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Haya Kaspi* and Jay Rosen†

1 Introduction

The local time process $(L_t^x)_{x\in S}$ for a Markov process with values in S measures, in a certain sense, the amount of time that the Markov process spends at each point up till time t. $(L_t^x)_{x\in S}$ is a family of continuous additive functionals which has been the subject of intensive investigation. Not all Markov processes have local times. In particular, Lévy processes can only have local times in one dimension since in higher dimensions they do not hit points. Nevertheless, one can study other families of continuous additive functionals and try to see which properties of the local time process $(L_t^x)_{x\in S}$ admit natural generalizations. The family of 'local times on lines' for the two-dimensional symmetric stable process X_t, which 'measures' the amount of time that X spends on each line up till time t, is in some ways the most natural extension of the family of local times at points. In this paper, the property of $(L_t^x)_{x\in R^1}$ which we plan to generalize is that of quadratic variation, or more generally p-variation, in the spatial variable. Aside from its intrinsic interest, we hope that this detailed study will pave the way for generalizations, both of other properties and to other families of continuous additive functionals.

The quadratic variation of the local time L_t^x of 1-dimensional Brownian motion W_s was studied in Bouleau and Yor [2] and Perkins [6]. They show that for any sequence of partitions π_n of $[a,b]$ with mesh size converging to zero,

$$\lim_{n\to\infty}\sum_{x_i\in\pi_n}(L_t^{x_i}-L_t^{x_{i-1}})^2=4\int_0^t 1_{[a,b]}(W_s)\,ds, \tag{1.1}$$

with convergence in probability. Similar results were obtained in [7] and [5] for the p-variation of the local times of 1-dimensional symmetric stable processes. The object of this paper is to generalize such results to the 'local times on lines' of 2-dimensional Brownian motion and symmetric stable processes. To better appreciate the results we shall obtain, we first reformulate (1.1). Let $c : [0,1]\mapsto R^1$ be a smooth curve, not necessarily monotone, and let $N(c\,|\,y)=\text{card}\{x\in[0,1]\,|\,c(x)=y\}$, the cardinality of the pre-image $c^{-1}(y)$. Using (1.1) on each interval of the complement of $\{s\,|\,c'(s)=0\}$ we see that for any sequence of partitions π_n of $[0,1]$ with mesh size converging to zero,

$$\lim_{n\to\infty}\sum_{s_i\in\pi_n}\left(L_t^{c(s_i)}-L_t^{c(s_{i-1})}\right)^2=4\int_0^t N(c\,|\,W_s)\,ds, \tag{1.2}$$

with convergence in probability.

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For any $0 \leq \theta < 2\pi$, let $e(\theta) = (\cos(\theta), \sin(\theta))$ denote the unit vector with angle θ, and $e(\theta^\perp) = (\sin(\theta), -\cos(\theta))$ denote the unit vector perpendicular to $e(\theta)$. We use $l_{a,\theta} = \{ae(\theta) + xe(\theta^\perp) \,|\, x \in R^1\}$ to denote the line such that $ae(\theta)$ is the foot of the perpendicular from the origin to $l_{a,\theta}$. If X_t denotes the symmetric stable process of index β in the plane, then $X_t^\theta = X_t \cdot e(\theta)$, the component of X_t in the direction of $e(\theta)$, is a real symmetric stable process of index β (just check the characteristic function). Let $L_t^{a,\theta}$ denote the local time of X_t^θ at $a \in R_+^1$.

Let $\gamma_s = (a_s, \theta_s)$ be a simple smooth curve, $\gamma : [0,1] \mapsto R_+^1 \times [0, 2\pi)$. For ease of notation we will sometimes write $\gamma(s)$ for γ_s.

Let $\Phi_\gamma : [0,1] \times R^1 \mapsto R^2$ be defined by $\Phi_\gamma(s,x) = a_s e(\theta_s) + xe(\theta_s^\perp))$ and let $N(\Phi_\gamma \,|\, y) = \text{card}\{(s,x) \in [0,1] \times R^1 \,|\, \Phi_\gamma(s,x) = y\}$, the cardinality of the pre-image $\Phi_\gamma^{-1}(y)$. Thus $N(\Phi_\gamma \,|\, y)$ is the number (possibly infinite) of parameter values s such that $y \in l_{\gamma_s}$.

Let $Q(0,1)$ denote the set of all partitions $\pi = \{0 = s_0 < s_1 \cdots < s_{k_\pi} = 1\}$ of $[0,1]$, and let $|\pi| = \sup_{1 \leq i \leq k_\pi}(s_i - s_{i-1})$ denote the length of the largest interval in π. ($|\pi|$ is called the mesh size of π).

Theorem 1 *If X_t is a planar Brownian motion, and $L_t^{a,\theta}$ denotes the local time of $X_t \cdot e(\theta)$, then*

$$\lim_{n\to\infty} \sum_{s_i \in \pi(n)} (L_t^{\gamma(s_i)} - L_t^{\gamma(s_{i-1})})^2 = 4 \int_0^t N(\Phi_\gamma \,|\, X_s)\, ds. \tag{1.3}$$

in L^2, uniformly both in $t \in [0,T]$ and $Q(0,1)$ as $|\pi(n)| \to 0$.

Note that for the special case of $\gamma(s) = (c(s), \theta_0)$ with θ_0 fixed, this reduces to (1.2) for the real Brownian motion $X_t \cdot e(\theta)$. When $\gamma(s) = (0, s)$, our theorem can be obtained from (1.1) using the skew-product representation for planar Brownian motion. However, already for the simple example $\gamma(s) = (1, s)$ our (1.3) is truly a two-dimensional result, and the formal similarity between Theorem 1 and (1.2) is thus rather striking.

Theorem 1 can essentially be proven using stochastic calculus and Tanaka's formula, and we will outline such a proof in section 4. However, we prefer to derive Theorem 1 as a special case of the Theorem 2 below for symmetric stable processes. We now introduce the notation needed for that Theorem. In what follows, X_t will denote the symmetric stable process in the plane of index β.

For $0 \leq \theta < 2\pi$, $a \in R_+^1$ let $\mu_{a,\theta}$ denote one-dimensional Lebesgue measure on $l_{a,\theta}$. Equivalently, if $h_{a,\theta} : R^1 \mapsto R^2$ is defined by $h_{a,\theta}(x) = ae(\theta) + xe(\theta^\perp)$ then $\mu_{a,\theta}$ is the measure induced by $h_{a,\theta}$ from Lebesgue measure λ^1 on the line: $\mu_{a,\theta}(A) = \lambda^1(h_{a,\theta}^{-1}(A))$. Thus

$$\int f(y)\, d\mu_{a,\theta}(y) = \int f(ae(\theta) + xe(\theta^\perp))\, dx.$$

We can easily check that $\mu_{a,\theta}$ is the Revuz measure of the CAF $L_t^{a,\theta}$ defined above.

Let $A_t^{\gamma,s}$ denote the CAF with Revuz measure $\widetilde{\mu}_{a,\theta,s}$ defined by

$$\int f(y)\, d\widetilde{\mu}_{a,\theta,s}(y) = \int f(a_s e(\theta_s) + xe(\theta_s^\perp))|a_s' + x\theta_s'|^{\beta-1}\, dx.$$

We note that

$$\int f(y)\,d\widetilde{\mu}_{a,\theta,s}(y) = \int f(y)|a'_s + y\cdot e(\theta_s^\perp)\theta'_s|^{\beta-1}\,d\mu_{a,\theta}(y) \tag{1.4}$$

so that

$$A_t^{\gamma,s} = \int_0^t |a'_s + X_r\cdot e(\theta_s^\perp)\theta'_s|^{\beta-1}\,dL_r^{a_s,\theta_s}. \tag{1.5}$$

In Lemma 1 below we show that $s\mapsto A_t^{\gamma,s}$ is continuous in L^{2k}, and therefore $\int_0^1 (A_t^{\gamma,s})^k\,ds$ is well defined in L^2.

Let $\bar{c}(\beta) = (2k)!!(4c(\beta))^k$ where

$$c(\beta) = -\frac{\cos(\pi(\beta-1)/2)}{\pi(\beta-1)}\Gamma(2-\beta)$$

for $1<\beta<2$, and $c(2)=1$.

Theorem 2 *If X_t is a symmetric stable process in the plane of index $\beta = 1+1/k$ with $k=1,2,\ldots$, and $L_t^{a,\theta}$ denotes the local time of $X_t\cdot e(\theta)$ then*

$$\lim_{n\to\infty}\sum_{s_i\in\pi(n)} (L_t^{\gamma(s_i)} - L_t^{\gamma(s_{i-1})})^{2k} = \bar{c}(\beta)\int_0^1 (A_t^{\gamma,s})^k\,ds \tag{1.6}$$

in L^2, uniformly both in $t\in[0,T]$ and $Q(0,1)$ as $|\pi(n)|\to 0$.

Remark 1. For the special case of $\gamma_s = (s,\theta)$, with θ fixed, so that $L_t^{\gamma(s_i)} = L_t^{s_i,\theta}$, this reduces to Theorem 1 of [7] for the p-variation of the local time of the 1-dimensional symmetric stable process X_t^θ.

For the case of planar Brownian motion, Theorem 2 says that

$$\lim_{n\to\infty}\sum_{s_i\in\pi(n)} (L_t^{\gamma(s_i)} - L_t^{\gamma(s_{i-1})})^2 = 4\int_0^1 A_t^{\gamma,s}\,ds. \tag{1.7}$$

In [1] it is shown that for planar Brownian motion we can choose an a.s. continuous version of $\{L_t^\theta\,;\,(\theta,t)\in[0,2\pi)\times R_+^1\}$, so that by (1.5) we can choose an a.s. continuous version of $\{A_t^{\gamma,s}\,;\,(s,t)\in[0,1]\times R_+^1\}$. We note that $\int_0^1 A_t^{\gamma,s}\,ds$ is the CAF with Revuz measure $\nu = \int_0^1 \widetilde{\mu}_{a,\theta,s}\,ds$. It is easily checked that $|a'_s + x\theta'_s|$ is $|J(\Phi_\gamma)(s,x)|$, the absolute value of the Jacobean of the map Φ_γ. We have that

$$\begin{aligned}\int f(y)\,d\nu(y) &= \int_0^1\int f(y)\,d\widetilde{\mu}_{a,\theta,s}(y)\,ds \\ &= \int_0^1\int f(a_s e(\theta_s) + xe(\theta_s^\perp))|a'_s + x\theta'_s|\,dx\,ds \\ &= \int_0^1\int f(a_s e(\theta_s) + xe(\theta_s^\perp))|J(\Phi_\gamma)(s,x)|\,dx\,ds \\ &= \int f(y)N(\Phi_\gamma\,|\,y)\,dy\end{aligned} \tag{1.8}$$

where the last step uses the Area Formula, see 3.2.3 of [3]. This shows that $d\nu(y) = N(\Phi_\gamma\,|\,y)\,dy$, which proves Theorem 1.

2 $A_t^{\gamma,s}$

Lemma 1 *If $\beta = 1 + 1/k$ with $k = 1, 2, \ldots$, then for any s, t, we have $A_t^{\gamma,s} \in L^{2k}$ and $s \mapsto A_t^{\gamma,s}$ is continuous in L^{2k}.*

Proof of Lemma 1: We have

$$
\begin{aligned}
(2.1)\qquad & E\left(\{A_t^{\gamma,s}\}^{2k}\right) \\
&= (2k)! \int \int \cdots \int_{\sum_{i=1}^{2k} t_i \le t} \prod_{i=1}^{2k} p_{t_i}(x_{i-1}, x_i) dt_i \, d\tilde{\mu}_{a,\theta,s}(x_i) \\
&\le C \int \prod_{i=1}^{2k} u^1(x_{i-1}, x_i) \, d\tilde{\mu}_{a,\theta,s}(x_i) \\
&= C \int \prod_{i=1}^{2k} u^1\left(a_s e(\theta_s) + (y_i - y_{i-1}) e(\theta_s^\perp)\right) |a_s' + y_i \theta_s'|^{1/k} \, dy_i \\
&\le C \int \prod_{i=1}^{2k} u^1\left(a_s e(\theta_s) + (y_i - y_{i-1}) e(\theta_s^\perp)\right) (1 + |y_i|^{1/k}) \, dy_i \\
&= C \int \prod_{i=1}^{2k} u^1\left(a_s e(\theta_s) + z_i e(\theta_s^\perp)\right) (1 + |\sum_{j=1}^{i} z_j|^{1/k}) \, dz_i \\
&\le C \int \prod_{i=1}^{2k} u^1\left(a_s e(\theta_s) + z_i e(\theta_s^\perp)\right) (1 + |z_i|^2) \, dz_i \\
&= C \prod_{i=1}^{2k} \int u^1\left(a_s e(\theta_s) + z_i e(\theta_s^\perp)\right) (1 + |z_i|^2) \, dz_i
\end{aligned}
$$

We claim that the one dimensional integral

$$(2.2)\qquad \int_{R^1} u^1\left(a e(\theta) + z e(\theta^\perp)\right) (1 + |z|^2) \, dz < \infty,$$

which would show that (2.1) is finite, establishing the first part of our Lemma. To show (2.2) we first recall that each component of $X_1 = (X_1^{(1)}, X_1^{(2)})$ is a real-valued symmentic stable process of index β, and therefore has moments of order β', for any $\beta' < \beta$, [4], p.578. Consequently,

$$(2.3)\qquad c_0 \stackrel{def}{=} E(|X_1|^{\beta'}) \le E(|X_1^{(1)}|^{\beta'}) + E(|X_1^{(2)}|^{\beta'}) < \infty.$$

Since, by scaling, $X_t = t^{1/\beta} X_1$ in law, we have $E(|X_t|^{\beta'}) = t^{\beta'/\beta} c_0$. Therefore

$$(2.4)\qquad \int_{R^2} |x|^{\beta'} u^1(x) \, dx = \int_0^\infty e^{-t} E(|X_t|^{\beta'}) \, dt < \infty.$$

$u^1(x)$ is spherically symmetric, and, by abuse of notation, we use $u^1(r)$ for the function on R_+^1 whose value is $u^1(re(\theta))$ for any θ. Using polar coordinates, we can now rewrite (2.4) as

$$(2.5)\qquad \int_0^\infty |r|^{1+\beta'} u^1(r) \, dr < \infty$$

which implies (2.2). In fact, (2.5) together with the monotonicity of $u^1(r)$ implies that for any $\delta > 0$

$$\sup_a \int_{R^1} u^1\left(ae(\theta) + ze(\theta^\perp)\right)(1 + |z|^{2+1/k-\delta})\,dz < \infty. \tag{2.6}$$

The fact that $s \mapsto A_t^{\gamma,s}$ is continuous in L^{2k} now follows from these considerations, using the bound (3.9) below with $\gamma > 0$ small.

3 A Second Moment Proof

Proof of Theorem 2: We write, for $\tau \in Q(0,1)$

$$\begin{aligned}
& E\left(\left\{\bar{c}(\beta)\int_0^1 (A_t^{\gamma,r})^k\,dr - \sum_{r_i\in\tau}\left(L_t^{\gamma(r_i)} - L_t^{\gamma(r_{i-1})}\right)^{2k}\right\}^2\right) \\
&= \bar{c}^2(\beta)\int_0^1\int_0^1 E\left\{(A_t^{\gamma,r})^k\left(A_t^{\gamma,r'}\right)^k\right\}dr\,dr' \\
&- 2\bar{c}(\beta)\int_a^b \sum_i E\left\{\left(L_t^{\gamma(r_i)} - L_t^{\gamma(r_{i-1})}\right)^{2k}\left(A_t^{\gamma,r'}\right)^k\right\}dr' \\
&+ \sum_{i,j} E\left\{\left(L_t^{\gamma(r_i)} - L_t^{\gamma(r_{i-1})}\right)^{2k}\left(L_t^{\gamma(r_j)} - L_t^{\gamma(r_{j-1})}\right)^{2k}\right\} \\
&\doteq A - 2B_\epsilon + C_\epsilon, \quad \text{where } \epsilon \doteq |\tau|
\end{aligned} \tag{3.1}$$

We will show that as $\epsilon \to 0$, each of $A, B_\epsilon, C_\epsilon$ converge to

$$\begin{aligned}
&[(2k)!(2c(\beta))^k]^2 \sum_{\tilde\pi}\int_0^1 dr\int_0^1 dr' \\
&\int\cdots\int_{\sum_{i=1}^{2k} t_i\le t} \prod_{i=1}^{2k} p_{t_i}(x^{\tilde\pi(i-1)}_{c_{\tilde\pi}(i-1)}, x^{\tilde\pi(i)}_{c_{\tilde\pi}(i)})dt_i \prod_{j=1}^{k} d\tilde\mu_{a,\theta,r}(x_j^1)\,d\tilde\mu_{a,\theta,r'}(x_j^2)
\end{aligned} \tag{3.2}$$

where the sum runs over all paths $\tilde\pi : \{1,\ldots,2k\} \to \{1,2\}$ which visit $1,2$ an equal number of times (i.e. k times each), and $c_{\tilde\pi}(i) = |\{j \le i \,|\, \tilde\pi(j) = \tilde\pi(i)\}|$.

The fact that A equals (3.2) is straightforward, so we turn to B_ϵ. We will write $\mu_{a,\theta,s}$ for μ_{a_s,θ_s}. We have

$$\begin{aligned}
&\bar{c}(\beta)E\left\{\left(L_t^{\gamma(r_l)} - L_t^{\gamma(r_{l-1})}\right)^{2k}(A_t^{\gamma,s})^k\right\} \\
&= ((2k)!)^2(2c(\beta))^k\sum_\pi\int\cdots\int_{\sum_{i=1}^{3k} t_i\le t}\prod_{i=1}^{3k} p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, x^{\pi(i)}_{c_\pi(i)})dt_i\,d\mu^{\pi(i)}_{a,\theta}(x^{\pi(i)}_{c_\pi(i)})
\end{aligned}$$

where the sum runs over all paths $\pi : \{1,\ldots,3k\} \longrightarrow \{1,2\}$ which visit 2 exactly k times, and

$$\begin{aligned}
\mu^1_{a,\theta} &\doteq \mu_{a,\theta,r_l} - \mu_{a,\theta,r_{l-1}} \\
\mu^2_{a,\theta} &\doteq \tilde\mu_{a,\theta,r'}
\end{aligned} \tag{3.3}$$

Fix such a π. We intend to rewrite

$$\int \cdots \int_{\sum_{i=1}^{3k} t_i \le t} \prod_{i=1}^{3k} p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, x^{\pi(i)}_{c_\pi(i)}) dt_i \, d\mu^{\pi(i)}_{a,\theta}(x^{\pi(i)}_{c_\pi(i)}) \tag{3.4}$$

as a sum of many terms, most of which will make 0 contribution to (3.1) in the $\epsilon \to 0$ limit. Eventually we will identify those terms that contribute to (3.1) in the $\epsilon \to 0$ limit, and show how they lead to (3.2). Our procedure involves three steps.

Step 1: Let $\phi : l(a_{r_l}, \theta_{r_l}) \mapsto l(a_{r_{l-1}}, \theta_{r_{l-1}})$ be defined by

$$\phi(a_{r_l} e(\theta_{r_l}) + x e(\theta^\perp_{r_l})) = a_{r_{l-1}} e(\theta_{r_{l-1}}) + x e(\theta^\perp_{r_{l-1}}).$$

With this notation we rewrite

$$\begin{aligned} &\int p_s(x,y) p_t(y,z) \left(d\mu_{a,\theta,r_l}(y) - d\mu_{a,\theta,r_{l-1}}(y) \right) \\ &= \int \{ p_s(x,y) p_t(y,z) - p_s(x,\phi(y)) p_t(\phi(y),z) \} \, d\mu_{a,\theta,r_l}(y) \\ &= \int \{ p_s(x,y) - p_s(x,\phi(y)) \} \, p_t(y,z) \, d\mu_{a,\theta,r_l}(y) \\ &\qquad + \int p_s(x,\phi(y)) \, \{ p_t(y,z) - p_t(\phi(y),z) \} \, d\mu_{a,\theta,r_l}(y) \\ &= \int \Delta_2 p_s(x,y) p_t(y,z) \, d\mu_{a,\theta,r_l}(y) \\ &\qquad + \int p_s(x,\phi(y)) \Delta_1 p_s(y,z) \, d\mu_{a,\theta,r_l}(y) \end{aligned} \tag{3.5}$$

where

$$\Delta_2 p_s(x,y) = p_s(x,y) - p_s(x,\phi(y)),$$
$$\Delta_1 p_s(y,z) = p_t(y,z) - p_t(\phi(y),z).$$

We proceed to rewrite the factors in (3.4), working with each factor in turn, in order of decreasing i. For the largest i with $\pi(i) = 1$ we use (3.5) to replace

$$\int p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, x^1_{c_\pi(i)}) p_{t_{i+1}}(x^1_{c_\pi(i)}, x^{\pi(i+1)}_{c_\pi(i+1)}) \, d\mu^1_{a,\theta}(x^1_{c_\pi(i)}) \tag{3.6}$$

by a sum of two terms, which we then view as replacing (3.4) by two terms. In case the largest such i is $3k$, we simply rewrite

$$\begin{aligned} &\int p_{t_{3k}}(x^{\pi(3k-1)}_{c_\pi(3k-1)}, x^1_{c_\pi(3k)}) \, d\mu^1_{a,\theta}(x^1_{c_\pi(3k)}) \\ &= \int p_{t_{3k}}(x^{\pi(3k-1)}_{c_\pi(3k-1)}, x^1_{c_\pi(3k)}) \left(d\mu_{a,\theta,r_l}(x^1_{c_\pi(3k)}) - d\mu_{a,\theta,r_{l-1}}(x^1_{c_\pi(3k)}) \right) \\ &= \int \Delta_2 p_{t_{3k}}(x^{\pi(3k-1)}_{c_\pi(3k-1)}, x^1_{c_\pi(3k)}) \, d\mu_{a,\theta,r_l}(x^1_{c_\pi(3k)}). \end{aligned} \tag{3.7}$$

We then proceed through decreasing i. At each stage that we find an i with $\pi(i) = 1$, we check to see if the previously handled factor, i.e. the factor involving $(x^1_{c_\pi(i)}, x^{\pi(i+1)}_{c_\pi(i+1)})$

is a p or a Δp. If it is a p, then we proceed as before using (3.5). If it is a Δp, we do not use (3.5), but rather simply write out $d\mu^1_{a,\theta}(x^1_{c_\pi(i)})$. In other words we use

$$\begin{aligned}(3.8)\qquad &\int p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, x^1_{c_\pi(i)})p_{t_{i+1}}(x^1_{c_\pi(i)}, x^{\pi(i+1)}_{c_\pi(i+1)})\, d\mu^1_{a,\theta}(x^1_{c_\pi(i)}) \\ &= \int p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, x^1_{c_\pi(i)})p_{t_{i+1}}(x^1_{c_\pi(i)}, x^{\pi(i+1)}_{c_\pi(i+1)})\, d\mu_{a,\theta,r_l}(x^1_{c_\pi(i)}) \\ &- \int p_{t_i}(x^{\pi(i-1)}_{c_\pi(i-1)}, \phi(x^1_{c_\pi(i)}))p_{t_{i+1}}(\phi(x^1_{c_\pi(i)}), x^{\pi(i+1)}_{c_\pi(i+1)})\, d\mu_{a,\theta,r_l}(x^1_{c_\pi(i)})\end{aligned}$$

to generate two terms.

After we have proceeded in this manner for all i, we will have replaced (3.4) by many terms. Each term will have at least k factors of the form Δp. We claim that any term with more that k factors of the form Δp will be $o(\Delta r_l)$, hence such terms will make 0 contribution to (3.1) in the $\epsilon \to 0$ limit. To see this, let $u_\alpha(x)$ denote a generic symmetric, positive, monotone decreasing (in $|x|$) function on R^2 such that $u_\alpha(x) \leq Cu^1(x/2)$ for $|x| \geq 1$ and $u_\alpha(x) \leq C|x|^{-\alpha}$ for $|x| \leq 1$. We then have that $\int_0^T p_t(x)\, dt \leq cu_{1-1/k}(x)$ while for any $0 \leq \gamma \leq 1$

$$(3.9)\qquad \int_0^T |p_t(x+a) - p_t(x)|\, dt \leq c|a|^\gamma(u_{1-1/k+\gamma}(x+a) + u_{1-1/k+\gamma}(x)).$$

(Consider seperately $|x| \leq 4|a|$ and $|x| > 4|a|$.) We then use these estimates to bound the integrals in the term we are studying, once again proceeding in order of decreasing i. (Note: our integration is always over lines.) If our term has $j > k$ factors of the form Δp, we choose $\gamma < 1/k$, so that $u_{1-1/k+\gamma}(x)$ will be integrable on any line, but with $1 < j\gamma < 1 + 1/k$. Using the fact that $|y - \phi(y)| \leq c(1 + |x|)\Delta r_l$ for $y = a_{r_l}e(\theta_{r_l}) + xe(\theta^\perp_{r_l})$, we see that our term can be bounded by $c|x|^{j\gamma}|\Delta r_l|^{j\gamma}$ which establishes our claim. (The condition $j\gamma < 1 + 1/k$ guarantees that the factors of $|x|$ do not mount up sufficiently to destroy integrability; see the proof of Lemma 1).

We will say that a path π is even if its visits to 1 occur in even runs. A path will be called odd if it is not even. It is easily seen that when our proceedure is applied to any odd π, all the resulting terms will have $> k$ factors of the form Δp, hence such terms will make 0 contribution to (3.1) in the $\epsilon \to 0$ limit. Similarly, if π is even, the only resulting terms with k factors of the form Δp, will be those terms in which for each i such that $\pi(i)$ is an even-numbered visit to 1, we replace the factor p_{t_i} by a $\Delta_2 p_{t_i}$, while for each i such that $\pi(i)$ is an odd-numbered visit to 1, we retain the factor p_{t_i}, more precisely we use (3.8). We note that for such terms, the k factors of the form $\Delta_2 p_{t_i}$ are seperated from each other by p factors

Step 2: Let π be a fixed even path, and consider one of the terms generated in Step 1 which does make a contribution to (3.1) in the $\epsilon \to 0$ limit. As described above, such a term has precisely k factors of the form $\Delta_2 p$ and these k factors are seperated from each other by p factors. Let us rewrite every pair of the form $\Delta_2 p_s(x,y)p_t(y,z)$ using

$$\begin{aligned}(3.10)\qquad &\Delta_2 p_s(x,y)p_t(y,z) \\ &= \Delta_2 p_s(x,y)p_t(x,z) + \Delta_2 p_s(x,y)\left(p_t(y,z) - p_t(x,z)\right).\end{aligned}$$

Once more this allows us to rewrite our term as a sum of many terms. We now show that any term which contains a pair of the form $\Delta_2 p_s(x,y)\left(p_t(y,z) - p_t(x,z)\right)$ will be

$o(\Delta r_l)$, hence such terms will make 0 contribution to (3.1) in the $\epsilon \to 0$ limit. To see this we proceed as above to bound our term by bounding the integrals, proceeding in order of decreasing i. We use (3.9) to bound the contribution of the z, t integrals of $|p_t(y,z) - p_t(x,z)|$ by $c|x-y|^\gamma$, with $\gamma = 3/(4k)$. We then use the bound

$$
\begin{aligned}
(3.11) \qquad & \int |y|^\gamma \left(\int_0^T |p_s(y+a) - p_s(y)|\, ds \right) d^1y \\
&= \int_{|y|\le 2|a|} |y|^\gamma \left(\int_0^T |p_s(y+a) - p_s(y)|\, ds \right) d^1y \\
&\quad + \int_{|y|>2|a|} |y|^\gamma \left(\int_0^T |p_s(y+a) - p_s(y)|\, ds \right) d^1y \\
&\le c|a|^\gamma \int_{|y|\le 2|a|} \left(u_{1-1/k}(y+a) + u_{1-1/k}(y) \right) d^1y \\
&\quad + c|a|^{\gamma'} \int_{|y|>2|a|} |y|^\gamma u_{1-1/k+\gamma'}(y)\, d^1y \\
&\le c|a|^{\gamma+1/k} + c|a|^{\gamma'} \int_{|y|>2|a|} u_{1-1/k+\gamma'-\gamma}(y)\, d^1y \\
&\le c(|a|^{7/(4k)} + |a|^{3/(2k)})
\end{aligned}
$$

by taking $\gamma' = 3/(2k)$. It is easily seen that this verifies our claim.

Step 3: We are now reduced to considering terms with precisely k factors of the form $\Delta_2 p_s(x, y_i)$, with no other factors containing the variable y_i. We now integrate each such factor with respect to $d\mu_{a,\theta,r_l}(y_i)$.

Note that

$$
\begin{aligned}
(3.12) \qquad & \int p_s(y,z)\, d\mu_{a,\theta,r}(z) \\
&= \int p_s(y - a_r e(\theta_r) - x e(\theta_r^\perp))\, dx \\
&= q_s(a_r - y \cdot e(\theta_r))
\end{aligned}
$$

where $q_s(x)$ denotes the density of the one dimensional symmetric stable process of index β. Setting

$$
(3.13) \qquad Q_s(y) = q_s\{a_{r_l} - y \cdot e(\theta_{r_l})\} - q_s\{a_{r_{l-1}} - y \cdot e(\theta_{r_{l-1}})\}
$$

we see from (3.12) that

$$
(3.14) \qquad \int \Delta_2 p_s(x_u^1, x_v^1) d\mu_{a,\theta,r_l}(x_v^1) = Q_s(x_u^1).
$$

In this manner we can see that the sum of all terms generated by (3.4) which contribute to (3.1) in the $\epsilon \to 0$ limit can be written as

$$
\begin{aligned}
(3.15) \qquad & \sum_{A \subseteq \{1,\dots,k\}} (-1)^{|A|} \int\cdots\int_{0 \le \sum_{i=1}^{2k} t_i + \sum_{j=1}^k \tau_j \le t} \prod_{i=1}^{2k} p_{t_i}(x^{\bar\pi(i)}_{c_{\bar\pi}(i),A}, x^{\bar\pi(i-1)}_{c_{\bar\pi}(i-1),A}) dt_i \\
& \prod_{j=1}^k Q_{\tau_j}(x^1_{j,A})\, d\mu_{a,\theta,r_l}(x_j^1)\, d\tilde\mu_{a,\theta,r'}(x_j^2)
\end{aligned}
$$

where $\bar{\pi}: \{1,\ldots,2k\} \longrightarrow \{1,2\}$ is the path visiting both 1 and 2 exactly k times induced by π as follows: since visits of π to 1 occur in pairs, we simply suppress one visit from each pair, and $x^2_{j,A} = x^2_j$ while $x^1_{j,A} = \phi(x^1_j)$ if $j \in A$ and $x^1_{j,A} = x^1_j$ if $j \notin A$. The methods of Step 1 show that we can replace this, up to terms which will contribute 0 to (3.1) in the $\epsilon \to 0$ limit by

$$\sum_{A\subseteq\{1\ldots,k\}} (-1)^{|A|} \int\cdots\int_{0\le\sum_{i=1}^{2k} t_i+\sum_{j=1}^{k}\tau_j\le t} \prod_{i=1}^{2k} p_{t_i}(x^{\bar{\pi}(i)}_{c_{\bar{\pi}}(i)}, x^{\bar{\pi}(i-1)}_{c_{\bar{\pi}}(i-1)})dt_i$$
$$\prod_{j=1}^{k} Q_{\tau_j}(x^1_{j,A})\, d\mu_{a,\theta,r_l}(x^1_j)\, d\tilde{\mu}_{a,\theta,r'}(x^2_j). \tag{3.16}$$

Note that

$$\begin{aligned} &Q_s(a_{r_l}e(\theta_{r_l}) + xe(\theta^{\perp}_{r_l}))\\ &= q_s\{0\} - q_s\{a_{r_{l-1}} - a_{r_l}e(\theta_{r_l})\cdot e(\theta_{r_{l-1}}) - xe(\theta^{\perp}_{r_l})\cdot e(\theta_{r_{l-1}})\}\\ &= q_s\{0\} - q_s\{a_{r_{l-1}} - a_{r_l}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - x\sin(\theta_{r_l}-\theta_{r_{l-1}})\}\end{aligned} \tag{3.17}$$

while

$$\begin{aligned} &Q_s(a_{r_{l-1}}e(\theta_{r_{l-1}}) + xe(\theta^{\perp}_{r_{l-1}}))\\ &= q_s\{a_{r_l} - a_{r_{l-1}}e(\theta_{r_{l-1}})\cdot e(\theta_{r_l}) - xe(\theta^{\perp}_{r_{l-1}})\cdot e(\theta_{r_l})\} - q_s\{0\}\\ &= q_s\{a_{r_l} - a_{r_{l-1}}\cos(\theta_{r_{l-1}}-\theta_{r_l}) - x\sin(\theta_{r_{l-1}}-\ th_{r_l})\} - q_s\{0\}\\ &= q_s\{a_{r_{l-1}}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - a_{r_l} - x\sin(\theta_{r_l}-\theta_{r_{l-1}})\} - q_s\{0\}\\ &= -Q_s(a_{r_l}e(\theta_{r_l}) + xe(\theta^{\perp}_{r_l})) + \Delta Q_s(a_{r_l}e(\theta_{r_l}) + xe(\theta^{\perp}_{r_l}))\end{aligned} \tag{3.18}$$

where

$$\begin{aligned} &\Delta Q_s(a_{r_l}e(\theta_{r_l}) + xe(\theta^{\perp}_{r_l}))\\ &= q_s\{a_{r_{l-1}}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - a_{r_l} - x\sin(\theta_{r_l}-\theta_{r_{l-1}})\}\\ &-q_s\{a_{r_{l-1}} - a_{r_l}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - x\sin(\theta_{r_l}-\theta_{r_{l-1}})\}.\end{aligned} \tag{3.19}$$

We record here Lemma 1 of [7].

Lemma 2

$$\int_0^t \mid q_t(x) - q_t(y) \mid\, dt \le c\, \Big|\, |x|^{\beta-1} - |y|^{\beta-1}\Big| \le c|x-y|^{\beta-1} \tag{3.20}$$

and

$$\int_0^T q_t(0) - q_t(x)dt = c(\beta)|x|^{\beta-1} + O\left(\frac{|x|^2}{T^{3/\beta-1}}\right) \tag{3.21}$$

where

$$c(\beta) = \int_0^\infty (q_t(0) - q_t(1))dt < \infty. \tag{3.22}$$

Actually, we need a slight refinement of (3.21). The error term $O\left(|x|^2/T^{3/\beta-1}\right)$ comes from a bound on $\int_T^\infty (q_t(0) - q_t(x))dt$. We can also bound this integral by

$2\int_T^\infty q_t(0)dt = C/T^{2/\beta-1}$. Interpolating these two bounds shows that for $\delta > 0$ sufficiently small we have

$$\int_0^T q_t(0) - q_t(x)dt = c(\beta)|x|^{\beta-1} + O\left(\frac{|x|^{\beta-1+\delta}}{T^{(3+\delta)/2\beta-1/2}}\right) \tag{3.23}$$

Using this we see that

$$\int_0^T |\Delta Q_\tau(a_{r_l}e(\theta_{r_l}) + xe(\theta_{r_l}^\perp))|\,d\tau = O\left(\frac{(1+|x|^{\beta-1+\delta})|\Delta r_l|^{\beta-1+\delta}}{T^{(3+\delta)/2\beta-1/2}}\right). \tag{3.24}$$

Thus, up to terms which will not contribute to (3.1) in the $\epsilon \to 0$ limit we can replace $Q_{\tau_j}(x^1_{j,A})$ by $(-1)^{1_A(j)}Q_{\tau_j}(x^1_j)$ in (3.16). Thus (3.16) can be replaced by

$$\begin{aligned}
&\sum_{A\subseteq\{1,\dots,k\}} \int\cdots\int_{0\le\sum_{i=1}^{2k}t_i+\sum_{j=1}^{k}\tau_j\le t} \prod_{i=1}^{2k} p_{t_i}(x^{\bar\pi(i)}_{c_{\bar\pi}(i)}, x^{\bar\pi(i-1)}_{c_{\bar\pi}(i-1)})dt_i \\
&\qquad \prod_{j=1}^{k} Q_{\tau_j}(x^1_j)\,d\mu_{a,\theta,r_l}(x^1_j)\,d\tilde\mu_{a,\theta,r'}(x^2_j) \\
&= 2^k \int\cdots\int_{0\le\sum_{i=1}^{2k}t_i+\sum_{j=1}^{k}\tau_j\le t} \prod_{i=1}^{2k} p_{t_i}(x^{\bar\pi(i)}_{c_{\bar\pi}(i)}, x^{\bar\pi(i-1)}_{c_{\bar\pi}(i-1)})dt_i \\
&\qquad \prod_{j=1}^{k} Q_{\tau_j}(x^1_j)\,d\mu_{a,\theta,r_l}(x^1_j)\,d\tilde\mu_{a,\theta,r'}(x^2_j)
\end{aligned} \tag{3.25}$$

Furthermore, (3.23) tells us that

$$\begin{aligned}
&\int_0^T Q_\tau(a_{r_l}e(\theta_{r_l}) + xe(\theta_{r_l}^\perp))\,d\tau \\
&= \int_0^T q_\tau\{0\} - q_\tau\{a_{r_{l-1}} - a_{r_l}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - x\sin(\theta_{r_l}-\theta_{r_{l-1}})\}\ ,d\tau \\
&= c(\beta)|a_{r_{l-1}} - a_{r_l}\cos(\theta_{r_l}-\theta_{r_{l-1}}) - x\sin(\theta_{r_l}-\theta_{r_{l-1}})|^{\beta-1} \\
&\qquad + O\left(\frac{(1+|x|^{\beta-1+\delta})|\Delta r_l|^{\beta-1+\delta}}{T^{(3+\delta)/2\beta-1/2}}\right) \\
&= c(\beta)|a'_{r_l} + x\theta'_{r_l}|^{\beta-1}(\Delta r_l)^{\beta-1} + O\left(\frac{(1+|x|^{\beta-1+\delta})|\Delta r_l|^{\beta-1+\delta}}{T^{(3+\delta)/2\beta-1/2}}\right)
\end{aligned} \tag{3.26}$$

Using Lemma 3 of [7] now completes the proof that B_ϵ converges to (3.2).

Finally, the fact that C_ϵ converges to (3.2) follows using the methods employed above for B_ϵ together with the methods of [7]. This completes the proof of Theorem 1.

4 A Stochastic Calculus Proof

In this section we outline a proof of Theorem 1 using stochastic calculus. In fact, we only deal with convergence in probability, and for a fixed sequence $\pi(n)$ of partitions with $|\pi(n)| \to 0$. To emphasize that we are dealing with planar Brownian motion we use the notation $B_s = (B_s^1, B_s^2)$ in place of X_s, and write $B_s^\theta = B_s \cdot e(\theta)$, etc. By Tanaka's formula

$$\frac{1}{2} L_t^{\gamma(s)} = \left(B_t^{\theta(s)} - a(s)\right)^+ - \left(B_0^{\theta(s)} - a(s)\right)^+ - \int_0^t 1_{\{B_u^{\theta(s)} > a(s)\}} dB_u^{\theta(s)} . \tag{4.1}$$

As we will explain below, the only term which will contribute to the quadratic variation in the limit is the stochastic integeral term. Let

$$\hat{B}_t^{\gamma(s)} = \int_0^t 1_{\{B_u^{\theta(s)} > a(s)\}} dB_u^{\theta(s)}$$

and consider

$$V_n = \sum_{s_i \in \pi(n)} \left(\hat{B}_t^{\gamma(s_{i+1})} - \hat{B}_t^{\gamma(s_i)}\right)^2 .$$

By Ito's formula

$$\begin{aligned}
&\left(\hat{B}_t^{\gamma(s_{i+1})} - \hat{B}_t^{\gamma(s_i)}\right)^2 \\
&= 2\int_0^t \left(\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)}\right) \left[1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1})\}} dB_u^{\theta(s_i)} - 1_{\{B_u^{\theta(s_i)} > a(s_i)\}} dB_u^{\theta(s_i)}\right] \\
&+ \int_0^t du \left(1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1})\}} + 1_{\{B_u^{\theta(s_i)} > a(s_i)\}}\right) \\
&-2\int_0^t du \cdot 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1})\}} 1_{\{B_u^{\theta(s_i)} > a(s_i)\}} [\cos\theta(s_{i+1})\cos\theta(s_i) + \sin\theta(s_{i+1})\sin\theta(s_i)] .
\end{aligned}$$

(4.2)

As we explain below, the only contribution to $\lim_{n\to\infty} V_n$ will come from the last two lines in (4.2) which we can rewrite as

$$\begin{aligned}
&\int_0^t \Big(1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1})\}} + 1_{\{B_u^{\theta(s_i)} > a(s_i)\}} \\
&\qquad -2\cos(\theta(s_{i+1}) - \theta(s_i)) 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}),\, B_u^{\theta(s_i)} > a(s_i)\}}\Big)\, du \\
&= \int_0^t \left(1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}),\, B_u^{\theta(s_i)} < a(s_i)\}} + 1_{\{B_u^{\theta(s_i)} > a(s_i),\, B_u^{\theta(s_{i+1})} < a(s_{i+1})\}}\right) du \\
&\qquad + O(s_{i+1} - s_i)^2 .
\end{aligned}$$

Writing

$$\begin{aligned}
C_{s_i, s_{i+1}} &= \left\{x \in R^2 \,|\, x \cdot e(\theta(s_{i+1})) > a(s_{i+1}),\, x \cdot e(\theta(s_i)) < a(s_i)\right\} \\
&\quad \bigcup \left\{x \in R^2 \,|\, x \cdot e(\theta(s_{i+1})) < a(s_{i+1}),\, x \cdot e(\theta(s_i)) > a(s_i)\right\}
\end{aligned}$$

we then see that

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \int_0^t \sum_{s_i \in \pi(n)} 1_{C_{s_i, s_{i+1}}}(B_u)\, du. \tag{4.3}$$

$C_{s_i,s_{i+1}}$ is, in general, the cone contained between the lines $l_{\gamma(s_i)}$ and $l_{\gamma(s_{i+1})}$. (If the lines are parallel, $C_{s_i,s_{i+1}}$ is the strip between them). $C_{s_i,s_{i+1}}$ is not quite the same as $\widetilde{C}_{s_i,s_{i+1}} = \cup_{s_i \le s \le s_{i+1}} l_{\gamma(s)}$, but a detailed trigonometric calculation shows that in the $n \to \infty$ limit we can replace $C_{s_i,s_{i+1}}$ by $\widetilde{C}_{s_i,s_{i+1}}$ in (4.4) to obtain

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \int_0^t \sum_{s_i \in \pi(n)} 1_{\widetilde{C}_{s_i,s_{i+1}}}(B_u)\, du \tag{4.4}$$

(It is mainly at this point that our proof is only an outline. We leave the trigonometric details to the dedicated reader). Now, if $N(\Phi_\gamma | x) = k < \infty$, it is easily seen that

$$\lim_{n\to\infty} \sum_{s_i \in \pi(n)} 1_{\widetilde{C}_{s_i,s_{i+1}}}(x) = N(\Phi_\gamma | x),$$

and since the arguments of section 2 and the end of section 1 show that $N(\Phi_\gamma | B_u) < \infty$ for a.e. u a.s., we see that

$$\lim_{n\to\infty} V_n = \int_0^t N(\Phi_\gamma | B_s) ds.$$

Noting the factor 1/2 in (4.1) will complete our proof, once we explain why the terms we have ignored do not contribute in the $n \to \infty$ limit.

To show that the stochastic integral term in (4.2) converges to 0 we write it as

$$\begin{aligned} &\int_0^t (\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)}) \left(1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1})\}} dB_u^{\theta(s_{i+1})} - 1_{\{B_u^{\theta(s_i)} > a(s_i)\}} dB_u^{\theta(s_i)} \right) \\ &= \int_0^t (\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)})\, 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}),\ B_u^{\theta(s_i)} > a(s_i)\}} \left(dB_u^{\theta(s_{i+1})} - dB_u^{\theta(s_i)} \right) \\ &+ \int_0^t (\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)})\, 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}),\ B_u^{\theta(s_i)} < a(s_i)\}} dB_u^{\theta(s_{i+1})} \\ &+ \int (\hat{B}_u^{\gamma(s_i)} - \hat{B}_u^{\gamma(s_{i+1})})\, 1_{\{B_u^{\theta(s_i)} > a(s_i),\ B_u^{\theta(s_{i+1})} < a(s_{i+1})\}} dB_u^{\theta(s_i)} . \end{aligned} \tag{4.5}$$

Summing over i in the first term, we get

$$\int_0^t \sum_{s_i \in \pi(n)} \left(\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)} \right) 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}), B_u^{\theta(s_i)} > a(s_i)\}}$$

$$(\cos\theta(s_{i+1}) - \cos\theta(s_i)) dB_u^1 + (\sin\theta(s_{i+1}) - \sin\theta(s_i)) dB_u^2 .$$

But, since $s \to \hat{B}_u^{\gamma(s)}$ is continuous, it follows that

$$\begin{aligned} &\left| \sum_{s_i \in \pi(n)} \left(\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)} \right) 1_{\{B_u^{\theta(s_{i+1})} > a(s_{i+1}), B_u^{\theta(s_i)} > a(s_i)\}} (\cos\theta(s_{i+1}) - \cos(\theta(s_i)) \right| \\ &\le \sum_{s_i \in \pi(n)} \left| \hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)} \right| \, |\cos\theta(s_{i+1}) - \cos(\theta(s_i))| \\ &\le \sup_i \left| \hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)} \right| \sum |\cos\theta(s_{i+1}) - \cos(\theta(s_i))| \end{aligned}$$

which is bounded by

$$\sup_i \left| \hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)} \right| \quad \underset{a \le s \le b}{\text{Variation of}} \; \cos\theta(s)$$

which converges to 0 as $|\pi_n| \to 0$. The same is true with $(\sin\theta(s_{i+1}) - \sin\theta(s_i))$ replacing the cosine. Thus, by the dominated convergence theorem for stochastic integrals (we may assume that B^1, B^2 are bounded by localization) the above expression converges to 0.

As to the remaining expression in (4.5), summing over i and writing $B_u^{\theta(s_{i+1})}$ and $B_u^{\theta(s_i)}$ explicitly we obtain

$$\int_0^t \sum_{s_i\in\pi(n)} (\hat{B}_u^{\gamma(s_{i+1})} - \hat{B}_u^{\gamma(s_i)}) 1_{\{B_u^{\theta(s_{i+1})}>a(s_{i+1}),\ \hat{B}_u^{\theta(s_i)}<a(s_i)\}} (\cos\theta(s_{i+1})dB_u^1 + \sin\theta(s_{i+1})dB_u^2)$$

$$+\int_0^t \sum_{s_i\in\pi(n)} (\hat{B}_u^{\gamma(s_i)} - \hat{B}_u^{\gamma(s_{i+1})})\ 1_{\{B_u^{\theta(s_i)}>a(s_i),\ B_u^{\theta(s_{i+1})}<a(s_{i+1})\}} (\cos\theta(s_i)dB_u^1 + \sin\theta(s_i)dB_u^2)$$

which by the dominated convergence theorem for stochastic integrals (by localization if necessary to bound $\hat{B}_u^{\theta(s_i)}, \hat{B}_u^{\theta(s_{i+1})}$) converges, as $|\pi_n| \to 0$, to

$$\int_0^t \sum_{k=1}^{N(\Phi(\gamma)\,|\,B_u)} \left(\hat{B}_u^{\bar{\gamma}_k(u)} - \hat{B}_u^{\gamma_k(u)}\right)\ (\cos\theta(\gamma_k(u))dB_u^1 + \sin\theta(\gamma_k(u))dB_u^2)$$

where $\gamma_k(u)$ is the k-th of the $N(\Phi(\gamma)\,|\,B_u)$ lines on which B_u lies and $\theta(\gamma_k(u))$ is its angle. If $\gamma_k(u) = \gamma(s)$ for some s, then $\hat{B}_u^{\bar{\gamma}_k(u)}$ is the limit of $\hat{B}_u^{\gamma(t)}$ for a sequence $\{t_n\}$ that converges to s. But since $t \to \hat{B}_u^{\gamma(t)}$ is continuous, $\hat{B}_u^{\gamma_k(u)} = \hat{B}_u^{\bar{\gamma}_k(u)}$ and the whole expression is equal to 0.

We now return to (4.1) to show that the only term which will contribute to the quadratic variation in the limit is the stochastic integeral term. If

$$\psi_t(s) = \left(B_t^{\theta(s)} - a(s)\right)^+ - \left(B_0^{\theta(s)} - a(s)\right)^+$$

$$= (B_t^1\cos\theta(s) + B_t^2\sin\theta(s) - a(s))^+ - (B_0^1\cos\theta(s) + B_0^2\sin\theta(s) - a(s))^+,$$

we note that $s \to \psi_t(s)$ is Liptschitz 1 (by the smoothness of $s \to \gamma(s)$) and therefore

$$\lim_{n\to\infty} \sum_{s_i\in\pi(n)} (\psi_t(s_{i+1}) - \psi_t(s_i))^2 = 0\ .$$

Furthermore, since $s \to \hat{B}_t^{\gamma(s)}$ is continuous,

$$\lim_{n\to\infty} \sum_{s_i\in\pi(n)} (\phi_t(s_{i+1}) - \phi_t(s_i))\ (\hat{B}_t^{\gamma(s_{i+1})} - \hat{B}_t^{\gamma(s_i)}) = 0\ .$$

This completes our outline.

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