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Almost Sure Path Properties of Branching Diffusion Processes

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Abstract

We consider a one-dimensional Branching Brownian Motion. We present a large deviations result concerning the almost sure number of particles along any given path. We then observe the implications of this result by studying Branching Integrated Brownian Motion.
Key Words: Strassen Law, Large Deviations, Branching Diffusion Processes, Reaction Diffusion Equations

1 Introduction

Branching Diffusion Processes (BDPs) have been studied extensively over the last decades. The behaviour in expectation is well understood, but to study the almost sure behaviour of a BDP, one must study the associated Reaction-Diffusion equation using martingales theory. We refer the reader to [Neveu] for an excellent exposition on the subject.
We analyse the almost-sure behaviour using large deviations techniques while concentrating on the study of a dyadic Branching Brownian Motion (BBM). We formulate a large deviations principle for the *almost sure* rate of growth of particles along any given path.

This work is divided into two sections.

- The derivation of the almost sure rate of growth function, measuring the number of BBM-particles along any path.
- An example of how this derivation can be utilised.

We begin by studying the rate of growth of the *expected* number of particles along any path of a BBM. The result follows directly from work by [Schilder] who first described the large-deviation principle associated with the paths of a single particle. We combine his result with a simple many-to-one picture to deduce the rate function for each BBM-path. This provides the upper bound for the *almost sure* rate function. We then pull together results by [Uchiyama] and [Chauvin] to prove the lower bound.

In the second section, we consider a system of breeding particles, whose velocity is given by a Brownian Motion. We arrive at a two dimensional point-process on the plane $\left(B_i(t), \int_0^t B_i(s)ds\right)$. We study the phase plane to discover that the behaviour almost surely and in expectation is extremely different.

The remainder of this introduction sets up the background with some definitions.

1.1 Constructing The Branching Brownian Motion Model

Let each Brownian particle wait an exponential time of rate 1 before dying while giving birth to $1 + C$ offspring. We excluded the possibility of death so that $1 + C$ is a $\mathcal{Z}^+$−valued random variable. We also impose that $E(C \log C) < \infty$. At birth, the parent particle and its offspring share the same spatial position, but from then on, each offspring follows an independent Brownian path. We follow Neveu's construction. A finite sequence i of numbers will label each particle, starting with the first particle labelled $\emptyset$. Each particle i has $1 + C_i$ descendants $i0, i1 \ldots iC_i$. Let $I = \cup_{n \in \mathcal{N}} \mathcal{Z}^{+n}$ be the space of labels. Let τ_i be the lifetime of particle i ($i \in I$). Particle i will thus be born at time

$$T_i = \sum_{k=0}^{k=n-1} \tau_{j_1 \ldots j_k} \quad \text{if } i = j_1 \ldots j_n.$$

The τ_i are assumed to be strictly positive random variables satisfying the non-explosion condition: $\{i : T_i \leq t\}$ is finite for all t. The trajectories of particles are continuous maps B_i of the time intervals $[T_i, T_i + \tau_i]$ into $\mathcal{R}$ such that $B_{ic}(T_{ic}) = B_i(T_i + \tau_i)$ for every $i \in I$ and $c \leq C_i$.
A point $\omega \in \Omega$ is a collection $\{\tau_i, B_i, C_i : i \in I\}$ satisfying the above conditions. Let $N_t(\omega) = \{i : T_i \leq t < T_i + \tau_i\}$ be the set of particles alive at time t. The filtration $\{\mathcal{F}_t : t \in \mathcal{R}^+\}$ on Ω is generated by $\{N_t, (B_i(t), i \in N_t)\}_{t \in \mathcal{R}^+}$. There exists a unique probability measure P on $(\Omega, \mathcal{F}_\infty)$ such that $\{B_i : i \in I\}$ is an independent family of Brownian-motion processes with each B_i started at $B_i(T_i)$, stopped after an exponential time τ_i of mean 1, and giving birth to C_i offspring at its time of death.
Although C need only satisfy $E(C \log C) < \infty$, we will restrict ourselves to the simple dyadic Branching Brownian Motion where $C = 2$ almost surely. Our results fail if $E(C \log C) = \infty$ because result 3 is no longer valid.

1.2 Scaling The Branching Brownian Motion

At time T (now a fixed time), let us scale the BBM by a factor of T in both the space and time coordinates. We get a branching process on the time-parameter set $[0, 1]$. Specifically, for every $i \in N_T$, let $x_i^T \in \mathcal{C}^0([0,1], \mathcal{R})$ – the space of continuous functions from $[0, 1]$ to $\mathcal{R}$ – be the T-scaled path of particle i, defined as

$$x_i^T(t) := \tfrac{1}{T} B_{a_{tT}(i)}(tT).$$

Here, $a_{tT}(i)$ denotes the unique ancestor of particle i at time tT. Clearly all T-scaled paths satisfy $x_i^T(0) = 0$. We let C_0 be the space of continuous paths started at 0 with the supremum norm $||x - z|| = \sup_{0 \leq t \leq 1} |x(t) - z(t)|$. Let C_1 be the space of paths which are also absolutely continuous with $\int_0^1 \dot{x}^2 dt < \infty$. If $D \subset C_0$, then let $M_D(T)$ denote the set of particles at time T whose T-scaled path is in D. Also if

$D|_\theta := \{x \in \mathcal{C}^0([0,\theta],\mathcal{R}) : \exists z \in D, \quad x(t) = z(t) \quad \forall t \in [0,\theta]\}$, let $M_D(T,\theta)$ denote set of particles whose T-scaled path is in $D|_\theta$ up to time $\theta \leq 1$.

$$\begin{aligned} M_D(T) &:= \{i \in N_T : x_i^T \in D\}, \\ M_D(T,\theta) &:= \{i \in N_{\theta T} : x_i^T|_\theta \in D|_\theta\}. \end{aligned}$$

The function $x|_\theta \in \mathcal{C}^0([0,\theta],\mathcal{R})$ is x truncated at time θ.

2 Rate of Growth in Expectation

We denote the law of a standard Brownian Motion run until time 1 as P_1. We also denote the law of an individual T-scaled path x_i^T path by $P_{1/T}$ which is the same in law as P_ϵ defined in [Varadhan, Section 5]. The large deviation principle associated with $\{P_{1/T} : T \in \mathcal{R}\}$ was first proved by [Schilder] with a rate function

$$I(x) := \begin{cases} \frac{1}{2}\int_0^1 \dot{x}^2 dt & \text{if } x \in C_1, \\ \infty & \text{otherwise.} \end{cases}$$

Of course, nothing stops us running the process only until time θT where $\theta \in [0,1]$. We get a slightly modified rate function $I(x,\theta) = \frac{1}{2}\int_0^\theta \dot{x}^2 dt$.

Let D be a subset of C_0. By conditioning on the first birth, we arrive at a many-to-one picture:

$$E(|M_D(T,\theta)|) = E(|N_{\theta T}|)P(x^T|_\theta \in D|_\theta),$$

whence

$$\begin{aligned} T^{-1}\log E(|M_D(T,\theta)|) &= T^{-1}\log\left\{E(|N_{\theta T}|)P(x^T|_\theta \in D|_\theta)\right\}, \\ &= \theta + T^{-1}\log P(x^T|_\theta \in D|_\theta), \\ &\approx \theta - I(x,\theta). \end{aligned}$$

In fact, combining [Varadhan] and the many-to-one particle picture, the following result is immediate:

Result 1. *Let $J(x,\theta) := \theta - I(x,\theta)$. If A and D are an open subset and a closed subset of $C_0|_\theta$ respectively, then*

$$\begin{aligned} \liminf_{T\to\infty} T^{-1}\log E(|M_A(T,\theta)|) &\geq \sup_{x\in A} J(x,\theta), \\ \limsup_{T\to\infty} T^{-1}\log E(|M_D(T,\theta)|) &\leq \sup_{x\in D} J(x,\theta). \end{aligned}$$

As a matter of convenience, for all θ and for all sets B, we let

$$I(B,\theta) := \inf_{x\in B} I(x,\theta),$$
$$J(B,\theta) := \sup_{x\in B} J(x,\theta).$$

We note that I is lower-semicontinuous while J is upper-semicontinuous in the sense that $\lim_{z\to x} I(z) \geq I(x)$ and $\lim_{z\to x} J(z) \leq J(x)$.

3 Rate of Growth Almost-Surely

We wish to transform the result in probability to an almost sure result, so that for some function $K(x)$ to be determined later (which we might hope looks like $J(x)$), we have almost surely;

$$\lim_{T\to\infty} T^{-1}\log|M_A(T)| \geq \sup_{x\in A} K(x),$$
$$\lim_{T\to\infty} T^{-1}\log|M_D(T)| \leq \sup_{x\in D} K(x).$$

We certainly expect $K(x) \leq J(x)$ for all $x \in C_1$. We can improve this upper bound by considering the following: Suppose that for some $\theta \in [0,1]$ we have $J(D,\theta) < 0$. Then, using result 1 and Chebychev inequality, we deduce that as T tends to infinity,

$$P(|M_D(T,\theta)| > 0) \leq \exp\{TJ(D,\theta)\} \to 0.$$

Intuitively, this implies that $\lim_{T\to\infty}|M_D(T,\theta)| = 0$, and consequently also $\lim_{T\to\infty}|M_D(T)| = 0$ almost surely. This is a better indication as to how J "controls" K. It turns out that this upper bound is actually tight and distinguishes exactly between the different rates of growth. We now begin the rigorous study.

3.1 Upper Bound

Lemma 1. *Let D be a closed subset of C_0. Then for every $\theta \in [0,1]$, we have almost surely:*

$$\limsup_{T\to\infty} T^{-1}\log|M_D(T,\theta)| \leq J(D,\theta).$$

Proof. Suppose that the result is false. Then there exists a θ and an event W with $P(W) > 0$ such that, for every $\omega \in W$, $\limsup_{T\to\infty} T^{-1}\log|M_D(T,\theta)| > J(D,\theta)$. Hence if

$$W_n := \{\omega \in \Omega : \limsup_{T\to\infty} T^{-1}\log|M_D(T,\theta)| > J(D,\theta) + n^{-1}\},$$

then $P(W_n) > 0$ for some n. It is now clear that

$$\limsup T^{-1} \log E(|M_D(T,\theta)|) \geq J(D,\theta) + n^{-1}$$

contradicting result 1.

□

In particular, we see that if for some $\theta \leq 1$ we have $J(D,\theta) < 0$, then, almost surely, $\lim_{T\to\infty} |M_D(T,\theta)| = 0$. Since $x_i^T \in D$ implies that $x_i^T|_\theta \in D|_\theta$ we must also have that $\lim_{T\to\infty} |M_D(T)| = 0$ almost surely. This leads us to the following definition and the upper bound result:

Definition (The Almost Sure Rate Function). *Let $\theta_0 \in [0,1] \cup \{\infty\}$ be the last time at which $J(x,\theta)$ is non-negative, $\theta_0 := \inf\{\theta \in [0,1] : J(x,\theta) < 0\}$. Define $K(x,\theta)$ as:*

$$K(x,\theta) := \begin{cases} J(x,\theta) & \text{if } \theta \leq \theta_0, \\ -\infty & \text{otherwise} \end{cases}$$

Result 2. *Let $\theta \in [0,1]$ and let $D \subset C_0$ be closed. Then*

$$\limsup_{T\to\infty} T^{-1} \log |M_D(T)| \leq \sup_{x\in D} K(x).$$

3.2 Lower Bound

We shall prove the lower bound in stages. We first consider open sets around linear functions, then open sets around piecewise-linear functions, and finally arbitrary open sets. We use the following definition of an open ϵ-neighbourhood:

$$A(x,\epsilon) := \{z \in C_0 : ||z - x|| < \epsilon\} = \{z \in C_0 : \sup_t |x(t) - z(t)| < \epsilon\}.$$

Lemma 2. *Let $x(t) = \lambda t$ be a linear function with $0 \leq \lambda < \sqrt{2}$. For every $\epsilon > 0$, we have almost surely,*

$$\liminf_{T\to\infty} T^{-1} \log |M_{A(x,\epsilon)}(T)| \geq 1 - \tfrac{1}{2}\lambda^2.$$

Our proof relies on work by [Uchiyama] with slight modifications using change of measure by [Warren]. Their result involves the convergence of expressions of the the form $e^{-t}\sqrt{t}\sum_{i\in N_t} g(B_i(t) - \lambda t)e^{\lambda B_i(t) - \frac{1}{2}\lambda^2}$. We consider a special case of their result, when $g(x) = 1_{[0,1]}e^{-\lambda x}$ to deduce the following:

Result 3. *Let $|\lambda| < \sqrt{2}$. Let $N_{[\lambda T,\lambda T+1]}(T) = \{i \in N_T : \lambda T \leq B_i(T) \leq \lambda T + 1\}$ represent the particles at time T with spacial position between λT and $\lambda T + 1$. Then, almost surely,*

$$\lim_{T\to\infty} \sqrt{T}e^{-(1-\frac{1}{2}\lambda^2)T}|N_{[\lambda T,\lambda T+1]}(T)| \to \text{constant} \times Z_\lambda(\infty),$$

where $Z_\lambda(\infty)$ is a strictly positive random variable.

Before we prove lemma 2, please note that this result is true for an arbitrary birth process C as long as $E(C \log C) < \infty$. It is for this reason that we imposed this condition on the birth process.

Proof of Lemma 2. Consider the total number of particles at time $T > 1/\delta$ in the interval $[\lambda T, (\lambda + \delta)T]$. Result 3 clearly implies that almost surely

$$\lim_{T \to \infty} T^{-1} \log |N_{[\lambda T, (\lambda+\delta)T]}| \geq 1 - \tfrac{1}{2}\lambda^2.$$

Next, define the closed sets

$$D_\delta := \{z \in C_0 \setminus A(x, \epsilon) : z(0) = 0, z(1) \in [\lambda, \lambda + \delta]\}.$$

If $\delta = 0$ then $I(D_0)$ is minimised by the piecewise-linear path $z(0) = 0, z(\frac{1}{2}) = \frac{1}{2}\lambda + \epsilon, z(1) = \lambda$ and so $I(D) = \frac{1}{2}\lambda^2 + 2\epsilon^2$. By definition, a lower semicontinuous function satisfies $\lim_{z \to x} I(z) \geq I(x)$. Since I is lower-semicontinuous $\lim_{\delta \downarrow 0} I(D_\delta) \geq I(D_0)$, so that for a sufficiently small δ, we can ensure that $I(D_\delta) > \frac{1}{2}\lambda^2$.

We now use the upper-bound (result 2) with the knowledge that D_δ is closed to deduce that, almost surely,

$$\limsup_{T \to \infty} T^{-1} \log |M_{D_\delta}(T)| < 1 - \tfrac{1}{2}\lambda^2.$$

Since $N_{[\lambda T, (\lambda+\delta)T]}(T) \subset M_{A(x,\epsilon)}(T) \cup M_{D_\delta}(T)$ the lemma is complete.

□

We now wish to glue together several linear functions.

Definition. *Let x be a piece-wise linear function. We say x satisfies the lower bound condition until time $\theta_1 > 0$, if for all $\epsilon > 0$, almost surely*

$$\liminf_{T \to \infty} T^{-1} \log |M_{A(x,\epsilon)}(T, \theta_1)| \geq J(x, \theta_1).$$

Suppose x satisfies the lower bound condition until θ_1. If from θ_1 until θ_2, x is a linear function satisfying $\dot{x} = \lambda$, then we wish to show that x satisfies the lower bound condition until θ_2. We first assume $|\lambda| < \sqrt{2}$. We will run the process until time $\theta_1 T$, arriving at $M_{A(x,\epsilon)}(T, \theta_1)$ particles. We will then run $M_{A(x,\epsilon)}(T, \theta_1)$ independent copies from time $\theta_1 T$ to time $\theta_2 T$, and add them all together. We require the following two definitions. The first simply introduces the change in the rate function over the interval $[\theta_1, \theta_2]$. The second defines a random variable, very much like $M_{A(x,\epsilon)}(T)$, for each $i \in N(\theta_1 T)$ which simply counts the offspring of i whose T-scaled paths follow x closely over the interval $[\theta_1, \theta_2]$. Formally,

$$J(x, \theta_1, \theta_2) := \theta_2 - \theta_1 - \tfrac{1}{2} \int_{\theta_1}^{\theta_2} \dot{x}^2(t) dt,$$

$$M^i_{A(x,\epsilon)}(T, \theta_1, \theta_2) :=$$

$$\left\{ j \in N(\theta_2 T) : a(j) = i, \left| (x_j^T(t) - x_j^T(\theta_1)) - (x(t) - x(\theta_1)) \right| < \epsilon \text{ for all } t \in [\theta_1, \theta_2] \right\}.$$

It is a simple matter to verify that since all particles are independent, M^i all share the same law and that

$$M_{A(x,\epsilon)}(T,\theta_2) \supseteq \sum_{i\in M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)} M^i_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1,\theta_2).$$

Also apparent is the additivity of the rate function:

$$J(x,\theta_2) = J(x,\theta_1) + J(x,\theta_1,\theta_2),$$

Lemma 3. *Let $x \in C_1$ be piecewise-linear satisfying the lower bound condition up until time θ_1. Let $\dot{x} = \lambda$ on $[\theta_1, \theta_2]$, with $|\lambda| < \sqrt{2}$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} T^{-1} \log |M_{A(x,\epsilon)}(T,\theta_2)| \geq J(x,\theta_2).$$

Proof. Let $\delta > 0$ be arbitrary. We use the Strong Law of Large Numbers. From previous discussion, we have

$$e^{-(J(x,\theta_2)-2\delta)T}|M_{A(x,\epsilon)}(T,\theta_2)| \geq e^{-(J(x,\theta_2)-2\delta)T} \sum_{i\in M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)} |M^i_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1,\theta_2)|$$

$$\geq e^{-(J(x,\theta_1)-\delta)T}|M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)| \dots$$
$$\times \frac{1}{|M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)|} \sum_{i\in M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)} e^{-(J(x,\theta_1,\theta_2)-\delta)T}|M^i_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1,\theta_2)|$$

The i.i.d. random variables $e^{-(J(x,\theta_1,\theta_2)-\delta)T}|M^i_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1,\theta_2)|$ inside the summation tend almost surely to infinity as T tends to infinity (using lemma 2). We average over *an independent* random number $|M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)|$ of particles, but this random variable tends to infinity almost surely as T tends to infinity, so that the SLLN still holds. By the induction hypothesis, x satisfies the lower bound condition until time θ_1, and thus the random variable $e^{-(J(x,\theta_1)-\delta)T}M_{A(x,\frac{1}{2}\epsilon)}(T,\theta_1)$ also tends almost surely to infinity. We conclude that the RHS (and hence the LHS) tends to infinity almost surely as T tends to infinity, and hence, almost surely,

$$\liminf_{T\to\infty} T^{-1} \log |M_{A(x,\epsilon)}(T,\theta_2)| \geq J(x,\theta_2) - 2\delta.$$

Letting $\delta \downarrow 0$ concludes the proof.

□

We turn our attention to the case where x satisfies the lower bound condition until time θ_1 , while between θ_1 and θ_2, the gradient $\dot{x} = \lambda > \sqrt{2}$. We of course insist on $J(x,\theta_2) > 0$.

Heuristics: The following proof is in principle the same as that of lemma 3 above. We run the process until time θ_1 arriving (using lemma 2) at an almost sure $M_{A(x,\epsilon)}(T,\theta_1)$

particles. We then run independent copies on $[\theta_1 T, \theta_2 T]$.

We replace the almost sure number of particles $M^i_{A(x,\epsilon)}(T, \theta_1, \theta_2)$ produced by an independent copy, with $P^i_{A(x,\epsilon)}(T, \theta_1, \theta_2)$, the probability of an independent copy with $x^T_i(\theta_1) = x(\theta_1)$, still remaining close to x by time $\theta_2 T$. Formally, we define

$$P^i_{A(x,\epsilon)}(T, \theta_1, \theta_2) := P\left(|M^i_{A(x,\epsilon)}(T, \theta_1, \theta_2)| > 0\right)$$

These probabilities are identical for all i, and are equal to the probability of finding a particle started at 0, at an ϵ-neighbourhood of $y = \lambda t$ at time $(\theta_2 - \theta_1)T$. [Chauvin] showed that the probability of the right-most particle starting at 0 ascending to level λT at time T decays at the rate $1 - \frac{1}{2}\lambda^2$. We will need to modify her result slightly to prove that $\liminf_{T\to\infty} P^i_{A(x,\epsilon)}(T, \theta_1, \theta_2) \geq J(x, \theta_1, \theta_2)$. This will be done by a method analogous to the one used in lemma 2.

We think of $e^{J(x,\theta_1)T}$ copies, each performing an independent trial, with probability of success $P^i \approx e^{J(x,\theta_1,\theta_2)T}$. We see that since $J(x,\theta_1) + J(x,\theta_1,\theta_2) = J(x,\theta_2) > 0$, the expected number of particles succeeding, increases exponentially. Using an estimate on the Binomial distribution, we show that the probability that the growth rate is less than $J(x,\theta_2) - \delta$, decays exponentially for all $\delta > 0$. Finally, this result is true only *in probability*. To get an almost sure result, we have to use some sort of Borel-Cantelli Lemma. Basically, we show that if we had a particle inside $A(x,r)$ at time t, for some $r < \epsilon$. Then the particle was inside $A(x,\epsilon)$ for some interval before t. This allows us to divide time into countably many intervals, and use BCL.

We state and prove the three supporting lemmas.

Lemma 4. *Let $x \in C_1$ be a piece-wise linear function. We claim that for every $\epsilon > 0$, there exists $r > 0$, such that, for all sufficiently large T, if $x^T \in A(x,r)$, then $x^\tau \in A(x,\epsilon)$ for all $\tau \in [T-1, T]$.*

Proof. We define the look-back transformation for all $\tau < T$:

$$L^T_\tau z(t) := \tfrac{T}{\tau} z(t\tfrac{\tau}{T}).$$

$L^T_\tau : A(x,r) \to A(L^T_\tau x, \frac{T}{\tau} r)$ and $\lim_{T\to\infty} \sup_{T-1<\tau<T} ||L^T_\tau x - x|| = 0$. We let $r = \frac{1}{4}\epsilon$. Pick T sufficiently large such that $\sup_{T-1<\tau<T} ||L^T_\tau y - y|| < r$ and $\frac{T}{T-1} < 2$. We deduce that for such T sufficiently large,

$$L^T_\tau A(x,r) \subseteq A(x,3r) \subset A(x,\epsilon) \quad \text{for all } \tau \in [T-1,T].$$

□

Result 4 (Right-Most Particle At The Subcritical Region - Chauvin). *Let $\lambda > \sqrt{2}$ and let R_T be the position of the right-most particle of a dyadic Branching Brownian Motion at time T. Then*

$$\liminf_{T\to\infty} T^{-1} \log P(R_T > \lambda T) = 1 - \tfrac{1}{2}\lambda^2$$

Corollary 1. *Let $y(t) = \lambda t$ where $\lambda > \sqrt{2}$. Then for all $r > 0$*

$$\liminf_{T\to\infty} T^{-1} \log P\left(|M_{A(y,r)}(T)| > 0\right) \geq 1 - \tfrac{1}{2}\lambda^2.$$

It follows that if $\dot{x} = \lambda$ on $[\theta_1, \theta_2]$, then for all $r > 0$

$$\liminf_{T\to\infty} T^{-1} \log P^i_{A(x,r)}(T, \theta_1, \theta_2) \geq J(x, \theta_1, \theta_2).$$

Proof. Define the closed set $D := \{z \in C_0 \setminus A(y,r) : z(1) \geq \lambda_2\}$. It is easy to show that $J(D) < 1 - \frac{1}{2}\lambda^2$ and hence

$$\limsup_{T\to\infty} T^{-1} \log P(|M_D(T)| > 0) < 1 - \tfrac{1}{2}\lambda^2.$$

Since $P(R_T \geq \lambda T) \leq P(|M_D(T)| > 0) + P(|M_{A(y,r)}(T)| > 0)$, the result follows. □

Finally, an estimate on the binomial distribution $\mathcal{B}(n,p)$.

Lemma 5. *Let $\alpha < 1$. Then $P(\mathcal{B}(n,p) < pn\alpha) < e^{-np\alpha}$.*

Proof. For $x \in [0,1]$ we know that $E(x^{\mathcal{B}(n,p)}) = (q + px)^n$. Since $\mathcal{B}(n,p) < np\alpha$ if and only if $x^{\mathcal{B}(n,p)} > x^{np\alpha}$, we deduce that

$$P(\mathcal{B}(n,p) < pn\alpha) < x^{-np\alpha}(q + px)^n$$

Picking $x = \alpha(1-p)/(1-p\alpha)$ which minimises the above expression we deduce that

$$\begin{aligned}
\log P(\mathcal{B}(n,p) < pn\alpha) &\leq -n\left\{p\alpha \log \alpha + (1-p\alpha)\log(1-p\alpha) - (1-p\alpha)\log(1-p)\right\}, \\
&\sim -n\left\{p\alpha \log \alpha - (1-p\alpha)p\alpha + (1-p\alpha)p\right\}, \\
&\sim -np\left\{\alpha \log \alpha + (1-\alpha)(1-p\alpha)\right\}, \\
&\sim -np, \\
&< -np\alpha.
\end{aligned}$$

□

Let us now state and prove the main lemma.

Lemma 6. *Let $x \in C_1$ be piece-wise linear satisfying the lower bound condition until time θ_1. Let $\dot{x} = \lambda$ on $[\theta_1, \theta_2]$, with $\lambda > \sqrt{2}$, but with $J(x, \theta_2) > 0$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} T^{-1} \log |M_{A(x,\epsilon)}(T, \theta_2)| \geq J(x, \theta_2).$$

Proof. Pick $r < \epsilon$ as in lemma 4. At integer times $T_m := m$ define the following events:

$$U_m := \{\omega \in \Omega : |M_{A(x,\frac{1}{2}r)}(T_m, \theta_1)| < e^{(J(x,\theta_1)-\delta)T_m}\},$$
$$V_m := \{\omega \in \Omega \setminus U_m : |M_{A(x,r)}(T_m, \theta_2)| < e^{(J(x,\theta_2)-3\delta)T_m}\}.$$

Since x satisfies the lower bound condition until time θ_1, we know that almost surely, U_m will not occur. To work out the probability of V_m we use lemma 5 with the values $n \geq e^{(J(x,\theta_1)-\delta)T_m}$, $p \geq e^{(J(x,\theta_1,\theta_2)-\delta)T_m}$ and $\alpha = e^{-\delta T_m}$. We take n to represent the number of particles which stayed within $A(x, \frac{1}{2}r)$ up to time $\theta_1 T_m$. Since we are not in U_m we know that n is large (i.e. $n > e^{(J(x,\theta_1)-\delta)T}$). We take p to represent the probability for each of these particles that we could find a descendent in $A(x,r)$ by time $\theta_2 T_m$. This probability is greater than $P^i_{A(x,\frac{1}{2}r)}(T, \theta_1, \theta_2)$ which was evaluated in corollary 1.
We deduce that $P(V_m)$ decays exponentially, and using BCL, V_m does not occur almost surely. Thus, almost surely,

$$\liminf_{m\to\infty} T_m^{-1} \log |M_{A(x,r)}(T_m)| \geq J(x, \theta_2) - 3\delta.$$

We now use lemma 4 to deduce that for all m and for all $\tau \in [T_{m-1}, T_m]$

$$\liminf_{\tau\to\infty} \tau^{-1} \log |M_{A(x,r)}(\tau)| \geq J(x, \theta_2) - 3\delta.$$

□

Corollary 2. *Let $x \in C_1$ be a piecewise-linear function such that $K(x) > 0$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} T^{-1} \log |M_{A(x,\epsilon)}(T)| \geq K(x).$$

Proof. Clearly, $\dot{x}(0) \leq \sqrt{2}$. Since $A(x, \epsilon)$ is open, there is no problem of finding a piecewise-linear function in $A(x, \epsilon)$ with $\dot{z}(0) < \sqrt{2}$. Now, proceed to piece together each linear segment of z using the previous lemmas. Please note that we avoided the case where $K(x) = 0$.

□

We now have the lower bound result for the almost sure rate function. We ignored the case where $K(x) = 0$ because if A is any open set, and $x \in A$ satisfies $K(x) = 0$, then for some $0 < \alpha < 1$ we have $\alpha x \in A$ and $K(\alpha x) > 0$.

Theorem 1. *Let A be an open subset in C_0. Then, almost surely,*

$$\liminf_{T\to\infty} T^{-1} \log |M_A(T)| \geq \sup_{x\in A} K(x).$$

Proof. Since the piecewise-linear functions are dense in C_0 and $I(x)$ is lower-semicontinuous in the supremum topology, the result follows directly from the above corollary. □

To conclude, we state and sketch-prove a more general result.
Consider a BBM with a birth process C satisfying $E(C) = \mu$. We assume that $E(C \log C) < \infty$. Let each particle die at an exponential rate $r\,(B_i(t)/t)$. The breeding rate $r \geq 0$ is assumed to be a continuous function. For every $x \in C_1$, the adjusted expectation rate function is defined as:

$$\bar{J}(x,\theta) := \int_0^\theta \mu r(x) - \tfrac{1}{2}\dot{x}^2 dt.$$

As before, let $\theta_0 := \inf\{\theta : \bar{J}(x,\theta) < 0\}$. Also let the almost sure rate function $\bar{K}$ be defined as

$$\bar{K}(x,\theta) := \begin{cases} \bar{J}(x,\theta) & \text{if } \theta \leq \theta_0, \\ -\infty & \text{otherwise.} \end{cases}$$

Theorem 2. *Let A, D be open and closed sets in C_0. Then*

$$\limsup_{T\to\infty} T^{-1} \log E(|M_D(T)|) \leq \bar{J}(D),$$
$$\liminf_{T\to\infty} T^{-1} \log E(|M_A(T)|) \geq \bar{J}(A).$$

Also, almost surely,

$$\limsup_{T\to\infty} T^{-1} \log |M_D(T)| \leq \bar{K}(D),$$
$$\liminf_{T\to\infty} T^{-1} \log |M_A(T)| \geq \bar{K}(A).$$

Proof: Almost-sure lower bound. We prove the lower bound for an open neighbourhood of a piece-wise linear function x. Take $\mathcal{D}$, a partition of $[0,1]$. $\mathcal{D} := \{0 = t_0 < t_1 \ldots < t_n = 1\}$. Over the interval $[t_i, t_{i+1}]$, along the path $\{x(t) : t_i < t < t_{i+1}\}$ the process "observes" breeding at a rate greater or equal to $\inf\{r(x(t)) : t_i \leq t \leq t_{i+1}\}$. Thus, using the lower bound lemmas 2, 3 and 5, almost surely,

$$\liminf_{T\to\infty} T^{-1} \log |M_A(T)| \geq \mu \sum_{i<n} (t_{i+1} - t_i) \inf_{t_i \leq t \leq t_{i+1}} r(x(t)) - \int_0^1 \tfrac{1}{2}\dot{x}^2 dt.$$

Since x is piece-wise continuous, by taking the supremum over all partitions we get the result. We cheated slightly, as we are only allowed to consider partitions which satisfy for all $j < n$,

$$\mu \sum_{i \leq j} (t_{i+1} - t_i) \inf_{t_i \leq t \leq t_{i+1}} r(x(t)) > \int_0^{t_{j+1}} \tfrac{1}{2}\dot{x}^2 dt.$$

Since $\bar{K}(x,\theta) > 0$ for all θ, it can be shown that this constraint does not matter.

□

4 Application: Branching Integrated Brownian Motion

4.1 An Integrated Brownian Motion

We take a Brownian path and use it to describe the velocity of a particle. The position is then defined as

$$Y(t) = Y(0) + \int_0^t B(s)ds.$$

We assume $Y(0) = 0$ for simplicity. $Y(t)$ is an integral with respect to a continuous path and is therefore a differentiable finite-variation process. It is also Gaussian and its variance is given by

$$2E\left(\int_0^t B_r dr \int_r^t B_s ds\right) = 2\int_0^t \int_r^t r ds dr = \frac{1}{3}t^3$$

We will proceed to show using current methods that there *must* be a difference between the behaviour in expectation and almost surely. We will do so by considering the wavefront speeds. We will then give the full phase-plane picture using the new techniques which we have developed.

4.2 The Expectation Wavefront

If we let $N_{[x,\infty)}(t) = \{i \in N_t : Y_i(t) \geq x\}$ by conditioning on the first birth we find the following many-to-one picture holds:

$$E(|N_{[u_t,\infty)}(t)|) = E(|N_t|)P(Y_t > u_t) \approx e^t \exp(-\frac{3}{2t^3}u_t^2),$$

and we deduce the expectation wavefront travels at the speed $t^2\sqrt{2/3}$ in the sense that

$$\lim_{t\to\infty} \frac{u_t}{t^2} = \sqrt{2/3}. \qquad (1)$$

4.3 The Almost-Sure Wavefront

[Neveu] observed that almost surely R_t, the rightmost particle of a branching Brownian Motion satisfies $\limsup_{t\to\infty} R_t - t\sqrt{2} = -\infty$. Because $\sup_{i\in N_t} Y_i(t) \leq \int_0^t R_s ds$ by integrating the bound on R_t we get an instant upper bound on v_t, the almost-sure wavefront speed.

$$\limsup_{t\to\infty} \frac{v_t}{t^2} \leq 1/\sqrt{2}. \qquad (2)$$

Before proving that equality holds in equation 2 we want to point out that the almost-sure wavefront speed is below the expectation wavefront speed already in the first order of magnitude! A more comprehensive explanation of this phenomenon will be offered when we study the phase plane.

Theorem 3. *Let v_t denote the rightmost particle's position of a branching Integrated Brownian Motion. Then, almost surely,*

$$\lim_{t\to\infty} \frac{v_t}{t^2} = 1/\sqrt{2}.$$

Proof (Lower Bound). We look at the Branching Brownian Motion. We follow [Neveu] and define Z_s^λ to be the number of particles which first among their ancestors crossed the line $x = s - \lambda t$.

$$Z_s^\lambda = |\{i \in I : \exists t \in [T_i, T_i + \tau_i] \quad B_i(t) > s - \lambda t, \quad \forall t < T_i \quad B_{a(i)}(t) < s - \lambda t\}|.$$

Definition (Infinitesimal Generator Function of a Galton-Watson Process). *Consider a Markovian birth-death process $Z : \mathcal{R}^+ \to \mathcal{N}$ representing the number of particles alive. Each particle lives for an exponential time of rate α and gives particle to $n \in \mathcal{Z}^+{}_\infty$ particles with probability a_n. The infinitesimal generator function is then defined as*

$$a(x) = \alpha\{\sum_{\mathcal{Z}^+{}_\infty} a_n x^n - x\}. \tag{3}$$

Note that $a(1) = 0$ while $\lim_{x\uparrow 1} a(x) = -\alpha a_\infty$. On $0 < x < 1$, $a(x)$ is convex and has a unique root $a(\sigma) = 0$ with $a(x) < 0$ on $(\sigma, 1)$. If $a_0 = 0$ then $\sigma = 0$. If a is also continuous at 1, then the solution of the equation

$$a = \psi' \circ \psi^{-1} \quad \text{on } (0,1)$$

has a unique (modulo translation) monotone decreasing solution $\psi : \mathcal{R} \to (0,1)$.

Result 5 (Neveu, Proposition 3). *For each $\lambda \geq \sqrt{2}$ the integer valued process $(Z_s^\lambda, s > 0)$ is a Galton-Walton process without extinction whose infinitesimal generating function a is given by*

$$a = \psi' \circ \psi^{-1} \quad \textit{on } (0,1)$$

where $\psi : \mathcal{R} \to (0,1)$ is the solution of Kolmogorov's equation

$$\tfrac{1}{2}\psi'' - \lambda\psi' = \psi - \psi^2. \tag{4}$$

We now consider what happens if $0 < \lambda < \sqrt{2}$. Z_s^λ can still be defined as a birth-death process. Since a Brownian Motion almost surely hits the downward sloping line $x(t) = s - \lambda t$ we see that Z_s^λ is without extinction. Reproducing [Neveu]'s proof we arrive at Kolmogorov's equation. From differential equations theory we know that Kolmogorov's equation 4 does not have a monotone solution on $(0,1)$. Looking at definition 3 this implies that a possesses a discontinuity at 1 which means $a_\infty > 0$ so that the process explodes almost surely. We let $T(\omega)$ denote the explosion time. Spatially this corresponds to there being, at all times, a particle below the line $x = T(\omega) - \lambda t$ whose all ancestors have also been below that line. (If after some time τ, there is no such particle, then $Z_T^\lambda \leq N_\tau < \infty$). Integrating the Brownian Path of this particle and its unique ancestors we deduce that almost surely $v_t(\omega) > \frac{1}{2}\lambda t^2 - T(\omega)t$ and hence, almost surely,

$$\liminf_{t\to\infty} \frac{v_t}{t^2} \geq \lambda/2.$$

We now let $\lambda \uparrow \sqrt{2}$ to complete the proof. □

The Phase Plane Picture

We use the projection from the BBM to the Branching point process, to project the space of paths of BBM to the phase plane. We deduce a large deviations principle for the phase plane, both in expectation and almost surely. We find the two rate functions to be different, and the difference explains the different wavefront speeds we observed earlier.

4.4 Scaling The Process

As before, at a fixed time T, let us scale the branching Brownian Motion by a factor of T in both the space and time coordinates. We get a branching process on $[0,1]$. For every $i \in N_T$ let $x_i^T \in \mathcal{C}^0([0,1], \mathcal{R})$ and $y_i^T \in \mathcal{C}^0([0,1], \mathcal{R})$ be defined as

$$\begin{aligned} x_i^T(t) &:= \frac{1}{T} B_{a(i)}(tT), \\ y_i^T(t) &:= \int_0^t x_i^T(s)ds = \frac{1}{T^2} Y_{a(i)}(tT). \end{aligned}$$

We define the projection map $\Pi : \mathcal{C}^0 \to \mathcal{R}^2$ as

$$\Pi(z) := (z(1), \int_0^1 z(t)dt)$$

which is clearly continuous in the $||z||_\infty$ norm. For every $D \subset \mathcal{C}^0$ we let $M_D(T)$ denote the particles at time T whose path is in D. For every $D \subset \mathcal{R}^2$ we define let $\Pi M_D(T)$ to be $M_{\Pi^{-1}D}(T)$.

$$\begin{aligned} M_D(T) &:= \{i \in N_T : z_i^T \in D\}, \\ \Pi M_D(T) &:= \{i \in N_T : \Pi z_i^T \in D\}. \end{aligned}$$

We must apologise to the reader for the slight change of notations which is about to occur. From now on, we will use $z \in C_1$ to denote a path of a BBM. x and y will now represent the coordinates in the phase plane.

4.5 The Expectation Picture

The Expectation large deviations result tells us that if $D \subset \mathcal{R}^2$ is closed and $A \subset \mathcal{R}^2$ is open then

$$\limsup_{T\to\infty} \tfrac{1}{T} \log E|\Pi M_D(T)| \leq \sup_{z\in\Pi^{-1}D} J(z) = \sup_{(x,y)\in D} \sup_{\Pi z=(x,y)} J(z),$$
$$\liminf_{T\to\infty} \tfrac{1}{T} \log E|\Pi M_A(T)| \geq \sup_{z\in\Pi^{-1}A} J(z) = \sup_{(x,y)\in A} \sup_{\Pi z=(x,y)} J(z).$$

We recall that $J(z) = J(z,1)$ is the expectation rate function for a Branching Brownian Motion

$$J(z,\theta) := \theta - \tfrac{1}{2}\int_0^\theta \dot{z}^2(t)dt.$$

For every $(x,y) \in \mathcal{R}^2$ we define $\Pi J(x,y) := \sup\{J(z) : \Pi z = (x,y)\}$ and use Calculus of Variation with Lagrange Multiplier optimising procedure (see section 4.7) to find that there is a unique z maximising $\Pi J(x,y)$:

$$z(t) = 3(x-2y)t^2 + 2(3y-x)t.$$

Accordingly, $\Pi J(x,y) = J(z) = 1 - \frac{1}{2}x^2 - 6(y-\frac{1}{2}x)^2$. We immediately have the following result.

Result 6. *Let $D \subset \mathcal{R}^2$ be closed and let $A \subset \mathcal{R}^2$ be open. Then*

$$\limsup_{T\to\infty} \tfrac{1}{T} \log E|\Pi M_D(T)| \leq \sup_{(x,y)\in D} \Pi J(x,y),$$
$$\liminf_{T\to\infty} \tfrac{1}{T} \log E|\Pi M_A(T)| \geq \sup_{(x,y)\in A} \Pi J(x,y).$$

Before carrying on please take time to consider how natural this formula is. For particles whose Brownian position at time T is xT we know that their rate function is given by $1 - \frac{1}{2}x^2$. Conditioning on their final position we know that the particles have the law of a Brownian Motion with drift x so that most of these particles arrive in a straight line $x_i^T(t) = xt$ yielding $y_i^T(1) = \frac{1}{2}x$. Some of them will deviate from that path and are penalised by the amount $6(y - \frac{1}{2}x)^2$.

Corollary 3. *Let u_t be the expectation wavefront speed. Then*

$$\lim_{t\to\infty} \frac{u_t}{t^2} = \sqrt{2/3}.$$

Proof. The boundary of the region $\{(x,y) : \Pi J(x,y) \geq 0\}$ defines the expectation wavefront. In particular, maximising y subject to $\Pi J(x,y) \geq 0$, we find $x = \sqrt{3/2}$ and $y = \sqrt{2/3}$. Since $y = \frac{1}{T^2}Y_i(T)$ the result follows.

□

4.6 The Almost-Sure Picture

We know from the large deviation result that almost surely

$$\limsup_{T\to\infty} \tfrac{1}{T}\log|\Pi M_D(T)| \le \sup_{z\in\Pi^{-1}D} K(z) = \sup_{(x,y)\in D} \ \sup_{\Pi z=(x,y)} K(z),$$
$$\liminf_{T\to\infty} \tfrac{1}{T}\log|\Pi M_A(T)| \ge \sup_{z\in\Pi^{-1}A} K(z) = \sup_{(x,y)\in A} \ \sup_{\Pi z=(x,y)} K(z).$$

Here $K(z) = K(z,1)$, is the almost-sure rate function for a Branching Brownian Motion. If $\theta_0 := \inf\{\theta \in [0,1] : J(z,\theta) < 0\}$ the K is defined as

$$K(z,\theta) := \begin{cases} J(z,\theta) & 0 < \theta \le \theta_0 \\ -\infty & \theta_0 \le \theta \le 1 \end{cases}$$

Finding $\Pi K(x,y) := \sup\{K(z) : \Pi z = (x,y)\}$ is more involved as in addition to $\Pi z = (x,y)$, we also impose that $J(z,\theta) \ge 0$ for all $\theta \in [0,1]$, but see section 4.8. We find that for the half-plane $\{y \ge \frac{1}{2}x\}$ the following holds.
Let $\alpha, \beta, \gamma \in \mathcal{R}^2$ be defined as

$$\alpha = (\sqrt{2}, 1/\sqrt{2}), \quad \beta = (-1/\sqrt{2}, 0), \quad \gamma = (-\sqrt{2}, -1/\sqrt{2}).$$

Let f, g, h be the functions defined as

$$f(x) = \tfrac{1}{3}x + \tfrac{1}{3\sqrt{2}}, \quad g(x) = \tfrac{1}{2}x + \tfrac{1}{6}\sqrt{6-3x^2}, \quad h(x) = \tfrac{1}{2}x.$$

Note that f links α to β, the function g links β to γ while clearly h links γ and α. So that they form a region D_1 (also see diagram)

$$D_1 = \left\{(x,y) : x \in [-\sqrt{2}, \sqrt{2}], y \in [h(x), f(x) \wedge g(x)]\right\}.$$

In addition let $l(x) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{9}(\sqrt{2}-x)^2$ be another function linking α and β and let D_2 be the region enclosed by $l(x)$ and $f(x)$.

$$D_2 = \left\{(x,y) : x \in [-1/\sqrt{2}, \sqrt{2}], y \in [f(x), l(x)]\right\}.$$

We find that in D_1 (and by symmetry in $-D_1$ too) the almost-sure behaviour and the behaviour in expectation agree so that the z which maximises the expectation satisfies $K(z) = J(z)$ and so $\Pi K = \Pi J$. Inside D_2 we find that the z maximising is given by

$$\dot{z}(t) = \begin{cases} \sqrt{2} & 0 \le t \le \theta \\ \sqrt{2} - \mu(t-\theta) & \theta \le t \le 1 \end{cases}$$

with

$$\theta = 1 - 3(\tfrac{1}{\sqrt{2}} - y)/(\sqrt{2} - x),$$
$$\mu = \frac{2}{9}(\sqrt{2} - x)^3/(\tfrac{1}{\sqrt{2}} - y)^2.$$

We conclude that for $(x, y) \in D_2$

$$\Pi K(x, y) = \sqrt{2}(\sqrt{2} - x)\left(1 - \frac{2}{9}(\sqrt{2} - x)^2/(1 - \sqrt{2}y)\right).$$

Otherwise we find $\Pi K(x, y) = -\infty$. We now do the same analysis for the other half plane $\{y \leq \frac{1}{2}x\}$ and get the almost-sure result.

Result 7. *Let $D \subset \mathcal{R}^2$ be closed and let $A \subset \mathcal{R}^2$ be open. Then*

$$\limsup_{T\to\infty} \tfrac{1}{T} \log |\Pi M_D(T)| \leq \sup_{(x,y)\in D} \Pi K(x, y),$$
$$\liminf_{T\to\infty} \tfrac{1}{T} \log |\Pi M_A(T)| \geq \sup_{(x,y)\in A} \Pi K(x, y).$$

Corollary 4. *Let v_t be the almost-sure wavefront speed, then*

$$\lim_{t\to\infty} \frac{v_t}{t^2} = 1/\sqrt{2}.$$

Proof. The boundary of the region $D_1 \cup D_2 = \{(x, y) : \Pi K(x, y) \geq 0\}$ defines the almost-sure wavefront. In particular, maximising y subject to $\Pi K(x, y) \geq 0$, we find that $\frac{dl}{dx}|_{x=\sqrt{2}} = 0$ and the supremum is attained at $x = \sqrt{2}$ and $y = 1/\sqrt{2}$. Since $y = \frac{1}{T^2}Y_i(T)$ the result follows.

□

Optimisation of The Rate Functions

In this section we explain briefly how the optimisations for $\Pi J(x, y)$ and $\Pi K(x, y)$ were derived.

4.7 The Expectation Rate Function

If $(x, y) \in \mathcal{R}^2$ we wish to maximise $\{J(z) : \Pi(z) = (x, y)\}$. Alternatively, we minimise $\frac{1}{2}\int_0^1 \dot{z}^2$ subject to the constraint $z(0) = 0, z(1) = x, \int_0^1 z(t)dt = y$. Using Lagrange multiplier we get the unconstrained problem of minimising F.

$$F(z, \dot{z}, t) = \int_0^1 \tfrac{1}{2}\dot{z}^2 - \lambda(y - z)dt.$$

From Calculus of Variations we have $F_z - \dot{F}_{\dot{z}} = 0$ from which we get that $\ddot{z} =$ constant and so $z(t) = (x-\alpha)t^2 + \alpha t$. Substituting $\int_0^1 z dt = y$ we arrive at the optimal path in expectations

$$z = 3(x-2y)t^2 + 2(3y-x)t,$$

from which we deduce that

$$J(z) = 1 - \tfrac{1}{2}\int_0^1 \dot{z}^2 dt = 1 - \tfrac{1}{2}x^2 - 6(y - \tfrac{1}{2}x)^2.$$

4.8 The Almost-Sure Rate Function

Throughout, we assume that $y \geq \frac{1}{2}x$. When $y < \frac{1}{2}x$ we use the symmetry $\Pi K(-x,-y) = \Pi K(x,y)$. Clearly if the z which optimises $\{J(z) : \Pi(z) = (x,y)\}$ also has $K(z) = J(z)$ we are done. This amounts to ensuring $\dot{z}(0) \leq \sqrt{2}$ and we find that if $(x,y) \in D_1$, this is indeed the case.

Outside D_1, although we can not follow the same optimising procedure, some points are clear. Keeping x fixed, as y increases ΠJ and ΠK are decreasing in y. To maximise y while keeping J constant, we must have $\dot{z}$ as a non increasing function. From the in-expectation optimisation procedure, another way of maximising y while keeping J constant is by ensuring $\dot{z}$ is piece-wise linear. Conditioning on the first time θ when $\dot{z} < \sqrt{2}$ we find that on $[\theta, 1]$, the in-expectation optimisation is also valid almost surely so that z must be of the form

$$\dot{z}(t) = \begin{cases} \sqrt{2} & 0 \leq t \leq \theta \\ \sqrt{2} - \mu(t-\theta) & \theta \leq t \leq 1 \end{cases}$$

which we integrate to get

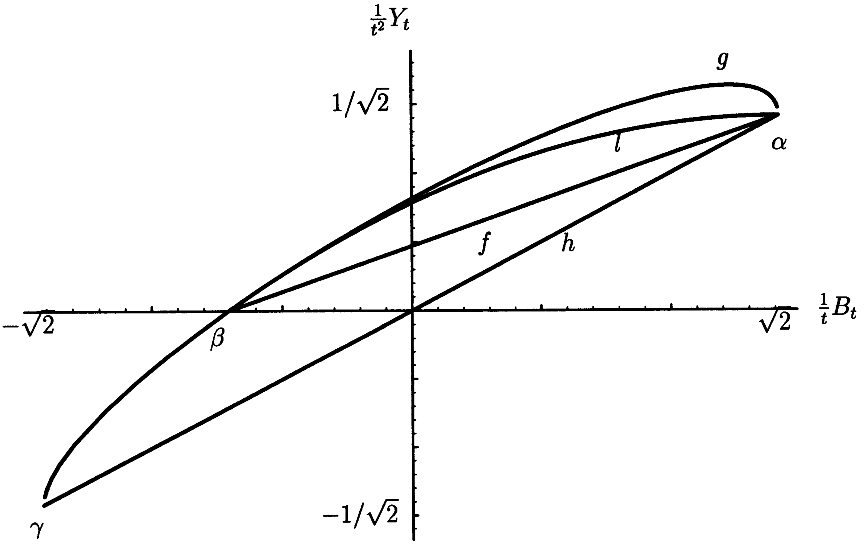
$$z(t) = \begin{cases} \sqrt{2}t & 0 \leq t \leq \theta \\ \sqrt{2}t - \frac{1}{2}\mu(t-\theta)^2 & \theta \leq t \leq 1 \end{cases}$$

and finally we deduce that

$$\int_0^1 z dt = \tfrac{1}{\sqrt{2}} - \tfrac{1}{6}\mu(1-\theta)^3.$$

We substitute boundary conditions $z(1) = x$ and $\int z = y$ to complete the analysis.

4.9 The Phase Plane Diagram



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A Final Note

Since submitting this manuscript, we discovered a book by P. Revesz (Random walks of infinitely many particles, 1994). He considered a split-at-integer-times Branching Brownian Motion and showed the space of paths to be the closure of $\{f : J(f) \geq 0\}$ without counting the actual growth rate along each path. His result is similar in nature although the methods he used are different.