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ON DEFORMATIONS OF MONOMIAL CURVES

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1. What is a monomial curve(-singularity) ?

1.1. Let C be a proper, reduced curve over an algebraically closed field of characteristic zero.

Let $s \in C(k)$ be an unibranched point of C.

Then let B be the formal ring of C at s.

B is a subring of $k[[t]]$ and $k[[t]] = \overline{B}$, the normalization of B.

B can be essentially generated as an k-algebra by polynomials $z_0(t),\dots,z_m(t)$, and choosing appropriate isomorphisms one can assume:

$$z_0(t) = t^{a_0} \quad , \quad z_i(t) = t^{a_i} + b_{i,1}t^{a_{i+1}} + \dots \quad , \; b_{i,j} \in k \; , \; i=1,\dots,m$$

$$a_0 < a_1 < \dots < a_m \quad .$$

1.2. $k[[t]]$ is a discrete valuation ring, so we have a valuation-map

$$v\colon k((t))^* \longrightarrow \mathbb{Z} \quad .$$

Let $\Gamma = v(B)$. Γ is a subsemigroup of $(\mathbb{N},+)$ with the properties:

a) $0 \in \Gamma$

b) $\exists\, c \in \mathbb{N} : c + \mathbb{N} \subseteq \Gamma$. The smallest such c is called the <u>conductor</u> of .

These properties imply that $\mathbb{N}-\Gamma$ is finite and the number of gaps g, $g = \text{card}(\mathbb{N}-\Gamma)$ is called the <u>genus</u> of Γ , $g = [\overline{B} : B]$.

Semigroups $\Gamma \subseteq \mathbb{N}$ with the properties a) and b) are called <u>numerical</u>.

1.3. To each ring B you can consider the corresponding affine ring

$$B_a = k[z_o, \ldots, z_m] \subseteq k[t]$$

(in the following we will not distinguish between B and B_a unless it is unavoidable)

Now there is obviously a special class of such rings: all the $z_i(t)$ are monomials, i. e. $z_i(t) = t^{a_i}$.

Such a ring B is called monomial.

Then we have:

(i) $B = k[\Gamma]$ for a numerical semigroup .

(ii) There exists a $\mathbb{G}_m$-action on spec B (the affine monomial curve):

$g \in G_m(k)$ operates as $g \cdot z_i = g \cdot t^{a_i} = g^{a_i} \cdot t^{a_i} = g^{a_i} \cdot z_i$

There is a partial converse to (ii):

If C is an affine curve over k with $\mathbb{G}_m$-action, then C is a Zariski-open subcheme of spec $k[\Gamma]$ for some Γ. ([K.])

(iii) B has an isolated singularity at the origin (if $\Gamma \neq \mathbb{N}$).

1.4. By what was said earlier we have:

Each ring $B \subseteq k[[t]]$ with value-semigroup Γ has a canonical filtration F given by

$$I_n = \{x \in B \mid v(x) \geq n\}$$

and we have

$$gr_F B \simeq k[\Gamma] .$$

2. What can we do with monomial curves?

We can deforme them.

For this let us first look at the $T^1(B/k, B) = T^1_B$, the cohomology groups of the cotangent-complex of $B = k[\Gamma]$.

2.1. Let $a_o,\dots,a_m$ be a generating set for Γ . Choose a presentation

$$0 \longrightarrow I \longrightarrow P = k[X_o,\dots,X_m] \xrightarrow{p} B = k[t^{a_o},\dots,t^{a_m}] \longrightarrow 0$$

with $p(X_i) = t^{a_i}$.

Then there is a $\mathbb{G}_m$-action on P compatible with p and we can give weights a_i to the X_i and speak about homogenous elements.

2.1.1. <u>Theorem</u> (Herzog [H])

I can be generated by homogenous elements

$$F_j = \prod_{i=o}^{m} X_i^{e_i} - \prod_{i=o}^{m} X_i^{f_i} \quad , \quad e_i.f_i = 0 \quad (1 \leqslant j \leqslant l) .$$

Set $h_j = v(F_j)$ and

$$z_j = (e_o-f_o , \dots ,e_m-f_m) \in \mathbb{Z}^m$$

2.2. There is the exact sequence:

$$\mathrm{Hom}_B(\Omega^1_{P/k} \otimes_P B,B) \xrightarrow{\varphi} \mathrm{Hom}_B(I/I^2,B) \longrightarrow T^1_B \longrightarrow 0$$

with ([P]):

(i) $\mathrm{Hom}_B(\Omega^1_{P/k} \otimes_P B,B)$: is a free B-module generated by the partial derivatives $\frac{\partial}{\partial X_i}$, $\mathbb{Z}$-graded by assigning the weights $v(\frac{\partial}{\partial X_i}) = -a_i$.

(ii) $\mathrm{Hom}_B(I/I^2,B)$:

Let $R_k \in P^l$ (k=1,...,l) be a set of generators for the relations between the F_j:

$R_k = (R_{1k},\dots,R_{lk})$ with $\sum_{i=1}^{l} R_{jk}F_j = 0$.

The R_k can also be chosen homogenous, i.e. such that

$v(R_{jk}) + v(F_j) = N_k = v(R_k)$.

The images r_k of R_k modulo I generate the relations between the F_i modulo I^2 in $I/_{I}2$ as a B-module.

An element of $\mathrm{Hom}_B(I/_{I}2,B)$ can now be identified with a column vector

$$f = (f_1,\dots,f_l)^T \quad (f_j \quad B) \text{ such that } r_k.f = 0 \quad .$$

f is <u>homogenous of weight a</u> iff

f_j is homogenous of weight h_j+a .

(iii) $\varphi(\frac{\partial}{\partial X_i}) = (\frac{\partial}{\partial t^{a_i}}(F_1),\dots,\frac{\partial}{\partial t^{a_i}}(F_l))$, where

$$\frac{\partial}{\partial t^{a_i}}(F_j) = \frac{\partial}{\partial X_i}(F_j) \text{ modulo } I \quad .$$

$(\frac{\partial}{\partial X_i})$ is homogenous of weight $-a_i$.

(iv) φ being homogenous of degree 0 , T^1_B is a $\mathbb{Z}$-graded k-vectorspace, which is finite-dimensional because the singularity of B is isolated.

Therefore: T^1_B can be decomposed

$$T^1_B = \bigoplus_{n\in\mathbb{Z}} T^1_B(n)$$

The same result is true for the higher cohomology groups T^i_B .

2.2.1. <u>Theorem</u>

$$\begin{aligned}\dim T^1_B(n) &= \mathrm{card}\{a_i : a_i+n \notin \Gamma\} - \mathrm{rank}\{a_i \in k : a_i+n \notin \Gamma\} \\ &\quad - \mathrm{rank}\{z_i : h_i+n \notin \Gamma\} \\ &= \max(0 , \mathrm{card}\{a_i : a_i+n \notin \Gamma\} - 1) - \mathrm{rank}\{z_i : h_i+n \notin \Gamma\}\end{aligned}$$

Proof:

Look at $\mathrm{Hom}_B(I/_{I}2,B)(n)$. The dimension of $\mathrm{Im}\,\varphi$ in degree n is $\mathrm{card}\{a_i : a_i+n \notin \Gamma\} - \mathrm{rank}\{a_i : a_i+n \notin \Gamma\}$, because there is the relation $\varphi(\sum a_i X_i \frac{\partial}{\partial X_i}) = 0$ - the "Euler-relation" . A vector f is in $\mathrm{Hom}_B(I/_{I}2,B)(n)$ iff $f_j \neq 0$ implies $h_j+n = v(f_j) \in \Gamma$.

2.2.2. Corollary: ([Ba],[B1],[V])

Monomial curve-singularities aren't rigid, i.e. $T^1_B \neq 0$.

Proof:

$\dim T^1(c-1-a_0-a_1) \geq 1$, because

$c-1-a_0-a_1+a_i \notin \Gamma$ for $i = 0,1$

but for every h_i: $c \leq c-1-a_0-a_1+h_i \in \Gamma$.

2.2.3. Corollary:

$\dim T^1(n) = 0 \quad \forall n < -(4g+2)$

and $\dim T^1(-4g-2) \neq 0$ iff Γ is hyperelliptic i.e.

$\Gamma = \langle 2,2g+1 \rangle$ (see 3.3.2.)

Proof:

Easy computation using a special generating set of Γ.

2.2.4. An example: $a_0 = 3$, $a_1 = 5$, $a_2 = 7$,

$\Gamma = \langle 3,5,7 \rangle$, $g(\Gamma) = 3$

$I = (F_{11}, F_{12}, F_{22})$ with

$F_{11} = X_1^2 - X_0^3X_2$, $F_{12} = X_1X_2 - X_0^4$, $F_{22} = X_2^2 - X_0X_1$,

these are the (2x2)-minors of the matrix:

	a_0	a_1	a_2	h_{ij}
z_{11}	-3	2	-1	14
z_{12}	-4	1	1	12
z_{22}	-1	-1	2	10

$\dim T^1(-7) = 1$.

2.3. By similar computations we obtain:

2.3.1. Theorem

$$\dim T^2(n) = \#\{h_j \mid h_j + n \notin \Gamma\} - \operatorname{rank}\{z_j \mid h_j + n \notin \Gamma\}$$

$$- \operatorname{rank}\{\bar{r}_k \mid v(r_k) + n \notin \Gamma\} ,$$

where $\bar{r}_k = (r_{1k},\ldots,r_{lk})(t = 1)$

Proof:

Look at the exact sequence

$$0 \longrightarrow B \xrightarrow{\cdot x} B \longrightarrow B/_{xB} \longrightarrow 0 \quad \text{with } x \in B .$$

Then you get the long exact cohomology-sequence:

$$T^1_B \xrightarrow{\cdot x} T^1_B \longrightarrow T^1(B/_k, B/_{xB}) \longrightarrow T^2_B \xrightarrow{\cdot x} T^2_B$$

If $v(x)$ is great enough, multiplication with x is the zero-map on T^1_B and T^2_B .

This yields:

$$T^1(B/_k, B/_{xB}) \simeq T^1_B \oplus T^2_B[v(x)]$$, where $[N]$ means shifting of degrees by N .

Following the lines of 2.2. it is now easy to obtain the result.

2.3.2. Corollary:

$$\dim T^2(n) = 0 \quad \text{for} \quad n \geq c - 2a_1$$

Proof:

$h_j \geq 2a_1$ for all j .

3. Now let's deforme B:

3.1. As one knows, there exists a formal, versal deformation of B over k:

$$(x) \quad \begin{array}{ccc} T & \longrightarrow & B \\ \downarrow & & \downarrow \\ S & \longrightarrow & k \end{array} \qquad \text{(or dually} \quad \begin{array}{ccc} V & \longrightarrow & C \\ \downarrow & & \downarrow \\ M & \longrightarrow & \text{spec } k \end{array} \quad \text{with} \quad \begin{array}{l} V = \text{spec } T, \\ M = \text{spec } S, \\ C = \text{spec } B.) \end{array}$$

M is called the base space or the moduli-space of the formal, versal deformation.

S is a local, complete noetherian k-algebra with:

$$\text{Hom}\,(S/_{m^2}, k[\varepsilon]) \cong T^1_B .$$

Let $t_1,\dots,t_r$ be a basis of T^1_B consisting of homogenous elements with the weights $e_1,\dots,e_r$ resp.

Then:

$$S/_{m^2} = k[[s_1,\dots,s_r]]/_{m^2}$$

and the s_i are the corresponding duals of the t_i and have weights $-e_i$ (!).

3.1.1. <u>Theorem</u> ([P])

The versal deformation is equivariant, that means there exist $\mathbb{G}_m$-actions on M and V making the maps in (x) equivariant.

3.1.2. <u>Proposition</u>

$$\dim M \geqslant 1 ,$$

i.e. there exists at least one unobstructed first-order deformation of C^Γ .

Proof:

1. If $T^1(n) = 0$ for $n \in \mathbb{N}$, then Γ is of a very special type ([B2]) and the prop. is easy to check.
2. So let's assume $\bigoplus_{n \in \mathbb{N}} T^1(n) \neq 0$.

   Then there exists $m \geq \max\{0, c-1-a_0-a_1\}$ s.t. $T^1(m) \neq 0$ (Corollary 2.2.2), but by (2.3.2.) $T^2(n) = 0$ for all $n \geq c-1-a_0-a_1$ and the obstructions to lift a deformation in $T^1(m)$ would lie in $T^2(m.\mathbb{N})$.

The moduli-space M has the important linear subspaces M^-, M^+ defined by

$$T^{1,-} = \bigoplus_{n \leq 0} T^1(n) \quad \text{and} \quad T^{1,+} = \bigoplus_{n \in \mathbb{N}} T^1(n) \quad \text{resp.}$$

3.2. The importance of M^+ (Zariski, Teissier):

A homogenous one-parameter deformation over M^+ corresponding to a homogenous element $t \in T^{1,+}$, is given by equations:

$$F_i^{\infty} = F_i + F_i^1 s + F_i^2 s^2 + \dots$$

and because the weight of s is lesser than 0 ($T^1(0) = 0$ by 2.2.4)), the weights of the F_i^j increase. So we have a section $M^+ \overset{\sigma}{\rightleftarrows} V^+$, σ picking out the point "all $X_i = 0$".

M^+ is a good candidate for equisingularity:

Let B' be any ring with value-semigroup Γ as in 1.1.

Then there exists a one-parameter deformation with general fiber B' and special fiber B and this deformation is induced over M^+.

(If B is a complete intersection, Teissier has shown that M^+ is the space parametrizing the curve-singularities with semigroup Γ.)

3.3. The importance of M^- (Pinkham):

One can projectivize the fibers over M^-:

A homogenous one-parameter deformation over M^- is given by:

$$F_i^{\infty} = F_i + F_i^1 s + F_i^2 s^2 + \ldots$$

and the weight of s is <u>positive</u>, so the weights of the F_i^j <u>decrease</u>. Fill in a homogenizing variable Z of weight 1:

$$F_i'^{\infty} = F_i + F_i^1 Z^e s + F_i^2 Z^{2e} s^2 + \ldots \qquad (v(s) = e)$$

and look at $\operatorname{Proj} k[X_i, Z] / (F_i'^{\infty})$.

Now there is also a section: $Z = 0$, picking out the point at infinity!

3.3.1. <u>Definition</u>:

Let C be a smooth projective curve (compact Riemann surface) over k of genus g, $x \in C(k)$. Then

$\Gamma_x = \{ n \in \mathbb{N} \mid$ there exists a meromorphic function f on C, holomorphic outside x, the degree of f at x being exactly $n \}$

is a numerical semigroup.

The Weierstrass-gap-theorem asserts: $|\mathbb{N} - \Gamma_x| = g$.

x is called a Weierstrass-point of C with semigroup Γ_x.

3.3.2. <u>Examples</u>

1. A point with semigroup Γ

   - $\Gamma = \{0, g+1, g+2, \ldots\}$ is called <u>ordinary</u>
   - $\Gamma = \{0, g, g+2, g+3, \ldots\}$ is called <u>normal</u>
   - Γ generated by 2 and $2g+1$ is called <u>hyperelliptic</u>.

2. The semigroup of example (2.4.) is normal of genus 3.

3.3.3. Let $\mathfrak{M}_{g,1}$ be the coarse moduli space of smooth projective curves of genus g with a section, or equivalently of pointed compact Riemann surfaces of genus g.

Set $W = \{ (X,x) \in \mathfrak{M}_{g,1} \mid \Gamma_x = \Gamma \}$.

Denoting by M^-_s the open subset of M^- given by the points with smooth fibers we have:

<u>Theorem</u> (Pinkham [P])

$$M^-_s/\mathbb{G}_m \simeq W_\Gamma .$$

This theorem can be generalized:

For $m \in \mathbb{N}$, let $M^{-,m}$ denote the linear subspace of M^- defined by $\bigoplus_{n \in \mathbb{N}} T^1(-m.n)$.

Set $A_n = \{ (X,x) \in \mathfrak{M}_{g,1}$: there exists $h \in \mathrm{Aut}\ ((X,x))$ with ord $h = n \}$.

Then we have

3.3.4. <u>Theorem</u>

$$M^{-,n}_s/\mathbb{G}_m \simeq A_n \cap W_\Gamma$$

Idea of proof:

Any point $x \in M^{-,n}_s$ has stabilizer $\mu_n \subsetneq \mathbb{G}_m$, acting as group of non-trivial automorphisms on V_x.

3.3.5. Corollaries:

1. $\Gamma = \langle 2,2g+1 \rangle$: Any hyperelliptic Riemann surface has an autmorphism of order 2 – "the interchanging of sheets"

   Proof:

   $T^1 \subsetneq T^1(2.\mathbb{Z})$.

2. An automorphism of a pointed Riemann surface of genus $g \geq 2$ has order at most $4g+2$.

Proof:

2.2.3.

3.3.6. <u>Example</u>

Let Γ be the semigroup of 2.2.4.

The fiber over $M^{-,7}$ is given by the (2x2)-minors of the matrix

$$\begin{vmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2+t & X_0^3 \end{vmatrix}$$

This curve is smooth for every $t \neq 0$, and so has an automorphism of order 7.

This can also be seen in the following way:

There is a canonical morphism to $\mathbb{P}^1$:

$$Y = \frac{X_1}{X_0} = \frac{X_2+t}{X_1} = \frac{X_0^3}{X_2}$$

Eliminating X_1 and X_2 and homogenizing we have the plane algebraic curve

$$Y^3X_0 - tYZ^3 - ZX_0^3 = 0.$$

This is the famous quartic of F. Klein with 168 automorphisms, so especially it has an automorphism of order 7.

4. Concerning the structure of the formal, versal deformation there are the following questions:

4.1. What is the dimension of the moduli-space M of the formal, versal deformation?

From (3.1.2.) we know $\dim M \geq 1$.

If there are smooth deformations then one knows more:

4.1.1. Theorem:

Let B be a monomial ring.

Let E be a non-empty component of M, s.t. the fiber over a generic point of E is smooth. Then

$$\dim E = 2g - 1 + r$$

with $r = \dim_k \mathrm{Ext}^1_B(k,B)$, the "type of B".

This theorem is a consequence of a result due to Deligne (see [SGA] and [B1]).

4.1.2. Remark:

If B is monomial with semigroup Γ and if

$m^{-1}(\Gamma) = \{ n \in \mathbb{N} : n+m \in \Gamma \text{ for all } m \in \Gamma - \{0\} \}$,

then $r = \mathrm{card}(m^{-1}(\Gamma) - \Gamma)$.

4.1.3. Corollary:

$$2g \overset{(1)}{\leq} \dim E \overset{(2)}{\leq} 2g - 2 + m(B)$$

where $m(B)$ denotes the multiplicity of B.

The equality sign holds in

(1) iff B is Gorenstein and in

(2) iff B is stable, i.e. $m(B) = \mathrm{edim}\,(B)$, the embedding dimension of B.

The next question follows almost immediatly from 4.1.1.:

4.2. Are there smooth deformations for monomial curves? The conjecture is that the answer is affirmative in general, but this is known only for very few cases:

4.2.1. (Schlessinger)

B is a complete intersection.

4.2.2. (Schlessinger, Schaps)

$$\operatorname{edim} B = 3$$

4.2.1. and 4.2.2. are true without assuming B monomial.

4.2.3. $\operatorname{edim} B = 4$ and the singularity of B is Gorenstein. (This is a consequence of [B-E] .)

In all the three foregoing cases $T_B^2 = 0$, i.e. there are no obstructions.

4.2.4. B is an Arf-ring

(that means: B is stable (4.1.3.), and the same is true for every infinitesimally-near overring)

In this case obstructions can occur:

Take for example Γ to be ordinary of genus $g \geqslant 3$ (3.3.2.).

Then $\dim T^1(B_\Gamma) = g^2+g-1$, but

$\dim E = 3g-1$, so there have to be obstructions.

So the question arises:

4.3. What are the possible singularities of the moduli-space? Very little is known about this question and we will only give an example to see what can happen.

4.3.1. Let Γ be generated by 12, 16, 20, 33, 37, 41 and $44+\mathbb{N}$,

$$0 \longrightarrow I \longrightarrow k[X_0, X_1, \ldots] \longrightarrow B = k[t^{12}, t^{16}, \ldots] \longrightarrow 0$$

be the presentation of $B = B_\Gamma$ (2.1.).

Then

$$\dim T^2_B(+2) = 1, \quad \dim T^2_B(+3) = 1$$

and these are the only non-vanishing eigenspaces of T^2_B for positive eigenvalues.

Let t_1 and t_2 be the non-trivial vectors of T^1_B which correspond to the elements

$$t^{13}\varphi(\frac{\partial}{\partial X_0}) \quad \text{and} \quad t^{17}\varphi(\frac{\partial}{\partial X_1}) - t^{13}\varphi(\frac{\partial}{\partial X_2})$$

in $\mathrm{Hom}\,(I/_{I}2, B)$ respectively.

Denoting by s_1 and s_2 the corresponding moduli of weight -1, we get after a tedious computation:

$$M^+ = \mathrm{spec}\,(k[[s_1, s_2]]/_{s_1^2 + s_2^2,\ s_1^2(2s_2 + s_1)}) \times W,$$

where W is smooth.

M^+ is obviously not reduced.

4.3.2. Let Γ' be the semigroup generated by Γ and one element of $\{35, 39, 43\}$.

Then $T^2(+3) = 0$, so the obstruction of weight -3 cancels and we get:

$$M^+ = \mathrm{spec}\,(k[[s_1, s_2]]/_{s_1^2 + s_2^2}) \times W'$$

with W' smooth.

M^+ is reduced, but singular.

4.3.3. Let Γ'' be the semigroup generated by Γ' and one element of $\{34, 38, 42\}$, then also $T^2(+2) = 0$, so that there are no obstructions at all in the positive and in consequence M^+ is smooth.

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