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GENERALIZED VARIATIONAL PRINCIPLES

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Abstract

In [7] *Weinan E, Y. G. Rykov,* and *Y. G. Sinai* have introduced a generalized variational principles in order to give a weak solution of the pressureless gas equations with initial velocity u_0 and distribution of masses given by a probability measure P. The aim of this work is to connect these generalized variational principles at each time $t > 0$ with the convex hull of any primitive of the map $m \in (0,1) \to F^{-1}(m) + tu_0(F^{-1}(m))$. Here F is the distribution function of the probability measure P and F^{-1} is its inverse. The latter convex hull is also used to obtain the solutions of the scalar conservation law and the Hamilton-Jacobi equation associated with the pressureless gas equations.

1 Motivation

First we recall the definition of the system of conservation law studied in [7].

Definition. *Let u_0 be a continuous and bounded real function. Let $(P_t : t \geq 0)$ be a family of probability measures on $\mathbf{R}$, weakly continuous with respect to t. For each $t \geq 0$, let $I_t = u(t, \cdot)P_t$ be a measure absolutely continuous with respect to P_t. $(P_t, I_t, u(\cdot, t) : t \geq 0)$ is a weak solution of the pressureless gas equations*

$$\begin{cases} \frac{\partial P_t}{\partial t} + \frac{\partial I_t}{\partial x} = 0 \\ \frac{\partial I_t}{\partial t} + \frac{\partial (uI_t)}{\partial x} = 0 \\ P_t \to P \, , uP_t \to u_0 P \quad \textit{weakly,} \end{cases}$$

if, for any $f \in C_0^1(\mathbf{R})$, the space of C^1-functions on $\mathbf{R}$ with compact support, and $0 < t_1 < t_2$,

$$\int f(x)dP_{t_2}(x) - \int f(x)dP_{t_1}(x) = \int_{t_1}^{t_2} \int f'(x)dI_t(x)dt \, ,$$

and

$$\int f(x)dI_{t_2}(x) - \int f(x)dI_{t_1}(x) = \int_{t_1}^{t_2} \int f'(x)u(x,t)dI_t(x)dt \, .$$

Weinan E, Rykov, Sinai in [7] have constructed a weak solution for the pressureless gas equations using sticky particle dynamics. Each particle is indexed by its initial position $x \in \mathbf{R}$, initial velocity $u_0(x)$, and the mass of the set of particles $(-\infty, x]$ is equal to $F(x) := P(-\infty, x]$. Before collision $x + tu_0(x)$ is the position of the particle x at time t. At collisions the colliding particles stick and form a new massive particle. The mass and the velocity of this new particle are given by the laws of conservation of mass and momentum. The method used in [7] is based on the construction of a partition ξ_t of $\mathbf{R}$, which divides the particles into ordered intervals (called clusters), so that each group of particles initially located in an interval $[\alpha, \beta] \in \xi_t$ are glued to a single one before or at time t, and different clusters are at different locations at time t. Each element $[\alpha, \beta]$ of ξ_t is then completely determined by its endpoints α, β. These endpoints are characterized by the following generalized variational principles, denoted in the sequel by (GVP).

(GVP1) α is the left endpoint of an element of ξ_t iff

$$\frac{\int_{[y_1,\alpha)}[\eta + tu_0(\eta)]dP(\eta)}{P[y_1, \alpha)} < \frac{\int_{[\alpha,y_2)}[\eta + tu_0(\eta)]dP(\eta)}{P[\alpha, y_2)},$$

for all $y_1 < \alpha < y_2$.

(GVP2) β is the right endpoint of an element of ξ_t iff

$$\frac{\int_{(y_1,\beta]}[\eta + tu_0(\eta)]dP(\eta)}{P(y_1, \beta]} < \frac{\int_{(\beta,y_2]}[\eta + tu_0(\eta)]dP(\eta)}{P(\beta, y_2]} y,$$

for all $y_1 < \beta < y_2$.

Having $(\xi_t : t \geq 0)$, *Weinan E, Rykov, Sinai* in [7] have defined the forward flow map associated to pressureless gas equations as follows:

$$\varphi(t, x, P, u_0) = \frac{\int_{[\alpha,\beta]}[\eta + tu_0(\eta)]dP(\eta)}{P[\alpha, \beta]}, \tag{1}$$

where $[\alpha, \beta]$ is the unique element of ξ_t which contains x. They showed that $(P_t = P \circ \varphi^{-1}(t, \cdot, P, u_0), I_t = u(\cdot, t)P_t : t \geq 0)$ is a weak solution of the pressureless gas equations. Here

$$u(x, t) = \frac{\int_{[\alpha,\beta]} u_0(\eta)dP(\eta)}{P[\alpha, \beta]}. \tag{2}$$

In the other hand *Dermoune* [3, 4, 5] has constructed, for any probability distribution P, a process $(X_t, t \geq 0)$ describing trajectories of sticky particle dynamics with initial velocity u_0, and masses distributed following the probability P. This process is solution of the non-linear stochastic differential equation

$$dX_t = \mathbf{E}[u_0(X_0) \,|\, X_t]dt, \quad \mathcal{L}(X_0) = P, \tag{3}$$

and

$$X_t = \mathbf{E}[X_0 + tu_0(X_0) \,|\, X_t] \,,\ \forall t \geq 0 \,. \tag{4}$$

Using a simple proof based on the formula of change of variables he showed that $(P_t = \mathcal{L}(X_t), I_t = u(\cdot, t)P_t)_{t\geq 0}$ is a weak solution of the pressureless gas equations. Here $u(x,t) = \mathbf{E}[u_0(X_0) \,|\, X_t = x]$. The formula (4) gives the relation between the trajectories $(X_t, t \geq 0)$ of sticky particles and the trajectories $(X_0 + tu_0(X_0), t \geq 0)$ of the particles without any interaction.

Now we make some remarks on the (GVP). If P is continuous then the partition ξ_t given by (GVP1) and (GVP2) is ambiguous. In fact, in this case, every left endpoint is also a right endpoint. Our aim here is to clarify this point and to give a precise definition of ξ_t.

The idea in our work is to index each particle by $m \in F(\mathbf{R})$, with initial position $F^{-1}(m)$ (**Fig. 1**), velocity $u_0(F^{-1}(m))$, and the mass of the set of particles $(0,m]$ (i.e. initially located in $(-\infty, F^{-1}(m)]$) is equal to m. Before collision $F^{-1}(m) + tu_0(F^{-1}(m))$ is the position of the particle m at time t (**Fig. 2**). After the collision the map $m \in (0,1) \to F^{-1}(m) + tu_0(F^{-1}(m))$ oscillates, and now the positions of massive particles are given by the derivative of the convex hull $H(\cdot, t)$ of any primitive of the map $m \in (0,1) \to F^{-1}(m) + tu_0(F^{-1}(m))$. The right inverse

$$M^*(x,t) := \inf\{m \in (0,1) : \partial_m H(m,t) > x\}$$

is the mass at time t of the set of clusters located in $(-\infty, x]$, and the left inverse

$$M_*(x,t) := \sup\{m \in (0,1) : \partial_m H(m,t) < x\} \,,$$

is the mass of the set of clusters located in $(-\infty, x)$ (see **Fig. 3**). The velocity at time t of the cluster located at x is given by

$$u(x,t) = \frac{\int_{M_*(x,t)}^{M^*(x,t)} u_0(F^{-1}(m))dm}{M^*(x,t) - M_*(x,t)} \,,$$

and the family $(\partial_x M^*(x,t), u(x,t)\partial_x M^*(x,t), u(x,t))_{t\geq 0}$ is a weak solution of our pressureless gas equations.

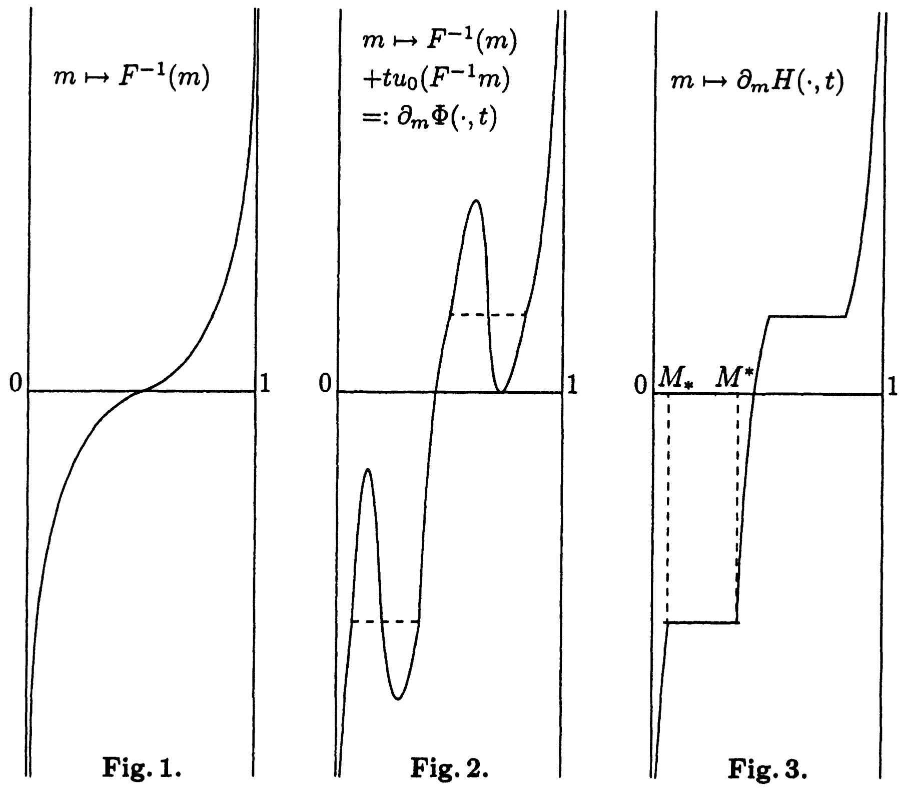

Fig. 1. The particle indexed by m is initially located at $F^{-1}(m)$.

Fig. 2. Before collision $F^{-1}(m) + tu_0(F^{-1}(m))$ is the position at time t of the particle indexed by m. The colliding particles before or at time t are represented by the graphic regions where the map $m \to F^{-1}(m) + tu_0(F^{-1}(m))$ oscillates.

Fig. 3. The ordinate x of a point (m, x) of the graph represents a location at time t of a cluster. The lenght $M^* - M_*$ of an horizontal segment belonging to the graph is the mass of a cluster at time t.

In [2] *Brenier* and *Grenier* have established a connection between the pressureless gas equations and the following scalar conservation law :

$$\partial_t M + \partial_x(A(M)) = 0 \ , \quad M(x, 0) = F(x) \ , \tag{5}$$

where A is any primitive of $m \in (0,1) \to u_0(F^{-1}(m))$. In our work we show that the entropy solution of (5) is given by the map $(x,t) \to M^*(x,t)$, defined above.

Plan of the paper. In Section 2 we give a precise definition of the (GVP). Section 3 is consecrated to the study of the extreme points of the convex hull of the map $z \in (0,1) \to \int_{\frac{1}{2}}^{z}[F^{-1}(m)+tu_0(F^{-1}(m))]dm$ as a function of the position of the clusters. We end our work by Section 4 which contains the connection between the (GVP), the scalar conservation law (5) and the Hamilton-Jacobi equation

$$\partial_t \Psi(x,t) + A(\Psi(x,t)) = 0 \ ,$$

with initial condition $\Psi(\cdot,0)$ is any primitive of F.

2 A precise definition of the (GVP)

The support $supp(P)$ of P is defined by

$$supp(P) = \{a \in \mathbf{R} : P(a-\varepsilon, a+\varepsilon) > 0 \ , \ \forall \varepsilon > 0\} \ .$$

Let $S_- = \{a \in \mathbf{R} : P(a-\varepsilon, a] > 0 \ , \ \forall \varepsilon > 0\}$, and $S_+ = \{a \in \mathbf{R} : P[a, a+\varepsilon) > 0 \ , \ \forall \varepsilon > 0\}$. Then $supp(P) = S_+ \cup S_-$. For $z \in (0,1]$ we define

$$F^{-1}(z) = \inf\{a : F(a) \geq z\} \ ,$$

where $F(a) = P(-\infty, a]$. We have $F(F^{-1}(z)) = z$ if $z \in F(\mathbf{R})$, and if $F(x-0) < z \leq F(x)$ for some $x \in \mathbf{R}$, then $F(F^{-1}(z)) = F(x)$. It is easy to show that F^{-1} is one to one from $F(\mathbf{R})$ into S_-.

Let us consider, for each fixed $t > 0$, a primitive of the map $m \in (0,1) \to F^{-1}(m) + tu_0(F^{-1}(m))$, for example

$$\Phi(z,t) = \int_{\frac{1}{2}}^{z} [F^{-1}(m) + tu_0(F^{-1}(m))]dm \ .$$

We denote by λ the Lebesgue measure on $[0,1]$. It follows from the equality $\lambda \circ (F^{-1})^{-1} = P$ that for all $\alpha, \beta \in \mathbf{R}$,

$$\Phi(F(\beta),t) - \Phi(F(\alpha),t) = \int_{(\alpha,\beta]} [\eta + tu_0(\eta)]dP(\eta) \ .$$

We suppose that $supp(P)$ is bounded or $\int_0^x \eta dP(\eta) \to +\infty$ as $|x| \to +\infty$. Under this assumption the convex hull $H(\cdot,t)$ of the map $\Phi(\cdot,t)$ is well defined.

Now we construct a partition ξ_t of $supp(P)$ as follows. An element of ξ_t has the form $[\alpha,\beta] \cap supp(P)$. The endpoints α, β are given by the extreme points of $H(z,t)$. More precisely let

$$F_l^{-1}(z) = \inf\{a : F(a) > z\} \ , \quad \text{and} \quad F_r^{-1}(z) = \sup\{a : F(a) < z\} \ .$$

If m is an extreme point of $H(\cdot,t)$, then $F_r^{-1}(m)$ a right endpoint of an element $G_1 \in \xi_t$ and $F_l^{-1}(m)$ is a left endpoint of an element $G_2 \in \xi_t$. The group of particles G_1 is just on the left of the group G_2. We can show that a right endpoint of ξ_t belongs to S_-, and a left endpoint belongs to S_+.

Now we give a detailed description of endpoints of ξ_t from extreme points of $H(\cdot,t)$, see **Fig.4** for a graphic illustration. We distinguish four cases.

A) m is isolated in the set of extreme points. Namely, there exist $p_1 < p_2$ and z_1, z_2 two extremes points such that

$$\frac{\Phi(t,m) - \Phi(t,z_1)}{m - z_1} = p_1 < p_2 = \frac{\Phi(t,z_2) - \Phi(t,m)}{z_2 - m} ,$$

and

$$\frac{\Phi(t,m) - \Phi(t,z')}{m - z'} \le p_1 < p_2 \le \frac{\Phi(t,z'') - \Phi(t,m)}{z'' - m} , \text{ for all } z' < m < z'' , \quad (6)$$

and there is no extreme point in $(z_1, m) \cup (m, z_2)$. We have two cases.
i) $F_r^{-1}(m) = F_l^{-1}(m)$. In this case there exist two sequences (z_1^n) and (z_2^n) in $F(\mathbf{R})$ such that $z_1^n \to m - 0$ and $z_2^n \to m + 0$. Using the continuity of u_0 in (6) we get

$$\alpha + tu_0(\alpha) \le p_1 < p_2 \le \alpha + tu_0(\alpha) ,$$

which is absurd.
ii) So necessarily $F_r^{-1}(m) < F_l^{-1}(m)$. $F_r^{-1}(m)$ is the right endpoint of a cluster with mass $m - z_1$ and $F_l^{-1}(m)$ is the left endpoint of a cluster with mass $z_2 - m$.

B) m is isolated from the right. Namely there exists $z_2 > m$ an extreme point such that (m, z_2) does not contain any extreme point, but for any $z' < m$, (z', m) contains an extreme point. In this case we have two situations.
i) $F_r^{-1}(m) = F_l^{-1}(m) := \alpha$. This situation implies that α is the left endpoint of the cluster with mass $z_2 - m$. And any interval $(\alpha - \varepsilon, \alpha)$ contains a cluster.
ii) $F_r^{-1}(m) < F_l^{-1}(m)$. This situation implies that $\{F_r^{-1}(m)\} \in \xi_t$. The cluster with mass $z_2 - m$ has $F_l^{-1}(m)$ as a left endpoint. Any interval $(F_r^{-1}(m) - \varepsilon, F_r^{-1}(m))$ contains a cluster.

C) z is isolated from the left. Namely there exists $z_1 < z$ an extreme point such that (z_1, z) does not contain any extreme point, but for any $z < z'$, (z, z') contains an extreme point.
i) $F_r^{-1}(m) = F_l^{-1}(m) := \alpha$. This situation implies that α is the right endpoint of the cluster with mass $m - z_1$. Any interval $(\alpha, \alpha + \varepsilon)$ contains a cluster.
ii) $F_r^{-1}(m) < F_l^{-1}(m)$. Then $F_r^{-1}(m)$ is a right endpoint of a cluster of mass $m - z_1$ and $\{F_l^{-1}(m)\} \in \xi_t$.

D) For all $z_1 < m < z_2$ there is an extreme point in (z_1, m) and an extreme point in (m, z_2). Then necessarily $F_r^{-1}(m) = F_l^{-1}(m) := \alpha$. In this case the particle α did not meet any other particle during the interval $[0, t]$, and any interval $(\alpha - \varepsilon, \alpha)$ or $(\alpha, \alpha + \varepsilon)$ contains a cluster.

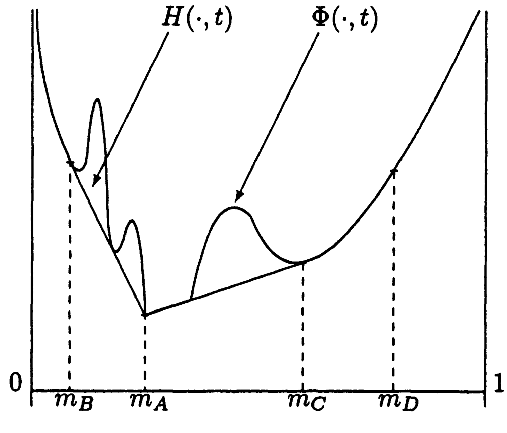

Fig. 4. The different cases of extremal points of $H(\cdot,t)$.
m_i represents the case i) studied above.

Remark 2.1. 1) From the construction of the partition ξ_t, each endpoint of an element of ξ_t satisfy (GVP1) and (GVP2).

2) It follows from D) that an endpoint, which is an accumulation of endpoints from the left, is necessarily a continuous point of F.

3) If F is increasing then there is no successive clusters with positive masses.

3 Extreme points as a function of the position of clusters

For $t > 0, x \in \mathbf{R}$ we set $G(x,z,t) = \int_{\frac{1}{2}}^{z}[F^{-1}(m) + tu_0(F^{-1}(m))]dm - x(z - \frac{1}{2})$, for $z \in (0,1)$. The properties refer to G as a function of z with x, t being fixed. G is continuous in z and has, according to the hypothesis on P, the property

$$\lim_{z \to 1-0} G(z)\ , \quad \lim_{z \to 0+} G(z) \quad \text{exist,}$$

or equal to $+\infty$. Hence G attains its smallest value for one or several values of z, the smallest and the largest of which are denoted by $M_*(x,t)$ and $M^*(x,t)$, respectively,

$$M_*(x,t) \leq M^*(x,t).$$

Following the same proof as in [6] we have,

Theorem 3.1. *The functions M_* and M^* have the properties*
(a) $M^*(x,t) \leq M_*(x',t)$ *if* $x < x'$,
(b) $M^*(x-0,t) = M_*(x,t), \quad M_*(x+0,t) = M^*(x,t)$
(c) As functions of (x,t), $M_*(x,t), M^*(x,t)$ are respectively lower and upper-semicontinuous. At a point where $M_*(x,t) = M^*(x,t)$ both functions are continuous.

3.1 The Physical meaning of $M_*(x,t), M^*(x,t)$

Let us consider the convex hull $H(\cdot,t)$ defined in Section 2. The left inverse $\sup\{z : \partial_z H(z,t) < x\}$ of $\partial_z H(\cdot,t)$ coincides with $M_*(x,t)$, and its right inverse

$$\inf\{z : \partial_z H(z,t) > x\}$$

coincides with $M^*(x,t)$. So $M_*(x,t)$ is the mass at time t of the set of clusters located in $(-\infty,x)$, and $M^*(x,t)$ is the mass of the set of clusters located in $(-\infty,x]$. More precisely let (X_t) be the process defined by (3), then

$$M^*(x,t) = \mathbf{E}[H(x-X_t)] , \tag{7}$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function.

The following result shows that collisions occur when $M_* < M^*$.

Proposition 3.1. *If $M_*(x,t) < M^*(x,t)$, then $M_*(x,t), M^*(x,t)$ are two extreme points of $H(\cdot,t)$, and x is a position of some cluster at time t and $M^*(x,t) - M_*(x,t)$ is the mass of this cluster. If $M_*(x,t) = M^*(x,t) := M(x,t)$, then x is not a position of any cluster at time t or $F_l^{-1}(M_*(x,t)) = F_r^{-1}(M^*(x,t)) := \alpha$ is a cluster situated at $x = \alpha + tu_0(\alpha)$ at time t. Moreover the velocity $u(x,t)$ defined by (2) is given by*

$$u(x,t) = \frac{\int_{M_*(x,t)}^{M^*(x,t)} u_0(F^{-1}(m))dm}{M^*(x,t) - M_*(x,t)} , \; \textit{if } M_*(x,t) < M^*(x,t) ,$$

and $u(x,t) = u_0(F^{-1}(M(x,t)))$ if $M_*(x,t) = M^*(x,t) := M(x,t)$.

3.2 Characteristics

We set $y_*(x_1,t_1) = F_l^{-1}(M_*(x_1,t_1))$, $y^*(x_1,t_1) = F_r^{-1}(M^*(x_1,t_1))$, and we define as in the inviscid Burgers equation [6] the segments

$$S_*(x_1,t_1) = [(y_*(x_1,t_1),0),(x_1,t_1)] , \quad S^*(x_1,t_1) = [(y^*(x_1,t_1),0),(x_1,t_1)].$$

Namely $(x,t) \in S_*(x_1,t_1)$, respectively $(x,t) \in S^*(x_1,t_1)$, if

$$x = x_1 + \frac{x_1 - y_*(x_1,t_1)}{t_1}(t-t_1) , \quad t \in (0,t_1) ,$$

respectively

$$x = x_1 + \frac{x_1 - y^*(x_1,t_1)}{t_1}(t-t_1) , \quad t \in (0,t_1) .$$

Theorem 3.2. *At every point x,t of the segment $S_*(x_1,t_1)$, respectively $S^*(x_1,t_1)$, $M_*(x,t) = M^*(x,t) = M_*(x_1,t_1)$, and $y_*(x,t) = y^*(x,t) = y_*(x_1,t_1)$ respectively $M_*(x,t) = M^*(x,t) = M^*(x_1,t_1)$, and $y_*(x,t) = y^*(x,t) = y^*(x_1,t_1)$ and they are continuous.*

Proof. Let $x = x_1 + \frac{x_1 - y_*(x_1,t_1)}{t_1}(t - t_1)$, for $t \in (0, t_1)$. First we have

$$G(x,z,t) = t\int_{\frac{1}{2}}^{z}\left[\frac{F^{-1}(m)-x}{t} + u_0(F^{-1}(m))\right]dm$$

and

$$\frac{F^{-1}(m)-x}{t} = \frac{F^{-1}(m)-x_1}{t_1} + \frac{t_1-t}{tt_1}(F^{-1}(m) - y_*(x_1,t_1))\ .$$

So $t^{-1}\{G(x,z,t) - G(x, M_*(x_1,t_1),t)\} =$

$$\int_{M_*(x_1,t_1)}^{z}\left[\frac{F^{-1}(m)-x}{t} + u_0(F^{-1}(m))\right]dm =$$

$$t_1^{-1}\{G(x_1,z,t_1) - G(x_1, M_*(x_1,t_1),t_1)\}$$
$$+\frac{t_1-t}{tt_1}\int_{M_*(x_1,t_1)}^{z}[F^{-1}(m) - y_*(x_1,t_1)]dm\ .$$

If $m > M_*(x_1,t_1)$, then $F^{-1}(m) > y_*(x_1,t_1)$, which implies that

$$\int_{M_*(x_1,t_1)}^{z}[F^{-1}(m) - y_*(x_1,t_1)]dm > 0, \quad \forall z > M_*(x_1,t_1)\ ,$$

and thus $G(x,z,t) - G(x, M_*(x_1,t_1),t) > 0$ for all $z > M_*(x_1,t_1)$.
If $z < M_*(x_1,t_1)$, then $G(x_1,z,t_1) - G(x_1,M_*(x_1,t_1),t_1) > 0$ and

$$\int_{M_*(x_1,t_1)}^{z}[F^{-1}(m) - y_*(x_1,t_1)]dm \geq 0\ , \quad \forall z < M_*(x_1,t_1)\ .$$

We conclude that $M_*(x_1,t_1)$ is the unique minimum of the map $z \in (0,1) \to G(x,z,t)$.

The proof of the case

$$x = x_1 + \frac{x_1 - y^*(x_1,t_1)}{t_1}(t-t_1)\ , \quad t \in (0,t_1)$$

is similar.

4 Pressureless gas equations and scalar conservation law

Let $P_n = \sum_j m_j\delta(x - x_j)$ be a sequence of finite probabilities such that $P_n \to P$. The particles $\{x_j : j\}$ move following the model of sticky particles. We denote by $x_j(t)$ the position of the particle x_j at time t. *Brenier* and *Grenier* [2] have proved that

$$\sum_j m_j\delta(x - x_j(t)) \to \partial_x M(x,t)\ ,$$

where M is the unique entropy solution of the scalar conservation (5).

First let us recall the definition of the entropy solution. Let f be a locally Lipschitz continuous function. The equation

$$\partial_t u(x,t) + \partial_x(f(u(x,t))) = 0\ , \quad u(0,x) = u_0(x) \text{ is given,} \tag{8}$$

is called a scalar conservation law. The entropy solution is a locally integrable function such that, for all positive smooth function ϕ,

$$\int\int \partial_t\phi(x,t)I(u(x,t)) + \partial_x\phi(x,t)F(u(x,t))dxdt + \int \phi(0,x)u_0(x)dx \geq 0\ , \tag{9}$$

where $I(u) = \int_0^u h(x)dx$,$F(u) = \int_0^u h(x)df(x)$ and h is any nondecreasing function.

In this part we show that $M_*(x,t)$ (respectively $M^*(x,t)$) is the left continuous version (respectively the right continuous) of the entropy solution of (5). We define the function M on the set of the points (x,t) such that $M_*(x,t) = M^*(x,t)$. So M is continuous at every point where $M_* = M^*$, and if $M_*(x_1,t_1) < M^*(x_1,t_1)$ then (x_1,t_1) is a discontinuity point of M. We have $\lim_{x\to x_1-0} M(x,t_1) = M_*(x_1,t_1)$ and $\lim_{x\to x_1+0} M(x,t_1) = M^*(x_1,t_1)$.

Now we show that M is the entropy solution of (5). It is known [1] (see also [2]) that the map

$$(x,t) \to \Psi(x,t) := \int^x m(y,t)dy\ ,$$

is a viscosity solution (in the sense of Crandall Lions) of the Hamilton-Jacobi equation

$$\partial_t\Psi + A(\partial_x\Psi) = 0\ ,$$

if and only if $(x,t) \to m(x,t)$ is an entropy solution of

$$\partial_t m(x,t) + \partial_x(A(m(x,t)) = 0\ .$$

Since the initial contition $\Psi(x,0) := \frac{1}{2}F^{-1}(\frac{1}{2}) + \int_{\frac{1}{2}}^x F(y)dy$ is convex, the second Hopf formula [1] asserts that the unique viscosity solution with $\Psi(\cdot,0)$ as initial conditions is given by

$$\Psi(x,t) = \sup_{m\in(0,1)} \{xm - \Psi(\cdot,0)^*(m) - tA(m)\}\ ,$$

where

$$\Psi(\cdot,0)^*(m) = \sup_{x\in\mathbf{R}}\{xm - \Psi(x,0)\}$$

is the Legendre-Fenchel transform of $\Psi(\cdot,0)$. It is known that for each $t \geq 0$ fixed, $\Psi(\cdot,t)^*$ is the convex hull of the map

$$m \in (0,1) \to \Psi(\cdot,0)^*(m) + tA(m)$$

and the inverse of $\partial_m\Psi(\cdot,t)^*$ coincides with $\partial_x\Psi(\cdot,t)$. We can show for

$$A(m) = \int_{\frac{1}{2}}^m u_0(F^{-1}(z))dz\ , \quad m \in (0,1)\ ,$$

and

$$\Psi(x,0) := \frac{1}{2}F^{-1}(\frac{1}{2}) + \int_{\frac{1}{2}}^{x} F(y)dy \ ,$$

that

$$\Psi(\cdot,0)^*(m) + tA(m) = \int_{\frac{1}{2}}^{m} F^{-1}(z) + tu_0(F^{-1}(z))dz \ , \quad \forall m \in (0,1).$$

In Section 3 we have denoted by $H(\cdot,t)$ the convex hull of the function

$$m \in (0,1) \to \int_{\frac{1}{2}}^{m} F^{-1}(z)dz + t\int_{\frac{1}{2}}^{m} u_0(F^{-1}(m'))dm' \ .$$

So, the inverse $M(\cdot,t)$ of the function $\partial_m H(\cdot,t)$ is equal to the function $\partial_x\Psi(\cdot,t)$. We derive that $M(x,t) = \partial_x\Psi(x,t)$ is the entropy solution of (5).

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