SÉMINAIRE DE THÉORIE DES NOMBRES DE BORDEAUX

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# On Sums of Squares in $Q^{\frac{1}{2}}(X)$ etc

*Séminaire de théorie des nombres de Bordeaux* (1970-1971), exp. nº 7, p. 1-5
<http://www.numdam.org/item?id=STNB_1970-1971____A7_0>

# ON SUMS OF SQUARES IN $Q^{1/2}(X)$ ETC

by

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Quantitative solutions to problems associated with Hilbert's problem on sums of squares of rational functions seem to be rather hard to find. Prof. Cassels has just described a theorem which shows that there are some positive definite functions in $\mathbb{R}(X, Y)$ which are not the sum of three squares in $\mathbb{R}(X, Y)$ and a theorem of Hilbert. Landau shows that every positive definite function in $\mathbb{R}(X, Y)$ is a sum of at most 4 squares in $\mathbb{R}(X, Y)$. Thus 4 is best possible.

To-day I shall describe a similar problem which is unsolved.

DEFINITION. Let $Q^{1/2}$ denote the real quadratic closure of the rationals i.e. $Q^{1/2}$ is the smallest subfield of $\mathbb{R}$ such that if $\alpha \in Q^{1/2}$ and $\alpha$ is totally positive then $\sqrt{\alpha} \in Q^{1/2}$.

LEMMA. (Hilbert) - Let K be a field with the following two properties

(I) If $\alpha \in K$ and $\alpha$ is totally positive, then $\alpha$ is a sum of at most 4 squares in $K$.

(II) For any totally imaginary extension $L$ of $K$, $-1$ is a sum of at most two squares in $L$.

Then every sum of squares in $K(X)$ can be written as a sum of two squares in $K(X)$.

Proof. - Denote a sum of squares in $K(X)$ by $f(X)$. Then we can assume that $f(X) \in K[X]$ and $f(X)$ is irreducible over $K$. The justification of these two assertions is a follows.

& First, if $\frac{p}{q}$ is a sum of squares in $K(X)$, then so is $pq$ and if $pq$ is a sum of squares in $K(X)$ then so is $p/q$.

Secondly if the polynomials $p_1, \ldots, p_t$ are each a sum of squares in $K(X)$ then their product is a sum of squares in $K(X)$.

Thirdly we must show that if $f(X) \in K[X]$ is a sum of squares in $K(X)$ then each monic irreducible factor which divides $f(X)$ to an odd power is a sum of squares in $K(X)$.

Without loss of generality we can suppose that $f(X)$ is monic and square free. Let $p(X)$ be a monic irreducible factor of $f(X)$. In order to show that $p(X)$ is a sum of squares in $K(X)$ it will suffice to show that for each ordering of $K$, $p(X)$ is a sum of squares in $R(X)$, where $R$ is a real closure of $K$ which extend the order on $K$. For if $p(X)$ is a sum of squares in $R(X)$ it implies $p(X)$ is a sum of squares in each real closure of $K(X)$ which extends the given ordering on $K$. This in turn implies that $p(X)$ is positive in all orderings of $K(X)$ and by a theorem of Artin this implies that $p(X)$ is a sum of squares in $K(X)$.

We now show that $p(X)$ is a sum of squares in $R(X)$, for each of the real closed fields mentioned above. It is trivial that $g(X) \in R[X]$ is a sum of two squares in $R[X]$ if and only if $g(a) \geq a$ for all $a \in R$. Now suppose that $p(a) \leq 0$ for some $a \in R$, since $p$ is monic it follows that $p(b) \geq 0$ for some $b \in R$. This implies that $p(X)$ has a root $\alpha$ in $R$. Hence $f(\alpha) = 0$ and this implies $\alpha$ is a double root of $f(X)$ which contradicts the fact that $f(X)$ has no square factors. Thus $p(a) > 0$ for all $a \in R$ and so is a sum of two squares in $R(X)$ .

We prove the lemma by induction on $\partial f$ , the degree of $f(X)$ . When $\partial f = 0$ the conclusion of the lemma is just hypothesis (I).

As an inductive hypothesis we assume the conclusion of the lemma for all polynomials of degree less than $f(X)$ which can be written as a sum of squares in $K(X)$ .

So we suppose $\partial f \geq 2$ and that $f(X)$ can be written

$$(1) \qquad f(X) = h_1^2 + \dots + h_m^2 \ .$$

By a theorem of Cassels we can assume $h_i \in K[X]$ for $1 \leq i \leq m$ . Consider the field $L = K[X] / \{f(X)\}$ , the above hypothesis on $f(X)$ implies that $-1$ is a sum of squares in $L$ . Thus $L$ is non-real and so by hypothesis (II) we have

$$-1 = a_1^2 + a_2^2 \quad , \quad a_i \in L \ .$$

Rewriting the above in terms of polynomials we have

$$(2) \qquad f(X) \cdot h(X) = 1 + a_1^2(X) + a_2^2(X)$$

where $h(X)$ , $a_i(X) \in K[X]$ and each of degree less than $f(X)$ . Equation (2) $\Rightarrow$ $h(X)$ can be written as a sum of squares in $K(X)$ and so by our induction hypothesis $h(X)$ can be written as a sum of 4 squares

$$(3) \qquad f(X) = \frac{1+a_1^2(X)+a_2^2(X)}{b_1^2(X)+\dots+b_4^2(X)} = g_1^2(X) + \dots + g_4^2(X) \ .$$

This completes the induction step and proves the lemma.

THEOREM 1. <u>If</u> $f(X)$ <u>is a totally positive element of</u> $Q^{1/2}(X)$ <u>then</u> $f(X)$ <u>can be written as a sum of at most</u> 4 <u>squares in</u> $Q^{1/2}(X)$ .

Proof. - We just have to check that the two hypotheses of the lemma are satisfied. Excercise.

How many squares do we really need ?

First of all, every totally definite quadratic polynomial is a sum of two squares in $Q^{1/2}[X]$, for we can write the polynomial in the form $f(X) = (X-\alpha)^2+\beta$ and $f(X)$ totally positive $\Rightarrow$ $\beta$ is totally positive $\Rightarrow$ $\sqrt{\beta} \in Q^{1/2}$ $\Rightarrow$ $f(X) = (X-\alpha)^2 + (\beta^{\frac{1}{2}})^2$.

LEMMA 1. There are totally positive quartic polynomials which are not the sum of two squares in $Q^{1/2}[X]$.

Proof. - Suppose that

$$x^4 + ax^2 + bx + c = (\alpha_1 x^2 + \beta_1 x + \gamma_1)^2 + (\alpha_2 x^2 + \beta_2 x + \gamma_2)^2 ,$$

then

$$1 = \alpha_1^2 + \alpha_2^2 \quad ; \quad 0 = \alpha_1\beta_1 + \alpha_2\beta_2 \quad ; \quad c = \gamma_1^2 + \gamma_2^2 \quad ;$$

$$a = 2(\alpha_1\gamma_1 + \alpha_2\gamma_2) + \beta_1^2 + \beta_2^2 \quad ; \quad b = \beta_1\gamma_1 + \beta_2\gamma_2 .$$

We can write these as a set of vector equations in $Q^{\frac{1}{2}}$ namely

$$1 = \mathfrak{a}.\mathfrak{a} \quad , \quad 0 = \mathfrak{a}.\mathfrak{b} \; ; \; a = 2\mathfrak{a}.\mathfrak{c} + \mathfrak{b}.\mathfrak{b} \; ; \; b = \mathfrak{b}.\mathfrak{c} \; ; \; c = \mathfrak{c}.\mathfrak{c} .$$

By making an orthogonal substitution we can assume that $\mathfrak{a} = (1,0)$ and the equations become

$$0 = \beta_1 \; , \; a = 2\gamma_1 + \beta_2^2 \; , \; b = \beta_2\gamma_2 \; ; \; c = \gamma_1^2 + \gamma_2^2 .$$

On eliminating $\beta_2$ and $\gamma_2$ we obtain

$$(a-2\gamma_1)(c-\gamma_1^2) = b^2 .$$

This equation must have at least one root in $Q^{1/2}$ if $f(X)$ can be written as a sum of two squares in $Q^{1/2}[X]$. Thus to find ar example of a quartic polynomial which is totally positive and not the sum of two squares in $Q^{1/2}[X]$ we must pick a, b, c $\in Q$ such that

(a) $x^4 + ax^2 + bx + c$ is a sum of squares in $Q^{1/2}(X)$

(b) The cubic equation $2t^3 - at^2 - 2ct + (ac - b^2) = 0$ has no roots in

in $Q^{1/2}$.

For example :

$$X^4 + 2X^2 + X + 1 = \left(\frac{X^2+X+1}{\sqrt{2}}\right)^2 + \left(\frac{X^2-X}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2$$

the cubic is

$$2t^3 - 2t^2 + 1 ,$$

which is irreducible over $Q$ and so over $Q^{1/2}$ .

LEMMA 2. Every totally positive quartic polynomial is a sum of 3 squares in $Q^{1/2}[X]$ .

Proof. Exercise

Thus, lemma 2 shows that the technique used by Cassels-Ellison-Pfister to prove that 4 was the correct solution when $K = R(X, Y)$ cannot be used.

PROBLEM : Can every sum of squares in $Q^{1/2}(X)$ be written as a sum of at most 3 squares in $Q^{1/2}(X)$ ?

I have no idea how to solve this problem.

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