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PATHWISE UNIQUENESS AND APPROXIMATION OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We consider stochastic differential equations for which pathwise uniqueness holds. By using Skorokhod's selection theorem we establish various strong stability results under perturbation of the initial conditions, coefficients and driving processes. Applications to the convergence of successive approximations and to stochastic control of diffusion processes are also given. Finally, we show that in the sense of Baire, almost all stochastic differential equations with continuous and bounded coefficients have unique strong solutions.

1 INTRODUCTION

We consider the following stochastic differential equation:

$$\begin{cases} dX_t = \sigma(t, X_t)\, dB_t + b(t, X_t)\, dt \\ X_0 = x \end{cases} \tag{1}$$

where $\sigma : \mathbf{R}_+ \times \mathbf{R}^d \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^r$and $b : \mathbf{R}_+ \times \mathbf{R}^d \longrightarrow \mathbf{R}^d$ are measurable functions, B is a given r-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions. Throughout this paper we assume that equation (1) has a unique strong solution $X_t(x)$ for each initial value $x \in \mathbf{R}^d$.

It is a well known fact that if the coefficients are Lipschitz continuous, then equation (1) has a unique strong solution $X_t(x)$, which is continuous with respect to the initial condition and coefficients. Moreover, the solution may be constructed by means of various numerical schemes.

Our purpose in this paper, is to study strong stability properties of the solution of (1) under pathwise uniqueness of solutions and a minimal assumption on the coefficients. Such minimal assumption ensures the existence of a weak solution, and is either the continuity of b, σ in the state variable [19], or the uniform ellipticity of the diffusion coefficient [15]. According to Yamada-Watanabe's theorem [22], existence of a weak solution and pathwise uniqueness imply existence of a unique strong solution.

The paper is organized as follows. In the second section we study the variation of the solution with respect to initial data and parameters. Extension of the above result to the Hölder space is the subject of section 3. The fourth section is devoted to the study of successive approximations. We know from the theory of ordinary differential equations with bounded continuous coefficients that, the uniqueness of a solution is not sufficient for successive approximations to converge. Under pathwise uniqueness, we give a necessary and sufficient condition for this convergence. Moreover, we introduce a class of moduli of continuity for which the method of successive approximations converges, covering the results of many authors. In the fifth section, we study the stability of solutions of stochastic differential equations driven by continuous semi-martingales, with respect to the driving processes. Note that we don't suppose pathwise uniqueness for the approximating equations, as it is usually done in the literature.

In section 6, we give an application to optimal control of diffusions. Namely we prove that under pathwise uniqueness, the trajectories associated to relaxed controls are approximated in L^2- sense by trajectories associated to ordinary controls. This result extends a theorem of S. Méléard [17] which is proved under Lipschitz condition. Extension of some of the previous results to the case where the coefficients are merely measurable, with uniformly elliptic diffusion matrix is the subject of section 7.

At the end of this work, we prove that in the sense of Baire, almost all stochastic differential equations with bounded continuous coefficients have unique strong solutions.

The main tool used in the proofs is the Skorokhod selection theorem given by the following

Lemma 1.1 ([11] page 9) *Let (S,ρ) be a complete separable metric space, $P_n, n = 1,2,...$ and P be probability measures on $(S,\mathcal{B}(S))$ such that $P_n \underset{n\to+\infty}{\longrightarrow} P$. Then, on a probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$, we can construct S-valued random variables X_n, $n = 1,2,...$, and X such that:*

(i) $P_n = \widehat{P}^{X_n}, n = 1,2,...$, and $P = \widehat{P}^X$.

(ii) X_n converges to X, $\widehat{P}$ almost surely.

We'll make use of the following result, which gives a criterion for tightness of sequences of laws associated to continuous processes.

Lemma 1.2 ([11] page 18) *Let* $(X_n(t))$, $n = 1, 2, \ldots$, *be a sequence of d-dimensional continuous processes satisfying the following two conditions:*

(i) There exist positive constants M *and* γ *such that* $E[|X_n(0)|^\gamma] \leq M$ *for every* $n = 1, 2, \ldots..$

(ii) There exist positive constants α, β, M_k, $k = 1, 2, \ldots$, *such that:* $E[|X_n(t) - X_n(s)|^\alpha] \leq M_k |t - s|^{1+\beta}$ *for every* n *and* $t, s \in [0, k]$, $(k = 1, 2, \ldots)$.

Then there exist a subsequence (n_k), *a probability space* $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ *and d-dimensional continuous processes* $\widehat{X}_{n_k}, k = 1, 2, \ldots$, *and* $\widehat{X}$ *defined on it such that*

1) The laws of $\widehat{X}_{n_k}$ *and* X_{n_k} *coincide.*

2) $\widehat{X}_{n_k}(t)$ *converges to* $\widehat{X}(t)$ *uniformly on every finite time interval* $\widehat{P}$ *almost surely.*

2 VARIATION OF SOLUTIONS WITH RESPECT TO INITIAL CONDITIONS AND PARAMETERS

Definition 2.3 *We say that pathwise uniqueness holds for equation* (1) *if whenever* $(X, B, (\Omega, \mathcal{F}, P), \mathcal{F}_t)$ *and* $(X', B, (\Omega, \mathcal{F}, P), \mathcal{F}'_t)$ *are weak solutions of equation* (1) *with common probability space and Brownian motion* B *(relative to possibly different filtrations) such that* $P[X_0 = X'_0] = 1$, *then* X *and* X' *are indistinguishable.*

In the theory of ordinary differential equations with continuous coefficients, uniqueness of solutions is sufficient for continuous dependence of the solution with respect to the initial condition [3]. The following theorem gives the analogue of the above result to the stochastic case.

Theorem 2.4 *Let* $\sigma(t, x)$ *and* $b(t, x)$ *be continuous functions satisfying the linear growth condition: for each* $T \geq 0$, *there exists* M *such that:*

$$|\sigma(t, x)| + |b(t, x)| \leq M \ (1 + |x|) \text{ for every } t \in [0, T] \tag{2}$$

Then, if pathwise uniqueness holds for equation (1), we get:

$$\lim_{x \to x_0} E\left[\sup_{t \leq T} |X_t(x) - X_t(x_0)|^2\right] = 0, \text{ for every } T \geq 0 \ .$$

Proof Suppose that the conclusion of our theorem is false, then there exist a positive number δ and a sequence (x_n) converging to x such that:

$$\inf_{n\in\mathbf{N}} E\left[\sup_{t\le T}|X_t(x_n)-X_t(x)|^2\right]\ge\delta. \qquad (3)$$

Let us denote by X^n (resp. X) the solution of (1) corresponding to the initial data x_n (resp. x).

By standard arguments from the theory of stochastic differential equations ([13], page 289) we can show that the sequence (X^n,X,B) satisfies conditions i) and ii) of lemma 1.2 with $\alpha=4$ and $\beta=1$. Then there exist a probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$ and a sequence $\left(\widehat{X}^n_t,\widehat{Y}^n_t,\widehat{B}^n_t\right)$ of stochastic processes defined on it such that:

α) The laws of (X^n,X,B) and $\left(\widehat{X}^n,\widehat{Y}^n,\widehat{B}^n\right)$ coincide for every $n\in\mathbf{N}$.

β) There exists a subsequence $\left(\widehat{X}^{n_k},\widehat{Y}^{n_k},\widehat{B}^{n_k}\right)$ converging to $\left(\widehat{X},\widehat{Y},\widehat{B}\right)$ uniformly on every finite time interval $\widehat{P}$-a.s.

If we denote by $\widehat{\mathcal{F}}^n_t=\sigma\left(\widehat{X}^n_s,\widehat{Y}^n_s,\widehat{B}^n_s;s\le t\right)$ and $\widehat{\mathcal{F}}_t=\sigma\left(\widehat{X}_s,\widehat{Y}_s,\widehat{B}_s;s\le t\right)$, then $\left(\widehat{B}^n_t,\widehat{\mathcal{F}}^n_t\right)$ and $\left(\widehat{B}_t,\widehat{\mathcal{F}}_t\right)$ are Brownian motions.

According to property α) and the fact that X^n_t and X_t satisfy (1) with initial data x_n and x, it can be proved ([15] page 89) that $\forall n\in\mathbf{N}$, $\forall t\ge 0$

$$E\left|\widehat{X}^n_t-x_n-\int_0^t\sigma\left(s,\widehat{X}^n_s\right)d\widehat{B}^n_s-\int_0^t b\left(s,\widehat{X}^n_s\right)ds\right|^2=0.$$

In other words, $\widehat{X}^n$ satisfies the stochastic differential equation:

$$\widehat{X}^n_t=x_n+\int_0^t\sigma\left(s,\widehat{X}^n_s\right)d\widehat{B}^n_s+\int_0^t b\left(s,\widehat{X}^n_s\right)ds.$$

Writing similar relations, we obtain:

$$\widehat{Y}^n_t=x+\int_0^t\sigma\left(s,\widehat{Y}^n_s\right)d\widehat{B}^n_s+\int_0^t b\left(s,\widehat{Y}^n_s\right)ds.$$

By using property (β) and a limit theorem of Skorokhod [19] page 32, it holds that

$$\int_0^t\sigma\left(s,\widehat{X}^{n_k}_s\right)d\widehat{B}^{n_k}_s\underset{k\to+\infty}{\longrightarrow}\int_0^t\sigma\left(s,\widehat{X}_s\right)d\widehat{B}_s$$

and $\displaystyle\int_0^t b\left(s,\widehat{X}^{n_k}_s\right)ds\underset{k\to+\infty}{\longrightarrow}\int_0^t b\left(s,\widehat{X}_s\right)ds$ in probability.

Therefore $\widehat{X}$ and $\widehat{Y}$ satisfy the same stochastic differential equation (1), on $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same Brownian motion $\widehat{B}_t$ and initial condition x. Then, by pathwise uniqueness, we conclude that $\widehat{X}_t = \widehat{Y}_t$, $\forall t \quad \widehat{P}$ a.s.

By uniform integrability, it holds that:

$$\delta \leq \liminf_{n\in\mathbf{N}} E\left[\sup_{t\leq T}\left|X_t(x_n) - X_t(x)\right|^2\right] \leq \liminf_{k\in\mathbf{N}} \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X}_t^{n_k} - \widehat{Y}_t^{n_k}\right|^2\right] =$$

$$= \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X}_t - \widehat{Y}_t\right|^2\right]$$

which contradicts (3). ■

We shall next state a variant of the first theorem. Let us consider a family of functions depending on a parameter λ, and consider the stochastic differential equation:

$$\begin{cases} dX_t^\lambda = \sigma\left(\lambda, t, X_t^\lambda\right) dB_t + b\left(\lambda, t, X_t^\lambda\right) dt \\ X_0^\lambda = \varphi(\lambda). \end{cases} \tag{4}$$

Theorem 2.5 *Suppose that $\sigma(\lambda, t, x)$ and $b(\lambda, t, x)$ are continuous. Further suppose that for each $T > 0$, and each compact set K there exists $L > 0$ such that*

i) $\sup_{t\leq T}\left(|\sigma(\lambda, t, x)| + |b(\lambda, t, x)|\right) \leq L\,(1 + |x|)$ uniformly in λ,

ii) $\lim_{\lambda\to\lambda_0} \sup_{x\in K} \sup_{t\leq T}\left(|\sigma(\lambda, t, x) - \sigma(\lambda_0, t, x)| + |b(\lambda, t, x) - b(\lambda_0, t, x)|\right) = 0$,

iii) $\varphi(\lambda)$ is continuous at $\lambda = \lambda_0$.

If pathwise uniqueness holds for equation (4) at λ_0, we have:

$$\lim_{\lambda\to\lambda_0} E\left[\sup_{t\leq T}\left|X_t^\lambda - X_t^{\lambda_0}\right|^2\right] = 0\ ,\ \textit{for every } T \geq 0\ .$$

Proof Similar to the proof of theorem 2.4. ■

Remark 2.6 *Though (4) need not have a pathwise unique solution for $\lambda \neq \lambda_0$, nevertheless its solutions are continuous in the parameter λ at λ_0.*

The same method may be applied to show the convergence of many approximation schemes such as Euler scheme, approximation by stochastic delay equations [6], the splitting up method [7] and polygonal approximation [12].

3 EXTENSION OF THE RESULTS TO THE HÖLDER SPACE

Let $\alpha > 0$ and denote by $C^{\alpha}\left([0,1]\,;R^d\right)$ the set of α-Hölder continuous functions equiped with the norm defined by:

$$\|f\|_{\alpha} = \sup_{0\leq t\leq 1} |f(t)| + \sup_{0\leq s\leq t\leq 1} \frac{|f(t)-f(s)|}{|t-s|^{\alpha}}$$

It follows from [13], page 53 that the solutions of (1) are α-Hölder continuous for any $\alpha \in \left[0, \frac{1}{2}\right[$. Let X (resp. X^n) denote the solution of (1) corresponding to the intial condition x (resp. x_n) and $Y_n = X - X_n$.

Lemma 3.7 *For any $p > 1$, $\delta > 0$ and any $\gamma < \dfrac{p-1}{2p}$ the following estimates hold:*

(i) $\sup_n E\left|Y_n(t) - Y_n(s)\right|^{2p} \leq c(p)\,|t-s|^p$;

(ii) $\sup_n P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\gamma}} > \delta\right) \leq c(p,\gamma)\,\delta^{-2p}$.

Proof Part (i) is a consequence of Ito's formula. (ii) is a simple consequence of the Garcia-Rodemich-Rumsey lemma ([20] page 49). ■

Proposition 3.8 *Under the hypothesis of theorem 2.4 and for any $\alpha \in \left[0; \frac{1}{2}\right[$, $\varepsilon > 0$ we have:*

$$\lim_{n\to+\infty} P\left(\|X_n - X\|_{\alpha} > \varepsilon\right) = 0$$

Proof $P\left(\|X_n - X\|_{\alpha} > \varepsilon\right) \leq P\left(\sup_{0\leq t\leq 1} |X_n(t) - X(t)| > \frac{\varepsilon}{2}\right) +$

$$+P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\alpha}} > \frac{\varepsilon}{2}\right).$$

According to theorem 2.4, the first term in the right hand side goes to 0 as n tends to $+\infty$.

Let $\eta > 0$ such that $\alpha + \eta < \dfrac{p-1}{2p}$ and let $\mu > 0$, we have

$$P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\alpha}} > \frac{\varepsilon}{2}\right) = P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\alpha}} > \frac{\varepsilon}{2};\, |t-s| < \mu\right) +$$

$$+P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\alpha}} > \frac{\varepsilon}{2};\, |t-s| > \mu\right)$$

$$\leq P\left(\sup_{s<t} \frac{|Y_n(t)-Y_n(s)|}{|t-s|^{\alpha+\eta}} > \frac{\varepsilon}{2}\mu^{-\eta}\right) + 2P\left(\sup_{0\leq t\leq 1} |X_n(t) - X(t)| > \frac{\varepsilon}{4}\mu^{\alpha}\right)$$

$$\leq c\left(\frac{\varepsilon}{2}\mu^{-\eta}\right)^{-2p} + 2P\left(\sup_{0\leq t\leq 1} |X_n(t) - X(t)| > \frac{\varepsilon}{4}\mu^{\alpha}\right).$$

By taking μ small enough and using theorem 2.4 we get the desired result. ■

4 PATHWISE UNIQUENESS AND SUCCESSIVE APPROXIMATIONS

Let σ and b as in theorem 2.4 and consider the stochastic differential equation (1). The sequence of successive approximations associated to (1) is defined as follows

$$\begin{cases} X_t^{n+1} = \int_0^t \sigma(s, X_s^n)\, dB_s + \int_0^t b(s, X_s^n)\, ds \\ X^0 = x. \end{cases} \tag{5}$$

If we assume that the coefficients are Lipschitz continuous, then the sequence (X^n) converges in quadratic mean and gives an effective way for the construction of the unique solution X of equation (1) (see [11]). Now if we drop the Lipshitz condition and assume only that equation (1) admits a unique strong solution, does the sequence (X^n) converge to X? The answer is negative even in the deterministic case, see ([4] pp.114-124).

The aim of the following theorem is to establish an additional necessary and sufficient condition which ensures the convergence of successive approximations.

Theorem 4.9 *Let σ and b as in theorem 2.4. Under pathwise uniqueness for s.d.e (1), (X^n) converges in quadratic mean to the unique solution of (1) if and only if $X^{n+1} - X^n$ converges to 0.*

Lemma 4.10 *Let (X^n) be defined by* (5), *then:*

1) For every $p > 1$, $\sup_n E\left[\sup_{t \le T} |X_t^n|^{2p}\right] < +\infty$.

2) For every $T > 0$ and $p > 1$, there exists a constant C independant of n such that for every $s < t$ in $[0, T]$, $E\left[|X_t^n - X_s^n|^{2p}\right] \le C\,|t - s|^p$.

Proof 1) For all $t > 0$ and $n > 1$, we have

$$|X_t^n|^{2p} \le C_1 \left[|x|^{2p} + \left| \int_0^t b\left(s, X_s^{n-1}\right) ds \right|^{2p} + \left| \int_0^t \sigma\left(s, X_s^{n-1}\right) dB_s \right|^{2p} \right].$$

By applying Hölder's inequality, it holds that

$$\left| \int_0^t b\left(s, X_s^{n-1}\right) ds \right|^{2p} \le \left[\sum_{i=1}^d \left(\int_0^t b_i\left(s, X_s^{n-1}\right) ds \right)^2 \right]^p$$

$$\le t^p \left[\int_0^t \left| b\left(s, X_s^{n-1}\right) \right|^2 ds \right]^p \le t^{2p-1}. \int_0^t \left| b\left(s, X_s^{n-1}\right) \right|^{2p} ds.$$

Burkholder Davis Gundy and Hölder inequalities provide the following estimate

$$E\left[\sup_{0 \le t \le T} \left| \int_0^t \sigma\left(s, X_s^{n-1}\right) dB_s \right|^{2p} \right] \le C_2 E\left[\left(\int_0^T \left| \sigma\left(s, X_s^{n-1}\right) \right|^2 ds \right)^p \right]$$

$$\le C_2 T^{p-1} E\left[\int_0^T \left| \sigma\left(s, X_s^{n-1}\right) \right|^{2p} ds \right].$$

Taking expectations, we obtain

$$E\left[\sup_{t\leq T}|X_t^n|^{2p}\right]\leq C_3\left[|x|^{2p}+C_4E\int_0^T\left(|\sigma|^{2p}+|b|^{2p}\right)\left(s,X_s^{n-1}\right)ds\right].$$

By using the linear growth condition we get

$$E\left[\sup_{t\leq T}|X_t^n|^{2p}\right]\leq C_5\left(1+|x|^{2p}\right)+C_5\int_0^T E\left[\sup_{s\leq t}|X_s^{n-1}|^{2p}\right]dt$$

where the various constants C_k depend only on T,m,d.

Iteration of the last inequality gives

$$E\left[\sup_{t\leq T}|X_t^n|^{2p}\right]\leq C_5\left(1+|x|^{2p}\right)\left[1+CT+\frac{(CT)^2}{2!}+...+\frac{(CT)^n}{n!}\right].$$

Then $\sup_n E\left[\sup_{t\leq T}|X_t^n|^{2p}\right]\leq C_5\left(1+|x|^{2p}\right)\exp(CT)$.

2) If we fix $s<t$ in $[0,T]$, we may proceed as before to obtain

$$E\left[|X_t^n-X_s^n|^{2p}\right]\leq C_6|t-s|^{p-1}\int_s^t\left(1+E\left[\sup_{v\leq u}|X_v^{n-1}|^{2p}\right]du\right).$$

Then by using the previous result, we get

$E\left[|X_t^n-X_s^n|^{2p}\right]\leq C_7|t-s|^p$, where C_7 depends on x,p,d,T. ■

Proof of theorem 4.9. Suppose that $X^{n+1}-X^n$ converges to 0 and there is some $\delta>0$ such that

$$\inf_n E\left[\max_{0\leq t\leq T}|X_t^n-X_t|^2\right]\geq\delta$$

According to lemma 4.10, the family (X^n,X^{n+1},X,B) satisfies conditions i) and ii) of lemma 1.2. Then by Skorokhod's selection theorem, there exists some probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$ carrying a sequence of stochastic processes $\left(\widehat{X}^n,\widehat{Z}^n,\widehat{Y}^n,\widehat{B}^n\right)$ with the following properties:

i) the laws of $\left(\widehat{X}^n,\widehat{Z}^n,\widehat{Y}^n,\widehat{B}^n\right)$ and (X^n,X^{n+1},X,B) coincide for each $n\in\mathbf{N}$,

ii) there exists a subsequence $\{n_k\}$ such that $\left(\widehat{X}^{n_k},\widehat{Z}^{n_k},\widehat{Y}^{n_k},\widehat{B}^{n_k}\right)$ converges to $\left(\widehat{X},\widehat{Z},\widehat{Y},\widehat{B}\right)$ uniformly on every finite time interval $\widehat{P}$ a.s.

But we know that $X^{n+1}-X^n$ converges to 0, then we can show easily that $\widehat{X}=\widehat{Z}$, $\widehat{P}$ a.s.

Proceeding as in the proof of theorem 2.4, we can show that

$$\widehat{Z}^{n_k}=x+\int_0^t\sigma\left(s,\widehat{X}^{n_k}\right)d\widehat{B}_s^{n_k}+\int_0^t b\left(s,\widehat{X}^{n_k}\right)ds;$$
$$\widehat{Y}^{n_k}=x+\int_0^t\sigma\left(s,\widehat{Y}_s^{n_k}\right)d\widehat{B}_s^{n_k}+\int_0^t b\left(s,\widehat{Y}_s^{n_k}\right)ds.$$

Taking the limit as k goes to $+\infty$, it holds that

$$\widehat{X}_t = x + \int_0^t \sigma\left(s, \widehat{X}_s\right) d\widehat{B}_s + \int_0^t b\left(s, \widehat{X}_s\right) ds;$$
$$\widehat{Y}_t = x + \int_0^t \sigma\left(s, \widehat{Y}_s\right) d\widehat{B}_s + \int_0^t b\left(s, \widehat{Y}_s\right) ds.$$

In other words, $\widehat{X}$ and $\widehat{Y}$ solve equation (1). Then by pathwise uniqueness we have $\widehat{X} = \widehat{Y}$, $\widehat{P}$ a.s.

Using uniform integrability, we obtain:

$$\begin{aligned}\delta &\leq \liminf_k E\left[\max_{0\leq t\leq T} \left|X_t^{n_k} - X_t\right|^2\right] = \liminf_k \widehat{E}\left[\max_{0\leq t\leq T} \left|\widehat{X}_t^{n_k} - \widehat{Y}_t^{n_k}\right|^2\right] \\ &= \widehat{E}\left[\max_{0\leq t\leq T} \left|\widehat{X}_t - \widehat{Y}_t\right|^2\right]\end{aligned}$$

which is a contradiction. ■

Remark 4.11 *Roughly speaking, under pathwise uniqueness, the series* $\sum\left(X^{n+1} - X^n\right)$ *converges if and only if* $\left(X^{n+1} - X^n\right)$ *converges to* 0.

As a generalization of the condition which S. Kawabata [14] has already considered, we'll assume the following:

Condition A 1) There exist measurable functions m and ρ such that:

$$|\sigma(t,x) - \sigma(t,y)|^2 + |b(t,x) - b(t,y)|^2 \leq m(t)\,\rho\left(|x-y|^2\right)$$

where m is in L^1_{loc}.

2) ρ is a continuous, non decreasing and concave function defined on $\mathbf{R}_+$ such that:

$$\int_{0+} \frac{du}{\rho(u)} = +\infty.$$

Theorem 4.12 *Assume that* σ *and* b *satisfy condition A. Then the successive approximations converge in quadratic mean to the unique solution of (1).*

Proof Under condition A, it is well known that pathwise uniqueness holds [22]. Now it is sufficient to prove that $X^{n+1} - X^n$ converges to 0 in quadratic mean. Let

$$\varphi_n(t) = E\left[\max_{0\leq s\leq t} \left|X_s^{n+1} - X_s^n\right|^2\right].$$

Using Doob and Schwarz inequalities, we have:

$$\varphi_{n+1}(t) \leq (8+2T)\,E\left[\int_0^t \left|\sigma\left(t, X_s^{n+1}\right) - \sigma\left(t, X_s^n\right)\right|^2 ds + \right.$$
$$\left. + \int_0^t \left|b\left(t, X_s^{n+1}\right) - b\left(t, X_s^n\right)\right|^2 ds\right].$$

Since ρ is increasing and concave, then:

$$\varphi_{n+1}(t) \leq (8+2T) \int_0^t \rho(\varphi_n(s)) .m(s)\, ds.$$

Let ψ_n the sequence defined by:

$$\psi_0(t) = u(t)$$
$$\psi_{n+1}(t) = (8+2T) \int_0^t \rho(\psi_n(s)) .m(s)\, ds \text{ for every } t \in [0,T].$$

By using a lemma in [4] page 114-124, see also [14], it is possible to choose a function u such that:

1) $\forall t \in [0,T]$, $u(t) \geq \varphi_0(t)$.

2) $u(t) \geq (8+2T) \int_0^t \rho(u(s)) .m(s)\, ds$

Hence by induction we have: $\varphi_n(s) \leq \psi_n(s)$ $\forall n \in \mathbf{N}$ and the sequence ψ_n is decreasing.

Let $\psi(t) = \lim_{n\to+\infty} \psi_n(t)$, note that this convergence is uniform and ψ is continuous and satisfies the equation:

$$\psi(t) = (8+2T) \int_0^t \rho(\psi(s)) .m(s)\, ds \text{ for every } t \in [0,T].$$

Condition A 2) implies that $\psi = 0$ and hence $\lim_{n\to+\infty} \varphi_n(t) = 0$. ■

Some examples of functions which are not Lipschitz but satisfy condition A2) are given by:

$$\rho(u) = u\,|\log u|^{\alpha} \ (0 < \alpha < 1)$$

or

$$\rho(u) = u\,|\log u|\,|\log|\log u||^{\alpha} \ (0 < \alpha < 1).$$

Let us introduce a different class of moduli of continuity $g(t,x)$ which are not necessarily written in the form $l(t) .m(x)$, covering the classes considered in [21], [8].

Let Ξ be the set of functions $g :]0,T] \times \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ satisfying:

i) g is continuous, non decreasing and concave with respect to the second variable.

ii) $\lim_{t\to 0} g(t,0) = 0$.

iii) If $F : [0,T] \longrightarrow \mathbf{R}_+$ is continuous such that $F(0) = 0$ and $F(t) \leq \int_0^t g(s, F(s))\, ds$, then $F = 0$ on $[0,T]$.

Theorem 4.13 *Let σ and b be continuous functions satisfying the linear growth condition (2). Moreover suppose that there exists $g \in \Xi$ such that:*

$$|\sigma(t,x) - \sigma(t,y)|^2 + |b(t,x) - b(t,y)|^2 \leq g\left(t, |x-y|^2\right)$$

Then pathwise uniqueness holds, and the sequence X^n converges in quadratic mean to the unique solution of (1).

Proof i) Pathwise uniqueness is an immediate consequence of the properties of the function g.

Let us prove the convergence of successive approximations. According to theorem 4.9, it is sufficient to prove that $\lim_{n\to+\infty} E\left[\max_{0\le s\le t}\left|X_s^{n+1}-X_s^n\right|^2\right]=0.$

Let

$$\varphi_n(t)=E\left[\max_{0\le s\le t}\left|X_s^{n+1}-X_s^n\right|^2\right]$$

and

$$A_T=\left\{t\in[0,T]\,;\lim_{n\to+\infty}\varphi_n(t)=0\right\}.$$

Since A_T is a non empty subset ($0\in A_T$), it is enough to establish that A_T is open and closed in $[0,T]$.

1) First step: A_T is closed.

Let $0<t_1\in\overline{A}_T$, $\varepsilon>0$ and $\delta\le\min(t_1,\varepsilon)$. Hence there exists $t_0\in A_T$ such that $t_1-t_0\le\delta$.

Since A_T is an interval which contains 0 (because $t\to\varphi_n(t)$ is an increasing function), it suffices to prove that:

$$\lim_{n\to+\infty} E\left[\max_{t_1-\delta\le s\le t_1}\left|X_s^{n+1}-X_s^n\right|^2\right]=0.$$

By Doob inequality and the non explosion condition, there exists $n_0\in\mathbf{N}$ such that for any $n\ge n_0$:

$$E\left[\max_{t_1-\delta\le s\le t_1}\left|X_s^{n+1}-X_s^n\right|^2\right]\le\varepsilon+12\,(8+2T)\,K_T\,(1+H)\,\delta$$

where $H=\sup_n E\left[\sup_{t\le T}\left|X_t^n\right|^2\right]<+\infty$ (see lemma 4.10). This proves that $t_1\in A_T$.

2) Second step: A_T is open.

Let $t_0\in A_T$; $t_0\ne 0$ and $t_0\ne T$.

We'll prove that $\exists r>0$ such that $t_0+r\in A_T$, which means that $\exists r>0$ such that $\lim_{n\to+\infty}\varphi_n(t)=0$ on $[0,t_0+r[$.

Since A_T is an interval (and $t_0\in A_T$), it is sufficient to show that:

$$\lim_{n\to+\infty} E\left[\max_{t_0\le s\le t}\left|X_s^{n+1}-X_s^n\right|^2\right]=0.$$

It is easy to see that there exists a sequence (ε_n) of positive real numbers decreasing to 0 as n goes to $+\infty$, such that:

$$E\left[\max_{t_0\le s\le t_0+r}\left|X_s^{n+1}-X_s^n\right|^2\right]\le 3\varepsilon_n+ct \qquad ,\forall t\le T-t_0$$

where c is a constant which depends only on T and x.

Let

$$M=\sup\{g(t,v):(t,v)\in[t_0,T]\times[0,3\varepsilon_0+2H]\}$$
$$r=\min\left(T-t_0,\frac{2H}{3(8+2T)\sup(c,M)}\right)$$

We consider the following ordinary differential equation:

$$(*)\begin{cases} u'(t)=g_1(t,u(t)) & t\in[t_0,t_0+r]\\ u(t_0)=0 \end{cases}$$

with $g_1(t,u(t))=3(8+2T)\,g(t,u(t))$.

Define the successive approximation for $(*)$ by:

$$\begin{cases} u_0(\tau)=3\varepsilon_0+3(8+2T)\sup(c,M)(\tau-t_0)\\ u_{n+1}(t)=3\varepsilon_{n+1}+\displaystyle\int_{t_0}^{\tau}g_1(t,u_n(t))\,dt. \end{cases}$$

It is obvious to see, by induction on $n\in\mathbf{N}$, that $(u_n(\tau))_{n\in\mathbf{N}}$ is a positive decreasing sequence. By using the monotone convergence theorem and the continuity of g_1, we obtain:

$$u(\tau)=\lim_{n\to+\infty}u_n(\tau)=\lim_{n\to+\infty}\varepsilon_n+\lim_{n\to+\infty}\int_{t_0}^{\tau}g_1(t,u_{n-1}(t))\,dt$$
$$=\int_{t_0}^{\tau}g_1(t,u(t))\,dt$$

Since $g_1\in\Xi$, then: $u(\tau)=0 \qquad \forall\tau\in[t_0,t_0+r]$.

Let

$$\psi_n(t)=E\left[\max_{t_0\le s\le t}\left|X_s^{n+1}-X_s^n\right|^2\right].$$

We remark that $\psi_n(t)$ is majorized by $u_n(t)$ on $[t_0,t_0+r]$. Therefore $\lim_{n\to+\infty}\psi_n(t)=0$, which implies that $\lim_{n\to+\infty}\varphi_n(t)=0$ on $[t_0,t_0+r]$.

This achieves the proof. ■

We recall a condition given by S. Nakao, which guarantees pathwise uniqueness, but under which the problem of convergence of successive approximations is not solved.

σ and b are $\mathbf{R}$-valued measurable, bounded functions and σ is of bounded variation such that $\sigma\ge\varepsilon$ for some $\varepsilon>0$.

The problem of convergence of successive approximations is still open for an important class of stochastic differential equations involving the local time of the unknown process.

5 STABILITY OF STOCHASTIC EQUATIONS DRIVEN BY CONTINUOUS SEMI MARTINGALES

In this section, we consider stochastic differential equations driven by continuous semi-martingales. We establish a continuity result with respect to the driving processes when pathwise uniqueness of solutions holds.

Let $b : [0,1]\times\mathbf{R}^d \longrightarrow \mathbf{R}^d$ and $\sigma : [0,1]\times\mathbf{R}^d \longrightarrow \mathbf{R}^{d\times r}$ be bounded continuous functions.

We consider the stochastic differential equation:

$$\begin{cases} dX_t = \sigma(t,X_t)\,dM_t + b(t,X_t)\,dA_t \\ X_0 = x \end{cases} \tag{6}$$

where A_t is an adapted continuous process of bounded variation and M_t is a continuous local martingale.

Definition 5.14 *Pathwise uniqueness property holds for equation (6) if whenever $(X, M, A, (\Omega,\mathcal{F},P), \mathcal{F}_t)$ and $(X', M', A', (\Omega,\mathcal{F},P), \mathcal{F}'_t)$ are two weak solutions such that $(M, A) = (M', A')$ P a.s, then $X = X'$ P a.s.*

Let (M^n) be a sequence of continuous $(\mathcal{F}_t, P)$ −local martingales and (A^n) be a sequence of $\mathcal{F}_t$-adapted continuous processes with bounded variation.

We consider the following equations:

$$\begin{cases} dX_t^n = \sigma(t,X_t^n)\,dM_t^n + b(t,X_t^n)\,dA_t^n \\ X_0^n = x. \end{cases} \tag{7}$$

Let us suppose that (A, A^n, M, M^n) satisfy the following conditions:

(H_1) The family (A, A^n, M, M^n) is bounded in probability in $C([0,1])^4$.

(H_2) $M^n - M \longrightarrow 0$ in probability in $C([0,1])$.

(H_3) $Var(A^n - A) \longrightarrow 0$ in probability.

(Var means the total variation).

Theorem 5.15 *If conditions H_1, H_2, H_3 are satisfied and if pathwise uniqueness holds for equation (6) then:*

$$\textit{For any } \varepsilon > 0,\ \lim_{n\to+\infty} P\left[\sup_{t\le 1} |X_t^n - X_t| > \varepsilon\right] = 0.$$

We need the following lemmas given in [9].

Lemma 5.16 *Let $\{f_n(t), f(t) : t \in [0,1]\}$ be a family of continuous processes and let $\{C_n(t), C(t) : t \in [0,1]\}$ be a family of continuous processes of bounded variation. Assume that:*

$\lim_{n\to+\infty} f_n = f$ *in probability in* $C([0,1])$.

$\lim_{n\to+\infty} C_n = C$ *in probability in* $C([0,1])$.

$\{Var(C_n)\ ;\ n\in\mathbf{N}\}$ *is bounded in probability.*

Then the following result holds:

$$\forall\varepsilon>0\ ,\qquad \lim_{n\to+\infty} P\left[\sup_{t\le 1}\left|\int_0^1 f_n dC_n - \int_0^1 f dC\right| > \varepsilon\right] = 0.$$

Lemma 5.17 *Consider a family of filtrations* $(F_t^n),(F_t)$ *satisfying the usual conditions. Let* $\{f_n(t), f(t) : t\in[0,1]\}$ *be a sequence of continuous adapted processes and let* $\{N_n(t), N(t) : t\in[0,1]\}$ *be a sequence of continuous local martingales with respect to* $(F_t^n),(F_t)$ *respectively. Suppose that*

$\lim_{n\to+\infty} f_n = f$ *in probability in* $C([0,1])$.

$\lim_{n\to+\infty} N_n = N$ *in probability in* $C([0,1])$.

Then

$$\forall\varepsilon>0\ ,\qquad \lim_{n\to+\infty} P\left[\sup_{t\le 1}\left|\int_0^1 f_n dN_n - \int_0^1 f dN\right| > \varepsilon\right] = 0.$$

Proof of theorem 5.15. Suppose that the conclusion of our theorem is false. Then there exists $\varepsilon>0$ such that

$$\inf_n P\left[\|X^n - X\|_\infty > \varepsilon\right] \ge \varepsilon\ .$$

It is clear that the family $Z^n = (X^n, X, A^n, A, M^n, M)$ is tight in $\left[C([0,T];\mathbf{R}^d)\right]^6$. Then by Skorokhod's theorem, there exist a probability space $(\Omega',\mathcal{F}',P')$ and $Z'^n = \left(X'^n,\widetilde{X}^n,A'^n,\widetilde{A}^n,M'^n,\widetilde{M}^n\right)$ which satisfy

i) law (Z^n) = law (Z'^n)

ii) There exists a subsequence (Z'^{n_k}) denoted also by (Z'^n) which converges P' a.s in $\left[C([0,T];\mathbf{R}^d)\right]^6$ to $Z' = \left(X',\widetilde{X},A',\widetilde{A},M',\widetilde{M}\right)$.

Let G_t^n denotes the completion of the σ-algebra generated by $Z_t'^n$ $(t\in[0,1])$ $\mathcal{F}_t'^n = \bigcap_{s>t} G_s^n$.

In an analogous manner we define the σ-algebra $(\mathcal{F}_t'; t\in[0,1])$ for the limiting process Z'. Then $(\Omega',\mathcal{F}',\mathcal{F}_t'^n,P')$ (resp. $(\Omega',\mathcal{F}',\mathcal{F}_t',P')$) are stochastic bases and $M_t'^n,\widetilde{M}^n$ (resp. $M_t',\widetilde{M}_t$) are $\mathcal{F}_t'^n$ (resp. $\mathcal{F}_t'$) continuous local martingales. The processes X'^n and $\widetilde{X}^n$ satisfy the following s.d.e :

$$\begin{cases} dX_t'^n = \sigma\left(t,X_t'^n\right)dM_t'^n + b\left(t,X_t'^n\right)dA_t'^n \\ X_0'^n = x \end{cases} \tag{8}$$

$$\begin{cases} d\widetilde{X}_t^n = \sigma\left(t,\widetilde{X}_t^n\right)d\widetilde{M}_t^n + b\left(t,\widetilde{X}_t^n\right)d\widetilde{A}_t^n \\ \widetilde{X}_0^n = x \end{cases} \tag{9}$$

on $(\Omega', \mathcal{F}', \mathcal{F}_t'^n, P')$.

By using lemmas 5.16 and 5.17, we see that the limiting processes satisfy the following equations:

$$\begin{cases} dX_t' = \sigma(t, X_t')\, dM_t' + b(t, X_t')\, dA_t' \\ X_0' = x \end{cases}$$
$$\begin{cases} d\widetilde{X}_t = \sigma\left(t, \widetilde{X}_t\right) d\widetilde{M}_t + b\left(t, \widetilde{X}_t\right) d\widetilde{A}_t \\ \widetilde{X}_0 = x. \end{cases}$$

By using hypothesis (H_2) and (H_3), it is easy to see that $M' = \widetilde{M}$ and $A' = \widetilde{A}$, P' *a.s.*

Hence by pathwise uniqueness, X' and $\widetilde{X}$ are indistinguishable. This contradicts our assumption. Therefore X^n converges to the unique solution X. ∎

6 AN APPROXIMATION RESULT IN STOCHASTIC CONTROL

In this section, we use the ideas developped in section 2, to establish an L^2-approximation result for relaxed control problems, where the controlled process evolves according to the Ito stochastic differential equation

$$\begin{cases} dX_t^u = \sigma(t, X_t^u)\, dB_t + b(t, X_t^u, u_t)\, dt \\ X_0^u = x \end{cases} \tag{10}$$

where u is a predictable process with values in a compact Polish space E.

The cost to be minimized over the class $\mathbf{U}$ of E-valued predictable processes is defined by:

$$J(u) = E\left[\int_0^1 l(t, X_t^u, u_t)dt + g(X_1^u)\right].$$

An optimal control u^* is a process belonging to $\mathbf{U}$, such that:

$$J(u^*) = \min\{J(u) : u \in \mathbf{U}\}.$$

Usualy an optimal control in the class $\mathbf{U}$ does not exist, unless some convexity assumptions are imposed ([5]). Thus, we transform the initial problem by embedding the class $\mathbf{U}$ into the class $\mathcal{R}$ of relaxed controls which has good compactness properties.

Let $\mathbf{V}$ be the set of probability measures on $[0,1] \times E$ whose projections on $[0,1]$ coincide with the Lebesgue measure dt. $\mathbf{V}$ is equiped with the topology of weak convergence of probability measures.

$\mathbf{V}$ is a compact metrisable set.

Definition 6.18 *A relaxed control* q *is a random variable with values in the set* $\mathbf{V}$.

Remark 6.19 *1) Every relaxed control q can be desintegrated as $q(\omega, dt, da) = dt.q(\omega, t, da)$, where $q(\omega, t, da)$ is a predictable process with values in the space of probability measures on E.*

2) The set $\mathbf{U}$ of ordinary controls is embedded into the set $\mathcal{R}$ of relaxed controls by the application $\Psi : \mathbf{U} \longrightarrow \mathcal{R}$, $u \longrightarrow \Psi(u)(dt, da) = dt.\delta_{u(t)}(da)$ where δ_a is the Dirac measure at a.

For a full treatment of relaxed controls see [5].

Lemma 6.20 *(Chattering lemma) Let q be a relaxed control, then there exists a sequence of predictable processes u^n with values in E such that the sequence $dt.\delta_{u^n(t)}(da)$ converges to $dt.q(\omega, t, da)$ P a.s.*

Proof. See [5]. ■

Let us now define the dynamic and the cost associated with a relaxed control $q \in \mathcal{R}$. For $q \in \mathcal{R}$, we denote by X^q the solution of:

$$\begin{cases} dX_t^q = \sigma(t, X_t^q)\, dB_t + \displaystyle\int_E b(t, X_t^q, a)\, q(t, da)\, dt \\ X_0^q = x. \end{cases} \tag{11}$$

The cost associated to (q, X^q) is given by:

$$J(q) = E\left[\int_0^1 \int_E l(t, X_t^q, a) q(t, da)\, dt + g(X_1^q)\right].$$

Because of the compactness of the space $\mathbf{V}$, it is proved in [5] that an optimal control exists in the class $\mathcal{R}$ of relaxed controls (even when the control enters in the diffusion coefficient σ). Moreover under uniqueness in law, it is established that the family of laws of $(dt.\delta_{u(t)}(da), X^u)$ is dense in the set of laws of $(dt.q(t, da), X^q)$ on $\mathcal{R} \times C(\mathbf{R}_+, \mathbf{R}^d)$ and:

$$\inf\{J(u) : u \in \mathbf{U}\} = \inf\{J(q) : q \in \mathcal{R}\}.$$

We give now our approximation result, extending a theorem proved in [17] (where the coefficients b, σ are supposed to be Lipschitz continuous in the space variable). The novelty of our result is that the approximation procedure remains valid under any conditions on σ and b ensuring pathwise uniqueness.

Assume the following conditions:

$b : \mathbf{R}_+ \times \mathbf{R}^d \times E \longrightarrow \mathbf{R}^d$

$\sigma : \mathbf{R}_+ \times \mathbf{R}^d \times E \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^r$

are continuous functions such that:

$\sup_{t \leq 1} (|\sigma(t, x)| + |b(t, x, a)|) \leq L(1 + |x|), \forall a \in E.$

Theorem 6.21 *Let q be a relaxed control and X^q be the corresponding solution of (11). Then if pathwise uniqueness holds for equation (11), there exists a sequence$(u^n)_{n \in \mathbf{N}}$ of E-valued predictable processes such that:*

1) $dt.\delta_{u^n(t)}(da)$ *converges to* $dt.q(t,da)$ *P a.s in* $\mathbf{V}$.

2) $\lim\limits_{n\to+\infty} E\left[\sup\limits_{t\le 1} |X_t^{u_n} - X_t^q|^2\right] = 0.$

Proof. 1) Let $q \in \mathcal{R}$, by lemma 6.20, there exists a sequence $(u^n) \subset \mathbf{U}$ such that $q^n = dt.\delta_{u^n(t)}(da)$ converges to $dt.q(t,da)$ P $a.s$ in $\mathbf{V}$.

2) Let $X_t^{u_n}, X_t^q$ the solutions of (10) and (11) associated with u^n and q. Suppose that 2) is false, then there exists $\delta > 0$ such that:

(H) $\inf\limits_n E\left[\sup\limits_{t\le 1} |X_t^{u_n} - X_t^q|^2\right] \ge \delta.$

From lemma 1.2 and compactness of $\mathbf{V}$, the family of processes $\gamma^n = (q^n, q, X^{u_n}, X^q, B)$ is tight in the space $\mathbf{V}^2 \times C^3$, where C denotes the space of continuous functions from $[0,1]$ into $\mathbf{R}^d$ endowed with the topology of uniform convergence. By Skorokhod's theorem, there exist a probability space $(\Omega', \mathcal{F}', P')$ carrying a sequence $\gamma'^n = (q'^n, v'^n, X'^n, Y'^n, B'^n)$ such that

i) For each $n \in \mathbf{N}$, the laws of $\gamma^n$and γ'^n coincide.

ii) There exists a subsequence γ'^{n_k} which converges to γ' P' $a.s$ on the space $\mathbf{V}^2 \times C^3$, where $\gamma' = (q', v', X', Y', B')$.

We assume without loss of generality that ii) holds for the whole sequence (γ'^n). By uniform integrability we have:

$\delta \le \liminf\limits_{n\in\mathbf{N}} E\left[\sup\limits_{t\le 1} |X_t^{u_n} - X_t^q|^2\right] = \liminf\limits_{n\in\mathbf{N}} E'\left[\sup\limits_{t\le 1} |X_t'^n - Y_t'^n|^2\right]$

$= E'\left[\sup\limits_{t\le 1} |X_t' - Y_t'|^2\right]$ where E' is the expectation with respect to P'.

By property i) we see that $X'^n$and Y'^n satisfy the following equations

$$\begin{cases} dX_t'^n = \sigma(t, X_t'^n)\, dB_t'^n + \int_E b(t, X_t'^n, a)\, q'^n(t,da)\, dt \\ X_0'^n = x \end{cases} \tag{12}$$

$$\begin{cases} dX_t'^n = \sigma(t, Y_t'^n)\, dB_t'^n + \int_E b(t, Y_t'^n, a)\, v'^n(t,da)\, dt \\ Y_0'^n = x. \end{cases} \tag{13}$$

By letting n going to infinity and using Skorokhod's limit theorem [19] page 32, we see that the processes X' and Y' satisfy equations (12) and (13) respectively, without the index n.

We know by 1) that $q^n \to q$ in $\mathbf{V}$, P $a.s$, then the sequence (q^n, q) converges to (q,q) in $\mathbf{V}^2$. Moreover law$(q^n, q) =$ law(q'^n, v'^n) and $(q'^n, v'^n) \longrightarrow (q', v')$, P' $a.s$ in $\mathbf{V}^2$. Therefore law$(q', v') =$ law(q,q) which is supported by the diagonal of $\mathbf{V}^2$. Then $q' = v'$ $\;P'$ $a.s.$

It follows that X' and Y' are solutions of the same stochastic differential equation driven by Brownian motion B'. Hence by pathwise uniqueness we have $X' = Y'$, P' $a.s$, which contradicts (H). ■

7 CASE WHERE THE COEFFICIENTS ARE NOT CONTINUOUS

In this section we drop the continuity assumption on the coefficients, nevertheless we suppose that $d = r$ and σ, b satisfy the following conditions.

a) σ and b are Borel bounded functions.

b) $\exists \lambda > 0$, such that $\forall (t, x, \xi) \in \mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d : \xi^* \sigma(t, x) \xi \geq \lambda |\xi|^2$.

Theorem 7.22 *Suppose that $\sigma\,(t, x)$ and $b(t, x)$ satisfy conditions a) and b). If pathwise uniqueness holds for equation* (1), *then the conclusion of theorem 2.4 remains valid without the continuity assumption.*

Proof. The proof goes as in theorem 2.4, the only difficulty (due to the lack of continuity of b and σ) is to show that

$$\int_0^t \sigma\left(s, \widehat{X}_s^{n_k}\right) d\widehat{B}_s^{n_k} \longrightarrow \int_0^t \sigma\left(s, \widehat{X}_s\right) d\widehat{B}_s \text{ in probability,}$$

$$\int_0^t b\left(s, \widehat{X}_s^{n_k}\right) ds \longrightarrow \int_0^t b\left(s, \widehat{X}_s\right) ds \text{ in probability.}$$

For $\varepsilon > 0$, we have

$$\begin{aligned}
&\lim_{k\to+\infty} P\left[\left|\int_0^t \sigma\left(s, \widehat{X}_s^{n_k}\right) d\widehat{B}_s^{n_k} - \int_0^t \sigma\left(s, \widehat{X}_s\right) d\widehat{B}_s\right| > \varepsilon\right] \\
&\leq \limsup_{k\to+\infty} P\left[\left|\int_0^t \left(\sigma\left(s, \widehat{X}_s^{n_k}\right) - \sigma\left(s, \widehat{X}_s\right)\right) d\widehat{B}_s^{n_k}\right| > \frac{\varepsilon}{2}\right] \\
&+ \limsup_{k\to+\infty} P\left[\left|\int_0^t \sigma\left(s, \widehat{X}_s\right) d\widehat{B}_s^{n_k} - \int_0^t \sigma\left(s, \widehat{X}_s\right) d\widehat{B}_s\right| > \frac{\varepsilon}{2}\right].
\end{aligned}$$

It follows according to the Skorokhod limit theorem [19] page 32, or lemma 3, chapter 2 in [15] that the second term in the right hand side is equal to 0.

Let $\sigma^\delta(t, x) = \delta^{-d} \varphi(x/\delta) * \sigma(t, x)$ where $*$ denotes convolution on $\mathbf{R}^d$ and φ an infinitely differentiable function with support in the unit ball such that $\int \varphi(x)\, dx = 1$.

Applying Chebyshev and Doob inequalities, we obtain

$$\begin{aligned}
&P\left[\left|\int_0^t \left(\sigma\left(s, \widehat{X}_s^{n_k}\right) - \sigma\left(s, \widehat{X}_s\right)\right) d\widehat{B}_s^{n_k}\right| > \frac{\varepsilon}{2}\right] \\
&\leq \frac{4}{\varepsilon^2} E\left[\int_0^t \left|\sigma\left(s, \widehat{X}_s^{n_k}\right) - \sigma\left(s, \widehat{X}_s\right)\right|^2 ds\right] \\
&\leq \frac{16}{\varepsilon^2} \left\{ E\left[\int_0^t \left|\sigma\left(s, \widehat{X}_s^{n_k}\right) - \sigma^\delta\left(s, \widehat{X}_s^{n_k}\right)\right|^2 ds\right] \right. \\
&+ E\left[\int_0^t \left|\sigma^\delta\left(s, \widehat{X}_s^{n_k}\right) - \sigma^\delta\left(s, \widehat{X}_s\right)\right|^2 ds\right] \\
&\left. + E\left[\int_0^t \left|\sigma^\delta\left(s, \widehat{X}_s\right) - \sigma\left(s, \widehat{X}_s\right)\right|^2 ds\right] \right\} = \frac{16}{\varepsilon^2}\left(I_1 + I_2 + I_3\right).
\end{aligned}$$

It follows from the continuity of σ^δ in x and the convergence of $\widehat{X}_s^{n_k}$ to $\widehat{X}_s$ uniformly $\widehat{P}$ a.s, that I_2 goes to 0 as $k \to +\infty$ for every $\delta > 0$.

On the other hand we know that for each $p > 1$, $\sup_k E\left[\sup_{t\leq T}|X_t^{n_k}|^p\right] < +\infty$ then $\lim_{M\to+\infty} P\left[\sup_{t\leq T}|X_t^{n_k}| > M\right] = 0$. Therefore, without loss of generality we may suppose that σ^δ, σ have compact support in $[0,T] \times B(0,M)$.

Applying Krylov's inequality ([15], chapter 2, theorem 3.4), we obtain $I_1 + I_3 \leq N.\left\|\sigma^\delta - \sigma\right\|_{d+1,M}$, where N does not depend on δ, k and $\|.\|_{d+1,M}$ denotes the norm in $L^{d+1}([0,T] \times B(0,M))$.

By letting $\delta \to 0$, we obtain the desired result.

A similar claim holds for the integrals involving the drift terms. This achieves the proof. ■

By using similar techniques, one can show the following

Theorem 7.23 *Suppose that $\sigma(t,x)$ and $b(t,x,a)$ satisfy conditions a) and b). Moreover suppose that $a \longrightarrow b(t,x,a)$ is continuous. If pathwise uniqueness holds for equation (11), then theorem 6.21 remains valid.*

8 GENERICITY OF EXISTENCE AND UNIQUENESS

As we have seen in previous sections, pathwise uniqueness plays a key role in the proof of many stability results. It is then quite natural to raise the question whether the set of all nice functions (σ, b) for which pathwise uniqueness holds for stochastic differential equation $e(x,\sigma,b)$ is larger than its complement, in a sense to be specified. To make the question meaningful let us recall what we mean by generic property.

A property P is said to be generic for a class of stochastic differential equations F, if P is satisfied by each equation in $F - A$, where A is a set of first category (in the sense of Baire) in F. Results on generic properties for ordinary differential equations seem to go back to an old paper of Orlicz [18] , see also [16]. The investigation of such questions for stochastic differential equations is carried out in [1] , [10]. In this section, we show that the subset of continuous and bounded coefficients for which pathwise uniqueness holds for equation $e(x,\sigma,b)$ is a residual set. The proof is based essentially on theorem 2.5. Moreover it does not use the oscillation function introduced by Lasota & Yorke in ordinary differential equations and used in stochastic differential equations (see[10]).

Let us introduce some notations.

$e(x,\sigma,b)$ stands for equation (1) corresponding to coefficients σ, b and initial data x.

$\mathbf{M}^2 = \left\{\xi : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}^d, \text{ continuous and } \forall T > 0,\ E\left[\sup_{t\leq T}|\xi_t|^2\right] < +\infty\right\}$

Define a metric on $\mathbf{M}^2$ by:

$$d(\xi_1,\xi_2) = \sum_{n=1}^{+\infty} 2^{-n} \frac{\left(E \sup\limits_{0\le t\le n} |\xi_t^1 - \xi_t^2|^2\right)^{\frac{1}{2}}}{1+\left(E \sup\limits_{0\le t\le n} |\xi_t^1 - \xi_t^2|^2\right)^{\frac{1}{2}}}$$

By using Borel-Cantelli lemma, it is easy to see that $(\mathbf{M}^2, d)$ is a complete metric space.

Let $\mathcal{C}_1$ be the set of functions $b : \mathbf{R}_+ \times \mathbf{R}^d \longrightarrow \mathbf{R}^d$ which are continuous and bounded. Define the metric ρ_1 on $\mathcal{C}_1$ as follows:

$$\rho_1(b_1, b_2) = \sum_{n=1}^{+\infty} 2^{-n} \frac{\|b_1 - b_2\|_{\infty,n}}{1 + \|b_1 - b_2\|_{\infty,n}}$$

where $\|h\|_{\infty,n} = \sup\limits_{|x|\le n, |t|\le n} |h(x)|$.

Note that the metric ρ_1 is compatible with the topology of uniform convergence on compact subsets of $\mathbf{R}_+ \times \mathbf{R}^d$.

Let $\mathcal{C}_2$ be the set of continuous bounded functions $\sigma : \mathbf{R}_+ \times \mathbf{R}^d \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^r$ with the corresponding metric ρ_2.

It is clear that the space $\mathfrak{R} = \mathcal{C}_1 \times \mathcal{C}_2$ endowed with the product metric is a complete metric space.

Let L be the subset of $\mathfrak{R}$ consisting of functions $h(t,x)$ which are Lipschitz in both their arguments.

Proposition 8.24 *L is a dense subset in $\mathfrak{R}$.*

Proof. By truncation and regularisation arguments. ■

The main result of this section is the following

Theorem 8.25 *The subset $\mathcal{U}$ of $\mathfrak{R}$ consisting of those (σ, b) for which pathwise uniqueness holds for $e(x,\sigma,b)$ is a residual set.*

Lemma 8.26 *For each $(\sigma, b) \in L$ and $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $(\sigma', b') \in B((\sigma,b), \delta)$ and every pair of solutions X, Y of $e(x, \sigma', b')$ (defined on the same probability space and Brownian motion), we have $d(X,Y) < \varepsilon$.*

Proof Let Z be the unique strong solution of $e(x,\sigma,b)$ defined on the same probability space and Brownian motion B.

$d(X,Y) \le d(X,Z) + d(Z,Y)$, the result follows from the continuity of Z with respect to the coefficients (see theorem 2.5). ■

Proof of theorem 8.25 We put $\mathfrak{F} = \bigcap\limits_{k\ge 1} \bigcup\limits_{(\sigma,b)\in L} B\left((\sigma,b), \delta\left(\frac{1}{k}\right)\right)$.

$\mathfrak{F}$ is a G_δ dense subset in the Baire space $(\mathfrak{R},\rho)$ and for every $(\sigma, b) \in \mathfrak{F}$ pathwise uniqueness holds for $e(x,\sigma,b)$. It follows that $\mathcal{U}$ is a residual subset in $\mathfrak{R}$. ■

Remark 8.26 M.T. Barlow [2] has shown that $\mathfrak{R} - \mathfrak{F}$ is not empty.

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