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PRIME MODELS OVER SUBSETS

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The following is basically a rewrite of a paper of S. SHELAH entitled "On uniqueness of prime models", in which he gives a simpler (in comparison to his book) proof of the result that if M is a model of a countable stable theory T, which is strictly prime over A, then M is the unique prime model over A. He also gives an example to show that we cannot omit the countability assumption. SHELAH's paper is on a rather more general level (in terms of generalised notions of prime, etc.). Here we work in terms of the usual notions. The example presented here is the same as SHELAH's, but the "proof" that it is the desired example is rather more simple, and due to B. POIZAT.

1. Strictly prime models.

T is a fixed complete theory. We work inside a big sufficiently saturated model of T. A, B, C etc. will denote subsets of this model, and M, N ... elementary submodels of it.

1.1. Définition.

(a) Let $A \subset B$. Then B is atomic over A, if every finite tuple of elements of B realises a principal type over A.

(b) $(a_i : i < \alpha)$ is a construction over A if for every $\beta < \alpha$, a_β is principal over (realises a principal type over) $A \cup \{a_i : i < \beta\}$.

(c) Let $A \subset B$. Then B is constructible over A if we can write the elements of $B - A$ as a sequence $(a_i : i < \alpha)$ which is a construction over A.

(d) Let $A \subset M$. Then M is prime over A if whenever f is an elementary map of A into N, then f can be extended to an elementary embedding of M in to N.

(e) Let $A \subset M$. Then M is strictly prime over A if M is constructible over A.

Note. - If B is constructible over A, then B is atomic over A.

We will now list some known results on prime models.

1.2. If M is strictly prime over A, then M is prime over A.

Proof. - If N is any other model containing A, we use the construction of M over A to elementarily embed M in N over A, element by element, using the fact that a principal type over what we have so far, is realised in N.

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1.3. Suppose that $A \subset B$ and B is atomic over A. Then for any finite subset C of B, B is atomic over $A \cup C$.

1.4. Let B be atomic over A and $B - A$ countable. Then B is constructible over A.

Proof. – By 1.3, any listing of $B - A$ of order type ω, will be a construction of B over A.

1.5. Suppose that for every A, the principal 1-types are dense in $S_1(A)$. Then for every A, there is a strictly prime model over A.

Proof. – Assume that for every A the principal 1-types are dense in $S_1(A)$. It follows that for any B, B is a model of T if and only if B realises all principal 1-types in $S_1(B)$. Now start with A, and realise principal 1-types over what we have got so far, until we arrive at a model.

1.6. Assume that T is countable, and $|A| \leq \omega_1$. Then there is an atomic model over A if and only if the principal types are dense in $S_n(A)$.

Proof. – The case in which A is countable is classically due to VAUGHT, and J. KNIGHT extended it to the case $|A| = \omega_1$, by expressing A as the union of a certain chain of countable subsets.

1.7. Assume that T is countable and that A is countable. Then the following are equivalent :

(i) M is countable and atomic over A.

(ii) M is prime over A.

(iii) M is strictly prime over A.

Proof. – The equivalence of (i) and (ii) is again classically due to VAUGHT. (iii) $\Rightarrow$ (ii) is by 1.2, and (i) $\Rightarrow$ (iii) is by 1.4.

1.8. Again assume that T and A are countable. Then there is at most one prime model over A, up to isomorphism over A.

Proof. – Again by VAUGHT. Let M and N be two prime models over A. They are both countable atomic over A, and we use 1.3 to build an isomorphism between them.

1.9. Let T be countable and ω-stable. Then

(i) for every A, there is a strictly prime model over A,

(ii) M is prime over A if and only if M is atomic over A and contains no uncountable set of indiscernables over A,

(iii) for every A there is a unique prime model over A.

Proof. - (i) is true because for a countable ω-stable theory, the hypothesis of 1.5 holds. (ii) and (iii) are by SHELAH (See next exposé).

We will now prove RESSAYRE's result that for any A there is at most one strictly prime model over A. This is essentially a generalisation of 1.8.

Let $A \subset C$, and C constructible over A. Let $(c_i : i < \alpha)$ be a construction of C over A. For $\beta \leq \alpha$ let us write $A_\beta = A \cup \{c_i : i < \beta\}$. For each $\beta < \alpha$ choose a formula $Q_\beta(x, \bar{b}_\beta)$ which determines the type of c_β over A_β. $\bar{b}_\beta$ is a tuple of elements from A_β. Relative to the above choice of formulae, we define a notion of closed.

1.10. Definition. - Let $X \subset C$. Then X is closed if whenever $c_\beta \in X$, then $\bar{b}_\beta \subseteq A \cup X$.

$T(B)$ denotes the true sentences (true in the big model) which have parameters in B.

1.11. LEMMA. - Let X be a closed subset of C. Then $T(A \cup X)$ is determined by $T(A) \cup \{Q_\beta(c_\beta, \bar{b}_\beta) : c_\beta \in X\}$.

Proof. - Let us write $X \restriction \delta$ to be $\{c_i \in X : i < \delta\}$ for $\delta \leq \alpha$. Each $X \restriction \delta$ is closed, and $X \restriction \alpha = X$. We prove that for each $\delta \leq \alpha$ the lemma is true for X replaced by $X \restriction \delta$, by induction on δ. It is trivial at the limit stage. So suppose the lemma is true for $X \restriction \delta$. Now $\{Q_\delta(x, \bar{b}_\delta)\} \cup T(A_\delta)$ determines the type of c_δ over A_δ. Namely $\{Q_\delta(c_\delta, \bar{b}_\delta)\} \cup T(A_\delta)$ determines $T(A_{\delta+1})$. So $\{Q_\delta(c_\delta, \bar{b}_\delta)\} \cup T(A \cup X \restriction \delta)$ determines $T(A \cup X \restriction \delta+1)$, and so by induction hypothesis we have the lemma for $X \restriction \delta+1$.

1.12. PROPOSITION. - Let X be a closed subset of C. Then C is atomic over $A \cup X$.

Proof. - Let X be closed. Let $a_1, \dots, a_n$ be in C. Let Y be the least closed set containing $a_1, \dots, a_n$. Y is clearly finite. Also $X \cup Y$ is closed.

It follows from the previous lemma that $T(A \cup X \cup Y)$ is determined by $T(A \cup X) \cup \{Q_\delta(c_\delta, \bar{b}_\delta) : c_\delta \in Y\}$. As Y is finite, it follows that the type of Y (or of a sequence that enumerates Y) over $A \cup X$ is principal. So clearly the type of $(a_1, \dots, a_n)$ over $A \cup X$ is principal. So C is atomic over $A \cup X$.

1.13. THEOREM. - Let M and N be strictly prime models over A. Then M and N are isomorphic over A.

Proof. - Clearly the cardinalities of $M - A$ and $N - A$ are the same, let us say K. If K is countable, then the result follows by the usual Vaught back and forth argument. So assume that K is uncountable. We will use Proposition 1.12 to generalise this back and forth argument. We define strictly increasing sequences C_α and D_α for $\alpha < K$, such that $A \cup \bigcup_{\alpha<K} C_\alpha = M$ and $A \cup \bigcup_{\alpha<K} D_\alpha = N$, $(M, A \cup C_\alpha) \equiv (N, A \cup D_\alpha)$ for each $\alpha < K$, and such that C_α is closed in M and D_α is closed in N (relative to some given constructions of M and N respectively over A). $C_{\alpha+1}$ will be got by adding a countable sequence on to C_α, and the same for $D_{\alpha+1}$. So suppose we have C_α and D_α satisfying the conditions. We define increasing finite sequences X_n and Y_n from M and N respectively for $n < \omega$, such that $(M, A \cup C_\alpha \cup X_n) \equiv (N, A \cup D_\alpha \cup Y_n)$. Assume that α is even or limit. Then X_0 is the next element of M not in A or C_α, and Y_0 is an element in N with the same type over $A \cup D_\alpha$ as the type of X_0 over $A \cup C_\alpha$ (as this type is principal). If n is even and bigger than 0, let X_n be the closure of X_{n-1}. Then as X_{n-1} is finite, $X_n - X_{n-1}$ is also finite. By the fact that C is closed and by proposition 1.12 and 1.3, the type of $X_n - X_{n-1}$ is principal over $A \cup C_\alpha \cup X_{n-1}$. So realise this type in N over $A \cup D_\alpha \cup Y_{n-1}$, augmenting Y_{n-1} to give us Y_n. For n odd, go through the same procedure, but starting with Y_n. For α not limit and odd, go through the same as above starting with Y_0 and making the obvious changes. Put $C_{\alpha+1} = C_\alpha \cup \bigcup_{n<\omega} X_n$, and $D_{\alpha+1} \cup \bigcup_{n<\omega} Y_n$. They clearly satisfy the conditions. The correspondence between the C_α's and the D_α's gives us an isomorphism between M and N over A.

2. Shelah's uniqueness theorem.

In this section we assume in addition that T is countable and stable. Stability enables us to define a notion of " p forks over A ", where $p \in S(B)$ and $A \subset B$, which has certain nice properties. $tp(\bar{a}, A)$ will denote the type of a (finite) sequence $\bar{a}$ over A. We will only give here the properties that we need.

2.1. PROPOSITION (Forking facts).

(A) If $p \in S(B)$, then there is a countable subset A of B such that p does not fork over A.

(B) If $tp(\bar{a}, A \cup \bar{b})$ does not fork over A, then $tp(\bar{b}, A \cup \bar{a})$ does not fork over A.

(C) Suppose that $p \in S(B)$ and p does not fork over A. Then if $B' \subset B$ and $A \subset A'$, $p \restriction B'$ does not fork over A'.

(D) If $p \in S(B)$ forks over A, then there is B' containing A and contained in B such that $B' - A$ is finite and $p \restriction B'$ does not fork over A.

(E) If $tp(\bar{a}, A)$ is principal and does not fork over $A' \subset A$, then $tp(\bar{a}, A')$ is principal.

Let us remark that only (A) above uses the countability of T; in fact $K(T) \leq \omega_1$ is sufficient (for instance if T is superstable), and the proof is still valid in that case.

2.2. LEMMA. - <u>Suppose that</u> $A \subset B \subset C$ <u>and</u> C <u>is constructible over</u> A . <u>Then</u> B <u>is constructible over</u> A .

<u>Proof</u>. - We prove the lemma by induction on the cardinality of $C - A$. We know anyway that C is atomic over A . So if $C - A$ is countable, then $B - A$ is also countable, B is atomic over A , and so B is constructible over A by 1.4.

So let $|C - A| = \mu > \aleph_0$. We will construct inductively, an increasing continuous sequence $(C_i : i < \mu)$ of subsets of C , such that

(1) $A \cup \bigcup_{i<\mu} C_i = C$

(2) $|C_i| \leqslant |i| + \aleph_0$

(3) Each C_i is closed (relative to some given construction of C over A)

(4) For each finite tuple $\overline{c}$ in C_i , $tp(\overline{c}, B)$ does not fork over $B \cap C_i$, for each $i < \mu$.

The idea is that C_{i+1} will be constructible over $A \cup C_i$, and then we will use the induction hypothesis and (4) to get a construction of B over A .

So suppose that we already have C_i defined. Let c be the next element in some listing of $C - A$. For each finite tuple $\overline{b}$ in $C_i \cup \{c\}$, let $B_{\overline{b}}$ be a countable subset of B such that $tp(\overline{b}, B)$ does not fork over $B_{\overline{b}}$ (by 2.1, (A)). Put $C_i^{(1)}$ equal to the closure of $C_i \cup \{c\} \cup \bigcup B_{\overline{b}}$. Again for each $\overline{b}$ in $C_i^{(1)}$ add countable $B_{\overline{b}}$ as above, and close to get $C_i^{(2)}$. Continue like this to get $C_i^{(n)}$ for all $n < \omega$, and put $C_{i+1} = \bigcup_{n<\omega} C_i^{(n)}$. C_{i+1} clearly satisfies conditions (2), (3) and (4). We also have (5) C_{i+1} is constructible over $A \cup C_i$. For list $C_{i+1} - (C_i \cup A)$ in the same order as the elements appear in the construction of C over A , say as $(d_\alpha : \alpha < \gamma)$. Then as C_i and C_{i+1} are closed, $C_i \cup \{d_\alpha : \alpha < \beta\}$ is closed for each $\beta \leqslant \gamma$. So by proposition 1.12, d_β is principal over $A \cup C_i \cup \{d_\alpha : \alpha < \beta\}$.

Note that for each i , $|C_{i+1} - (C_i \cup A)| < \mu$. Now using (5) apply the induction hypothesis to $A \cup C_i \subset A \cup C_i \cup (B \cap C_{i+1}) \subset A \cup C_{i+1}$. We conclude that $A \cup C_i \cup (B \cap C_{i+1})$ is constructible over $A \cup C_i$. The idea now is to piece together these constructions to give a construction of B over A . As A and the C_i's cover C we can define for each $i < \mu$, ordinals γ_i , such that $(b_j : j < \bigcup_{i<\mu} \gamma_i)$ is a listing of $B - A$, and $(b_j : \gamma_i \leqslant j < \gamma_{i+1})$ is a construction of $A \cup C_i \cup (B \cap C_{i+1})$ over $A \cup C_i$ for each $i < \mu$. Now take an arbitrary b_j . We want to show that b_j is principal over $A \cup (b_\gamma : \gamma < j)$. Then the b_j's will give us a construction of B over A . Let this j be between γ_i , γ_{i+1} . So $tp(b_j, A \cup C_i \cup \{b_\gamma : \gamma_i \leqslant \gamma < \gamma_{i+1}\})$ is principal. <u>We now show that this type does not fork over</u> $A \cup \{b_\gamma : \gamma < j\}$.

Let $\bar{c} \in C_i$. Then $tp(\bar{c}, B)$ does not fork over $B \cap C_i$, by construction. So by forking fact (C), $tp(\bar{c}, A \cup \{b_\gamma : \gamma \leqslant j\})$ does not fork over $A \cup \{b_\gamma : \gamma < j\}$. By (B), $tp(b_j, A \cup \bar{c} \cup \{b_\gamma : \gamma < j\})$ does not fork over $A \cup \{b_\gamma : \gamma < j\}$. Finally by (D), $tp(b_j, A \cup C_i \cup \{b_\gamma : \gamma < j\})$ does not fork over $A \cup \{b_\gamma : \gamma < j\}$, as the above is true for all $\bar{c}$ in C_i . But as mentioned before this type is principal, so by (E) $tp(b_j, A \cup \{b_\gamma : \gamma < j\})$ is principal. So we are finished.

2.3. THEOREM (Remember T is stable and countable). - <u>If there is a strictly prime model over</u> A , <u>then it is the unique prime model over</u> A (<u>up to isomorphism over</u> A).

<u>Proof</u>. - Let M be the strictly prime model over A . Let N be another prime model over A . So we have $A \subset N \prec M$. By the above lemma, N is constructible over A , and thus also strictly prime over A . The theorem follows by theorem 1.13.

3. The example.

We will describe a theory T , which is stable but whose language is uncountable, such that T has a strictly prime model (over the empty set) which is <u>not</u> the unique prime model of T .

L the language of T , consists of, for each $a < \omega_1$, a binary relation symbol E_s . T says that (i) each E_s is an equivalence relation with infinitely many equivalence classes, each of which is infinite, and (ii) for $s < t$, $xE_t\, y \rightarrow xE_s\, y$, and actually E_t partitions E_s into infinitely many equivalence classes.

It is routine to show that T has elimination of quantifiers and that it is a complete theory.

3.1. PROPOSITION. - T <u>is stable</u>.

<u>Proof</u>. - Let M be a model of T , and $A \subset M$. Put $T(A)$ to be $Th(M, a)_{a \in A}$. We just count the complete 1-types of $T(A)$. By elimination of quantifiers, any such type $p(x)$ is determined by its atomic and negated atomic formulae. We can ignore the case in which p is realised by an element of A , for there are just $|A|$ many such types. By completeness p depends just on the atomic formulae it contains, i. e. formulae of the kind $xE_s\, a$ for a in A and $s < \omega_1$. For each s , choose a_s in A such that $xE_s\, a_s \in p$, if such an a_s exists. Then for any $a \in A$, $xE_s\, a \in p$ if and only if $T(A) \models aE_s\, a_s$. So p depends, over $T(A)$ on such a choice of the a_s . Thus each type corresponds to a partial map from ω_1 to A . Thus the number of types is $\leqslant |A|^{\aleph_1}$, and so T is stable.

We now describe the strictly prime model of T . Let the universe of the model M consist of those functions f from ω_1 to ω, which have value 0 at all but a finite number of points. We put $M \models f_1 E_s f_2$ if and only if $f_1 \upharpoonright s = f_2 \upharpoonright s$. It is easy to verifiy that M is a model of T.

Let us define for each $s < \omega_1$, a subset $X_s = \{f$ in M : $f(t) = 0$ for all $t \geq s\}$.

It is clear that each X_s is countable and nonempty, and that $M = \bigcup_{s<\omega_1} X_s$.

3.2. LEMMA. - X_0 <u>is atomic over the empty set, and each</u> X_{s+1} <u>is atomic over</u> X_s.

<u>Proof</u>. - X_0 consists only of the function which is zero everywhere, and its type is determined over T by ' $x = x$ '.

Let now $f_1, \dots, f_n$ be in X_{s+1}. We want to show that the type of $(f_1, \dots, f_n)$ is principal over X_s. We may assume that the f_i are pairwise distinct and not in X_s. For each i, choose g_i in X_s such that $M \models f_i E_s g_i$ (g_i will just be the same as f_i but with $f_i(s)$ replaced by 0). Then it is easily seen that the open type, and thus by elimination of quantifiers, the complete type of $(f_1, \dots, f_n)$ over X_s is determined over $T(X_s)$ by the following finite set of formulae.

$$\{x_i \neq g_i : i = 1,\dots,n\} \cup \{x_i E_s g_i : i = 1,\dots,n\} \cup \{x_i \neq x_j : i < j\}$$
$$\cup \{x_i E_{s+1} x_j : \text{whenever } i<j \text{ and } M \models f_i E_{s+1} f_j\} \cup \{\neg x_i E_{s+1} x_j : i<j \text{ and } M \models \neg f_i E_{s+1} f_j\}$$

So the lemma is proved.

3.3. PROPOSITION. - M <u>is strictly prime</u> (<u>over the empty set</u>).

<u>Proof</u>. - As $X_{s+1} - X_s$ is countable, by lemma 3.2, X_{s+1} is constructible over X_s. Putting together these constructions gives a construction of M.

Let $(f_i : i < \omega_1)$ be a sequence of elements of M. We say that this is a Cauchy sequence if for every $s < \omega_1$, there is i_s such that $j \geq i_s$ implies that $M \models f_j E_s f_{i_s}$. We say that an element f is the limit point of the sequence $(f_i : i < \omega_1)$ if for every s there is i_s such that $j \geq i_s$ implies $M \models f_j E_s f$.

3.4. PROPOSITION. - <u>Every Cauchy sequence in</u> M <u>has a limit point in</u> M.

<u>Proof</u>. - Let $(f_i : i < \omega_1)$ be a Cauchy sequence in M. We can assume that $s < t$ implies that $i_s < i_t$. It is thus consistent to define a function f such that $f \upharpoonright s = f_{i_s} \upharpoonright s$ for all $s < \omega_1$. We must show that f is in M, i. e. that f is 0 except at a finite number of points. If not, then there is countable s such that $f \upharpoonright s$ is already nonzero at an infinite number of points. But then by the definition of f, f_{i_s} is nonzero at an infinite number of points, which is impossible, as it is in M. f is clearly a limit point of the sequence

$(f_i : i < \omega_1)$.

Let now N be the substructure of M whose universe is that of M minus the function which is 0 everywhere. It is easily checked that N is a model of T , and thus by elimination of quantifiers, N is an elementary substructure of M . So as M is a prime model of T , N must also be a prime model of T . We will show that N is not isomorphic to M , by showing that N does not satisfy proposition 3.4.

Let us define elements f_i of N , for $i < \omega_1$, where $f_i(i) = 1$, and for $j \neq i$ and $j < \omega_1$, $f_i(j) = 0$. Then $(f_i : i < \omega_1)$ is a Cauchy sequence in N . However the only possible limit point of this sequence is the function which is 0 everywhere, but this is not in the model N . So N has a Cauchy sequence with no limit point. So we have proved

3.5. THEOREM. - There is a stable theory, which has a strictly prime model which is not the unique prime model.

The above example can be actually coded up to give us also a countable unstable theory, which has a strictly prime model over a set A , which is again not the unique prime model over A (The equivalence relations E_s are now indexed by elements s of a total ordering).

Remark to theorem 3.5. - If N' is an arbitrary substructure of M , and $\langle f_i ; i < \omega_1 \rangle$, $\langle g_i ; i < \omega_1 \rangle$ are Cauchy sequences of elements of N' , we can say that they are equivalent if the obvious thing holds (i. e. for each $s < \omega_1$, there is $i_s < \omega_1$ such that for $i , j > i_s$ $f_i \, E_s \, g_j$).

Then in the model N (obtained by dropping the 0-sequence from M), there is exactly one equivalence class of Cauchy sequences which have no limit point. We can obtain other substructures of M , which are prime models of T , by dropping n points for each $n < \omega$. In this case there will be exactly n equivalence classes of Cauchy sequences with no limit point.

In this way we actually get at least $\aleph_0$ pairwise non isomorphic prime models of T .