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APPROXIMATING THE FOCK SPACE WITH THE TOY FOCK SPACE

Stéphane Attal

ABSTRACT . — We show how the toy Fock space can be embedded into the usual Fock space of quantum stochastic calculus. This embedding gives rise to a rigorous discrete approximation of the Fock space and its natural noise operators. We recover the quantum Ito table from the discrete one. We finally show that the quantum Brownian motion and Poisson process can be simultaneously approached by quantum Bernoulli random walks.

I. The toy Fock space.

Let us realise a Bernoulli random walk on its canonical space. Let $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{F}$ be the σ-field generated by finite cylinders. One denotes by ν_n the coordinate mapping : $\nu_n(\omega) = \omega_n$, for all $n \in \mathbb{N}$.

Let $p \in]0,1[$ and $q = 1-p$. Let μ_p be the probability measure on $(\Omega, \mathcal{F})$ which makes the sequence $(\nu_n)_{n\in\mathbb{N}}$ a sequence of independent, identically distributed Bernoulli random variables with law $p\delta_1 + q\delta_0$. Let $\mathbb{E}_p[\ \cdot\ ]$ denote expectation with respect to μ_p. We have $\mathbb{E}_p[\nu_n] = \mathbb{E}_p[\nu_n^2] = p$. Thus the random variables

$$X_n = \frac{\nu_n - p}{\sqrt{pq}} ,$$

satisfy the following:

i) the X_n are independent,

ii) X_n takes the value $\sqrt{q/p}$ with probability p and $-\sqrt{p/q}$ with probability q,

iii) $\mathbb{E}_p[X_n] = 0$ and $\mathbb{E}_p[X_n^2] = 1$.

Let $\widetilde{\Phi}_p$ be the space $L^2(\Omega, \mathcal{F}, \mu_p)$. We define particular elements of $\widetilde{\Phi}_p$ by

$$\begin{cases} X_\emptyset = 1, & \text{in the sense } X_\emptyset(\omega) = 1 \text{ for all } \omega \in \Omega \\ X_A = X_{i_1} \cdots X_{i_n} & \text{if } A = \{i_1, \ldots, i_n\} \text{ is any finite subset of } \mathbb{N}. \end{cases}$$

Let $\mathcal{P}_f(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$. From *i)* and *iii)* above it is clear $\{X_A\ ;\ A \in \mathcal{P}_f(\mathbb{N})\}$ is an orthonormal set of vectors of $\widetilde{\Phi}_p$.

PROPOSITION 1. — *The family* $\{X_A\ ;\ A \in \mathcal{P}_f(\mathbb{N})\}$ *is an orthonormal basis of* $\widetilde{\Phi}_p$.

Proof. — We just have to prove that $\{X_A,\ A \in \mathcal{P}_f(\mathbb{N})\}$ forms a total set in $\widetilde{\Phi}_p$. In the same way as for the X_A, define

$$\begin{cases} \nu_\emptyset = 1 \\ \nu_A = \nu_{i_1} \cdots \nu_{i_n} \quad \text{for } A = \{i_1, \ldots, i_n\}. \end{cases}$$

It is sufficient to prove that the set $\{\nu_A\ ;\ A \in \mathcal{P}_f(\mathbb{N})\}$ is total.

The space $(\Omega, \mathcal{F}, \mu_p)$ can be identified to $\big([0,1], \mathcal{B}([0,1]),\ \tilde{\mu}_p\big)$ for some probability measure $\tilde{\mu}_p$, via the base 2 decomposition of real numbers. Note that

$$\nu_n(\omega) = \omega_n = \begin{cases} 1 & \text{if} \quad \omega_n = 1 \\ 0 & \text{if} \quad \omega_n = 0 \end{cases}$$

thus $\nu_n(\omega) = 1_{\omega_n = 1}$. Consequently $\nu_A(\omega) = 1_{\omega_{i_1}=1} \cdots 1_{\omega_{i_n}=1}$. Now let $f \in \widetilde{\Phi}_p$ be such that $\langle f, \nu_A \rangle = 0$ for all $A \in \mathcal{P}_f(\mathbb{N})$. Let $I = \big[k2^{-n}, (k+1)2^{-n}\big]$ be a dyadic interval with $k < 2^n$. The base 2 decomposition of $k2^{-n}$ is of the form $(\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)$. Thus

$$\int_I f(\omega) d\tilde{\mu}_p(\omega) = \int_{[0,1]} f(\omega) 1_{\omega_1 = \alpha_1} \cdots 1_{\omega_n = \alpha_n}\ d\tilde{\mu}_p(\omega)\ .$$

The function $1_{\omega_1=\alpha_1} \cdots 1_{\omega_n=\alpha_n}$ can be clearly written as a linear combination of the ν_A. Thus $\int_I f\ d\tilde{\mu}_p = 0$. The integral of f vanishes on every dyadic interval, thus on all intervals. It is now easy to conclude that $f \equiv 0$. ■

We have proved that every element $f \in \widetilde{\Phi}_p$ admits a unique decomposition

$$f = \sum_{A \in \mathcal{P}_f(\mathbb{N})} f(A) X_A \tag{1}$$

with

$$\|f\|^2 = \sum_{A \in \mathcal{P}_f(\mathbb{N})} |f(A)|^2 < \infty\ . \tag{2}$$

We can now define the toy Fock space. The ***toy Fock space*** is the separable Hilbert space $\widetilde{\Phi}$ whose orthonormal basis is chosen to be indexed by $\mathcal{P}_f(\mathbb{N})$. Let $\{X_A\ ;\ A \in \mathcal{P}_f(\mathbb{N})\}$ be this basis. As a consequence there is a natural isomorphism between $\widetilde{\Phi}$ and $\widetilde{\Phi}_p$. For each $p \in]0,1[$, the space $\widetilde{\Phi}_p$ is called the ***p-probabilistic interpretation*** of $\widetilde{\Phi}$.

The only property that allows to make a difference between $\widetilde{\Phi}$ and $\widetilde{\Phi}_p$, or between different $\widetilde{\Phi}_p$'s, is the product. Indeed, as $\widetilde{\Phi}_p$ is a L^2 space it admits a natural product. The way we have chosen the basis of $\widetilde{\Phi}_p$ makes the product being determined by the value of X_n^2, $n \in \mathbb{N}$.

PROPOSITION 2. — *In* $\tilde{\Phi}_p$ *we have*

$$X_n^2 = 1 + c_p X_n$$

where $c_p = \frac{q-p}{\sqrt{pq}}$.

Proof.

$$\begin{aligned} X_n^2 &= \frac{1}{pq}(\nu_n^2 + p^2 - 2p\nu_n) = \frac{1}{pq}\big(p^2 + (1-2p)\nu_n\big) \\ &= \frac{1}{pq}\big(p^2 + (q-p)\nu_n\big) = 1 + \frac{p^2 - qp}{qp} + \frac{q-p}{qp}\nu_n \\ &= 1 - \frac{pc_p}{\sqrt{pq}} + \frac{c_p}{\sqrt{pq}}\nu_n = 1 + c_p\frac{\nu_n - p}{\sqrt{pq}} \ . \end{aligned} \qquad \blacksquare$$

The product that the p-probabilistic interpretation $\tilde{\Phi}_p$ determines in $\tilde{\Phi}$ is called *p-product.*

On $\tilde{\Phi}$, one defines the creation, annihilation and conservation operators by

$$\begin{aligned} a_n^+ X_A &= X_{A\cup\{n\}} 1_{n\notin A} \\ a_n^- X_A &= X_{A\setminus\{n\}} 1_{n\in A} \\ a_n^\circ X_A &= X_A 1_{n\in A} \ . \end{aligned}$$

Note that a_n^+, a_n°, a_n^- are completely determined by

i) their value on 1 and X_n,

ii) the fact they act trivialy on X_m, $m \neq n$.

What we mean exactly is the following. If H_n denotes the closed subspace generated by 1 and X_n, then there exists a natural isomorphism between $\tilde{\Phi}$ and $\bigotimes_{n\in\mathbf{N}} H_n$ (where the countable tensor product is understood to be associated to the stabilizing sequence $(u_n)_{n\in\mathbf{N}}$ such that $u_n = 1$ for all n) given by

$$X_A \mapsto 1 \otimes \cdots \otimes 1 \otimes X_{i_1} \otimes 1 \otimes \cdots \otimes 1 \otimes X_{i_2} \otimes \cdots \quad \text{if} \quad A = \{i_1, \ldots, i_n\} \ .$$

The definitions of a_n^+, a_n^-, a_n° show that these operators act only on H_n and act as the identity everywhere else. In particular a_n^ε commutes with a_m^η for all $n \neq m$ and all $\varepsilon, \eta \in \{+, -, 0\}$. The compositions $a_n^\varepsilon a_n^\eta$ are given by the following *discrete quantum Ito table.*

PROPOSITION 3. — *The products $a_n^\varepsilon a_n^\eta$ are given by*

$a_n^\varepsilon \backslash a_n^\eta$	a_n^+	a_n^-	a_n°
a_n^+	0	a_n°	0
a_n^-	$I-a_n^\circ$	0	a_n^-
a_n°	a_n^+	0	a_n° .

Proof. — Straightforward. ∎

PROPOSITION 4. — *The operator $M_{X_n}^p$ of p-multiplication by X_n is given by*

$$M_{X_n}^p = a_n^+ + a_n^- + c_p a_n^\circ \,.$$

Proof.

$$\begin{aligned} X_n X_A &= X_{A\cup\{n\}} 1_{n\notin A} + X_{A\setminus\{n\}}(1+c_p X_n) 1_{n\in A} \\ &= a_n^+ X_A + a_n^- X_A + c_p a_n^\circ X_A \,. \end{aligned}$$ ∎

II. The Fock space.

We here give a short presentation of the Fock space and its quantum stochastic calculus; one can find all details in [Att].

Let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$. Then $\mathcal{P} = \bigcup_n \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of n-elements subsets of $\mathbb{R}^+$. The set $\mathcal{P}_n$ can be identified to the increasing simplex $\Sigma_n = \{0 < t_1 < \cdots < t_n\}$ of $\mathbb{R}^n$. Thus $\mathcal{P}_n$ inherits a measured space structure from the Lebesgue measure on $\mathbb{R}^n$. This also gives a measure structure on $\mathcal{P}$ if we specify that on $\mathcal{P}_0 = \{\emptyset\}$ we put the measure $\delta_\emptyset$. Elements of $\mathcal{P}$ are often denoted by σ, the measure on $\mathcal{P}$ is denoted by $d\sigma$. The σ-field obtained this way on $\mathcal{P}$ is denoted by $\mathcal{F}$.

The *Fock space* Φ is the space $L^2(\mathcal{P},\mathcal{F},d\sigma)$. An element f of Φ is thus a measurable function $f : \mathcal{P} \to \mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma < \infty \,.$$

One can define, in the same way, $\mathcal{P}_{[a,b]}$ and $\Phi_{[a,b]}$ by replacing $\mathbb{R}^+$ with $[a,b] \subset \mathbb{R}^+$. There is a natural isomorphism between $\Phi_{[0,t]} \otimes \Phi_{[t,+\infty[}$ given by $h\otimes g \mapsto f$ where $f(\sigma) = h(\sigma \cap [0,t]) g(\sigma \cap (t,+\infty[)$. Define $\chi_t \in \Phi$ by

$$\chi_t(\sigma) = \begin{cases} 0 & \text{if } |\sigma| \neq 1 \\ 1_{[0,t]}(s) & \text{if } \sigma = \{s\}. \end{cases}$$

Then χ_t belongs to $\Phi_{[0,t]}$. We even have $\chi_t - \chi_s \in \Phi_{[s,t]}$ for all $s \leq t$. This last property allows to define an *Ito integral* on Φ. Indeed, let $(g_t)_{t\geq 0}$ be a family in Φ such that

i) $t \mapsto \|g_t\|$ is measurable,

ii) $g_t \in \Phi_{[0,t]}$ for all t,

iii) $\int_0^\infty \|g_t\|^2 \, dt < \infty$,

then one defines $\int_0^\infty g_t \, d\chi_t$ to be the limit in Φ of

$$\sum_i \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} P_{t_i} g_s \, ds \otimes \left(\chi_{t_{i+1}} - \chi_{t_i}\right) \tag{3}$$

where P_t is the orthogonal projection onto $\Phi_{[0,t]}$ and $\{t_i, \; i \in \mathbb{N}\}$ is a partition of $\mathbb{R}^+$ which is understood to be refining and to have its diameter tending to 0. Note that $\frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} P_{t_i} g_s \, ds$ belongs to $\Phi_{[0,t_i]}$, which explains the tensor product symbol in (3).

We get that $\int_0^\infty g_t \, d\chi_t$ is an element of Φ with

$$\left\| \int_0^\infty g_t \, d\chi_t \right\|^2 = \int_0^\infty \|g_t\|^2 \, dt \, . \tag{4}$$

Let $f \in L^2(\mathcal{P}_n)$, one can easily define the *iterated Ito integral* on Φ.

$$I_n(f) = \int_{0<t_1<\cdots<t_n} f(t_1, \ldots, t_n) \, d\chi_{t_1} \cdots d\chi_{t_n}$$

by iterating the definition of the Ito integral. We put

$$I_n(f) = \int_{\mathcal{P}_n} f(\sigma) \, d\chi_\sigma \, .$$

We have the following important representation.

THEOREM 5 ([Att]). — *Any element f of Φ admits an* abstract chaotic representation

$$f = \int_{\mathcal{P}} f(\sigma) \, d\chi_\sigma$$

with

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma$$

and an abstract predictable representation

$$f = f(\emptyset)1 + \int_0^\infty D_t f \, d\chi_t$$

with

$$\|f\|^2 = |f(\emptyset)|^2 + \int_0^\infty \|D_s f\|^2 \, ds$$

where $[D_s f](\sigma) = f(\sigma \cup \{s\}) 1_{\sigma \subset [0,s[}$ *is well-defined for* $ds \times d\sigma$ *almost all* (s, σ).

Let us now recall the definitions of the *creation, annihilation* and *conservation* processes in Φ. They are respectively defined by

$$[a_t^+ f](\sigma) = \sum_{\substack{s \in \sigma \\ s \leq t}} f(\sigma \setminus \{s\}), \tag{5}$$

$$[a_t^- f](\sigma) = \int_0^t f(\sigma \cup \{s\}) \, ds, \tag{6}$$

$$[a_t^\circ f](\sigma) = |\sigma \cap [0,t]| \, f(\sigma) \, . \tag{7}$$

There is a good common domain to all these operators, namely

$$\mathcal{D} = \left\{ f \in \Phi \, ; \quad \int_{\mathcal{P}} |\sigma| \, |f(\sigma)|^2 \, d\sigma < \infty \right\} .$$

We also recall an equivalent definition taken from [A-M], which can be easily recovered from (5), (6) and (7). Let $f = f(\emptyset)1 + \int_0^\infty D_t f \, d\chi_t$ be an element of $\mathcal{D}$. Then $P_t f = f(\emptyset)1 + \int_0^t D_s f \, d\chi_s$ also belongs to $\mathcal{D}$ and

$$a_t^+ P_t f = \int_0^t a_s^+ D_s f \, d\chi_s + \int_0^t P_s f \, d\chi_s, \tag{8}$$

$$a_t^- P_t f = \int_0^t a_s^- D_s f \, d\chi_s + \int_0^t D_s f \, ds, \tag{9}$$

$$a_t^\circ P_t f = \int_0^t a_s^\circ D_s f \, d\chi_s + \int_0^t D_s f \, d\chi_s. \tag{10}$$

Finally, let us recall Hudson-Parthasarathy's quantum Ito table. We give it under a formal way only, we refer to [H-P] or [A-M] for a rigorous statement.

$\frown$	da_t^+	da_t^-	da_t°
da_t^+	0	0	0
da_t^-	$dt \; I$	0	da_t^-
da_t°	da_t^+	0	da_t° .

III. Embedding the toy Fock space into the Fock space.

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a partition of $\mathbb{R}^+$ and $\delta(\mathcal{S}) = \sup_i |t_{i+1} - t_i|$ be the diameter of $\mathcal{S}$. For $\mathcal{S}$ being fixed, define $\Phi_i = \Phi_{[t_i, t_{i+1}]}$, $i \in \mathbb{N}$. We then have $\Phi \simeq \bigotimes_{i \in \mathbb{N}} \Phi_i$ (with respect to the stabilizing sequence $(1)_{n \in \mathbb{N}}$).

For all $i \in \mathbb{N}$, define

$$X_i = \frac{\chi_{t_{i+1}} - \chi_{t_i}}{\sqrt{t_{i+1} - t_i}} \in \Phi_i \, ,$$

$$a_i^- = \frac{a_{t_{i+1}}^- - a_{t_i}^-}{\sqrt{t_{i+1} - t_i}} ,$$

$$a_i^\circ = a_{t_{i+1}}^\circ - a_{t_i}^\circ ,$$

$$a_i^+ = P_{1]}\frac{a_{t_{i+1}}^+ - a_{t_i}^+}{\sqrt{t_{i+1} - t_i}} \text{ on } \Phi_i, \quad a_i^+ = I \text{ on } \Phi_j \text{ for } j \neq i$$

where $P_{1]}$ is the orthogonal projection onto $L^2(\mathcal{P}_1)$.

PROPOSITION 6. — *We have*

$$\begin{cases} a_i^- X_i = 1 \\ a_i^- 1 = 0 \end{cases}, \qquad \begin{cases} a_i^\circ X_i = X_i \\ a_i^\circ 1 = 0 \end{cases}, \qquad \begin{cases} a_i^+ X_i = 0 \\ a_i^+ 1 = X_i \end{cases}.$$

Proof. — As $a_t^- 1 = a_t^\circ 1 = 0$ it is clear that $a_i^- 1 = a_i^\circ 1 = 0$. Furthermore, $a_t^+ 1 = \chi_t$ thus

$$a_i^+ 1 = P_{1]}\frac{\chi_{t_{i+1}} - \chi_{t_i}}{\sqrt{t_{i+1} - t_i}} = X_i .$$

Furthermore, by (8), (9) and (10) we have

$$\begin{aligned} a_i^- X_i &= \frac{1}{t_{i+1} - t_i}\left(a_{t_{i+1}}^- - a_{t_i}^-\right)\int_{t_i}^{t_{i+1}} 1 \, d\chi_t \\ &= \frac{1}{t_{i+1} - t_i}\left[\int_{t_i}^{t_{i+1}} (a_t^- - a_{t_i}^-) \, 1 \, d\chi_t + \int_{t_i}^{t_{i+1}} 1 \, dt\right] \\ &= \frac{1}{t_{i+1} - t_i}(0 + t_{i+1} - t_i) = 1 ; \end{aligned}$$

$$\begin{aligned} a_i^\circ X_i &= \frac{1}{t_{i+1} - t_i}\left(a_{t_{i+1}}^\circ - a_{t_i}^\circ\right)\int_{t_i}^{t_{i+1}} 1 \, d\chi_t \\ &= \frac{1}{t_{i+1} - t_i}\left[\int_{t_i}^{t_{i+1}} (a_t^\circ - a_{t_i}^\circ) \, 1 \, d\chi_t + \int_{t_i}^{t_{i+1}} 1 \, d\chi_t\right] \\ &= \frac{1}{t_{i+1} - t_i}\left(\chi_{t_{i+1}} - \chi_{t_i}\right) = X_i ; \end{aligned}$$

$$\begin{aligned} a_i^+ X_i &= \frac{1}{t_{i+1} - t_i}P_{1]}\left(a_{t_{i+1}}^+ - a_{t_i}^+\right)\int_{t_i}^{t_{i+1}} 1 \, d\chi_t \\ &= \frac{1}{t_{i+1} - t_i}P_{1]}\left[\int_{t_i}^{t_{i+1}} (a_t^+ - a_{t_i}^+) \, 1 \, d\chi_t + \int_{t_i}^{t_{i+1}}\int_{t_i}^{t} 1 \, d\chi_s \, d\chi_t\right] \\ &= \frac{2}{t_{i+1} - t_i}P_{1]}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} 1 \, d\chi_s \, d\chi_t \\ &= 0 . \end{aligned}$$

Thus the action of the operators a_i^ε on the X_i is similar to the action of the corresponding operators on the toy Fock spaces. We are now going to construct the toy Fock space inside Φ.

We are still given a fixed partition $\mathcal{S}$. Define $T\Phi(\mathcal{S})$ to be the space of $f \in \Phi$ which are of the form

$$f = \sum_{A \in \mathcal{P}_f(\mathbb{N})} f(A) X_A$$

(with $\|f\|^2 = \sum_{A \in \mathcal{P}_f(\mathbb{N})} |f(A)|^2 < \infty$).

The space $T\Phi(\mathcal{S})$ is thus clearly identifiable to the toy Fock space $\tilde{\Phi}$; the operators a_i^ε, $\varepsilon \in \{+,-,0\}$, act on $T\Phi(\mathcal{S})$ exactly in the same way as the corresponding operators on $\tilde{\Phi}$. We have completely embedded the toy Fock space into the Fock space.

IV. Projections on the toy Fock space.

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a fixed partition of $\mathbb{R}^+$. The space $T\Phi(\mathcal{S})$ is a closed subspace of Φ. We denote by $P_\mathcal{S}$ the operator of orthogonal projection from Φ onto $T\Phi(\mathcal{S})$.

PROPOSITION 7. — *If $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ and if $f \in \Phi$ is of the form*

$$f = \int_{0<s_1<\cdots<s_m} f(s_1, \ldots, s_m) d\chi_{s_1} \cdots d\chi_{s_m}$$

then

$$P_\mathcal{S} f = \sum_{i_1<\cdots<i_m \in \mathbb{N}} \frac{1}{\sqrt{t_{i_1+1} - t_{i_1}} \cdots \sqrt{t_{i_m+1} - t_{i_m}}} \int_{t_{i_1}}^{t_{i_1+1}} \cdots \int_{t_{i_m}}^{t_{i_m+1}} f(s_1, \ldots, s_m)\, ds_1 \cdots ds_m\, X_{i_1} \cdots X_{i_m}\,. \tag{11}$$

Proof. — The quantity f_n on the right handside of (11) is clearly an element of $T\Phi(\mathcal{S})$. We have, for $A = \{i_1, \ldots, i_k\}$

$$\langle f, X_A \rangle = \frac{\delta_{k,m}}{\sqrt{t_{i_1+1}-t_{i_1}} \cdots \sqrt{t_{i_m+1}-t_{i_m}}} \langle \int_{0<s_1<\cdots<s_m} f(s_1, \ldots, s_m)\, d\chi_{s_1} \cdots d\,\chi_{s_m}, \int_{t_{i_1}}^{t_{i_1+1}} \cdots \int_{t_{i_m}}^{t_{i_m+1}} 1\, d\chi_{s_1} \cdots d\chi_{s_m} \rangle$$

$$= \frac{\delta_{k,m}}{\sqrt{t_{i_1+1}-t_{i_1}} \cdots \sqrt{t_{i_m+1}-t_{i_m}}} \int_{t_{i_1}}^{t_{i_1+1}} \cdots \int_{t_{i_m}}^{t_{i_m+1}} \bar{f}(s_1, \ldots, s_m)\, ds_1 \cdots ds_m.$$

But on the other hand we have

$$\begin{aligned}\langle f_n, X_A\rangle &= \delta_{k,m}\frac{1}{(t_{i_1+1}-t_{i_1})^{3/2}\cdots(t_{i_m+1}-t_{i_m})^{3/2}}\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}}\\ &\quad\times \bar{f}(s_1,\dots,s_m)\, ds_1\cdots ds_m \left\|\left(\chi_{t_{i_1+1}}-\chi_{t_{i_1}}\right)-\left(\chi_{t_{i_m+1}}-\chi_{t_{i_m}}\right)\right\|^2\\ &= \delta_{k,m}\frac{1}{\sqrt{t_{i_1+1}-t_{i_1}}\cdots\sqrt{t_{i_m+1}-t_{i_m}}}\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}}\\ &\quad\times \bar{f}(s_1,\dots,s_m)\, ds_1\cdots ds_m\ .\end{aligned}$$

This proves our proposition. ■

Note that the following identities could have been used as natural definitions of the operators a_i^ε on $T\Phi(\mathcal{S})$.

PROPOSITION 8. — *For any partition $\mathcal{S}$ and any $f\in\mathcal{D}$ we have*

$$a_i^\circ\, P_{\mathcal{S}}f = P_{\mathcal{S}}(a_{t_{i+1}}^\circ - a_{t_i}^\circ)f$$
$$\sqrt{t_{i+1}-t_i}\; a_i^\pm P_{\mathcal{S}}f = P_{\mathcal{S}}(a_{t_{i+1}}^\pm - a_{t_i}^\pm)f\ .$$

Proof. — Let us take f of the form

$$f = \int_{0<s_1<\cdots<s_m} f(s_1,\dots,s_m)\, d\chi_{s_1}\cdots d\chi_{s_m}\ .$$

Then

$$(a_{t_{i+1}}^\circ - a_{t_i}^\circ)f = \int_{0<s_1<\cdots<s_m} \left|\{s_1,\dots,s_m\}\cap[t_i,t_{i+1}]\right| f(s_1,\dots,s_m)\, d\chi_{s_1}\cdots d\chi_{s_m}$$

$$\begin{aligned}P_{\mathcal{S}}(a_{t_{i+1}}^\circ - a_{t_i}^\circ)f &= \sum_{j_1<\cdots<j_m\in\mathbb{N}}\frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_m+1}-t_{j_m}}}\int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_m}}^{t_{j_m+1}}\\ &\quad\times\left|\{s_1,\dots,s_m\}\cap[t_i,t_{i+1}]\right| f(s_1,\dots,s_m)\\ &\quad\times ds_1\cdots ds_m\, X_{j_1}\cdots X_{j_m}\\ &= \sum_{j_1<\cdots<j_m\in\mathbb{N}}\frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_m+1}-t_{j_m}}}1_{i\in\{j_1,\dots,j_m\}}\\ &\quad\int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_m}}^{t_{j_m+1}} f(s_1,\dots,s_m)\, ds_1\cdots ds_m\, X_{j_1}\cdots X_{j_m}\\ &= a_i^\circ\sum_{j_1<\cdots<j_m\in\mathbb{N}}\frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_m+1}-t_{j_m}}}\\ &\quad\int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_m}}^{t_{j_m+1}} f(s_1,\dots,s_m)\, ds_1\cdots ds_m\, X_{j_1}\cdots X_{j_m}\\ &= a_i^\circ\, P_{\mathcal{S}}f\ .\end{aligned}$$

In the same way

$$(a^-_{t_{i+1}} - a^-_{t_i})f = \int_{0<s_1<\cdots<s_{m-1}} \int_{t_i}^{t_{i+1}} f(\{s_1,\ldots,s_{m-1}\}\cup s)\ ds\ d\chi_{s_1}\cdots d\chi_{s_{m-1}}$$

$$\begin{aligned}
P_S(a^-_{t_{i+1}} - a^-_{t_i})f = &\sum_{j_1<\cdots<j_{m-1}\in\mathbb{N}} \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_{m-1}+1}-t_{j_{m-1}}}}\\
&\times \int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_{m-1}}}^{t_{j_{m-1}+1}}\int_{t_i}^{t_{i+1}} f(\{s_1,\ldots,s_{m-1}\}\cup s)\\
&\times ds\ ds_1\cdots ds_{m-1}\ X_{j_1}\cdots X_{j_{m-1}}\\
= &\sum_{j_1<\cdots<j_{m-1}\in\mathbb{N}}\sum_{k=0}^{m-1} 1_{0<j_1<\cdots<j_k<i<j_{k+1}<\cdots<j_{m-1}}\\
&\times \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_{m-1}+1}-t_{j_{m-1}}}}\int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_k}}^{t_{j_k+1}}\int_{t_i}^{t_{i+1}}\\
&\times \int_{t_{j_{k+1}}}^{t_{j_{k+1}+1}}\cdots\int_{t_{j_{m-1}}}^{t_{j_{m-1}+1}} f(s_1,\ldots,s_k,s,s_{k+1},\ldots,s_{m-1})\\
&\times ds_1\cdots ds_k\ ds\ ds_{k+1}\cdots ds_{m-1}\ X_{j_1}\cdots X_{j_{m-1}}\\
= &\sqrt{t_{i+1}-t_i}\sum_{j_1<\cdots<j_m\in\mathbb{N}} \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_m+1}-t_{j_m}}}\\
&\times \int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_m}}^{t_{j_m+1}} f(s_1,\ldots,s_m)\ ds_1\cdots ds_m\\
&1_{i\in\{j,\ldots,j_m\}}X_{j_1}\cdots\widehat{X}_i\cdots X_{j_m}\\
= &\sqrt{t_{i+1}-t_i}\ a^-_i\ P_S f\ .
\end{aligned}$$

Finally,

$$\begin{aligned}
(a^+_{t_{i+1}} - a^+_{t_i})f = &\sum_{k=0}^{n}\int_{0<s_1<\cdots<s_k<s<s_{k+1}<\cdots<s_m} 1_{[t_i,t_{i+1}]}(s)\\
&\times f(s_1,\ldots,s_m)\ d\chi_{s_1}\cdots d\chi_{s_k}\ d\chi_s\ d\chi_{s_{k+1}}\cdots d\chi_{s_m}\ .
\end{aligned}$$

$$\begin{aligned}
P_S(a^+_{t_{i+1}} - a^+_{t_i})f = &\sum_{j_1<\cdots<j_{m+1}\in\mathbb{N}} \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_{m+1}+1}-t_{j_{m+1}}}}\sum_{k=0}^{n}\\
&\times \int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_{m+1}}}^{t_{j_{m+1}+1}} 1_{[t_i,t_{i+1}]}(t_{j_{k+1}})f(s_1,\ldots,\widehat{s_{k+1}},\ldots,s_{m+1})\\
&\times ds_1\cdots ds_{m+1}\ X_{j_1}\cdots X_{j_{m+1}}
\end{aligned}$$

$$= \sum_{j_1<\cdots<j_{m+1}\in\mathbb{N}} \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_{m+1}+1}-t_{j_{m+1}}}} \, 1_{i\in\{j_1,\ldots,j_{m+1}\}}$$
$$\int_{t_{j_1}}^{t_{j_1+1}}\cdots\int_{t_{j_{m+1}}}^{t_{j_{m+1}}} f(s_1,\ldots,\widehat{s_i},\ldots,s_{m+1})$$
$$ds_1\cdots ds_{m+1}\; X_{j_1}\cdots X_{j_{m+1}}$$
$$= \sum_{j_1<\cdots<j_m\in\mathbb{N}} \sqrt{t_{i+1}-t_i}\, \frac{1}{\sqrt{t_{j_1+1}-t_{j_1}}\cdots\sqrt{t_{j_m+1}-t_{j_m}}} \int_{t_{j_1}}^{t_{j_1+1}}$$
$$\cdots\int_{t_{j_m}}^{t_{j_m+1}} f(s_1,\ldots,s_m)\, ds_1\cdots ds_m\; 1_{i\notin\{j_1,\ldots,j_m\}} X_{j_1}\cdots X_{j_m} X_i$$
$$= \sqrt{t_{i+1}-t_i}\;\; a_i^+\, P_S f\;.$$

■

V. Approximations.

We are now going to prove that the Fock space Φ and its basic operators a_t^+, a_t^-, a_t° can be approached by the toy Fock spaces $T\Phi(\mathcal{S})$ and their basic operators a_i^+, a_i^-, a_i°.

We are given a sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of partitions which are getting finer and finer and whose diameter $\delta(\mathcal{S}_n)$ tends to 0 when n tends to $+\infty$. Let $T\Phi(n) = T\Phi(\mathcal{S}_n)$ and $P_n = P_{\mathcal{S}_n}$, for all $n\in\mathbb{N}$.

THEOREM 9.

i) For every $f\in\Phi$ the sequence $f_n = P_n f$, $n\in\mathbb{N}$, satisfies $f_n \in T\Phi(n)$ for all $n\in\mathbb{N}$ and converges to f in Φ.

ii) If $\mathcal{S}_n = \{0 = t_0^n < t_1^n < \cdots < t_k^n < \cdots\}$, then for all $t\in\mathbb{R}^+$, the operators

$$\sum_{i;t_i^n\le t} a_i^\circ,\; \sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\, a_i^- \text{ and } \sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\, a_i^+$$

converge strongly on $\mathcal{D}$ to a_t°, a_t^- and a_t^+ respectively.

iii) With the same notations as in ii), for all $t\in\mathbb{R}^+$, the operators

$$\sum_{i;t_i^n\le t} a_i^\circ\, P_n,\; \sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\, a_i^-\, P_n \text{ and } \sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\, a_i^+\, P_n$$

converge strongly on $\mathcal{D}$ to a_t°, a_t^- and a_t^+ respectively.

Proof. — *i)* As the $\mathcal{S}_n$ are refining the $(P_n)_n$ form an increasing family of orthogonal projections in Φ. Let $P_\infty = \bigvee_n P_n$. Clearly, for all $s \le t$, we have that $\chi_t - \chi_s$ belongs to $\mathrm{Ran}P_\infty$. But by the construction of the Ito integral and by Theorem 5, we have that the $\chi_t - \chi_s$ generate Φ. Thus $P_\infty = I$. Consequently if $f \in \Phi$, the sequence $f_n = P_n f$ satisfies the statements.

ii) The convergence of $\sum_{i,t_i^n \leq t} a_i^\circ$ and $\sum_{i,t_i^n \leq t} \sqrt{t_{i+1}^n - t_i^n} a_i^-$ to a_t° and a_t^- respectively is clear from the definitions. Let us check the case of a^+. We have, for $f \in \mathcal{D}$

$$\left[\sum_{i;t_i^n \leq t} \sqrt{t_{i+1}^n - t_i^n} a_i^+ f\right](\sigma) = \sum_{i;t_i^n \leq t} 1_{|\sigma \cap [t_i^n, t_{i+1}^n]|=1} \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} f(\sigma \setminus \{s\}) .$$

Put $t^n = \inf\{t_i^n \in \mathcal{S}_n \,; t_i^n \geq t\}$. We have

$$\begin{aligned}
&\Big\| \sum_{i;t_i^n \leq t} \sqrt{t_{i+1}^n - t_i^n} a_i^+ - a_t^+ f \Big\|^2 \\
&= \int_{\mathcal{P}} \Big| \sum_{i;t_i^n \leq t} 1_{|\sigma \cap [t_i^n, t_{i+1}^n]|=1} \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} f(\sigma \setminus \{s\}) - \sum_{s \in \sigma \cap [0,t]} f(\sigma \setminus \{s\}) \Big|^2 \, d\sigma \\
&\leq 2 \int_{\mathcal{P}} \Big| \sum_{s \in \sigma \cap [t,t]} f(\sigma \setminus \{s\}) \Big|^2 \, d\sigma + 2 \int_{\mathcal{P}} \Big| \sum_{i;t_i^n \leq t} 1_{|\sigma \cap [t_i^n, t_{i+1}^n]| \geq 2} \\
&\qquad\qquad \times \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} f(\sigma \setminus \{s\}) \Big|^2 \, d\sigma.
\end{aligned}$$

For any fixed σ, the terms inside each of the integrals above converge to 0 when n tends to $+\infty$. Furthermore we have, for n large enough,

$$\begin{aligned}
\int_{\mathcal{P}} \Big| \sum_{s \in \sigma \cap [t,t^n]} f(\sigma \setminus \{s\}) \Big|^2 \, d\sigma &\leq \int_{\mathcal{P}} |\sigma| \sum_{\substack{s \in \sigma \\ s \leq t+1}} |f(\sigma \setminus \{s\})|^2 \, d\sigma \\
&= \int_0^{t+1} \int_{\mathcal{P}} (|\sigma| + 1) |f(\sigma)|^2 \, d\sigma \, ds \\
&\leq (t+1) \int_{\mathcal{P}} (|\sigma| + 1) |f(\sigma)|^2 \, d\sigma
\end{aligned}$$

which is finite for $f \in \mathcal{D}$;

$$\begin{aligned}
&\int_{\mathcal{P}} \Big| \sum_{i;t_i^n \leq t} 1_{|\sigma \cap [t_i^n, t_{i+1}^n]| \geq 2} \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} f(\sigma \setminus \{s\}) \Big|^2 \, d\sigma \\
&\qquad \leq \int_{\mathcal{P}} \Big(\sum_{i;t_i^n \leq t} 1_{|\sigma \cap [t_i^n, t_{i+1}^n]| \geq 2} \Big| \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} f(\sigma \setminus \{s\}) \Big| \Big)^2 \, d\sigma \\
&\qquad \leq \int_{\mathcal{P}} \Big(\sum_{i;t_i^n \leq t} \sum_{s \in \sigma \cap [t_i^n, t_{i+1}^n]} |f(\sigma \setminus \{s\})| \Big)^2 \, d\sigma \\
&\qquad = \int_{\mathcal{P}} \Big(\sum_{\substack{s \in \sigma \\ s \leq t^n}} |f(\sigma \setminus \{s\})| \Big)^2 \, d\sigma
\end{aligned}$$

$$= \int_{\mathcal{P}} |\sigma| \sum_{\substack{s\in\sigma \\ s\le t^n}} |f(\sigma \setminus \{s\})|^2 \, d\sigma$$

$$\le (t+1) \int_{\mathcal{P}} (|\sigma|+1)\left|f(\sigma)\right|^2 \, d\sigma$$

in the same way as above. So we can apply Lebesgue's theorem. This proves *ii)*.

iii) By Proposition 8, we have for all $f\in\mathcal{D}$

$$\sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\; a_i^+ \; P_n f = P_n a_{t^n}^+ f \; .$$

Consequently

$$\begin{aligned} \Big\| \sum_{i;t_i^n\le t} \sqrt{t_{i+1}^n - t_i^n}\; a_i^+ \; P_n f - a_t^+ f\Big\|^2 & \\ &\le 2\left\|a_t^+ f - P_n a_t^+ f\right\|^2 + 2\left\|P_n(a_t^+ f - a_{t^n}^+ f)\right]\right\|^2 \\ &\le 2\left\|a_t^+ f - P_n a_t^+ f\right\|^2 + 2\left\|a_t^+ f - a_{t^n}^+ f\right\|^2 \end{aligned}$$

which tends to 0 as n tends to $+\infty$.

The cases of a° and a^- are obtained in the same way. ∎

VI. Probabilistic interpretations.

It is not the aim of this article to give a complete course about probabilistic interpretations of the Fock space Φ (see [Att] for details) ; but we recall that in the same way as $\widetilde{\Phi}$, the space Φ is naturally isomorphic to the L^2 space of the canonical space $(\Omega, \mathcal{F}, \mathbb{P})$ of some basic processes. Namely, the Brownian motion, the Poisson process, the Azéma martingales, and some other ones.

Again the multiplication of random variables will make a difference between the different interpretations. What we need to know here is that the operator of Brownian multiplication by the Brownian motion is the operator

$$W_t = a_t^+ + a_t^-$$

and the operator of Poisson multiplication by the Poisson process is

$$N_t = a_t^+ + a_t^- + a_t^\circ + tI \; .$$

Let us consider an approximation of the Fock space Φ by toy Fock spaces $T\Phi(n)$, $n\in\mathbb{N}$.

THEOREM 10. — *On $T\Phi(n)$, let $X_i = a_i^+ + a_i^-$, $i\in\mathbb{N}$. Then, for all $t\in\mathbb{R}^+$; we have that*

$$\sum_{i;t_i\le t} \sqrt{t_{i+1} - t_i}\; X_i$$

converges strongly to W_t.

Proof. — The proof is immediate from Theorem 9. ∎

Let $\mathcal{S}_n = \{i/n\ ;\ i\in\mathbb{N}\}$.

THEOREM 11. — *On $T\Phi(n)$, let $X_i = a_i^+ + a_i^- + c_n a_i^\circ$, $i\in\mathbb{N}$ be associated to the coefficient $p_n = 1/n$. Then, for all $t\in\mathbb{R}^+$, we have that*

$$\frac{1}{\sqrt{n}} \sum_{i;t_i\leq t} X_i$$

converges strongly to $X_t = N_t - tI$, the operator of multiplication by the compensated Poisson process.

Proof. — If $p_n = 1/n$, then $q_n = 1 - 1/n$ and $c_n = \frac{1-2/n}{\sqrt{1/n-1/n^2}} = \frac{n-2}{\sqrt{n-1}}$. Thus $c_n/\sqrt{n}$ converges to 1. Now,

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{i;t_i\leq t} X_i &= \sum_{i;t_i\leq t} \frac{1}{\sqrt{n}}\, a_i^+ + \frac{1}{\sqrt{n}}\, a_i^- + \frac{c_n}{\sqrt{n}}\, a_i^\circ \\ &= \sum_{i;t_i\leq t} \sqrt{t_{i+1}-t_i}\,(a_i^+ + a_i^-) + \frac{c_n}{\sqrt{n}} \sum_{i;t_i\leq t} a_i^\circ\end{aligned}$$

which clearly converges to $a_t^+ + a_t^- + a_t^\circ$ by Theorem 9. ∎

The two results above are stronger than the usual approximations of the Brownian motion (resp. Poisson process) by Bernoulli random walks. Not only they give an approximation of the trajectories but of the multiplication operators. And this is obtained all together, in a single approximation theorem, the Theorem 9.

VII. The Ito tables.

This section is heuristic, but it gives a good idea of why the discrete quantum Ito table is a discrete approximation of the usual one, though they seem different. Let $\mathcal{S}_n = \{i/n\ ;\ i\in\mathbb{N}\}$. Let $\tilde{a}_i^+ = 1/\sqrt{n}\ a_i^+$, $\tilde{a}_i^- = 1/\sqrt{n}\ a_i^-$ and $\tilde{a}_i^\circ = a_i^\circ$. The Theorem 9 shows that $\tilde{a}_i^\varepsilon$ is a good approximation of da_t^ε, where $t = t_i$. Now the discrete Ito table becomes

$\frown$	$\tilde{a}_i^+$	$\tilde{a}_i^-$	$\tilde{a}_i^\circ$
$\tilde{a}_i^+$	0	$\frac{1}{n}\tilde{a}_i^\circ$	0
$\tilde{a}_i^-$	$\frac{1}{n}I - \frac{1}{n}\tilde{a}_i^\circ$	0	$\tilde{a}_i^-$
$\tilde{a}_i^\circ$	$\tilde{a}_i^+$	0	$\tilde{a}_i^\circ$.

But

1) $\frac{1}{n}\tilde{a}_i^\circ$ is not an infinitesimal for $\sum_{i;t_i\le t} \frac{1}{n}\tilde{a}_i^\circ$ is almost $\frac{1}{n}a_t^\circ$ which converges to 0. Thus $\frac{1}{n}\tilde{a}_i^\circ$ can be considered to be 0 in this table;

2) $\frac{1}{n}I$ is simply $dt\ I$, that is $(t_{i+1} - t_i)I$. Thus at the limit this table becomes

$\frown$	da_t^+	da_t^-	da_t°
da_t^+	0	0	0
da_t^-	$dt\ I$	0	da_t^-
da_t°	da_t^+	0	da_t° .

That is the usual Ito table.

Remark. — The above heuristic approximation of the Ito table as been made rigourous by Y. Pautrat [Pau].

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