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An Itô type isometry for loops in $\mathbf{R}^d$ via the Brownian bridge

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Summary. We show that iterated stochastic integrals (in the Itô-sense) with respect to the Brownian bridge on $\mathbf{R}^d$ give an explicit unitary isomorphism between the symmetric Fock space over the $\mathbf{C}^d$-valued square-integrable functions on the unit interval having zero mean and the space of complex valued L^2-functions on based continuous loops on $\mathbf{R}^d$.

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Introduction

The theory of Gaussian processes yields an identification of the space of square integrable functions on a given probability space provided with a Gauss process $(Z_t)_{t\in T}$ with the sum of the symmetric tensor powers of the "Gauss space", the closure of the span of the random variables Z_t (see e.g. [HT], Kap. 5.8 or [N], Ch.7). In the case of "normal martingales" the above identification is concretely realized by means of multiple stochastic integration (see [DMM], pp.199). Motivated by the important rôle recently played by loop spaces, especially with values in $\mathbf{R}^d$ or compact Lie groups, and "holomorphic L^2-sections" in appropriate line bundles over them in mathematics and physics (see e.g. [BR] or [PS]), we consider here the case of the Brownian bridge, which is only a semimartingale.

We show that iterated stochastic integrals with respect to the Brownian bridge exist and yield a unitary isomorphism between the Fock space over square-integrable, $\mathbf{C}^d$-valued functions on the unit interval having mean zero and the space of $\mathbf{C}$-valued square-integrable functions on based paths in $\mathbf{R}^d$ ("Wiener space"). Since the Brownian bridge is an adapted stochastic process that induces an isomorphism between paths and loops, transporting the Wiener measure to the conditioned Wiener measure, this result implies an orthogonal decomposition of L^2-functions on based loops in $\mathbf{R}^d$. We apply this Itô isometry in [GW] to give an analytically rigorous derivation of the Virasoro anomaly in quantization of strings.

The article is organised as follows:
In the first section we recall some notation and elementary facts about Gauss spaces and associated reproducing kernel Hilbert spaces, adapted to the case of a Brownian bridge. The second section relates the stochastic integrals of deterministic functions with respect to a Brownian motion to those with respect to the particular Brownian bridge, that we use to simulate based continuous loops.
In the final section 3 we state and prove the isometry between "mean-zero functions" on simplices and the L^2-functions on Wiener space by means of iterated stochastic integrals with respect to the Brownian bridge of the second section.

Section 1: The Gauss space of a Brownian bridge

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a $\mathbf{R}$-valued Brownian bridge $(X_t)_{t\in[0,1]}$ from 0 to 0. On the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}) = (\Omega \times \mathbf{R}, \mathcal{F} \otimes \mathcal{B}(\mathbf{R}), \mathbf{P} \otimes \nu)$, where ν denotes the probability measure $(2\pi)^{-1/2} exp(-z^2/2)dz$ on $\mathbf{R}$, we have the additional random variable Z, defined by $Z(\omega, z) = z$, who is independent of the variables $\tilde{X}_t$, given by $\tilde{X}_t(\omega, z) = X_t(\omega)$. A direct calculation shows that the process $W_t = \tilde{X}_t + tZ$ $(t \in [0,1])$ is a Brownian motion on $\tilde{\Omega}$.
Denoting the Gauss spaces of $\tilde{X}_t$ respectively W_t by $G_{\tilde{X}}$ respectively G_W, we find the following orthogonal decomposition:

$$G_W = G_{\tilde{X}} \oplus \mathbf{C} \cdot Z$$

(See e.g. [N] for the general theory of Gauss spaces). The map

$$u : G_W \to \mathbf{C}^{[0,1]}, \ u(V)(t) = \mathbf{E}[V \cdot W_t]$$

is a unitary isomorphism of the closed subspace G_W of $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ onto its image $u(G_W) = \mathcal{H}_W$, the associated reproducing kernel Hilbert space (referred to as the "reproducing Hilbert space" in the sequel). Since W_t is a Brownian motion, one knows that sending h in $\mathcal{H}_W$ to $\dot{h} = f$ is a unitary isomorphism onto $L^2([0,1])$. Furthermore the inverse map u^{-1} is given as follows:

$$u^{-1}(h) = \int_0^1 \dot{h}(t) dW_t = J^W(\dot{h}) = J^W(f),$$

the notation J^W designing the stochastic integral with respect to W (see e.g. [N] for this fact).
It follows that $u(Z)(t) = \mathbf{E}[ZW_t] = t$ corresponds to the constant function 1 in $L^2([0,1])$. Thus $G_{\tilde{X}}$ is isomorphic to $L^2_0([0,1])$, the space of L^2-functions having mean zero. Since $d\tilde{X}_t = dW_t - W_1 dt$, the isomorphism between $L^2_0([0,1])$ and $G_{\tilde{X}}$ is given by a map $J^{\tilde{X}}$, which consists in integrating a deterministic function with respect to $d\tilde{X}_t$. This operator $J^{\tilde{X}}$ equals in fact the stochastic integral with respect to $\tilde{X}_t$, as can be shown by either the general theory of enlargements of filtrations (see e.g. [JY]) or by the estimates given below (see Proposition 2.2).

Remark. The above considerations extend easily to processes with values in $\mathbf{R}^d$ with $d > 1$. For the sake of clarity of the exposition, we will nevertheless work in the case $d = 1$.

Section 2: Stochastic integration with respect to the "adapted" bridge

On the Wiener space $\Omega = \{\omega \in C^0([0,1], \mathbf{R}^d) \mid \omega(0) = 0\}$ one has the natural filtration $\mathcal{F}_t = \sigma(\{B_s | 0 \leq s \leq t\})$ associated to the Brownian motion $B_t(\omega) = \omega(t)$, and the Wiener measure $\mathbf{P}$. We define a ("adapted") Brownian bridge by setting $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$ for $0 \leq t < 1$ and $X_1 = 0$. Let us recall that X_t is a continuous semi-martingale and that $B_t = X_t + \int_0^t \frac{X_s}{1-s} ds$. It follows notably that the X_t generate the same filtration as the B_t.
Since the law of X_t equals the conditioned Wiener measure, Ω together with X can be looked upon as a substitute for the space of based, continous loops in $\mathbf{R}$.
The above inversion formula implies that the Gauss spaces G_X and G_B are equal. "Pulling-back" $G_{\tilde{X}}$ to G_X, we know that G_X is isomorphic to $L_0^2([0,1])$ by means of an integration operator J^X with respect to X. The Gauss space G_B being given by stochastic integrals with respect to B based on $L^2([0,1])$, we aim in this section for an explicit description of the relation between the two "realizations" of $G_X = G_B$.
We prepare ourselves first with a lemma concerning deterministic L^2-functions :

Lemma 2.1
(i) Let f be in $L_0^2([0,1])$, then $\lim_{\xi \nearrow 1}\left(\frac{\int_0^\xi f(t)dt}{\sqrt{1-\xi}}\right) = 0$.
(ii) Let $\tilde{f} \in L^2([0,1])$ and $\gamma(\tilde{f})(t) = \int_0^t \frac{\tilde{f}(s)}{1-s} ds$, then $\|\gamma(\tilde{f})\|_{L^2} \leq 2\|\tilde{f}\|_{L^2}$.
(iii) The operator $\alpha : L_0^2([0,1]) \to L^2([0,1])$, defined by

$$\alpha(f)(s) = f(s) + \frac{1}{1-s}\int_0^s f(t)dt,$$

is a unitary isomorphism, whose inverse is given by $\beta(\tilde{f}) = \tilde{f} - \gamma(\tilde{f})$.

We omit the proof, who boils down to Hardy type estimates for the parts (i) and (ii) (variants of the estimates can be found in the classical reference [HLP] and in [JY] for related probabilistic purposes as in the text) and uses partial integration plus convergence arguments on the boundary terms based on (i) and (ii) for the third part.
We can now prove the fundamental relation.

Proposition 2.2
(i) For all f in $L^2([0,1])$, the stochastic integral $J^X(f) = \int_0^1 f(t)dX_t$ exists and J^X is a unitary isometry from $L_0^2([0,1])$ to the Gauss space G_X.
(ii) A random variable Y in G_B, realized as $\int_0^1 \tilde{f} dB_t$ with $\tilde{f}$ in $L^2([0,1])$, equals $\mathbf{P}$-almost everywhere the random variable $\int_0^1 f(t)dX_t$ with $f(t) = \beta(\tilde{f})(t)$ $= \tilde{f}(t) - \int_0^t \frac{\tilde{f}(s)}{1-s} ds$.

Proof. Since $\int_0^1 1 \cdot dX_t = 0$, we can restrict ourselves to f in $L_0^2([0,1])$.
For $0 \leq \xi < 1$ integration-by-parts implies then:

$$\int_0^\xi f(t)dX_t = \int_0^\xi \alpha(f)(t)dB_t + \left(\int_0^\xi f(s)ds\right)\left(\int_0^\xi \frac{dB_s}{1-s}\right).$$

The limit $\lim_{\xi \nearrow 1}$ of the first term exists in L^2 by the Itô isometry for B and the fact that $\alpha(f)$ is in $L^2([0,1])$ by Lemma 2.1. The same lemma shows, again together with the Itô isometry for B, that the limit of the second term vanishes. This shows the existence of $\int_0^1 f(t)dX_t$ as well as its $\mathbf{P}$-a.e. equality to $\int_0^1 \alpha(f)(t)dB_t$. Since α and β are mutually inverse unitary isomorphisms and J^B is isometric as well, the other assertions of the proposition follow immediately. □

Remarks.
(1) The existence of J^X as well as its unitarity can of course also be derived from the analogous properties of $J^{\tilde{X}}$ (see Section 1). We state the result nevertheless in the above proposition since it follows effortlessly as a by-product of the proof of the assertion (ii).
(2) The second assertion above can be restated by saying that the isomorphism

$$L_0^2([0,1]) \cong \mathcal{H}_X \cong G_X = G_B \cong \mathcal{H}_B \cong L^2([0,1])$$

is realized by α and the respective stochastic integrals yield the corresponding representations of elements of the Gauss space.

Section 3: Wiener chaos decomposition

The general theory of Gaussian processes gives an isomorphism between the symmetric Fock space over the Gauss space (or reproducing Hilbert space) of a Gaussian process and the space of L^2-functions of the given probability space (see again e.g. [N]). We will show in this section that, for the Brownian bridge X_t considered in Section 2, this unitary isomorphism onto $L^2(\Omega, \mathcal{F}, \mathbf{P})$ (where $\mathcal{F} = \sigma(\{B_s | 0 \leq s \leq 1\})$) can be realized by means of iterated stochastic integrals with respect to X and based on appropriate L^2-functions on simplices. Similar results for normal martingales are well-known (see e.g. [DMM]). We begin by assuring the existence of the iterated integrals with respect to X_t.
Let $\sum_n$ denote the n-simplex $\{(s_1, \dots, s_n) \in \mathbf{R}^n | 0 \leq s_1 \leq \dots \leq s_n \leq 1\}$ in the sequel.

Lemma 3.1 *Let F_n in $L^2(\sum_n)$ be the restriction to $\sum_n$ of the n-fold symmetrization $\underbrace{f \otimes \cdots \otimes f}_{n \text{ times}}$ of a function f in $L^2([0,1])$. Then the iterated integral*

$$J_t^X(F_n) = \int_0^t \left(\int_0^{s_n} \cdots \left(\int_0^{s_2} F_n(s_1, \dots, s_n) dX_{s_1} \right) dX_{s_2} \dots dX_{s_{n-1}} \right) dX_{s_n}$$

exists and is a continuous semimartingale.

Proof. The first chaos $Y_t = \int_0^t f(s)dX_s$ is already known to exist and is a continuous semi-martingale. Writing now $J_t^X(F_n)$ as

$$\int_0^t \left(\int_0^{s_n} \cdots \left(\int_0^{s_2} f(s_1) dX_{s_1} \right) dY_{s_2} \dots dY_{s_{n-1}} \right) dY_{s_n}$$

and recalling that the stochastic integral of a continuous semi-martingale with respect to a continuous semi-martingale is again a continuous semi-martingale (see e.g. [HT]), the assertion follows by induction on n.

□

Let us recall that the solution of the stochastic differential equation

$$d(\mathcal{E}_t^B(h)) = \mathcal{E}_t^B(h)h(t)dB_t$$

for h in $L^2([0,1])$ is explicitely given by

$$\mathcal{E}_t^B(h) = \exp\left(\int_0^t h(s)dB_s - \frac{1}{2}\int_0^t h^2(s)ds\right)$$

(see e.g. [DMM], XXI §1.11). This process is called the "exponential process with respect to B based on h" and the random variable $\mathcal{E}^B(h) = \mathcal{E}_1^B(h)$ a "stochastic exponential vector".

The exponential process $\mathcal{E}_t^X(h) = \exp\left(\int_0^t h(s)dX_s - \frac{1}{2}\int_0^t h^2(s)ds\right)$ and the exponential vector $\mathcal{E}^X(h) = \mathcal{E}_1^X(h)$ enjoy similar properties as $\mathcal{E}_t^B(h)$ and $\mathcal{E}^B(h)$ (see [DMM], §1.9-1.11 for the case of the Brownian motion).

Lemma 3.2
Let h be in $L^2([0,1])$, then

(i) $d(\mathcal{E}_t^X(h)) = \mathcal{E}_t^X(h)h(t)dX_t$

(ii) $\mathcal{E}_t^X(h) = 1 + \int_0^t \mathcal{E}_s^X(h)h(s)dX_s$

(iii) $\mathcal{E}^X(\lambda h) = \sum_{n\geq 0}\lambda^n\left(\int_{\Sigma_n} h(s_1)\cdots h(s_n)dX_{s_1}\cdots dX_{s_n}\right)$ *for* $\lambda \in \mathbf{C}$.

Proof.
(i) This assertion follows directly from $(dX_t)^2 = dt$ and Itô's formula.
(ii) Since $\mathcal{E}_t^X(0) = 1$, the stochastic differential equation in Assertion (i) yields the result.
(iii) This assertion is given by applying $N+1$ times the Picard iteration scheme to the equality in Assertion (ii) and the fact that $\mathcal{E}_1^X(\lambda h)$ is analytic in the parameter λ. □

Lemma 3.3
For $\tilde{f}$ in $L^2([0,1])$ and $f = \beta(\tilde{f})$ in $L_0^2([0,1])$ (see Lemma 2.1 (iii) for the definition of the map β), we have equality of the corresponding exponential vectors :

$$\mathcal{E}^B(\tilde{f}) = \mathcal{E}^X(f).$$

Proof. Since $\int_0^1 \tilde{f}(s)dB_s = \int_0^1 f(s)dX_s$ by Proposition 2.2, it remains only to show that $\int_0^1 \tilde{f}^2(s)ds = \int_0^1 f^2(s)ds$. The latter equality follows now by observing that the unitary isometry α is already defined over the reals. □

Remark. Of course, we also have $\mathcal{E}^X(f) = \mathcal{E}^B(\tilde{f})$ with $\tilde{f} = \alpha(f)$.

We can now prove that the map J^X is isometric :

Lemma 3.4
Let f and g be in $L_0^2([0,1])$ and $F_n = \underbrace{f \otimes \cdots \otimes f}_{n \text{ times}}$ (resp. $G_m = \underbrace{g \otimes \cdots \otimes g}_{m \text{ times}}$) their n-fold (respectively m-fold) symmetrisation, viewed as functions on Σ_n (respectively Σ_m). Then we have
(i) $\langle J^X(F_n), J^X(G_m)\rangle_{L^2(\Omega,\mathbf{P})} = 0$ for $n \neq m$
(ii) $\langle J^X(F_n), J^X(G_m)\rangle_{L^2(\Omega,\mathbf{P})} = \frac{1}{n!}\left(\langle f, g\rangle_{L_0^2([0,1])}\right)^n = \langle F_n, G_m\rangle_{L^2(\Sigma_n)}$ for $n = m$.

Proof. Let λ, μ be in $\mathbf{R}$. We have, by Lemma 3.2 (iii)

$$J^X(F_n) = \int_{\Sigma_n} f(s_1)\cdots f(s_n) dX_{s_1}\ldots dX_{s_n} = \frac{1}{n!}\frac{d^n}{d^n\lambda}\bigg|_{\lambda=0} \mathcal{E}^X(\lambda f)$$

and analogously for $J^X(G_m)$.
By Lemma 3.3, the Itô-isometry for B, and Lemma 2.1(iii), we have

$$\begin{aligned}\langle \mathcal{E}^X(\lambda f), \mathcal{E}^X(\mu g)\rangle &= \langle \mathcal{E}^B(\lambda \tilde{f}), \mathcal{E}^B(\mu \tilde{g})\rangle \\ &= \exp(\langle \lambda \tilde{f}, \mu \tilde{g}\rangle) = \exp(\lambda\mu\langle \tilde{f}, \tilde{g}\rangle) = \exp(\lambda\mu\langle f, g\rangle).\end{aligned}$$

As a consequence, using again Lemma 3.2(iii), we get

$$\begin{aligned}\langle J^X(F_n), J^X(G_m) &= \frac{1}{n!}\frac{1}{m!}\frac{d^n}{d^n\lambda}\bigg|_{\lambda=0}\frac{d^m}{d^m\mu}\bigg|_{\mu=0} \langle \mathcal{E}^X(\lambda f), \mathcal{E}^X(\mu g)\rangle \\ &= \frac{1}{n!}\frac{1}{m!}\frac{d^n}{d^n\lambda}\bigg|_{\lambda=0}\frac{d^m}{d^m\mu}\bigg|_{\mu=0} \left(\sum_{k\geq 0}\frac{(\lambda\mu)^k}{k!}\langle f, g\rangle^k\right) \\ &= \delta_{n,m}\left(\frac{1}{n!}\langle f, g\rangle^n\right) = \delta_{n,m}\left(\langle F_n, G_m\rangle_{L^2(\Sigma_n)}\right). \quad \square\end{aligned}$$

To complete the description of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ by stochastic integrals with respect to X, we define a suitable space of functions.

Definition 3.5
Let $L^2(\sum_n)$ be identified with the restrictions to $\sum_n$ of symmetric functions in $L^2(C_n)$, where $C_n = [0,1]^n$ denotes the unit cube. The space $L_0^2(\sum_n)$ is defined as the restrictions of those symmetric F_n in $L^2(C_n)$ such that

$$\int_0^1 F_n(s_1, s')ds_1 = 0$$

Lebesgue-almost-everywhere in the parameter $s' = (s_2, ..., s_n)$ in C_{n-1}.

Since finite sums of symmetric products of functions in $L_0^2([0,1])$ are dense in $L_0^2(\sum_n)$ and J^X preserves scalar products of symmetrizations by Lemma 3.4, J^X is automatically extended to an isometry defined on all of $L_0^2(\sum_n)$. One complements this by setting $L_0^2(\sum_0) = \mathbf{C}$ and $J^X(F_0) = F_0$ for a constant F_0 in $L_0^2(\sum_0)$. It follows that J^X is defined on $\overline{\bigoplus_{n\geq 0} L_0^2(\sum_n)}$, the Hilbert space completion of $\bigoplus_{n\geq 0} L_0^2(\sum_n)$.

Proposition 3.6
The map $J^X : \overline{\bigoplus_{n\geq 0} L_0^2(\sum_n)} \to L^2(\Omega, \mathbf{P})$ *, given by multiple stochastic integration with respect to the process* X*, is a surjective isometry.*

Proof. By Lemma 3.4 and the preceding discussion J^X is an isometry, thus having closed image. It suffices therefore to show that J^X has dense image.
Let us recall that the set $\{\mathcal{E}^B(\tilde{f}) | \tilde{f} \in L^2([0,1])\}$ is dense in $L^2(\Omega, \mathbf{P})$ (see e.g. [DMM]). By Lemma 3.3 each exponential vector $\mathcal{E}^B(\tilde{f})$ is in the image of J^X, indeed, by Lemma 3.2, we have

$$\mathcal{E}^B(\tilde{f}) = \mathcal{E}^X(f) = J^X\left(\sum_{n\geq 0} F_n\right),$$

where $f = \beta(\tilde{f})$ (notation of Lemma 2.1) and $F_n = f \otimes \cdots \otimes f$ is, as in Lemma 3.4, viewed as a function on Σ_n. Thus J^X is surjective. □

Remarks.
Taking care of scalar products on Fock spaces (see e.g. [M] for the right conventions) one identifies $\overline{\bigoplus_{n\geq 0} L_0^2(\sum_n)}$ unitarily with the symmetric Fock space over $L_0^2([0,1])$. Thus Proposition 3.6 gives the desired realization of the isomorphism of the Fock space over the reproducing Hilbert space $\mathcal{H}_X$ and $L^2(\Omega, \mathcal{F}, \mathbf{P})$ by means of stochastic integrals.

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