Nathalie Eisenbaum

A gaussian sheet connected to symmetric Markov chains

Séminaire de probabilités (Strasbourg), tome 36 (2002), p. 331-334
<http://www.numdam.org/item?id=SPS_2002__36__331_0>

A Gaussian sheet connected to symmetric Markov chains

Nathalie Eisenbaum

Laboratoire de Probabilités - Université Paris VI - CNRS , UMR 7599
4, Place Jussieu - Case 188 - 75252 Paris Cedex 05 - France

Abstract : A large family of Gaussian sheets arises in connection with second order limit theorems for continuous time symmetric Markov processes. We show here that results of the same nature can also be written for symmetric Markov chains.

1 - Introduction

As a variant of a result of Papanicolaou, Stroock and Varadhan [PSV], Yor [Y] has shown that the Brownian sheet can be obtained as a second order limit in law under the following form :

$$\left(\frac{L^x_{\tau_{\lambda t}} - \lambda t}{\sqrt{\lambda}}; x \in \mathbb{R}^+, t \geq 0\right) \xrightarrow{\text{(law)}} (W_t(x); x \in \mathbb{R}^+, t \geq 0)$$

as λ tends to ∞, where $(L^x_t, x \in \mathbb{R}, t \geq 0)$ denotes the family of local times of a linear Brownian motion starting from 0, $(\tau_s, s > 0)$ the right inverse of the local time at 0, and $(W_t(x), x \in \mathbb{R}^+, t \geq 0)$ a Brownian sheet, i.e., a centered Gaussian process with covariance given by :

$$\mathbb{E}(W_s(x)W_t(y)) = (s \wedge t)(x \wedge y).$$

This result has been then extended under various forms. Rosen [R] gives an analogue for the symmetric stable processes (see also [E]) and obtains the so-called "fractional Brownian sheet" as the limit in law of the corresponding expression. Csáki, Shi and Yor show in [CSY] that when replacing the local times process by the fractional derivatives of Brownian motion, one obtains a limit also equal to the fractional Brownian sheet.

We present here an analogue of theses results for recurrent symmetric Markov chains. We consider a recurrent Markov chain $(U_n)_{n\geq 0}$ defined on a regular lattice in a d-dimensional Euclidian space. We denote by $\mathcal{S}$ the state space of U and assume that 0 belongs to $\mathcal{S}$. We set : $\mathcal{S}^* = \mathcal{S} \setminus \{0\}$. The process of the number of visits is denoted by $(\ell^x_n, x \in \mathcal{S}, n \in \mathbb{N})$. We define for $n \in \mathbb{N}^*$:

$$\tau_n = \inf\{k \geq 1 : \sum_{p=1}^{k} 1_{\{0\}}(U_p) \geq n\}$$

and for x, y elements of $\mathcal{S}^*$:

$$g_0(x, y) = \mathbb{E}_x(\ell^y_{\tau_1}).$$

We have the following theorem. For any x in $\mathbb{R}^+$, $[x]$ denotes the integer part of x.

Theorem 1 *Let U be a symmetric recurrent Markov chain with state space $\mathcal{S}$. Then under $I\!P_0$, we have the following convergence as λ tends to ∞:*

$$\left(\frac{\ell^x_{\tau_{[\lambda t]}} - [\lambda t]}{\sqrt{\lambda}}; x \in \mathcal{S}^*, t \geq 0\right) \overset{\text{(law)}}{\longrightarrow} (D_t(x); x \in \mathcal{S}^*, t \geq 0)$$

where $(D_t(x); x \in \mathcal{S}^*, t \geq 0)$ is a centered Gaussian process whose covariance is given by :

$$I\!E(D_s(x)D_t(y)) = (s \wedge t)(2g_0(x,y) - 1_{(x=y)} - 1)$$

The convergence in law considered here corresponds to the convergence in law in $D_E([0,+\infty])$ where E is the metric space of the real sequences indexed by $\mathcal{S}^*$.

2 - Proof of Theorem 1

Thanks to the Central Limit Theorem and the Markov property of U, we remark that for any function f on $\mathcal{S}^*$ with compact support, setting :
$A_t = \sum_{x \in \mathcal{S}^*} f(x)\ell^x_t$, we have as λ tends to ∞ :

$$\frac{A_{\tau_n} - I\!E_0(A_{\tau_n})}{\sigma\sqrt{n}} \overset{\text{(law)}}{\longrightarrow} N$$

where N is a standard Gaussian variable and σ is defined by :
$\sigma^2 = I\!E_0((A_{\tau_1} - I\!E_0(A_{\tau_1}))^2)$.
At this level, several authors (see Kesten [Ke], Kasahara [KA], Csáki et al. [CCFR]) make some assumptions on the sequence $(\tau_n)_{n>0}$ in order to establish second order limit laws for $(A_n)_{n>0}$ instead of $(A_{\tau_n})_{n>0}$. Our goal here is to put in evidence the Gaussian nature of the limit in the general case.
We write :

$$\sigma^2 = \sum_{x \in \mathcal{S}^*} \sum_{y \in \mathcal{S}^*} f(x)f(y)\{I\!E_0(\ell^x_{\tau_1}\ell^y_{\tau_1}) - I\!E_0(\ell^x_{\tau_1})I\!E_0(\ell^y_{\tau_1})\}$$

But for any x, y distinct from 0

$$I\!E_0(\ell^x_{\tau_1}\ell^y_{\tau_1}) = 2I\!E_x(\ell^y_{\tau_1}) - 1_{(x=y)}.$$

The above identity can be obtained by a simple computation, exactly as Csáki et al. do it in the case of a random walk [CCFR]. Moreover it is well known that for any x distinct from 0 : $I\!E_0(\ell^x_{\tau_1}) = 1$. Consequently, we obtain :

$$\sigma^2 = \sum_{x \in \mathcal{S}^*} \sum_{y \in \mathcal{S}^*} f(x)f(y)(2g_0(x,y) - 1_{(x=y)} - 1).$$

Since under $I\!P_x$, τ_1 is the first hitting time of 0, we easily obtain that the function g_0 is symmetric. The recurrence of U ensures that σ^2 is strictly positive. Hence the function $(2g_0(x,y) - 1_{(x=y)} - 1, (x,y) \in \mathcal{S}^* \times \mathcal{S}^*)$ is definite positive.
Let $(\psi_x)_{x \in \mathcal{S}^*}$ be a centered Gaussian process with a covariance equal to $(2g_0(x,y) - 1_{(x=y)} - 1, (x,y) \in \mathcal{S}^* \times \mathcal{S}^*)$. Then note that :

$$\sigma^2 = I\!E((\sum_{x \in \mathcal{S}} f(x)\psi_x)^2)$$

Hence the above convergence can be rewritten as follows :

$$\sum_{x\in\mathcal{S}^*} f(x)\frac{\ell^x_{\tau_n}-n}{\sqrt{n}} \xrightarrow{\text{(law)}} \sum_{x\in\mathcal{S}^*} f(x)\psi_x$$

as λ tends to ∞. This implies that :

$$\mathbb{E}(\exp\{i\sum_{x\in\mathcal{S}^*} f(x)\frac{\ell^x_{\tau_n}-n}{\sqrt{n}}\}) \longrightarrow \mathbb{E}(\exp\{i\sum_{x\in\mathcal{S}^*} f(x)\psi_x\})$$

Since this convergence holds for any function f with compact support, we obtain as λ tends to ∞ :

$$(\frac{\ell^x_{\tau_n}-n}{\sqrt{n}}, x\in\mathcal{S}^*) \xrightarrow{\text{(law)}} (\psi_x, x\in\mathcal{S}^*).$$

Then one can easily deduce the above convergence for any $t>0$, as λ tends to ∞ :

$$(\frac{\ell^x_{\tau_{[t\lambda]}}-[t\lambda]}{\sqrt{\lambda}}, x\in\mathcal{S}^*) \xrightarrow{\text{(law)}} (\sqrt{t}\psi_x, x\in\mathcal{S}^*)$$

We introduce now $(D_t(x); x\in\mathcal{S}^*, t\geq 0)$ a centered Gaussian process whose covariance is given by :

$$\mathbb{E}(D_s(x)D_t(y)) = (s\wedge t)(2g_0(x,y) - 1_{(x=y)} - 1).$$

This allows to interpret the above convergence as follows :

$$(\frac{\ell^x_{\tau_{[t\lambda]}}-[t\lambda]}{\sqrt{\lambda}}, x\in\mathcal{S}^*) \xrightarrow{\text{(law)}} (D_t(x), x\in\mathcal{S}^*).$$

We set :

$$Z_\lambda(t) = (\frac{\ell^x_{\tau_{[t\lambda]}}-[t\lambda]}{\sqrt{\lambda}}, x\in\mathcal{S}^*).$$

We have for any $x\in\mathcal{S}$,

$$\ell^x_{\tau_{[(t+s)\lambda]}} - \ell^x_{\tau_{[t\lambda]}} = \ell^x_{\tau_{[(t+s)\lambda]-[t\lambda]}} \circ \theta_{\tau_{[t\lambda]}}.$$

Consequently for a fixed λ, the process $(Z_\lambda(t), t\geq 0)$ is a process with independent increments and hence is Markovian. Since for each fixed $t>0$, $Z_\lambda(t)$ converges to $D_t(.)$ as λ tends to ∞, we can use the result of Ethier and Kurtz ([EK], Theorem 2.5 p.167) to claim that as λ tends to ∞:

$$(Z_\lambda(t), t\geq 0) \xrightarrow{\text{(law)}} (D_t(.), t\geq 0). \quad \square$$

References

[CSY] Csáki E., Shi Z. and Yor M. : Fractional Brownian motions as "higher-order" fractional derivatives of Brownian local times.*Preprint.*

[CCF] Csáki E., Csorgö M., Foldes A. and Révész P. : Strong approximation of additive functionals.*Jour.Theor.Probab. Vol 5,4,707-726,1992.*

[E] Eisenbaum N. Théorèmes limites pour les temps locaux d'un processus stable symétrique. *Séminaire de Probabilités XXXI. Lect.Notes in Maths. 1655,216-224 (1997). Errata dans Séminaire de Probas XXXIII, Lect. Notes in Maths 1709,p.417 (1999).*

[EK] Ethier N. and Kurtz T. *Markov Processes. Characterization and Convergence. John Wiley & Sons, Inc., New York. 1986*

[Ka] Kasahara Y. : A limit theorem for sums of random number of i.i.d. rnadom variables and its application to occupation times of markov chains.*J. Math. Soc. Japan 37, 197-205 (1985).*

[Ke] Kesten H. : Occupation times for Markov and semi-Markov chains. *Trans. Amer. Math. Soc. 103, 82-112 (1962).*

[PSV] Papanicolaou G.C., Stroock D.W. and Varadhan S.R.S. : Martingale approach to some limit theorems.*Duke Univ. Maths. Series III, Statistical Mechanics and Dynamical Systems (1977).*

[R] Rosen J. : Second order limit laws for the local times of stable processes.*Sém. de Probab. XXV,Lect. Notes Math. 1485,407-425, Berlin Springer,1991.*

[Y] Yor M. : Le drap brownien comme limite en loi des temps locaux d'un mouvement brownien linéaire. *Sém. de Probabilités XVII - Lect.Notes Math. 986,89-105- Berlin Springer (1983).*