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Simple strategies in exponential utility maximization

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Abstract

This short note contains a ramification of one of the main results of [3] and answers a question formulated by Michel Émery. Assuming that the price process is locally bounded and admits an equivalent local martingale measure with finite entropy, we show that in the case of exponential utility the optimal value can always be attained on a sequence of uniformly bounded portfolios with simple bounded integrands.

Key words: exponential utility, semimartingale topology, stochastic integrals

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1. Approximating stochastic integrals.

We are given a probability space $(\Omega, \mathcal{F}, P)$, a fixed time horizon $T \in (0, +\infty)$ and a filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right continuity and completeness. All our processes are defined on $[0, T]$. We fix throughout the paper an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{0 \le t \le T}$ and think of this as the discounted prices of d assets in a financial market. A process H is said to be **simple predictable** if H has a representation

$$H_t = \sum_{i=0}^{n} \xi_i 1_{(T_i, T_{i+1}]}(t)$$

where $0 = T_0 \le T_1 \le \ldots \le T_{n+1} \le T$ is a finite sequence of stopping times, ξ_i an $\mathbb{R}^d$-valued $\mathcal{F}_{T_i}$-measurable random variable, $0 \le i \le n$. The collection of simple predictable processes is denoted by $\mathcal{S}$ and the subset of $\mathcal{S}$ formed by bounded processes is denoted by $\mathcal{S}_b$. We denote by X^* the maximal function of a process X, i.e. $X_t^* = \sup_{s \le t} |X_s|$. The semimartingale topology, introduced by Émery [2], is defined by the distance

$$d(X, Y) := \sup_{|H| \le 1} E(1 \wedge [H \cdot (X - Y)]_T^*)$$

where the supremum is taken over all predictable processes bounded by 1. The semimartingale topology can be shown to make the space of semimartingales a topological vector space which is complete. For a general treatment of the semimartingale topology we refer to [2] or [4]. Observe that if the sequence (X^n) converges to X in the semimartingale topology, then $(X^n - X)^*_T$ converges to 0 in probability. We denote by $\mathcal{H}^1$ the space of one-dimensional special semimartingales X such that the canonical decomposition $X = M + A$ satisfies $\|X\|_{\mathcal{H}^1} := E([M,M]_T^{1/2} + \int_0^T |dA_s|) < +\infty$. Here, as usual, M is a martingale and A a predictable process of integrable variation. It is easy to show that if (X_n) converges in $\mathcal{H}^1$, then it converges in the semimartingale topology. Moreover we have the following result (see Th. VIII.58 in [1]).

Theorem 1 *Let X be a semimartingale. There exits a probability measure $Q \sim P$ with bounded density such that X is in $\mathcal{H}^1(Q)$.*

Now we are in a position to show the main result of this section.

Proposition 2 *Let $H = (H^1, \dots, H^d)$ be a d-dimensional predictable process with all components bounded by a constant c. Then there exists a sequence of simple predictable processes (H^n) with all components bounded by c such that $H^n \cdot S$ converges to $H \cdot S$ in the semimartingale topology.*

To prove Proposition 2 we need the following lemma.

Lemma 3 *Let H be a one-dimensional predictable process bounded by a constant c and $X \in \mathcal{H}^1$. Then there exists a sequence of simple predictable processes (H^n) bounded by c such that $H^n \cdot X$ converges to $H \cdot X$ in $\mathcal{H}^1$.*

Proof. It is well-known that there exists a sequence of processes $K^n \in \mathcal{S}_b$ such that $K^n \cdot X$ converges to $H \cdot X$ in $\mathcal{H}^1$ (see for instance [4]). Set

$$H^n := K^n 1_{\{|K^n| \le c\}} + c1_{\{K^n > c\}} - c1_{\{K^n < -c\}}.$$

Since H is bounded by c, we get $|H^n - H| \le |K^n - H|$ and therefore by definition of the norm in $\mathcal{H}^1$

$$\|H^n \cdot X - H \cdot X\|_{\mathcal{H}^1} \le \|K^n \cdot X - H \cdot X\|_{\mathcal{H}^1}.$$

It follows that $\|H^n \cdot X - H \cdot X\|_{\mathcal{H}^1}$ converges to 0. □

Proof of Proposition 2. By Theorem 1 there exists $Q \sim P$ such that all components of S are in $\mathcal{H}^1(Q)$. By applying Lemma 3 to each component H^i of H and $X = S^i$ we get the result since the semimartingale topology is invariant under an equivalent change of measure. □

2. An application to Mathematical Finance.

From now on we assume that S is locally bounded and that there exits a probability measure $Q \sim P$ such that S is a Q-local martingale and the relative entropy $H(Q,P) := E(\frac{dQ}{dP} \ln \frac{dQ}{dP})$ is finite. Denote by $\mathcal{Z}$ the set of positive random variables ξ with $E(\xi) = 1$ such that S is a local martingale under the probability Q with density $\frac{dQ}{dP} = \xi$. Then there exits an optimal density $\underline{\xi}$ for the minimization problem

$$E(\xi \ln \xi) \to \min \quad \text{on } \mathcal{Z}.$$

Moreover, $\underline{\xi}$ is of the form $\underline{\xi} = e^{\underline{I} - H^o \cdot S_T}$ where the integral $H^o \cdot S$ is a true martingale with respect to the optimal measure $\underline{P}$ (see [3]).

Let $\mathcal{A}$ (resp., $\mathcal{A}_b$) be the set of all integrands such that the process $H \cdot S$ is a $\underline{P}$-martingale (resp., $H \cdot S$ and H are bounded) and let $\mathcal{R} := \{\eta : \ \eta = H \cdot S_T, \ H \in \mathcal{A}\}$.

Recall Theorem 2.1 of [3].

Theorem 4 *(a) We have* $\underline{\xi} = e^{\underline{J} - H^o \cdot S_T}$ *where* $H^o \in \mathcal{A}$,

$$J^o := \sup_{\eta \in \mathcal{R}} E(1 - e^{-\eta}) = 1 - e^{-\underline{J}},$$

and the supremum is attained at the (unique) point $\eta^o = H^o \cdot S_T$*;*

(b) $H^o \cdot S$ *is a* Q*-martingale for all* $Q = \xi P$ *such that* $\xi \in \mathcal{Z}$ *and* $E(\xi \ln \xi) < \infty$*;*

(c) there are $H^n \in \mathcal{A}_b$ *such that* $E(1 - e^{-H^n \cdot S_T}) \to J^o$.

In fact, in assertion (c), we can replace the set of strategies $\mathcal{A}_b$ by the set $\mathcal{S}_b$ that is more relevant in the context of Mathematical Finance.

Theorem 5 *There are* $H^n \in \mathcal{S}_b$ *such that* $E(1 - e^{-H^n \cdot S_T}) \to J^o$.

Its proof is based on the following extension of Lemma 5.1 in [3].

Lemma 6 *Let* $H \in \mathcal{A}_b$*. Then there exist integrands* $H^n \in \mathcal{S}_b$ *such that* $H^n \cdot S \to H \cdot S$ *in the semimartingale topology and the sequence* $(H^n \cdot S)$ *is uniformly bounded.*

Proof. Let $H \in \mathcal{A}_b$. Then $H \cdot S$ and all components of H are bounded by some constant c_1. For any stopping time τ we have

$$HI_{[0,\tau]} \cdot S = (H \cdot S)^\tau = H \cdot S^\tau.$$

If (τ_n) is a sequence of stopping times such that $P(\tau_n = T) \to 1$, it is easy to show that the uniformly bounded sequence $(H \cdot S)^{\tau_n}$ converges to $H \cdot S$ in the semimartingale topology. Hence we may assume without loss of generality that S is not only locally bounded but bounded, say, by a constant c_2. By Proposition 2 there exits a sequence $(\tilde{H}^n)$ in $\mathcal{S}$ with all components bounded by c_1 such that $H^n \cdot S \to H \cdot S$ in the topology of semimartingales. Thus $((H - \tilde{H}^n) \cdot S)^*_T \to 0$ in probability as $n \to \infty$ and for the sequence of stopping times

$$\sigma_n := \inf\{t : \ ((H - \tilde{H}^n) \cdot S)^*_t \ge 1\} \wedge T$$

we have that $P(\sigma_n = T) \to 1$. Let $H^n := \tilde{H}^n I_{[0,\sigma_n]}$. Then $H^n \cdot S \to H \cdot S$ in the semimartingale topology and we have by definition of σ_n that

$$|H^n \cdot S| \le 1 + c_2 \ \text{ on } [0, \sigma_n[$$

and

$$|H^n_{\sigma_n} \Delta S_{\sigma_n}| \le 2dc_1c_2.$$

Hence

$$|H^n| \le c_1 \ \text{ and } \ |H^n \cdot S| \le 1 + c_2 + 2dc_1c_2,$$

and we get the assertion. □

Proof of Theorem 5. By assertion (c) of Theorem 4 there are $K^j \in \mathcal{A}_b$ such that $E(1 - e^{-K^j \cdot S_T}) \to J^o$. We apply Lemma 6 to each K^j: for each integer j there are integrands $H^{n,j} \in \mathcal{S}_b$ such that the sequence $(H^{n,j} \cdot S)$ is uniformly bounded and $H^{n,j} \cdot S \to H \cdot S$ in the semimartingale topology. Thus, for each j we have $E(1 - e^{-H^{n,j} \cdot S_T}) \to E(1 - e^{-K^j \cdot S_T})$. We conclude by the diagonal procedure. □

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