frac{1}{\sqrt{N}}\, S_+\phi_k = N^{-1/2}(Nk-k^2+k)^{1/2}\phi_{k-1} \to \sqrt{k}\ \phi_{k-1}$$

$$\frac{1}{\sqrt{N}}\, S_-\phi_k = N^{-1/2}(N(k+1)-k^2-k)\phi_{k+1} \to \sqrt{k+1}\ \phi_{k+1}.$$

For $N \to \infty$ the operators $N^{-1/2}S_\pm$ become b and b^+ resp., and shifting the total energy by $N/2$ we arrive at the expression of H_0.

By the considerations above it is clear that the Hamiltonian H_0 describes only the initial behavior of superradiance before saturation has to be considered. We calculate in II.6 the occupation numbers of the oscillator state when there is no influx of photons. The probabilities that the oscillator is in state $|k\rangle\langle k|$ are described by a classical Markov process $X(t)$, which can jump by $+1$ and where the jumping rate is proportional to $k+1$, if $X(t) = k$.

By the way, if we started here by $\psi_{-N/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N/2}$, all atoms are in the lower state, we would have arrived at the equation for the non-inversed oscillator.

We split H_0 (eq. (1)) into two commuting operators $H_0 = H+H'$:

(2) $$H = \sum\omega_\lambda a_\lambda^+ a_\lambda + \sum(g_\lambda a_\lambda b + \overline{g_\lambda} a_\lambda^+ b^+)$$

(3) $$H' = \omega_0(-b^+b + \sum a_\lambda^+ a_\lambda).$$

The time dependence due to H' is trivial, it describes a fast oscillation, modulated by H. We consider H alone and introduce the interaction picture with respect to $\sum\omega_\lambda a_\lambda^+ a_\lambda$ and obtain for the time development operator.

$$\frac{d}{dt}\, U_{t,s} = -iH(t)U_{t,s} = (-i(\sum g_\lambda a_\lambda e^{-i\omega_\lambda t})b - i(\sum g_\lambda a_\lambda^+ e^{i\omega_\lambda t})b^+)U_{t,s}.$$

Now interpret

(4) $$F(t) = \sum g_\lambda a_\lambda e^{-i\omega_\lambda t}$$

as quantum noise

(5) $$[F(t), F^+(s)] = k(t-s) = \sum |g_\lambda|^2 e^{-i\omega_\lambda(\tau-\sigma)}$$

and perform the so-called singular coupling limit where you replace $k(t-s)$ by $\delta(t-s)$. Calling $F(t)dt = da_t$ we arrive to the quantum stochastic differential equation

(6) $$\frac{d}{dt} U_{t,s} = (-ida_t b - ida_t^+ b^+ - \frac{1}{2} bb^+ dt) U_{t,s},$$

where correction $-\frac{1}{2} bb^+ dt$ is the so-called Itô creation term.

II. 2. The Heisenberg equation

As the Hamiltonian is quadratic, the Heisenberg equations are linear. We shall establish them in a non-rigorous way and discuss them. They will later in II.5 come out of the exact theory in a rigorous way.

We return to equation II.1(2) and calculate the Heisenberg operators

$$b^+(t) = e^{iHt} b^+ e^{-iHt}$$
$$a_\lambda(t) = e^{iHt} a_\lambda e^{iHt}.$$

We have

$$\frac{d}{dt} b^+(t) = i \sum g_\lambda\, a_\lambda(t)$$
$$\frac{d}{dt} a_\lambda(t) = i\, \omega_\lambda a_\lambda(t) - i\, \overline{g_\lambda}\, b^+(t).$$

Solve the second equation

$$a_\lambda(t) = e^{-i\omega_\lambda t} a_\lambda - i\, \overline{g_\lambda} \int_0^t e^{-i\omega_\lambda(t-\tau)}\, b^+(\tau) d\tau$$

and insert into the first one

$$\frac{d}{dt} b^+(t) = i \sum g_\lambda\, a_\lambda\, e^{-i\omega_\lambda t} + \int_0^t k(t-\tau) b^+(\tau) d\tau$$

with

$$k(t) = \sum |g_\lambda|^2 e^{-i\omega_\lambda t}$$

as in II.1(5). Using II.1(4) we write

$$\frac{d}{dt} b^+(t) = iF(t) + \int_0^t k(t-\tau) b^+(\tau) d\tau).$$

Performing the singular coupling limit we obtain

$$d_t b^+(t) = i\, da_t + \frac{1}{2}\, b^+(t) dt.$$

Remark that we have to go with $k(t)$ to $\delta(t)$ in a symmetric way. We integrate and arrive at

$$b^+(t) = e^{t/2}b^+ + i\int_0^t e^{(t-t')/2}\, da_{t'}.$$

Multiplying with $e^{-t/2}$ we see that

$$e^{-t/2}\, b^+(t) \to b^+ + i\int_0^\infty e^{-t'/2}\, da_t = B^+$$

for $t \to \infty$. So $e^{-t/2}\, b^+(t)$ does not converge to zero, but to some quantity which can be interpreted as a classical quantity as

$$[B, B^+] = 0.$$

One may get the idea that the inverse oscillator acts as some photon multiplier where after amplification a classical quantity comes out [2].

Let us investigate the stochastical behavior of B and B^+, when the initial density matrix ρ of the oscillator and the bath is given. We calculate

$$\Phi(z) = \mathrm{Tr}\, \rho\, e^{i(zB+\bar{z}B^+)}.$$

By Bochner's theorem $\Phi(z)$ is the Fourier transform of a probility measure on $\mathbf{C}^2$.

$$\Phi(z) = \int p(d\xi)\, e^{i(\xi z+\bar{\xi}\bar{z})},$$

so $p(d\xi)$ describes the statistical behavior after amplification.

Assume $\rho = \rho_0 \otimes |\emptyset\rangle\langle\emptyset|$, where ρ_0 is the initial density matrix for the oscillator and $|\emptyset\rangle\langle\emptyset|$ is the vacuum of the heat bath. Then

$$\Phi(z) = e^{-|z|^2/2}\, \mathrm{Tr}\, \rho_0\, e^{i(zb+\bar{z}b^+)}.$$

Now $\mathrm{Tr}\, \rho_0\, e^{i(zb+\bar{z}b^+)}$ is the Fourier-Weyl transform of ρ_0, its Fourier transform is the Wigner transform of ρ_0, we call it $W(\rho_0,\xi)$. Then

$$p(d\xi) = \frac{2}{\pi}\int W(\rho_0, \eta)\exp(-2(\xi-\eta)^2)d\eta d\xi;$$

so $p(d\xi)$ is the Wigner transform of ρ_0 smeared out by a Gaussian distribution.

Assume $\rho_0 = |0\rangle\langle 0|$, the ground state of the oscillator, which is here the state of highest energy, then

$$p(d\xi) = \frac{1}{\pi}\, e^{-|\xi|^2/2}d\xi.$$

Assume a coherent state

$$\psi = e^{-|\beta|^2/2}\, e^{\beta b^+}\, |\, 0\, \rangle,$$

and $\rho_0 = |\psi\rangle\langle\psi|$, then

$$p(d\xi) = \frac{1}{\pi}\exp(-|\xi-\beta|^2)d\xi.$$

So we recover the value β with an additional incertainty.

II. § 3. An inequality for two oscillators

We will derive an inequality for the space of two oscillators crucial for the treatment of the inverse oscillator in a heat bath. We do not attempt to derive the inequality in its strongest form, but only in the special case needed below.

Assume the Hilbert space $\ell^2(\mathbf{N}^2)$, the subspace $\ell_0(\mathbf{N}^2) = \ell_0$ of linear combinations of the standard basis vectors $|\ell, m\rangle$, $\ell, m = 0,1,\ldots$, and define the usual creation and annihilation operators a, a^+, b, b^+ with

(1) $$[a, a^+] = [b, b^+] = 1, \quad [a, a] = [a, b] = [b, b] = 0$$

and

(2) $$\begin{aligned} a|\ell, m\rangle &= \sqrt{\ell}\, |\ell-1, m\rangle \\ a^+|\ell, m\rangle &= \sqrt{\ell+1}\, |\ell+1, m\rangle \\ b|\ell, m\rangle &= \sqrt{m}\, |\ell, m-1\rangle \\ b^+|\ell, m\rangle &= \sqrt{m+1}\, |\ell, m+1\rangle. \end{aligned}$$

Define the quadratic operators

(3) $$\begin{aligned} A &= a^+a + bb^+ \\ B &= ab + a^+b^+ \\ C &= ab - a^+b^+. \end{aligned}$$

Theorem: Assume $\alpha,\beta \in \mathbf{R}$, $\alpha \geq 0$ and $\alpha^2 - \beta^2 = 1$. Then there exist constants $\gamma_0 = 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots$, such that for all $\xi \in \ell_0(\mathbf{N}^2)$

$$\langle \xi, (\alpha A+\beta B)^k \xi \rangle \leq e^{\gamma_k t} \langle \xi, A^k \xi \rangle$$

with $t = \frac{1}{2} \log(\alpha+\beta)$.

The proof will be the result of several lemmata. We will use the fact that the operators A,B,C have the same commutation relations as the traceless 2×2 matrices, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. One has

$$[B,A] = 2C, \ [C,A] = 2B, \ [C,B] = 2A.$$

Define $\ell^{(m)} = \ell^{(m)}(\mathbf{N}^2)$ to be the set of linear combinations of $|\ell,n\rangle$ with $\ell+n \leq n$. Then

$$C : \ell^{(m)} \longrightarrow \ell^{(m+1)}$$

$$\|C\xi\| \leq 2(m+1)\|\xi\|$$

for $\xi \in \ell^{(m)}$. Define

$$\ell_\tau = \{\xi \in \ell^2 : \langle \xi, A^k \xi \rangle < \infty \ \text{for} \ k = 0, 1, 2, \ldots\}.$$

Lemma 1: The sum $\sum \frac{t^m}{m!} C^m \xi = e^{Ct}$ converges in norm to a vector in $\mathcal{L}_T$ for $\xi \in \mathcal{L}_0$ for $|t| < \frac{1}{2}$. More precisely,

$$\sum_{m=0}^{\infty} \|A^{k/2} \frac{C^m}{m!} t^m \xi\| \leq \sum_{m=0}^{\infty} (n+m)^{k/2}\, 2^m |t|^m \frac{(n+1)\dots(n+m)}{m!} \|\xi\| < \infty$$

for $\xi \in \mathcal{L}^{(n)}$ and $k = 0,1,2,\dots$.

Proof: Apply the quotient criterion.

The algebra $\mathcal{W} = \mathcal{W}(a^\#, b^\#)$ generated by $a^\#$ and $b^\#$ can be described as the algebra generated by four indeterminates with the commutation relations (1). More precise: Define the free algebra $\mathbf{C}\langle x_1,x_1^+, x_2,x_2^+\rangle$, divide by the ideal J generated by

$$x_i x_j - x_j x_i$$
$$x_i^+ x_j^+ - x_j^+ x_i^+$$
$$x_i x_j^+ - x_j^+ x_i - \delta_{ij}$$

and call

$$x_1^\# + J = a^\#$$
$$x_2^\# + J = b^\#.$$

Define

$$a_t = (\cosh t)a + (\sinh t)b^+$$
$$b_t^+ = (\cosh t)b^+ + (\sinh t)a$$
$$a_t^+ = (\cosh t)a^+ + (\sinh t)b$$
$$b_t = (\cosh t)b + (\sinh t)a^+.$$

As the $a_t^\#$ and $b_t^\#$ have the same commutation relations as the $a^\#$ and $b^\#$, replacing in a polynomial in $\mathcal{W}(a^\#,b^\#)$ the $a^\#$ and $b^\#$ by the $a_t^\#$ and $b_t^\#$ defines an isomorphism $\eta_t : \mathcal{W} \to \mathcal{W}$.

The polynomials $p \in \mathcal{W}$ can be defined as operators on $\mathcal{L}_T$. The mapping $a \to a^+$, $a^+ \to a$, $b \to b^+$, $b^+ \to b$ defines an involution in $\mathcal{W}$. One has

$$\langle \xi, p\zeta \rangle = \langle p^+\xi, \zeta \rangle$$

for $\xi, \zeta \in \mathcal{L}_T$.

Lemma 2: Assume $p \in \mathcal{W}(a^\#,b^\#)$ and $|t| < 1/2$, $\xi, \zeta \in \mathcal{L}_0$. Then

$$(*) \qquad \langle e^{-Ct}\xi,\, p\, e^{-Ct}\zeta \rangle = \langle \xi, (\eta_t p)\zeta \rangle.$$

Proof: The inequality in Lemma 1 shows that e^{Ct} is differentiable. As $C^+ = -C$, one obtains

$$\frac{d}{dt} \langle e^{-Ct}\xi,\, p\, e^{-Ct}\xi \rangle = \langle e^{-Ct},\, [C,p]\, e^{-Ct}\zeta \rangle.$$

On the other hand

$$\frac{d}{dt} a_t = \eta_t([C,a]), \ldots, \frac{d}{dt} b_t^+ = \eta_t([C,b^+]).$$

Hence

$$\frac{d}{dt} \eta_t(p) = \eta_t([C,p]).$$

The mapping $p \to [C,p]$ has the property that it maps the subspace $\mathcal{W}_n$ spanned by the monomials of degree $\leq n$ into itself. Let $p_1, \ldots, p_N$ be a basis of $\mathcal{W}_n$, then

$$[C,p_i] = \sum_{k-1}^{N} c_{ik} p_k, \ c_{ik} \in \mathbf{C};$$

hence

$$\frac{d}{dt} \left\langle e^{-Ct}\xi, \, p_i \, e^{-Ct}\zeta \right\rangle = \sum c_{ik} \left\langle e^{-Ct}\xi, \, p_k \, e^{-Ct}\zeta \right\rangle;$$

on the other hand

$$\frac{d}{dt} \left\langle \xi, \eta t(p_i)\zeta \right\rangle = \sum c_{ik} \left\langle \xi, \eta_t(p_k)\zeta \right\rangle.$$

So both sides of (*) obey the same system of differential linear equations. As they coincide for $t = 0$ they must be equal.

Lemma 3: $\alpha A + \beta B = \eta_t A$ with $t = \frac{1}{2} \log(\alpha + \beta)$.

Proof by direct calculation.

We define a polynomial $p \in \mathcal{W}$ to be positive $p \geq 0$, if

$$\left\langle \xi, p \, \xi \right\rangle \geq 0$$

for all $\xi \in \mathcal{L}_r$. We shall use the following inequality again and again:

For $p,q \in \mathcal{W}$

$$p^+q + q^+p \leq p^+p + q^+q.$$

That is a direct consequence of

$$p^+p + q^+q - p^+q - q^+p = (p-q)^+(p-q) \geq 0.$$

Lemma 4: There exist constants $\gamma_0 = 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots$, such that

$$[C,A^k] \leq \gamma_k A^k.$$

Proof: We prove the lemma in the following way. Call E_k and E_k' the assertions

(E_k). There exists κ_k such that

$$BA^k + A^kB \leq \kappa_k A^{k+1}.$$

(E_k') There exist γ_k such that

$$[C,A^k] \leq \gamma_k A^k.$$

Now

(E_0): $B = ab + a^+b^+ \leq a^+a + bb^+ = A.$

By the inequality above

$$BA + AB \leq A^2 + B^2$$

and

$$\begin{aligned} B^2 &= a^2b^2 + a^{+2}b^{+2} + aa^+bb^+ + a^+ab^+b \\ &\leq a^{+2}a^2 + b^2b^{+2} + aa^+bb^+ + a^+ab^+b \\ &= a^+aa^+a + bb^+bb^+ + 2a^+abb^+ - 2a^+a + 2bb^+ \\ &\leq A^2 + 2A \leq 3A^2 \end{aligned}$$

as $A \leq A^2$. Finally

(E_1) $BA + AB \leq 4\,A^2.$

(E_0') is trivial. Now

$$\begin{aligned} BA^k + A^kB &= ABA^{k-1} + A^{k-1}BA + [B,A]A^{k-1} + A^{k-1}[A,B] \\ &= A(BA^{k-2} + A^{k-2}B)A + 2[C,A^{k-1}] \\ &\leq \kappa_{k-2}A^{k+1} + 2\gamma_{k-1}A^{k-1} \\ &\leq (\kappa_{k-2} + 2\gamma_{k-1})A^{k+1} \end{aligned}$$

as $A^{k-1} \leq A^{k+1}$. This proves (E_k) out of (E_{k-2}) and (E_{k-1}'). On the other hand

$$\begin{aligned} [C,A^k] &= 2\sum_{i=0}^{k-1} A^iBA^{k-1-i} = 2((BA^{k-1} + A^{k-1}B) + A(BA^{k-2} + A^{k-2}B)A + \ldots) \\ &\leq 2(\kappa_{k-1} + \kappa_{k-2} + \ldots)A^k. \end{aligned}$$

This proves (E_k') out of (E_k), (E_{k-1}), ...

Let us collect facts. We have already proven (E_0), (E_1) and (E_0'). Then (E_0) implies (E_1'), and (E_0) and (E_1') imply (E_2), and E_0 and E_1 imply (E_2'), and (E_1) and (E_2') imply (E_3), (E_0) implies (E_1'), (E_0) and (E_1') imply (E_2), (E_0) and (E_1) imply (E_2'), (E_1) and (E_2') imply (E_3), (E_0), (E_1), (E_2) imply (E_3') and so on.

Lemma 5: For $\xi \in \mathcal{L}_0$ and $|t| < \frac{1}{2}$ one has

$$\left\langle e^{-Ct}\xi, A^k e^{-Ct}\xi \right\rangle \leq e^{\gamma_k |t|} \left\langle \xi, A^k \xi \right\rangle.$$

Proof: We differentiate the left-hand side and obtain

$$\left\langle e^{-Ct}\xi, [C,A^k]\, e^{-Ct}\xi \right\rangle \leq \gamma_k \left\langle e^{-Ct}\xi, A^k e^{-Ct}\xi \right\rangle.$$

Integrating this differential inequality yields the result for $t > 0$. Negative t means replacing C by -C, or b by a and a by b. So we have the same result.

Corollary: The mapping $|t| < \frac{1}{2} \to e^{-Ct}$ can be extended to a unitary group of operators on **R**. These operators map the subspace $\mathcal{L}_f^{(k)}$ of vectors with the property $\langle \xi, A^k \xi \rangle < \infty$ into itself and one has

$$\langle e^{-Ct}\xi, A^k e^{-Ct}\xi \rangle \le e^{\gamma_k |t|} \langle \xi, A^k \xi \rangle$$

for all t.

Proof: Unitarity follows from Lemma 2 for p = 1 for small t. From there extension to all t by iterated application of lemma 4, as do the other statements. The theorem is the direct consequence of the corollary and lemma 3.

§ II. 4. The kernel solution of the stochastic differential equation

The equation

$$d_t u_{t,s} = (-ibda_t - ib^+da_t^+ - \frac{1}{2} bb^+\delta_{\emptyset,\emptyset}dt) * u_{t,s}$$

can be interpreted as an equation for matrix elements, which we write

$$\langle m|u_{t,s}|n \rangle \qquad m, n \in \mathbf{N}.$$

So $\langle m|u_{t,s}|n \rangle$ is a kernel with values in **C**, the vectors |m>, n> are the eigenstates of the inverse oscillator in the usual Dirac notation. Then the equation becomes

$$(1) \qquad d_t\langle m|u_{t,s}|n \rangle = \sum_{\ell} \left(-i \langle m|b|\ell \rangle da_t - i \langle m|b^+|\ell \rangle da_t^+ - \frac{1}{2}\langle m|bb^+|\ell \rangle \delta_{\emptyset,\emptyset}dt\right) * \langle \ell|u_{t,s}|u \rangle.$$

Remark that the sum over ℓ is finite due to the special character of $b^{\#}$.

Proposition 1: There exists exactly one family of solutions

$$\langle m|u_{t,s}|n \rangle, \quad m, n \in \mathbf{N}, \quad t > s$$

of (1), which is $\mathbf{C}^1$ and foward adapted and obeys the condition

$$\langle m|u_{s,s}|n \rangle = \delta_{mn}\, \delta_{\emptyset,\emptyset}$$

and it is given by

$$\langle m|u_{t,s}|n \rangle (\sigma,\tau) = (-i)^{\#\sigma+\#\tau} \langle m|e^{-(t-t_k)bb^+/2} b^{\varepsilon_k} e^{-(t_k-t_{k-1})bb^+/2} b^{\varepsilon_{k-1}} \ldots b^{\varepsilon_2} e^{-(t_2-t_1)bb^+/2} b^{\varepsilon_1} e^{-(t_1-s)bb^+}|n \rangle$$

for $\tau \cup \sigma \subset [s,t]$ and 0 otherwise, where

$$\#\tau + \#\sigma = k \text{ and } \tau \cup \sigma = \{t_1, \ldots, t_k\} \text{ with } t_1 < \ldots < t_k$$

and

$$b^\varepsilon = \begin{cases} b & \text{if } \varepsilon = 0 \\ b^+ & \text{if } \varepsilon = 1 \end{cases}$$

and
$$\varepsilon_i = \begin{cases} 0 & \text{if } t_i \in \tau \\ 1 & \text{if } t_i \in \sigma \end{cases}.$$

Proof: Assume at first $\tau = \sigma = \emptyset$. Then $\langle m|u_{t,s}|n\rangle(\emptyset,\emptyset)$ does not have any jumps. As $t \to \langle m|u_{t,s}|n\rangle(\emptyset,\emptyset)$ is $\mathbf{C}^1$ we have the differential equation

$$\frac{d}{dt}\langle m|u_{t,s}|n\rangle(\emptyset,\emptyset) = -\frac{1}{2}\langle m|bb^+|m\rangle\langle m|u_{t,s}|n\rangle(\emptyset,\emptyset)$$

and

$$\langle m|u_{t,s}|n\rangle(\emptyset,\emptyset) = \langle m|e^{-(t-s)bb^+/2}|m\rangle\,\delta_{m,n}.$$

Assume now

$$\tau \cup \sigma \neq \emptyset \text{ and } t > \max(\sigma \cup \tau) = t'.$$

Then

$$\frac{d}{dt}\langle m|u_{t,s}|n\rangle(\sigma,\tau) = -\frac{1}{2}\langle m|bb^+|m\rangle\langle m|u_{t,s}|n\rangle(\sigma,\tau).$$

Assume e.g. $t' \in \sigma$, then

$$\langle m|u_{t'+0,s}|n\rangle(\sigma,\tau) = -i\,\langle m|b^+|m-1\rangle\langle m-1|u_{t',s}|n\rangle(\sigma\backslash\{t'\},\tau)$$

and $\langle m|u_{t'-0,s}|m\rangle(\sigma,\tau) = 0$ because of the forward adaptedness. So

$$\langle m|u_{t,s}|n\rangle(\sigma,\tau) = -i\langle m|e^{-(t-t')bb^+/2}|m\rangle\langle m|b^+|m-1\rangle\langle m-1|u_{t',s}|n\rangle(\sigma\backslash\{t'\},\tau)$$

$$= (-i)\,\langle m|e^{-(t-t')bb^+/2}b^+|m-1\rangle\langle m-1|u_{t',s}|u\rangle(\sigma\backslash\{t'\},\tau).$$

This shows that an induction is possible with respect to $\#\sigma + \#\tau$.
Inspection of the solution and the theorem 1 of I.3 show

Proposition 2: The processes $s \to \langle m|u_{t,s}|n\rangle$ are backward adapted and $\mathbf{C}^1$ and

$$d_s u_{t,s} = u_{t,s} * (ib\, da_s + ib^+\, da_s^+ + \frac{1}{2}bb^+ds\,\delta_{\emptyset,\emptyset})$$

or more precise

$$d_s\langle m|u_{t,s}|n\rangle = \sum_{\ell}\langle m|u_{t,s}|\ell\rangle * \langle \ell|ib\,da_s + ib^+da_s^+ + \frac{1}{2}bb^+ds\,\delta_{\emptyset,\emptyset}|n\rangle.$$

Remark 1: The matrix element $\langle m|u_{t,s}|n\rangle(\sigma,\tau)$ are $\neq 0$ only if $m + \#\tau = n + \#\sigma$.

Proposition 3: Let $r < s < t$. Then

$$u_{t,s} * u_{s,r} = u_{t,r}$$

or

$$\Big(\sum_{\ell}\langle m|u_{t,s}|\ell\rangle * \langle \ell|u_{s,r}|n\rangle\Big)(\sigma,\tau) = \langle m|u_{t,r}|n\rangle(\sigma,\tau).$$

For fixed σ,τ the sum consists out of at most one term.

Proof: By proposition 5 of I.4

$$\Big(\sum_{\ell} \langle m|u_{t,s}|\ell\rangle * \langle \ell|u_{s,r}|n\rangle\Big)(\sigma,\tau)$$

$$= \sum_{\ell} \langle m|u_{t,s}|\ell\rangle\,(\sigma \cap [s,\infty[,\ \tau \cap [s,\infty[)\,\langle \ell(u_{s,r}|n\rangle\,(\sigma \cap]-\infty,s],\ \tau \cap]-\infty,s]).$$

So by the remark 1 there is at most one term in the sum over ℓ.

Differentiate the left-hand side of the last equation with respect to s and apply proposition 5 of I.4, then

$$\sum_{\ell} \frac{d}{ds}\langle m|u_{t,s}|\ell\rangle * \langle \ell|u_{s,r}|n\rangle = 0,$$

that means that that expression is independent of s, so

$$\sum_{\ell} \langle m|u_{t,s}|\ell\rangle * \langle \ell|u_{s,r}|n\rangle = \langle m|u_{t,r}|n\rangle\ .$$

Define the kernel valued matrix $u^{+}_{t,s}$

$$\langle \ell|u^{+}_{t,s}|m\rangle = \langle m|u_{t,s}|\ \ell\rangle^{+},\ \text{ or}$$
$$\langle \ell|u^{+}_{t,s}|m\rangle\,(\sigma,\tau) = \langle m|u_{t,s}|\ n\rangle^{-}\,(\tau,\sigma).$$

Lemma 1: We have the estimate

$$e^{-(t-s)/2(\ell+p+1)}\frac{(\ell+p)!}{\sqrt{\ell!}\,\sqrt{m!}} \leq \big\|\langle \ell\,|\,u^{\#}_{t,s}\,|\,m\rangle\big\|_{p,q} \leq \frac{(\ell+p)!}{\sqrt{\ell!}\,\sqrt{m!}}$$

if ℓ+p = m+q. If ℓ+p ≠ m+q, then

$$\langle \ell\,|\,u^{\#}_{t,s}\,|\,m\rangle\,(\sigma,\tau) = 0$$

for $\#\sigma = p$ and $\#\tau = q$.

Proof: We have by proposition 1

$$|\,\langle \ell\,|\,u^{\#}_{t,s}\,|\,m\rangle\,| = \langle \ell\,|\,e^{-(t-t_k)bb^+/2}\,b^{\varepsilon_k}\ldots b^{\varepsilon_1(t_1-s)bb^+/2}\,|m\rangle \leq \langle \ell\,|\,b^{\varepsilon_k}\ldots b^{\varepsilon_1}|m\rangle \leq \langle \ell\,|\,b^{q}\ b^{+p}|m\rangle$$

From there one gets immediately the second part of the inequality. The first one follows by a similar reasoning.

Lemma 1 shows that $\langle \ell\,|\,u^{\#}_{t,s}\,|\,m\rangle$ does not satisfy Maassen's condition (cf. I.2, Remark 3). But the pairs $\langle \ell\,|\,u^{\varepsilon}_{t,s}\,|\,m\rangle$ and $\langle \ell'\,|\,u^{\varepsilon'}_{t,s}\,|\,m'\rangle$ where $\varepsilon = 0,1$ and $u^0 = u$, $u^1 = u^+$ have the finite product property as

$$\| \langle \ell | u^{\varepsilon}_{t,s} | m \rangle \|_{p,q+r} = 0$$

and

$$\| \langle \ell | u^{\varepsilon}_{t,s} | m \rangle \|_{p+r,q} = 0$$

for sufficiently big r by lemma 1.

Lemma 2: The sum

$$\sum_{\ell=0}^{\infty} \int d\gamma \, \ell^k \langle \ell | u^{\varepsilon}_{t,s} | m \rangle (\alpha'+\gamma,\beta') \langle \ell+j | u^{\varepsilon'}_{t,s} | n \rangle (\alpha''+\gamma,\beta'')$$

converges uniformly for all $\alpha', \alpha'', \beta', \beta''$ with fixed $\#\alpha' = p'$, $\#\alpha'' = p''$, $\#\beta' = q'$ and $\#\beta'' = q''$ for fixed j and k and $0 < t-s < 1$.

Proof: Assume $\#\gamma = r$, $\#\alpha' = p'$, $\#\alpha'' = p''$, $\#\beta' = q'$, $\#\beta'' = q''$. This means

$$m + p' + r = \ell + q'$$
$$n + p'' + r = \ell + j + q''.$$

So $r = \ell + c$ for some constant c, or $r = 0$. So the terms in the sum can be estimated by

$$\sum_{\ell=0}^{\infty} \frac{(t-s)^{(\ell+c)}}{(\ell+c)!} \ell^k \frac{(\ell+q')! \, (\ell+j+q'')!}{\sqrt{\ell!} \sqrt{(\ell+j)!} \sqrt{m!} \sqrt{n!}} < \infty$$

using the quotient criterion.

Lemma 3: Let **C**<x,y> be the free algebra over **C** generated by x and y and define the mapping

$$\Delta: \mathbf{C}\langle x,y \rangle \to \mathbf{C}\langle x,y \rangle$$
$$P \to xyP - 2xPy + Pxy.$$

Then for two elements $P, Q \in \mathbf{C}\langle x,y \rangle$

$$\Delta PQ = (\Delta P)Q + P(\Delta Q) + 2[x,P][y,Q]$$

and for n elements $P_1, \ldots, P_n$

$$\Delta(P_1 \ldots P_n) =$$
$$(\Delta P_1)P_2 \ldots P_n + P_1(\Delta P_2)P_3 \ldots P_n + \ldots + P_1 \ldots P_{n-1}\Delta P_n$$
$$+ 2 \sum_{1 \leq i < j \leq n} P_1 \ldots P_{i-1} [x,P_i] P_{i+1} \ldots P_{j-1} [y,P_j] P_{j+1} \ldots P_n.$$

Proof: By direct calculation

$$\Delta(PQ) - (\Delta P)Q - P(\Delta Q) = 2 [x,P][y,Q].$$

From there proceed by induction.

Denote for $s \leq t$

$$b_{t,s} = b\, e^{(t-s)/2}\, \delta_{\emptyset,\emptyset} - \int_s^t e^{(t-t')/2} da^+_{t'}$$

(1)

$$b^+_{t,s} = b^+ e^{(t-s)/2} \delta_{\emptyset,\emptyset} + i \int_s^t e^{(t-t')/2} da_{t'}$$

and

(2) $$b_{s,t} = b\, e^{(t-s)/2} \delta_{\emptyset,\emptyset} + i \int_s^t e^{(t-t')/2} da^+_{t'}$$

$$b^+_{s,t} = b^+ e^{(t-s)/2} \delta_{\emptyset,\emptyset} - i \int_s^t e^{(t-t')/2} da_{t'}.$$

Call as in § II.3 $\mathcal{W}(b,b^+)$ the algebra generated by b and b^+ with the defining relation $[b,b^+]= 1$. As $[b_{t,s}, b^+_{t,s}] = 1$ and $[b_{s,t}, b^+_{s,t}] = 1$, replacing in a polynomial p in b, b^+ the elements b, b^+ by $b_{t,s}$ and $b^+_{t,s}$ (resp. $b_{s,t}$ and $b^+_{s,t}$) defines an isomorphism $\eta_{t,s}$ (resp. $\eta_{s,t}$) from $\mathcal{W}$ into the algebra of kernels. Multiplication of these kernels is not a problem as in the integral term $\int d\gamma$ only finitely many $\#\gamma$ occur.

Proposition 4: Assume $p \in \mathcal{W}(b, b^+)$ and $0 \le t-s < 1$.
Then

$$\sum_{\ell, \ell'=0}^{\infty} \left(\langle m|u^+_{t,s}|\ell\rangle * \langle \ell|p|\ell'\rangle \delta_{\emptyset,\emptyset} * \langle \ell'|u_{t,s}|n\rangle\right)(\sigma,\tau) = \langle m|\eta_{t,s}p|n\rangle(\sigma,\tau)$$

and

$$\sum_{\ell, \ell'=0}^{\infty} \left(\langle m|u_{t,s}|\ell\rangle * \langle \ell|p|\ell'\rangle \delta_{\emptyset,\emptyset} * \langle \ell'|u^+_{t,s}|n\rangle\right)(\sigma,\tau) = \langle m|\eta_{s,t}p|n\rangle(\sigma,\tau).$$

The sums converge uniformly for fixed $\#\sigma$ and $\#\tau$.

Proof: Due to the structure of $b^\#$ we have that

$$\langle \ell|p|\ell'\rangle = 0 \text{ for } |\ell-\ell'| \ge c$$

where c is some constant dependent on p and for fixed ℓ

$$|\langle \ell|p|\ell'\rangle| \le \sum_{k=0}^{K} a_k \ell^k$$

for all ℓ' and with fixed constants a_k. So using lemma 2 we show that the sum as well as its derivative with respect to t converge uniformly for fixed $\#\sigma$ and $\#\tau$.
Using the fact that $(\langle m|u^+_{t,s}|\ell\rangle, \langle \ell'|u_{t,s}|m\rangle)$ have the property (FP) of definition 4 of I.3, we can apply proposition 4 of I.4 and justify the following calculations in matrix form.

One has

$$d_t u^+_{t,s} = u^+_{t,s} * (ib\, da_t + ib^+ da^+_t - \tfrac{1}{2} bb^+ dt)$$

and

$$d_t u^+_{t,s} * p\delta_{\emptyset,\emptyset} * u_{t,s} = u^+_{t,s} * (i[b,p]da_t + i[b^+,p]da^+_t - \frac{1}{2}\Delta p\ \delta_{\emptyset,\emptyset}dt) * u_{t,s}$$

with

$$\Delta p = bb^+p - 2bpb^+ + pbb^+.$$

Now

$$d_t b_{t,s} = \frac{1}{2} b_{t,s}dt - ida^+_t$$

$$d_t b^+_{t,s} = \frac{1}{2} b^+_{t,s}dt + ida_t.$$

As the pair $(b^{\#}_{t,s}, b^{\#}_{t,s})$ has the property (FP) we can apply this theorem repeatedly and obtain

$$d_t(b^{\varepsilon_1}_{t,s} * \dots * b^{\varepsilon_n}_{t,s}) = \frac{n}{2} b^{\varepsilon_1}_{t,s} * \dots * b^{\varepsilon_n}_{t,s}dt$$

$$+ i \sum_{1\le j\le n} b^{\varepsilon_1}_{t,s} * \dots * \widehat{b^{\varepsilon_j}_{t,s}} * \dots * b^{\varepsilon_u}_{t,s} * (c_{\varepsilon_j}\, da_t + c'_{\varepsilon_j}\, da^+_t)$$

$$+ \sum_{1\le j<k\le n} c_{\varepsilon_j\varepsilon_k}\, b^{\varepsilon_1}_{t,s} * \dots * \widehat{b^{\varepsilon_j}_{t,s}} * \dots * \widehat{b^{\varepsilon_n}_{t,s}} * \dots * b^{\varepsilon_n}_{t,s}\ dt.$$

The ^ signifies that this factor has to be deleted $\varepsilon_i = \pm 1$, and $b^{(+1)} = b^+$, $b^{(-1)} = b$. $c_1 = 1$, $c_{-1} = 0$; $c'_1 = 0$, c'_{-1} and $c_{1,1} = c_{-1,-1} = c_{-1,1} = 0$ and $c_{+1,-1} = 1$.
Now

$$[b,b^{\varepsilon_i}] = c_{\varepsilon_i}$$

$$[b^+,b^{\varepsilon_i}] = c'_{\varepsilon_i}$$

$$\Delta b = -b$$

$$\Delta b^+ = -b^+.$$

So

$$d_t b^{\varepsilon_1}_{t,s} * \dots * b^{\varepsilon_n}_{t,s} = d_t\eta_{t,s}(b^{\varepsilon_i} \dots b^{\varepsilon_n}) =$$

$$i\eta_{t,s}([b,b^{\varepsilon_i} \dots b^{\varepsilon_n}]) * da_t + i\eta_{t,s}([b^+,b^{\varepsilon_i} \dots b^{\varepsilon_n}]) * da^+_t - \frac{1}{2}\eta_{t,s}(\Delta(b^{\varepsilon_i} \dots b^{\varepsilon_n}))dt.$$

So for all $p \in \mathcal{W}(b,b^+)$

$$(*)\quad d_t(\eta_{t,s}p) = i\,\eta_{t,s}([b,p]) * da_t + i\,\eta_{t,s}([b^+,p]) * da^+_t - \frac{1}{2}\eta_{t,s}(\Delta p)dt.$$

For $p = 1$ we have

$$d_t(\eta_{t,s}1) = 0.$$

So $\eta_{t,s}1 = \eta_{s,s}1 = 1$.

Let $\mathcal{W}_n(b,b^+)$ be the span of all polynomials in $b^{\#}$ of degree $\le n$. Then $p \to [b^{\#},p]$ is a mapping from $\mathcal{W}_n$ onto $\mathcal{W}_{n-1}$ and $p \to \Delta p$ is a mapping from $\mathcal{W}_n$ into itself. We assume $\eta_{t,s}p$ to be calculated for $p \in \mathcal{W}_{n-1}$. Find a basis $p_1, \ldots, p_N$ of $\mathcal{W}_n$, then $\Delta p_i = \sum D_{ik}p_k$ and for $\sigma = \tau = \emptyset$ the function $t \to \eta_{t,s}p(\emptyset,\emptyset)$ is differentiable and

$$\frac{d}{dt}\eta_{t,s}p_i(\emptyset,\emptyset) = \frac{1}{2}\sum D_{ik}\eta_{t,s}p_k(\emptyset,\emptyset).$$

So it is uniquely determined by $\eta_{s,s}p_k(\emptyset,\emptyset) = p_k$. Explicitly

$$(\eta_{t,s}p_i)(\emptyset,\emptyset) = \sum_k (e^{-1/2(t-s)D})_{ik}p_k.$$

Assume $\tau \cup \sigma \neq \emptyset$, and e.g. $\max(\tau \cup \sigma) = t' \in \sigma$. Then for $t > t'$:

$$t \to \eta_{t,s}(p)(\tau, \sigma)$$

is C^1 and

$$\frac{d}{dt}\eta_{t,s}p_i(\tau,\sigma) = \frac{1}{2}\sum D_{ik}\eta_{t,s}p_k(\tau,\sigma).$$

For $t < t'$ one has $\eta_{t,s}p_i(\tau,\sigma) = 0$ and for $\eta_{t'+0,s}p_i(\tau,\sigma) = i\eta_{t,s}([b^+,p])(\tau\backslash\{t'\},\sigma)$.
So $\eta_{t,s}p$ is uniquely defined by the differential equation (*) and the initial conditions. As the

$$u^+_{t,s} * p\delta_{\emptyset,\emptyset} * u_{t,s}, \quad p \in \mathcal{W}_n$$

obey to the same differential equation and to the same initial conditions they coincide.

A similar reasoning applies to the second half of the proposition. Differentiate now with respect to s!

§ II. 5. The solution as unitary operator

We introduce

$$\mathcal{C} = \mathcal{C}(\Omega(I) \times \mathbf{N})$$

the space of all functions $\xi : \Omega(I) \times \mathbf{N} \to \mathcal{C}$; $(\omega,\ell) \to \xi(\omega,\ell)$, such that

$$\sup_{\#\omega+\ell=p} |\xi(\omega,\ell)| = \|\xi\|_p < \infty.$$

We denote by

$$\mathcal{C}_0 = \mathcal{C}_0(\Omega(I) \times \mathbf{N})$$

the subspace of these ξ such that $\xi(\omega,\ell) = 0$ for $\#\omega+\ell$ sufficiently big.

Let $\xi \in \mathcal{C}_0$, we define $\xi_m(\omega) = \xi(\omega,m)$. Let $u_{t,s}$ and $u^+_{t,s}$ be the kernel valued matrices of the last chapter.

Define

$$(U_{t,s}\xi)(\omega,\ell) = \Big(\sum_m \langle \ell | u_{t,s} | m\rangle * \xi_m\Big)(\omega)$$

or

$$(U(t,s)\xi)(\omega,\ell) = \sum_m \sum_{\sigma\subset\omega} \int d\tau \, \langle \ell | u_{t,s} | m\rangle (\sigma,\tau)\xi((\omega\backslash\sigma) + \tau, m).$$

Similar

$$(U^+_{t,s}\xi)(\omega,\ell) = \sum_m \left(\langle \ell | u^+_{t,s} | m\rangle * \xi_m\right)(\omega).$$

So $U_{t,s}$ and $U^+_{t,s}$ are operators $\mathcal{C}_0 \to \mathcal{C}$ by proposition 1 of I.2. They are adjoint in the sense that

$$\left\langle U^+_{t,s}\xi,\ \zeta\right\rangle = \left\langle \xi,\ U_{t,s}\zeta\right\rangle \text{ for } \xi, \zeta \in \mathcal{C}_0.$$

Define the operators $N = b^+b$ and Λ as in § I.5, so

$$(N\xi)(\omega,\ell) = \ell\xi(\omega,\ell)$$
$$\Lambda\xi(\omega,\ell) = (\#\omega)\xi(\omega,\ell).$$

Proposition 1: We have

$$(N\text{-}\Lambda)U_{t,s} = U_{t,s}(N\text{-}\Lambda)$$
$$(N\text{-}\Lambda)U^+_{t,s} = U^+_{t,s}(N\text{-}\Lambda).$$

Proof: We have

$$\langle \ell | u_{t,s} | m\rangle(\sigma,\tau)(N-\Lambda)\xi((\omega\backslash\sigma) + \tau, m)$$
$$= (m - \#\tau - \#\omega + \#\sigma)\langle \ell | u_{t,s} | m\rangle(\sigma,\tau)\xi((\omega\backslash\sigma) + \tau, m)$$
$$= (\ell - \#\omega)\langle \ell | u_{t,s} | m\rangle(\sigma,\tau)\xi((\omega\backslash\sigma) + \tau, m),$$

as $\langle \ell | u_{t,s} | m\rangle(\sigma,\tau)$ vanishes unless $\ell + \#\tau = m + \#\sigma$ (see II. 4, Remark 1). Denote by $\mathcal{C}_r(\Omega(I) \times N) = \mathcal{C}_r$ the subset of $\mathcal{C}$ given by

$$\left\langle \xi,(N+\Lambda)^k\xi\right\rangle = \sum_{r,\ell=0}^{\infty} (\ell+r)^k \int_{u_1<\dots<u_r}\!\!\int |\xi(u_1, \dots, u_r\}, \ell)|^2 du_1 \dots du_r < \infty$$

for all $k = 0,1,2, \dots$.

Proposition 2: The operators $U^\#_{t,s}$ for $0 \leq t\text{-}s<1$ map $\mathcal{C}_0$ into $\mathcal{C}_r$.

Proof: Assume $\xi \in \mathcal{C}_0$, then

$$\left\langle U_{t,s}\xi, N^k U(t,s)\xi\right\rangle$$

$$= \sum_{\ell,m_1,m_2} \xi^+_{m_1} * \langle m_1 | u^+_{t,s} | \ell\rangle * \ell^k\delta_{\emptyset,\emptyset} * \langle \ell | u_{t,s} | m_2\rangle * \xi_{m_2}.$$

Following the discussions of § I.1 we obtain

$$= \sum_{\ell,m_1,m_2} \iiint d\gamma_{12}\, d\gamma_{13}\, d\gamma_{23}\, \overline{\xi_{m_1}}(\gamma_{12} + \gamma_{13})$$

$$\left(\langle m_1 | u^+_{t,s} | \ell\rangle * \ell^k \langle \ell | u_{t,s} | m_2\rangle\right)(\gamma_{12}, \gamma_{23})\xi_{m_2}(\gamma_{13} + \gamma_{23}).$$

By lemma 2 of II.4 the sum

$$\Big(\sum_{\mathcal{L}} \langle m|u^{+}_{t,s}|\mathcal{L}\rangle * \mathcal{L}^{k}\langle \mathcal{L}|u_{t,s}|m_2\rangle\Big)\ (\gamma_{12}, \gamma_{23})$$

converges uniformly for all γ_{12}, γ_{23} with fixed $\#\gamma_{12}$ and $\#\gamma_{13}$ to some bounded function. As $\#\gamma_{12}$ and $\#\gamma_{13}$ stay bounded, we have finally that

$$\left\langle U_{t,s}\xi, N^{k}U_{t,s}\xi\right\rangle < \infty$$

for all $\xi \in \mathcal{C}_0$ and all k. Now

$$\left\langle U_{t,s}\xi,\ (N+\Lambda)^{k}U_{t,s}\xi\right\rangle = \left\langle U_{t,s}\xi,\ (2N+(N-\Lambda))^{k}U_{t,s}\xi\right\rangle$$

$$= \sum_{j=0}^{k} \binom{k}{j} 2^{j} \left\langle U_{t,s}\xi,\ N^{j}\ U_{t,s}(N-\Lambda)^{k-j}\xi\right\rangle < \infty$$

using Schwarz's inequality.

b and b^{+} can be defined as operators on $\mathcal{C}_r$ and are mutually adjoint.

Proposition 3: Let $\xi, \zeta \in \mathcal{C}_0$, $0 \le t - s < 1$ and $p \in \mathcal{W}(b,b^{+})$. Then

$$\left\langle U_{t,s}\xi, pU_{t,s}\zeta\right\rangle = \left\langle \xi, \eta_{t,s}(p)\zeta\right\rangle$$

$$\left\langle U^{+}_{t,s}\xi, pU^{+}_{t,s}\zeta\right\rangle = \left\langle \xi, \eta_{t,s}(p)\zeta\right\rangle$$

interpreting $b^{\#}_{t,s}$ and $b^{\#}_{s,t}$ as operators on $\mathcal{C}_o$.

Proof: We have

$$\left\langle U_{t,s}\xi, pU_{t,s}\zeta\right\rangle = \sum_{\mathcal{L},\mathcal{L}',m_1,m_2} \xi^{+}_{m_1} * \langle m_1|u^{+}_{t,s}|\mathcal{L}\rangle * \langle \mathcal{L}|p|\mathcal{L}'\rangle\delta_{\emptyset,\emptyset} * \langle \mathcal{L}'|u_{t,s}|m_2\rangle * \zeta_{m_1}.$$

We use again lemma 2 of II.4 in order to ensure the convergence of the sums and apply then proposition 4 of II.4. The last expression becomes

$$= \sum_{m_1,m_2} \xi^{+}_{m_1} * \langle m_1|\eta_{t,s}p|m_2\rangle * \zeta_{m_2} = \left\langle \xi, (\eta_{t,s}p)\zeta\right\rangle.$$

Proposition 4: Let $0 \le s\text{-}r<1$ and $0 \le t\text{-}s<1$, then for $\xi,\zeta \in \mathcal{C}_0$

$$\left\langle U^{+}_{t,s}\xi, U_{s,r}\zeta\right\rangle = \left\langle \xi, U_{t,r}\zeta\right\rangle.$$

Proof: The convergence of the sum

$$\sum_{\mathcal{L},m_1,m_2} \xi^+_{m_1} * \langle m_1 | U_{t,s} | \mathcal{L}\rangle * \langle \mathcal{L} | U_{s,r} | m_2\rangle \; * \; \zeta m_2$$

follows from lemma 2 of II.4. Apply proposition 3 and proposition 4 of II.4 for $p = 1$.
Proposition 5: Let $\xi,\zeta \in \mathcal{C}_0$. For fixed s and $t \downarrow s$

$$\left\langle (U^{\varepsilon}_{t,s}\xi - \xi), (\Lambda+N)^k (U^{\varepsilon}_{t,s}\xi - \xi)\right\rangle \to 0$$

where $\varepsilon = \pm 1$ and $U^{-1} = U^+$, $U^{-1} = U$.

Proof: By proposition 2 it is sufficient to show that

$$\left\langle U^{\varepsilon}_{t,s}\xi - \xi,\; N^k (U^{\varepsilon}_{t,s}\xi - \xi)\right\rangle \to 0.$$

The left side is equal to

$$\sum_{m_1,m_2,\mathcal{L}} \xi^+_{m_1} * \left(\langle m_1 | u^{\varepsilon}_{t,s} | \mathcal{L}\rangle - \delta_{\mathcal{L}m_1}\delta_{\emptyset,\emptyset}\right) * \mathcal{L}^k\left(\langle \mathcal{L} | u^{\varepsilon}_{t,s} | m_2\rangle - \delta_{\mathcal{L}m_2}\delta_{\emptyset,\emptyset}\right) * \xi_{m_1}.$$

Now

$$z_{t,s}(\mathcal{L},\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2) = \left(\langle m | u^{\varepsilon}_{t,s} | \mathcal{L}\rangle - \delta_{\mathcal{L}m}\delta_{\emptyset,\emptyset}\right)(\alpha_1,\beta_1+\gamma)\mathcal{L}^k\left(\langle \mathcal{L} | u^{\varepsilon}_{t,s} | m_2\rangle - \delta_{\mathcal{L}m_2}\delta_{\emptyset,\emptyset}\right)(\alpha_2+\gamma,\beta_2)$$

has the property that for $\gamma \neq \emptyset$

$$|z_{t,s}(\mathcal{L},\gamma)| \leq \left\langle m_1 | b^{\#(\beta_1)+\#\gamma}(b^+)^{\#\alpha_1} | \mathcal{L}\right\rangle \mathcal{L}^k \left\langle \mathcal{L} | b^{\#(\beta_2)}(b^+)^{\#\alpha_2 + ||\#\gamma||} | m_2\right\rangle \mathbf{1}(\gamma \subset]s,t[) =$$
$$= C(\mathcal{L},\gamma)\mathbf{1}(\gamma \subset]s,t[).$$

So if $s < t \leq t_0$ and $t_0 - s < 1$,

$$|z_{t,s}(\mathcal{L}, g)| \leq C(\mathcal{L},\gamma)\, \mathbf{1}(\gamma \subset]s,t_0[)$$

and by the reasoning applied in lemma 2 of II.4, we have that

$$\int d\gamma \sum_{\mathcal{L}} C(\mathcal{L},\gamma)\, \mathbf{1}(\gamma \subset]s,t_0[) < \infty.$$

On the other hand, all $z_{t,s}(\mathcal{L},\gamma) \to 0$ for $t \downarrow s$. Apply the theorem of Lebesgue.

Proposition 6: There exist constants $\gamma_0 = 0 \leq \gamma_1 \leq \gamma_2 \leq \dots$ such that for $0 \leq t-s < 1$, $\varepsilon = \pm 1$

$$\left\langle U^{\varepsilon}_{t,s}\xi, (\Lambda+N+1)^k U^{\varepsilon}_{t,s}\xi\right\rangle \leq \exp\left(\gamma_k \rho(t-s)\right)\left\langle \xi, (\Lambda+N+1)^k \xi\right\rangle$$

with

$$\rho(t) = \frac{1}{2}(t + \log(1 + \sqrt{1-e^{-t}})) \leq \frac{1}{2}(t + \log 2).$$

Proof: We have by (1) of II.4

$$b_{t,s} = e^{(t-s)/2}b + \sqrt{e^{t-s}-1}\ a^+(f_{t,s})$$
$$b^+_{t,s} = e^{(t-s)}b^+ + \sqrt{e^{t-s}-1}\ a(f_{t,s})$$

where $\quad f_{t,s}(t') = \frac{i}{\sqrt{e^{t-s}-1}}\ \mathbb{1}_{[s,t]}(t')\ e^{(t-t')/2}$

has been chosen such that

$$\int f_{t,s}(t')^2 dt' = 1.$$

Then

$$b_{t,s}b^+_{t,s} = e^{t-s}bb^+ + e^{(t-s)/2}\sqrt{e^{t-s}-1}\ (b^+a^+(f_{t,s}) + ba(f_{t,s})) + (e^{t-s}-1)a^+(f_{t,s})a(f_{t,s}).$$

We have

$$[N-\Lambda,\ b_{t,s}b^+_{t,s}] = 0$$

and recall

$$bb^+ = N+1.$$

Use proposition 1

$$\left\langle U_{t,s}\xi,\ (\Lambda+N+1)^k U_{t,s}\xi\right\rangle = \left\langle U_{t,s}\xi,\ (2(N+1) + (\Lambda-N))^k U_{t,s}\xi\right\rangle$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left\langle U_{t,s}\xi,\ (2(N+1) + (\Lambda-N))^j U_{t,s}\xi\right\rangle = \sum_{j=0}^{k} \binom{k}{j} \left\langle \xi,\ 2(\eta_{t,s}bb^+)^j(\Lambda-N)^{k-j}\xi\right\rangle$$

$$= \left\langle \xi,\ (2b_{t,s}b^+_{t,s} + \Lambda-N)^k\xi\right\rangle = \left\langle \xi,\ (\Lambda_0 + M_{t,s})^k\xi\right\rangle$$

with

$$\Lambda_0 = \Lambda - a^+(f_{t,s})a(f_{t,s})$$
$$M_{t,s} = 2b_{t,s}b^+_{t,s} - bb^+ + a^+(f_{t,s})a(f_{t,s}).$$

So

$$M_{t,s} = (2e^{t-s} - 1)bb^+ + 2e^{(t-s)/2}\sqrt{e^{t-s}-1}\ (b^+a^+(f_{t,s}) + ba(f_{t,s})) + (2e^{t-s} - 1)a^+(f_{t,s})a(f_{t,s}).$$

Let $\xi \in \mathcal{C}_0$, we can write it in the form

$$\xi = \sum \frac{b^{+\ell}}{\sqrt{\ell!}}|0> \otimes \xi_\ell = \sum |\ell> \otimes \xi_\ell$$

with $\xi_\ell \in \mathcal{C}_0(\Omega(I))$. Apply the results of I.5. We obtain

$$\xi = \sum_{\ell,m} \frac{b^{+\ell}}{\sqrt{\ell!}}|0> \otimes \frac{a^+(f_{t,s})}{\sqrt{m!}}\ \xi_{\ell,m}$$

with $a(f_{t,s})\xi_{\ell,m} = 0.$

We establish so an isomorphism between $\mathcal{C}_0$ and $\ell_0(N^2) \otimes \mathcal{K}$

$$\xi \rightarrow \sum_{\ell,m} \frac{b^{+\ell} a^{+m}}{\sqrt{\ell!}\sqrt{m!}} |0\rangle \otimes \xi_{\ell,m} = \sum_{\ell,m} |\ell,m\rangle \otimes \xi_{\ell,m}.$$

The operator $M_{t,s}$ works only on the first factor, whereas Λ_0 is the number operator on the second factor.

So $M_{t,s}$ can be represented as a polynomial in $a^\#$ and $b^\#$. Recalling the notations of II.3 we obtain

$$M_{t,s} = \alpha A + \beta B$$

with $\alpha = 2e^{(t-s)} - 1$, $\beta = 2e^{(t-s)/2}\sqrt{e^{t-s}-1}$. We obtain

$$\left\langle \xi, (\Lambda_0 + M_{t,s})^k \xi \right\rangle = \sum_{\substack{\ell,m \\ \ell',m'}} \binom{k}{j} \left\langle \ell, m | M^j_{t,s} | \ell', m' \right\rangle \left\langle \xi_{\ell,m}, \Lambda_0^{k-j} \xi_{\ell',m'} \right\rangle,$$

as $\alpha^2 - \beta^2 = 1$ we apply the theorem of II.3 and use the fact that both ℓ,m-matrices are positive definite

$$\leq \sum_{j=0}^{k} \binom{k}{j} \left\langle \ell, m | A^j | \ell', m' \right\rangle \left\langle \xi_{\ell,m}, \Lambda_0^{k-j} \xi_{\ell',m'} \right\rangle e^{\gamma_k u}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left\langle \xi, A^j \Lambda_0^{k-j} \xi \right\rangle = \left\langle \xi, (A + \Lambda_0)^k \xi \right\rangle e^{\gamma_k u} = \left\langle \xi, (1 + \Lambda + N)^k \xi \right\rangle e^{\gamma_k u}$$

where

$$e^{2u} = \alpha + \beta = (e^{(t-s)/2} - \sqrt{e^{t-s}-1})^2$$

or

$$u = \frac{1}{2}(t-s) + \frac{1}{2}\log\sqrt{1-e^{-(t-s)}}.$$

For U^+ we have nearly the same developments changing $f_{t,s}$ by $f_{s,t} = -f_{t,s}$.

Theorem: There exists a unique family $\widetilde{U}_{t,s}$ of unitary operators on $L^2(\Omega(I) \times \mathbf{N})$ for $t,s \in \mathbf{R}$ with the properties

$$\widetilde{U}_{t,t} = 1$$
$$\widetilde{U}^+_{t,s} = U_{s,t}$$
$$\widetilde{U}_{t,s}\widetilde{U}_{s,r} = \widetilde{U}_{t,r}$$

for $s, t, r \in \mathcal{R}$ such that

$$\widetilde{U}_{t,s}\xi = U_{t,s}\xi$$
$$\widetilde{U}_{s,t}\xi = U^+_{t,s}\xi$$

for $\xi \in \mathcal{C}_0$.

Denote by $\mathcal{D}_k$ the subspace of all $\xi \in L^2(\Omega(I) \times \mathbf{N})$ such that

$$\left\langle \xi, (\Lambda+N+1)^k \xi \right\rangle = \|\xi\|_k^2 < \infty.$$

Then

$$\widetilde{U}t,s : \mathcal{D}_k \to \mathcal{D}_k$$

and

$$\|\widetilde{U}_{t,s}\xi\|_k \leq C_k e^{\Gamma_k |t-s|}$$

for some constants C_k and Γ_k. Furthermore

$$\left\langle \widetilde{U}_{t,s}\xi, p\, \widetilde{U}_{t,s}\xi \right\rangle = \left\langle \xi, \eta_{t,s} p \xi \right\rangle$$

for all $p \in \mathcal{W}(b, b^+)$ and $\xi \in \mathcal{D}_k$ with $k \geq$ the degree of p. The function $t,s \to U_{t,s}\xi$ is continuous in $\mathcal{D}_k$-norm for $\xi \in \mathcal{D}_k$.

Proof: In order to establish unitarity apply proposition 3 for $p = 1$ and obtain

$$\left\langle U_{t,s}\xi, U_{t,s}\xi \right\rangle = \left\langle \xi, \zeta \right\rangle$$

$$\left\langle U_{t,s}^+\xi, U_{t,s}^+\xi \right\rangle = \left\langle \xi, \zeta \right\rangle.$$

As $U_{t,s}$ and $U_{t,s}^+$ are formally adjoint, we conclude that they are restrictions of unitary operators on $\mathcal{C}_0$. In order to obtain the other results apply the propositions 4, 5 and 6. In future we shall delete the ~ and write $U_{t,s}$ instead of $\widetilde{U}_{t,s}$.

§ II.6. The classical Markov process of the occupation numbers

We want to investigate the time behavior of the occupation states

$$\Phi_m = |m\rangle \langle m|, \quad m = 0, 1, 2, \ldots$$

of the inverse oscillator under the assumption that at initial time the heat bath is in the vacuum state. Denote by $\Phi_m(t,s) = U_{t,s}^+ \Phi_m U(t,s)$. We are interested e.g. in

$$\mathrm{Tr}((|\emptyset\rangle \langle\emptyset| \otimes \rho_0)\, \Phi_m(t,s) = \mathrm{Tr}\rho_0 P_\emptyset \Phi_m(t,s) P_\emptyset,$$

where $|\emptyset\rangle \langle\emptyset|$ is the vacuum state of the Fock space, ρ_0 is some initial density matrix of the oscillator and $P_\emptyset$ is the projector

$$P_\emptyset : L^2(\Omega(\mathbf{R}) \times \mathbf{N}) \to L^2(\mathbf{N})$$
$$\sum \xi_m |m\rangle \to \sum \xi_m(\emptyset)\, |m\rangle.$$

Definition: Let $X(t)$ be the classical Markov process on $\mathbf{N} = \{0, 1, 2, \ldots\}$ with possible jumps by $+1$ and the transition rates given by

$$\mathbf{P}(X(t+dt) = n+1 \mid X(t) = n) = (n+1)dt$$
$$\mathbf{P}(X(t+dt) = n \mid X(t) = n) = 1 - (n+1)dt.$$

Denote by

$$p_{mn}(t) = \mathbf{P}(X(t) = m \mid X(0) = n)$$

the transition probability coming from n to m during the time t.

Lemma 1: For $s < t$

$$P_\emptyset U^+_{t,s} \Phi_m U(t,s) P_\emptyset = \sum_{\ell=0}^{m} p_{m,\ell}(t-s)\, \Phi_\ell.$$

Proof: Using kernels

$$\langle \ell \mid P_\emptyset U^+_{t,s} \Phi_m U_{t,s} P_\emptyset \mid \ell' \rangle = \left(\langle \ell \mid u^+_{t,s} \mid m \rangle * \langle m \mid u_{t,s} \mid \ell' \rangle \right)(\emptyset,\emptyset)$$

$$= \int d\gamma \langle \ell \mid u^+_{t,s} \mid m \rangle (\emptyset,\gamma) \ \langle m \mid u_{t,s} \mid \ell' \rangle (\gamma,\emptyset)$$

$$= \delta_{\ell\ell'} \left(\delta_{m\ell} + \sum_{k=1}^{\infty} \int_{s \le t_1 < \ldots < t_n \le t} \int \left| \langle m | e^{-(t-t_n)bb^+/2}\, b^+ e \ldots b^+ e^{-(t_1-s)bb^+/2} | \ell \rangle \right|^2 dt_1 \ldots dt_k = \right.$$

$$d_{\ell\ell'} \left(\delta_{m\ell} + \sum_{k=1}^{\infty} \delta_{m-k,\ell} \int_{s \le t_1 < \ldots < t_k \le t} \int e^{-(t-t_k)m} \cdot m\, e^{-(t_k-t_{k-1})(m-1)} (m-1) \ldots (\ell+1) e^{-(t_1-s)(\ell+1)} dt_1 \ldots dt_k \right)$$

and the integral equals $p_{m,\ell}(t-s)$.

Theorem 1: Call

$$\Phi_m(t) = \Phi_m(t,0) = U^+_{t,0}\, \Phi_m\, U_{t,0},$$

then

$$\mathrm{Tr}(\rho_0 \otimes |\emptyset\rangle\langle\emptyset|)\, \Phi_{m_1} \ldots \Phi_{m_p}(t_p)) = \mathbf{P}_\pi (X(t_1) = m_1, \ldots, X(t_p) = m_p)$$

provided that $t_1, \ldots, t_p$ are pyramidally ordered, i.e. there exists a q with $1 \le q \le p$ such that

$$t_1 < \ldots < t_{q-1} < t_q > t_{q+1} > \ldots > t_p.$$

Here π is the initial distribution of the Markov process

$$\pi(n) = \mathbf{P}(X(0) = n) = \langle n | \rho_0 | n \rangle.$$

Proof: We shall show that

$$(*) \qquad P_\emptyset \Phi_{m_1}(t_1) \ldots \Phi_{m_p}(t_p) P_\emptyset = \sum \Phi_\ell\, \mathbf{P}_\ell (X(t_1) = m_1, \ldots, X(t_p) = m_p)$$

where $\mathbf{P}_\ell$ is the probability distribution of Markov process $X(t)$ starting with $X(0) = \ell$. For $p = 1$ this is a direct consequence of lemma 1. Assume now $p \ge 1$ and call

$$s = \max(t_{q-1}, t_{q+1}).$$

Then the left-hand side of (*) equals

$$P_\emptyset U_{0,t_1} \Phi_{m_1} U_{t_1,t_2} \Phi_{m_2} \cdots U_{t_{p-1}t_p}\Phi_{m_p} U_{t_p,C} P_\emptyset$$

$$= (P_\emptyset U_{0,t_1} \Phi_{m_1} \ldots \Phi_{m_{q-1}} U_{t_{q-1},s}) (U_{s,t_q} \Phi_{m_q} U_{t_q,s}) (U_{s,t_{q+1}} \Phi_{m_p} U_{t_p,C} P_\emptyset).$$

Using I.1 and remark 1 of I.4 we write the $\mathcal{L},\mathcal{L}'$-matrix element of this expression in the form of kernels

$$\sum_{j,j'} (\langle \mathcal{L}|u_{0,t_1}|m_1\rangle * \cdots * \langle m_{q-1}|u_{t_{q-1},s}|j\rangle * \langle j|u_{s,t_q}|u_q\rangle * \langle m_q|U_{t_q,s}|j'\rangle *$$

$$\langle j'|u_{t_q t_{q+1}}|m_{q+1}\rangle * \cdots * \langle m_p|u_{t_p,0}|\mathcal{L}'\rangle)(\emptyset, \emptyset)$$

$$= \sum_{j,j'} \int d\gamma \langle \mathcal{L}|u_{0,t_1}|m_1\rangle (\emptyset, \gamma\cap[0,t_1]) \ldots \langle m_{q-1}|u_{t_{q-1},s}|j\rangle (\emptyset, \gamma\cap[t_{q-1},s])$$

$$\langle j|u_{s,t_q}|m_q\rangle \ (\emptyset, \gamma\cap [\, s,t_q]) \langle m_q|u_{t_q,s}|j'\rangle (\gamma\cap [s, t_q], \emptyset)$$

$$\langle j'|u_{s,t_{q+1}}|m_{q+1}\rangle (\gamma\cap [t_{q+1}, t_q], \emptyset) \ldots \langle m_p|u_{t_p,0}|\mathcal{L}'\rangle (\gamma\cap [0, t_p], \emptyset).$$

Split the integral

$$\int d\gamma \ldots = \int d\gamma_1 \int d\gamma_2 \ldots$$

with $\gamma_1 = \gamma \cap [0,s]$ and $\gamma_2 = \gamma \cap [s,t_q]$, and perform the integral over γ_2

$$\int d\gamma_2 \langle j|u_{s,t_q}|m_q\rangle (\emptyset, \gamma_2) \langle m_q|u_{t_q,s}|j'\rangle (\gamma_2, \emptyset) = \delta_{jj'} \ p_{mq,j}(t_q\text{-}s).$$

Assume now that $t_{q-1} = s$, then

$$\langle m_{q-1}|u_{t_{q-1},s}|j\rangle = \delta_{m_{q-1},j}\ \delta_{\emptyset,\emptyset}$$

and the matrix element becomes

$$\int d\gamma \langle \mathcal{L}\ |u_{0,t_1}|m_1\rangle (\emptyset, \gamma\cap [0, t_1]) \ldots \langle m_{q-2}|u_{t_{q-2},t_{q-1}}|m_{q-1}\rangle (\emptyset, \gamma \cap [t_{q-2}, t_{q-1}])$$

$$\langle m_{q-1}|u_{t_{q-1},t_{q+1}}|m_{q+1}\rangle (\gamma \cap [t_{q+1}, t_{q-1}], \emptyset) \langle m_p|u_{t_p,0}|\mathcal{L}'\rangle (\gamma \cap [0, t_p], \emptyset)\ p_{m_q,m_{q-1}}(t_q\text{-}t_{q-1})$$

$$= \langle \mathcal{L}\ |\ P_\emptyset U_{0,t_1}\Phi_{m_1} \cdots \Phi_{q-2}U_{t_{q-2},\, t_{q-1}}\Phi_{m_{q-1}}\, U_{t_{q-1},t_{q+1}}\, \Phi_{m_{q+1}} \cdots \Phi_{m_{t_p}}U_{t_p,0}\mathcal{L}' \rangle\, p_{m_q,m_{q-1}}(t_q\text{-}t_{q-1})$$

$$= \left\langle \mathcal{L} \mid P_\emptyset\, \Phi_{m_1}(t_q) \ldots \Phi_{m_{q-1}}(t_{q-1})\, \Phi_{m_{q+1}}(t_{q+1}) \cdots \Phi_{m_{t_p}}\, P_\emptyset | \mathcal{L}' \right\rangle p_{m_q, m_{q-1}}(t_q - t_{q-1}).$$

If $(*)$ is true for $p' = p-1$, the last expression equals

$$\delta_{\mathcal{L}\mathcal{L}'}\, \mathbf{P}_{\mathcal{L}}(X(t_1) = m_1, \ldots, X_{t_{q-1}} = m_{q-1}, X_{t_{q+1}} = m_{q+1}, \ldots, X_{t_q} = m_q)\, p_{m_q, m_{q-1}}(t_q - t_{q-1})$$

$$= \delta_{\mathcal{L}\mathcal{L}'}\, \mathbf{P}_{\mathcal{L}}(X(t_1) = m_1, \ldots, X_{t_p} = m_p)$$

using the Markov property.
If $t_{q+1} = s$, we have a similar argument.

Theorem 2: Assume a density matrix ρ_0 with $\operatorname{Tr} \rho_0 N^k < \infty$ for all k, where $N = b^+ b$. Call

$$N(t) = U_{t,0}^+ N^k U_{t,0}.$$

Then

$$\operatorname{Tr}(\rho_0 \otimes |0\rangle\langle 0|)\, N(t_1)^{k_1} \ldots N(t_{t_p})^{k_p} = \mathbf{E}_\pi\, X(t_1)^{k_1} \ldots X(t_p)^{k_p}$$

provided that $t_1, \ldots, t_p$ are pyramidally ordered where $k_1 \geq 0, \ldots, k_p \geq 0$ are some integers.

Proof: Recall

$$\mathcal{D}_k = \{\xi \in L^2(\Omega(I) \times N) : \|\xi\|_k^2 = \left\langle \xi, (1+\Lambda+N)^k \xi \right\rangle < \infty\}.$$

Then

$$\|N^k \xi\|_{\mathcal{L}}^2 = \left\langle N^k \xi, (1+N+\Lambda)^{\mathcal{L}} N^k \xi \right\rangle \leq \|\xi\|_{2k+\mathcal{L}}.$$

Denoting by

$$\|X\|_{k,\mathcal{L}} = \sup\{\|X\xi\|_k : \|\xi\|_{\mathcal{L}} \leq 1\}$$

we have

$$\|N^k\|_{2k+\mathcal{L},\mathcal{L}} \leq 1.$$

Denoting by

$$N_M = \sum_{m=0}^{M} m\, \Phi_m = \sum_{m=0}^{M} m|m\rangle\langle m|$$

we have for $M \to \infty$

$$N_M^k\, \xi \to N^k\, \xi$$

in $\mathcal{L}$-norm for all $\xi \in \mathcal{D}_{2k+\mathcal{L}}$.
The $U_{s,t}$ are bounded operators in $\mathcal{D}_k$ for all k. Write

$$N(t_1)^{k_1} \cdots N(t_p)^{k_p} = U_{0,t_1} N^{k_1} U_{t_1,t_2} N^{k_p} U_{t_p,0}.$$

So for $\xi \in \mathcal{D}_{2(k_1+\ldots+k_p)}$ we have

$$\left\langle \xi, N(t_1)^{k_1} \cdots N(t_p)^{k_p} \xi \right\rangle \leq C \, \|\xi\|^2_{2(k_1+\cdots+k_p)} = C \left\langle \xi, (1+N+\Lambda)^{2(k_1+\cdots+k_p)} \xi \right\rangle$$

and

$$\left\langle \xi, N_M(t_1)^{k_1} \cdots N_M(t_p)^{k} \xi \right\rangle \to \left\langle \xi, N(t_1)^{k_1} \cdots N(t_p)^{k} \xi \right\rangle \text{ for } M \to \infty.$$

Write

$$\rho_0 = \sum_{r=0}^{\infty} \vartheta_r \, |\phi_r\rangle\langle\phi_r|$$

where ϑ_r are the eigenvalues > 0 of ρ_0 and ϕ_r are the eigenvectors, then

$$T_r \, \rho_0 \, N^k = \sum \vartheta_r \left\langle \phi_r \, |N^k| \, \phi_r \right\rangle.$$

So $|\phi_r\rangle \otimes |\emptyset\rangle$ are in $\mathcal{D}_k$ for all k and r.
We calculate

$$\sum \left(\rho_0 \otimes |\emptyset\rangle\langle\emptyset\right)\left(N(t_1)^{k_1} \cdots N(t_p)^{k_p}\right)$$

$$= \sum \vartheta_r \left\langle \phi_r, \emptyset | N(t_1)^{k_1} \cdots N(t_p)^{k_p} | \phi_r, \emptyset \right\rangle$$

$$\leq \sum_r \vartheta_r \left\langle \phi_r, \emptyset | (1+\Lambda+N)^{2(k_1+\cdots+k_p)} | \, \phi_r, \emptyset \right\rangle$$

$$= \sum_r \vartheta_r \left\langle \phi_r | (1+N)^{2(k_1+\cdots+k_p)} | \phi_r \right\rangle$$

$$= \mathrm{Tr} \, \rho_0 \, (1+N)^{2(k_1+\cdots k_p)} < \infty$$

and

$$\mathrm{Tr}\left(\rho_0 \otimes |\emptyset\rangle\langle\emptyset|\right)\left(N_M(t_1)^{k_1} \cdots N_M(t_p)^{k_p}\right) \to \mathrm{Tr}\left(\rho_0 \otimes |\emptyset\rangle\langle\emptyset|\right)\left(N(t_1)^{k_1} \cdots N(t_p)^{k_p}\right).$$

Call

$$X_M(t) = X(t) \, \mathbf{1} \, \{X(t) \leq M\},$$

then

$$X_M(t) \uparrow X(t)$$

for $M \to \infty$ and

$$\mathbf{E}_\pi\left(X_M(t_1)^{k_1} \cdots X_M(t_p)^{k_p}\right) \uparrow \mathbf{E}_\pi(X|t_1)^{k_1} \cdots X(t_p)^{k_p}).$$

On the other side by theorem 1

$$\mathbf{E}_\pi\left(X_M(t_1)^{k_1} \cdots X_M(t_p)^{k_p}\right)$$

$$= \sum_{m_1,\ldots,m_p=0}^{M} m_1^{k_1} \ldots m_p^{k_p} \, \mathbf{P}_\pi \left(X(t_1) = m_1, \ldots, X(t_p) = m_p\right)$$

$$= \sum_{m_1,\ldots,m_p=0}^{M} m_1^{k_1} \ldots m_p^{k_p} \, \mathrm{Tr}\left(\rho \otimes |\emptyset\rangle\langle\emptyset|\right)\left(\phi_{m_1}(t_1) \cdots \phi_{m_p}(t_p)\right)$$

$$= \mathrm{Tr}\left(\rho_0 \otimes |\emptyset\rangle\langle\emptyset|\right)\left(N_M(t_1)^{k_1} \cdots N_M(t_p)^{k_p}\right).$$

Giving on both sides with $M \to \infty$ we obtain the theorem.

Lemma 2: One has

$$\mathbf{E}_{\ell}((X(t) + k)(X(t) + k-1) \ldots (X(t) + 1))$$

$$= \mathrm{Tr}(|\ell\rangle\langle\ell| \otimes |\emptyset\rangle\langle\emptyset|)\,(N(t)+k)\,(N(t)+(k-1)) \ldots (N(t)+1)$$

$$= e^{kt}(\ell+k) \ldots (\ell+1).$$

Proof: Differentiate

$$\frac{d}{dt}\sum_m (m+k)\,(m+k-1) \ldots (m+1)\, p_{m,\ell}(t)$$

$$= \sum_m (-(m+k) \ldots (m+1)\,(m+1)\, p_{m,\ell}(t)$$

$$+ \sum_m (m+k) \ldots (m+1)m\, p_{m-1,\ell}(t)$$

$$= k \sum_m (m+k) \ldots (m+1)\, p_{m,\ell}(t).$$

Lemma 2 gives the possibility of a check $b^k b^{+k} = (N+1) \ldots (N+k)$ so using equation (1) of II.4

$$\mathrm{Tr}(|\ell\rangle\langle\ell| \otimes |\emptyset\rangle\langle\emptyset|)\,(N(t_1)+k) \ldots (N(t_1)+1)$$

$$= \langle\ell, \emptyset|\, b_{t,0}^{k} \left(b_{t,0}^{+}\right)^{k} |\ell, \emptyset\rangle$$

$$= e^{kt}\left\langle \ell\, |b^k b^{+k}| \ell\right\rangle$$

$$= e^{kt}\,(\ell+1) \ldots (\ell+k).$$

Remark: If r, s, t are not pyramidally ordered, then in general

$$\mathrm{Tr}(|0\rangle\langle 0| \otimes |\emptyset\rangle\langle\emptyset|)\, N(r)N(s)N(t) \neq \mathbf{E}_0\, X(r)X(s)X(t).$$

Proof: Using again equation (1) of II.4 we have

$$\langle 0, \emptyset|(N(r)+1)\,(N(s)+1)\,(N(t)+1)|0, \emptyset\rangle = \langle 0, \emptyset| b_{r,0} b_{r,0}^{+} b_{s,0} b_{s,0}^{+} b_{t,0} b_{t,0}^{+} |0, \emptyset\rangle.$$

Now

$$b_{t,0} = e^{t/2}\,(b + a^{+}(f_t))$$

with

$$f_t(t') = -i\; 1_{[0,t]}(t')\; e^{-t'/2}$$

Then we obtain

$$\langle 0, \emptyset | (N(r)+1)\ (N(s)+1)\ (N(t)+1) | 0, \emptyset \rangle$$

$$= e^{r+s+t}\left(6-2e^{-r\cap s} - 2e^{-r\cap t} - e^{-s\cap t} + e^{-r\cap s - r\cap t}\right)$$

where

$$s\cap t = \min(s,t).$$

For r, s, t pyramidally ordered, we have for the last expression

$$e^{t_1+t_2+t_3}\left(6-4e^{-t_1} - 2e^{-t_2} + e^{-t_1-t_2}\right)$$

with

$$\{r,s,t\} = \{t_1, t_2, t_3\},\ t_1 < t_2 < t_3.$$

If r, s, t are not pyramidally ordered, e.g. $r > s$ and $t > s$, then we have

$$e^{t_1+t_2+t_3}\left(6-4e^{-t_1} - 2e^{-t_2} + e^{-2t_1}\right)$$

which is different. So it is

$$\neq \mathbf{E}_0\ X(r)X(s)X(t).$$

Literature

[1] Belavkin, V.P.: A quantum non adapted Ito formula and non stationary evolution in Fock scale. Quantum probability and applications. VI, p. 137-180. World Scientific (Singapore) (1992)

[2] Glauber, R.J.: Amplifiers, Attenuators and the Quantum Theory of Measurement. In Frontiers in Quantum Optics, ed. by E.R. Pike and S. Sarkar, Vol. X of Malveru Physics Theories (Adam Hilger), Bristol, 1986.

[3] Haake, F., Walls, D.F.: Overdamped and Amplifying Meters in the Quantum Theory of Measurement. Phys. Rev. A. **36** (1987), p. 730-739.

[4] Hepp, K., Lieb, E.H.: Phase Transitions in Reservoirdriven Open Systems with Applications to Lasers and Superconductors. Helv. Phys. Acta. **46** (1973), p. 573-603.

[5] Hudson, R.L., Parthasarathy, K.R.: Construction of Quantum Diffusions. Lecture Notes in Mathematics **1055**, Springer (1984), p. 173-205.

[6] Lindsay, J.M.: Quantum and non-causal stochastic calculus. Prob. Theory Relat. Fields **97**, (1993), p. 65-80.

[7] Lindsay, J.M., Maassen, H.: The Stochastic Calculus of Bose Noise. Preprint, Nijmwegen (1988).

[8] Lindsay, M., Maassen, H.: An Integral Kernel Approach to Noise. Lecture Notes in Mathematics **1303**, Springer (1988), p. 192-208.

[9] Maassen, H.: Quantum Markov Processes on Fock Space Described by Integral Kernels. Lecture Notes in Mathematics **1136**, Springer (1985), p. 361-374.

[10] Meyer, P.A.: Quantum Probability for Probabilists. Lecture Notes in Mathematics

1538, Springer (1993).

[11] Palma, G.M., Vaglica, A., Leonardi, C., De Oliveira, F.A.M., Knight, P.L.: Effects of Broadband Squeezing on the Quantum Onset of Superradiance. Optics Communications, **79** (1990), p. 377-380.

[12] Robinson, P., Maassen, H.: Quantum Stochastic Calculus and the Dynamical Stark Effect. Reports an Math. Phys. Vol. 30 (1991).

[13] Waldenfels, W.v.: Spontaneous Light Emission Described by a Quantum Stochastic Differential Equation. Lecture Notes in Mathematics **1136**, Springer (1985), p. 515-534.

[14] Waldenfels, W.v.: The Inverse Oscillator in a Heat Bath as a Quantum Stochastic Process. Preprint 630. 1991. SFB 123 (Heidelberg).