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Surjective limits of locally convex spaces and their application to infinite dimensional holomorphy

SURJECTIVE LIMITS OF LOCALLY CONVEX SPACES
AND THEIR APPLICATION TO INFINITE DIMENSIONAL
HOLOMORPHY

BY

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ABSTRACT. — A locally convex space, F, is a surjective limit of the locally convex spaces, $(E_\alpha)_{\alpha \in A}$ if there exists, for each α in A, a continuous linear mapping, π_α, from E onto E_α and the inverse images of the neighbourhoods of zero in E_α, as α ranges over A, form a basis for the neighbourhood system at zero in E. In this article, the theory of surjective limits of locally convex spaces is systematically developed and applied to a variety of topics in the theory of holomorphic functions of infinitely many variables. These topics include pseudo-convex domains, domains of holomorphy, Zorn spaces, holomorphically complete and paracomplete spaces, weak holomorphy, hypoanalytic functions, extensions of Hartogs' theorem and locally convex topologies on spaces of holomorphic function.

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1. Introduction

In infinite dimensional holomorphy, we are interested in characterising collections of locally convex spaces in which certain holomorphic properties are true. For example, we would like to know in which locally convex spaces the pseudo-convex and the holomorphically convex open

sets coincide and we are interested in determining which locally convex spaces are holomorphically complete. The classical theory of several complex variables is concerned with holomorphy in a finite dimensional setting, in other words, it limits its investigation to the simplest collection of locally convex spaces — the locally compact topological vector spaces.

Infinite dimensional holomorphy proceeds by investigating the next simplest and most interesting collection of locally convex spaces, i. e., the locally bounded or normed linear spaces. We may then investigate many different collections of locally convex spaces, e. g. Baire, metrizable, barrelled, bornological, nuclear, etc. spaces. All of these collections have proved themselves of interest, within the context of linear functional analysis, so it is quite natural that we should consider them. However, within the context of holomorphic functional analysis, they have proved inadequate so we are forced to find a new method of classifying locally convex spaces.

In this paper, we introduce the concept of surjective limit and use it to generate collections of locally convex spaces which are of holomorphic interest. *A locally convex space, E, is a surjective limit of the locally convex spaces, $(E_i)_{i \in A}$, if there exists a continuous linear mapping π_i from E onto E_i for each $i \in A$ and the inverse images of the neighbourhoods of 0 in E_i, as i ranges over A, form a basis for the neighbourhood system at 0 in E.* We use this definition and the classical Theorem of Liouville concerning bounded entire functions to obtain our results.

This approach was first used by HIRSCHOWITZ [16] and RICKART [38] in their studies of holomorphic functions over the Cartesian product of complex planes. NACHBIN [32] essentially uses this method in his study of uniform holomorphy and subsequently further applications to the study of pseudo-convex domains were made by DINEEN ([9], [13]) and NOVERRAZ [35].

In section 2, we define, discuss and give examples of various kinds of surjective limits and representations.

In section 3, we introduce the different definitions of holomorphic function that we shall use in our work. We are mainly interested in the locally bounded and the continuous holomorphic functions but hypo-analytic, Silva holomorphic and weakly holomorphic functions are useful in proving results (e. g. by using hypoanalytic mappings we show that most results proved for dual of Fréchet-Schwartz spaces can be extended to dual of Fréchet-Montel spaces (*see* sections 4, 5 and 6)).

Section 4 is devoted to pseudo-convex domains and domains of holomorphy in locally convex spaces. In contrast to the other sections many of the proofs in this section have appeared previously [13]. We include this section, however, as some of the results are new (Example 4.7 and Proposition 4.8), some previous proofs have been simplified, and we are using a new definition of surjective limit.

Zorn's theorem [44] states that a Banach valued Gateaux holomorphic function defined on a connected open subset of a Banach space is every where continuous if it is continuous at one point. In section 5, we extend this result to different collections of spaces and to different definitions of holomorphy. The results proved greatly help to understand the various counterexamples connected with Zorn's theorem ([5] and [18]). We also prove various factorization theorems for holomorphic functions.

In section 7, we extend Zorn's theorem by replacing the requirement of continuity at any one point and we also extend Hartogs' theorem concerning separately holomorphic functions. To obtain these results, we had to place many conditions on our surjective limits. These conditions are sufficient and we show by counterexample that they are not unnecessary. In section 6, we discuss holomorphically complete and paracomplete locally convex spaces, two concepts introduced by HIRSCHOWITZ [17]. Holomorphic completion involves extending holomorphic functions from E into F to holomorphic functions from E_1 into F where E is dense in E_1. We find, in the case of continuous holomorphic functions, that it is necessary to assume that the range space is a very strongly complete space [10]. We show that a locally convex space is very strongly complete if and only if it is a complete surjective limit of normed linear spaces. Using this concept and surjective limits we generate holomorphically complete locally convex spaces and simplify the proof of one of the main results of [10].

If V is a C-holomorphic extension of U paracompleteness involves finding what F-valued holomorphic functions on U can be extended to F-valued holomorphic functions on V. Again we need to place a completeness condition on F (which we call R completeness). Sequentially complete and holomorphically complete spaces are R-complete. Various spaces are shown to be paracomplete.

In section 8, we briefly investigate locally convex topologies on spaces of holomorphic functions and apply results from [12] to show that

certain spaces of holomorphic functions are quasi-complete and barrelled. Theorem 8.3 gives conditions under which various definitions of holomorphic functions coincide.

Many of the results in this paper were announced previously ([9], [11] and [14]), and we apologise for any confusion that the many changes in definitions and notation may cause. We adopted our present definition of surjective limit as it enjoys great generality and enabled us to simplify many proofs. The term projective limit was previously used to denote surjective limit ([9], [10]), N-projective limit was used to denote open surjective limit ([9], [10], [11], [13]) and the locally convex spaces of [32] and [35] with property (C) coincide with our open surjective limits of normed linear spaces. We have recently learned that E. LIGOCKA was engaged in a study similar to ours and that some of her results [28] coincided with our results in [11] (*see* also section 5). This paper went through many versions before reaching its final form and the author would like to thank all those who showed an interest in the various stages of development of this work.

2. Surjective Limits

All vector spaces considered are over the complex numbers, **C**, and all topological vector spaces considered are locally convex spaces. $\hat{E}$ will denote the completion of the locally convex space E and in general the ^ notation will denote some sort of completion.

$\mathscr{F}, \mathscr{R}, \mathscr{M}, \mathscr{N}, \mathscr{S}, \mathscr{RF}, \mathscr{FM}, \mathscr{FN}$ and $\mathscr{FS}$ will denote respectively the collection of all Fréchet, reflexive, Montel, nuclear, Schwartz, reflexive Fréchet, Fréchet-Montel, Fréchet-nuclear and Fréchet-Schwartz locally convex spaces ([20], [22]). The strong dual of a θ space is a $\mathscr{D}\theta$ space and E, a locally convex space, is a $D\theta$ space if E'_β (the strong dual of E) is a θ-space ($\theta = \mathscr{F}, \mathscr{R}, \mathscr{M}, \mathscr{N}, \mathscr{S}, \mathscr{RF}, \mathscr{FM}, \mathscr{FN}, \mathscr{FS}$).

A collection of locally convex spaces and linear mappings, $(E_i, \pi_i)_{i \in A}$, is called a surjective representation of the locally convex space E if π_i is a continuous linear mapping from E onto E_i for each $i \in A$ and $(\pi_i^{-1}(V_i))_{i \in A}$ forms a base for the filter of neighbourhoods at 0 in E when V_i ranges over the neighbourhoods of 0 in E_i and i ranges over A.

Remark. — A is merely an indexing set and the neighbourhood requirement implies that for each neighbourhood of 0 in E, W, there exists an $i \in A$ and V a neighbourhood of 0 in E_i such that $(\pi_i)^{-1}(V) \subset W$.

E is called a surjective limit of $(E_i, \pi_i)_{i \in A}$ and we write $E = \varprojlim_{i \in A} (E_i,$ $\pi_i)$. When there is no possibility of confusion or when the existence of the mappings π_i, $i \in A$, is asserted but not explicitly given we say E is a surjective limit of $(E_i)_{i \in A}$, and we write $E = \varprojlim_{i \in A} E_i$. Let $\mathcal{O}(E)$ (resp. $\mathcal{B}$ $(E), \mathcal{K}(E), \mathcal{S}(E))$ denote the set of all open subsets (resp. bounded subsets, compact subsets, convergent sequences) of the locally convex space E. $(E_i, \pi_i)_{i \in A}$ is called an open (resp. bounded, compact, sequential) surjective representation of E if π_i maps $\mathcal{O}(E)$ (resp. $\mathcal{B}(E), \mathcal{K}(E), \mathcal{S}(E))$ onto $\mathcal{O}(E_i)$ (resp. $\mathcal{B}(E_i), \mathcal{K}(E_i), \mathcal{S}(E_i))$ for each $i \in A$ and $E = \varprojlim_{i \in A}(E_i, \pi_i)$ is called an open (resp. bounded, compact, sequential) surjective limit ([1]). Thus $E = \varprojlim_{i \in A}(E_i, \pi_i)$ is an open surjective limit if, and only if, π_i is an open mapping for each $i \in A$ and $E = \varprojlim_{i \in A}(E_i, \pi_i)$ is a bounded (resp. compact, sequential) surjective limit if and only if each bounded subset (resp. compact subset, convergent sequence) of E_i is the image of some bounded subset (resp. compact subset, convergent sequence) of E for each $i \in A$. If $\mathcal{C}$ is a collection of locally convex spaces then the smallest collection of locally convex spaces which contains $\mathcal{C}$ and which is closed under the operation of surjective limit (a trivial application of Zorn's lemma shows that such a collection exists) consists of all locally convex spaces which are surjective limits of elements of $\mathcal{C}$. This follows immediately from the fact that if

$$E = \varprojlim_{\lambda \in B}(E^\lambda, \rho_\lambda) \qquad \text{and} \qquad E^\lambda = \varprojlim_{i_\lambda \in A_\lambda}(E_i^\lambda, \pi_{i_\lambda}),$$

are surjective limits then

$$E = \varprojlim_{\lambda \in B, \, i_\lambda \in A_\lambda}(E_i^\lambda, \pi_{i_\lambda} \circ \rho_\lambda),$$

is also a surjective limit ([2]). Similar results hold for open, bounded, compact and sequential surjective limits.

([1]) For the remainder of this paper we will, where appropriate and without explanation unless there is a possibility of confusion, interchange terminology between surjective limits and representations in the same obvious way as in this sentence.

([2]) If we had restricted ourselves to directed sets in the definition of surjective representation (see [9], [10]) then this result would not necessarily be true.

If $\mathscr{C}$ is a collection of locally convex spaces which is closed under the operations of taking arbitrary products and subspaces (^{3}) then $\mathscr{C}$ is also closed under the operation of surjective limit. Hence $\mathscr{N}$, $\mathscr{S}$ and the locally convex spaces in which every bounded set is precompact are examples of collections of locally convex spaces closed under the operation of surjective limit ([22] p. 275-278, and [36]).

EXAMPLE 2.1.

(a) Let $\mathscr{C}\mathscr{S}(E)$ denote the collection of all continuous semi-norms on the locally convex space E. For each $p \in \mathscr{C}\mathscr{S}(E)$, we let E_p denote the vector space E endowed with the topology generated by p and π_p will denote the canonical surjection from E onto $E_{p/p^{-1}(0)}$. $(E_{p/p^{-1}(0)}, \pi_p)_{p \in \mathscr{C}\mathscr{S}(E)}$ is called the canonical normed surjective representation of E.

(b) If E is a locally convex space, we let $\mathscr{M}(E)$ denote the set of all locally convex semi-metrizable topologies on E weaker than or equal to the topology of E. For $m \in \mathscr{M}(E)$ we let E_m denote the vector space E endowed with the topology m, $E_{m/\{\bar{0}\}^m}$ will denote the associated Hausdorff topology and π_m is the canonical surjection from E onto $E_{m/\{\bar{0}\}^m}$. $(E_{m/\{\bar{0}\}^m}, \pi_m)_{m \in \mathscr{M}(E)}$ is called the canonical metrizable surjective representation of E.

EXAMPLE 2.2. — Nuclear spaces are surjective limits of separable inner product spaces [36] and Schwartz spaces are surjective limits of separable normed linear spaces [42]. A topological space X is said to be Lindelöf if each open cover of X contains a countable subcover. The continuous surjective image of a Lindelöf space is a Lindelöf space. Hence if E is a locally convex Lindelöf space and p is a continuous seminorm on E, then $E_{p'p^{-1}(0)}$ is a normed Lindelöf space. Since every metrizable Lindelöf space is separable it follows that E is a surjective limit of separable locally convex spaces.

EXAMPLE 2.3. — A locally convex space E possesses the Banach-Grothendieck approximation property if for each compact subset K

(^{3}) We refer to DIESTEL (J.), MORRIS (S. A.) and SAXON (S. A.), Varieties of linear topological spaces, *Trans. Amer. math. Soc.* (to appear) for a discussion of such collections.

of E and each neighbourhood V of 0 in E there exists a continuous linear mapping π from E into E with finite dimensional range such that $\pi(x) - x \in V$ for all $x \in K$. Let $E = \lim_{i \in A} (E_i, \pi_i)$ and suppose each E_i possesses the Banach-Grothendieck approximation property. There exists an $i \in A$ and W a neighbourhood of 0 in E_i such that $V \supset \pi_i^{-1}(W)$. Choose μ a continuous linear operator from E_i into E_i with finite dimensional range such that $\mu \circ \pi_i(x) - \pi_i(x) \in W$ for all $x \in K$. Let F denote a finite dimensional subspace of E such that π_i is an isomorphism from F onto $\mu \circ \pi_i(E)$ and let G denote a topological complement of $\mu \circ \pi_i(E)$ in E_i. Now let ω denote the mapping from E_i into E which takes $x = x_1 + x_2$, $x_1 \in \mu \circ \pi_i(E)$ and $x_2 \in G$, onto $(\pi_i)^{-1}(x_1) \cap F$. $\omega \circ \mu \circ \pi_i$ is a continuous linear mapping from E into E with finite dimensional range. If $x \in K$, then

$$\pi_i^{-1}(\mu \circ \pi_i(x) - \pi_i(x)) \in \pi_i^{-1}(W) \subset V.$$

Hence $\pi_i^{-1}(\mu \circ \pi_i(x)) \subset V$ and since $\mu(\pi_i(x)) \in \mu \circ \pi(E)$ it follows that

$$\pi_i^{-1}(\mu \circ \pi_i(x)) - \omega(\mu \circ \pi_i(x)) \subset \pi_i^{-1}(0)$$

and

$$\omega \circ \mu \circ \pi_i(x) - x \in V.$$

We have thus shown that the collection of locally convex spaces which possess the Banach-Grothendieck approximation property is closed under the operation of surjective limit.

EXAMPLE 2.4. — A Schauder basis, $(e_n)_{n=1}^{\infty}$, in a locally convex space E is an equi-Schauder basis if there exists a family of continuous semi-norms on E, $(p_\alpha)_{\alpha \in \Gamma}$, which define the topology of E and is such that

$$(2.1) \qquad p_\alpha(\sum_{n=1}^{\infty} x_n e_n) = \sup_n p_\alpha(\sum_{m=1}^{n} x_m e_m)$$

for all $\alpha \in \Gamma$ and all $\sum_{n=1}^{\infty} x_n e_n \in E$ (we may assume that Γ is directed). If E is a locally convex space with a Schauder basis, $(e_m)_{m=1}^{n}$, and π_n is the projection of E onto the subspace spanned by $(e_n)_{n=1}^{\infty}$ then $(\pi_n)_{n=1}^{\infty}$ is an equicontinuous family of mappings if and only if $(e_n)_{n=1}^{\infty}$ is an equi-Schauder basis for E. An equi-Schauder basis is sometimes called a strong basis. Any Schauder basis in a barrelled locally convex space is an equi-Schauder basis. If $(e_n)_{n=1}^{\infty}$ is an equi-Schauder basis for E then it is also an equi-Schauder basis for $\hat{E}$ and

$$\hat{E} = \{ \sum_{n=1}^{\infty} x_n e_n \mid \sup_m p_\alpha(\sum_{n=1}^{m} x_n e_n) < \infty \text{ for all } \alpha \in \Gamma \}.$$

Let p_α denote a continuous seminorm on E which satisfies (2.1). Let

$$Z(\alpha) = \{ n \in Z \mid p_\alpha(e_n) = 0 \}.$$

It is immediate that

$$\ker p_\alpha = \{ x \in E, \ x = \textstyle\sum_{n \in Z(\alpha)} x_n e_n \}.$$

For $x = \sum_{n=1}^{\infty} x_n e_n \in E$ we define $\pi_\alpha(x)$ as the formal sum $\sum_{n \notin Z(\alpha)} x_n e_n$. We let

$$\pi_n(\pi_\alpha(x)) = \textstyle\sum_{m \leqslant n, \ m \notin Z(\alpha)} x_m e_m.$$

Hence

$$\pi_n(\pi_\alpha(x)) = \pi_\alpha(\pi_n(x)).$$

Now

$$p_\alpha(\pi_n(\pi_\alpha(x)) = p_\alpha(\textstyle\sum_{m \leqslant n, \ m \notin Z(\alpha)} x_m e_m) = p_\alpha(\textstyle\sum_{m \leqslant n} x_m e_m) = p_\alpha(\pi_n(x)).$$

Hence

(2.2) $$p_\alpha(x) = \sup_n p_\alpha(\pi_n(x)) = \sup_n p_\alpha(\pi_n(\pi_\alpha(x))).$$

E induces on $F_\alpha = \{ \pi_\alpha(x), \ x \in E \}$ a vector space structure.

Let $\tilde{p}_\alpha(\pi_\alpha(x)) = \sup_n p_\alpha(\pi_n(\pi_\alpha(x)))$, $\tilde{p}_\alpha$ is a well defined norm on F_α and the mapping π_α is an isometry from $E_{p_\alpha / p_\alpha^{-1}(0)}$ onto $(F_\alpha, \tilde{p}_\alpha)$.

Since $p_\alpha(\pi_n(x) - x) \to 0$ as $n \to \infty$ it follows that

$$\tilde{p}_\alpha(\pi_n(\pi_\alpha(x)) - \pi_\alpha(x)) \to 0$$

as $n \to \infty$.

Now $\tilde{p}_\alpha(\pi_\alpha(x)) = 0$ if and only if $x_n = 0$ for all $n \notin Z(\alpha)$ and thus $(e_n)_{n \notin Z(\alpha)}$ is an equi-Schauder basis for $(F_\alpha, \tilde{p}_\alpha)$. We have thus proved that E is a surjective limit of normed linear spaces each of which has an equi-Schauder basis.

Now for $\beta \in \Gamma$, $\beta \geqslant \alpha$ and $\pi_\alpha(x) \in F_\alpha$ we let

$$p_\beta(\pi_\alpha(x)) = \inf_{y \in E, \ \pi_\alpha(y) = 0} p_\beta(x + y).$$

Let G_α denote the vector space F_α endowed with the locally convex topology generated by the seminorms $(\tilde{p}_\beta)_{\beta \geqslant \alpha}$. π_α is an open continuous mapping from E onto G_α and G_α is isomorphic to $E/p_\alpha^{-1}(0)$. G_α also has $(e_n)_{n \notin Z(\alpha)}$ as a basis. For any integer n

$$\inf_{y \in E, \ \pi_\alpha(y) = 0} p_\beta(x + y) \geqslant \inf_{y \in E, \ \pi_\alpha(y) = 0} p_\beta(\pi_n(x) + \pi_n(y))$$
$$= \inf_{y \in E, \ \pi_\alpha(y) = 0} p_\beta(\pi_n(x) + y).$$

Hence $p_\beta (\pi_\alpha (x)) = \sup_n \tilde{p}_\beta (\pi_n (\pi_\alpha (x)))$. Hence G_α has $(e_n)_{n \notin Z(\alpha)}$ as an equi-Schauder basis and we have shown that a locally convex space with an equi-Schauder basis is an open surjective limit of locally convex spaces each of which has an equi-Schauder basis and admits continuous norms.

EXAMPLE 2.5.

(a) $\prod_{i \in A} E_i$ is an open, bounded, compact and sequential surjective limit of $(F_i)_{i \in \mathscr{F}(A)}$ where $\mathscr{F}(A)$ consists of all finite subsets of A and $F_j = \prod_{i \in j} E_i$ for all $j \in \mathscr{F}(A)$. (($E_1 \times E_2$ is not a surjective limit of $(E_i)_{i=1,2}$.) In particular $\sum_{n=1}^\infty \mathbf{C} \times \prod_{n=1}^\infty \mathbf{C}$ is an open surjective limit of $(\sum_{n=1}^\infty \mathbf{C} \times \prod_{n=1}^k \mathbf{C} \cong \sum_{n=1}^\infty \mathbf{C})_{k=1}^\infty$.

(b) If E is a locally convex space then $(E, \sigma(E, E'))$ is a surjective limit of finite dimensional spaces.

If $(E_i, \pi_i)_{i \in A}$ is a surjective representation of E then E *defines a preorder on the indexing set* A *in the following manner;* $i \geqslant j$ *if for each neighbourhood of* 0 *in* E_j, W, *there exists a neighbourhood of* 0 *in* E_i, V, *such that* $\pi_i^{-1}(V) \subset \pi_j^{-1}(W)$ [4]. When E_i and E_j are Hausdorff this implies the existence of a continuous linear mapping π_j^i from E_i onto E_j, defined for each $x \in E_i$ by the formula $\pi_j^i(x) = \pi_j(Z)$ where $x = \pi_i(Z)$ for some Z in E, and of the following commutative diagram

(2.3)

$$
\begin{array}{ccc}
E & \xrightarrow{\pi_i} & E_i \\
 & \pi_j \searrow & \downarrow \pi_j^i \\
 & & E_j
\end{array}
$$

i. e. $\pi_j = \pi_j^i \circ \pi_i$ [5].

Moreover if $i \geqslant j \geqslant k$ then $\pi_k^j \circ \pi_j^i = \pi_k^i$, i. e. the following diagram commutes;

(2.4)

$$
\begin{array}{ccc}
E_i & \xrightarrow{\pi_j^i} & E_j \\
 & \pi_k^i \searrow & \downarrow \pi_k^j \\
 & & E_k
\end{array}
$$

(2.5) and π_i^i is the identity mapping for each $i \in A$.

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[4] We shall only discuss orders and preorders on the indexing set which arise in this fashion.

[5] If $i \geqslant j$ and $j \geqslant i$ then, if E_i and E_j are Hausdorff, it follows that π_j^i is a linear isomorphism from E_i onto E_j with inverse π_i^j (this does not, however, imply that $E_i = E_j$).

If the system $(A, \geqslant)$ is directed (i. e. for each $i, j \in A$ there exists a $k \in A$ such that $k \geqslant i$ and $k \geqslant j$) and E_i is Hausdorff for each $i \in A$ then we say that $(E_i, \pi_i)_{i \in A}$ is a directed surjective representation of E. A collection of locally convex Hausdorff spaces and linear surjections $(E_i, \pi_j^i)_{i, j \in A, i \geqslant j}$ is called a surjective system if A is a directed set and (2.4) and (2.5) hold. A surjective representation of E, $(E_i, \pi_i)_{i \in A}$, is said to be associated with the surjective system $(E_i, \pi_j^i)_{i, j \in A; i \geqslant j}$ if (2.3) is satisfied.

We have thus shown that a surjective representation is directed if and only if it can be associated with a surjective system. Most of the surjective limits and representations we encounter are in fact directed. It is trivial to show that any surjective limit of normed linear spaces is a directed surjective limit. We write $E = \lim_{i, j \in A, i \geqslant j} (E_i, \pi_j^i, \pi_i)$ to show that the surjective limit $E = \lim_{i \in A} (E_i, \pi_i)$ is associated with the surjective system $(E_i, \pi_j^i)_{i, j \in A; i \geqslant j}$.

PROPOSITION 2.6. — $E = \lim_{i, j \in A; i \geqslant j} (E_i, \pi_j^i, \pi_i)$ *is an open surjective limit if and only if the associated surjective system $(E_i, \pi_j^i)_{i, j \in A, i \geqslant j}$ is open (i. e. π_j^i is an open mapping for all $i, j \in A$, $i \geqslant j$).*

Proof. — Suppose $(E_i, \pi_j^i)_{i, j \in A, i \geqslant j}$ is an open surjective system. Let V and j denote respectively a neighbourhood of 0 in E and an element of A. Choose $i \in A$, $i \geqslant j$ such that $\pi_i (V)$ is a neighbourhood of 0 in E_i. Hence $\pi_j (V) = \pi_j^i \circ \pi_i (V)$ is a neighbourhood of 0 in E_j and E is an open surjective limit. Conversely, suppose E is a directed open surjective limit. Let $i, j \in A$, $i \geqslant j$, and let V denote a neighbourhood of 0 in E_i. By hypothesis $\pi_j (\pi_i^{-1} (V))$ is a neighbourhood of 0 in E_j. Now if $x \in \pi_j (\pi_i^{-1} (V))$ then $x = \pi_j (w)$ where $\pi_i (w) \in V$. Hence

$$\pi_j^i \circ \pi_i (w) = \pi_j (w) = x \in \pi_j^i (V),$$

i. e. $\pi_j (\pi_i^{-1} (V)) \subset \pi_j^i (V)$. This implies that $\pi_j^i (V)$ is a neighbourhood of 0 in E_j and completes the proof.

The same method of proof may be used if $E = \lim_{i \in A, i \geqslant j} (E_i, \pi_i)$ is an open surjective limit to show that $i \geqslant j$ if and only if $\pi_i (x) = 0$ implies $\pi_j (x) = 0$ for any x in E.

EXAMPLE 2.7.

(a) Directed surjective limits of Fréchet spaces (resp. $\mathscr{DRF}$ spaces) are open surjective limits. It suffices to note that Fréchet spaces

and $\mathscr{DRF}$ spaces are fully complete and barrelled ([22], p. 300) and hence we may apply the open mapping Theorem and Proposition 2.6.

(b) A surjective limit of Banach spaces is an open surjective limit. We have already noted that it is a directed surjective limit and hence we may apply (a).

(c) Let X denote a completely regular Hausdorff topological space. $\mathscr{C}(X)$, the space of all continuous complex valued functions on X endowed with the topology of uniform convergence on compact sets, is a directed surjective limit of $\mathscr{C}(K)$ ([6]), K compact in X (the compact sets are directed by set inclusion and the restriction mappings are used between $\mathscr{C}(K)$ and $\mathscr{C}(K_1)$, $K_1 \subset K$), where $\mathscr{C}(K)$ is endowed with the topology of uniform convergence on K. Since each $\mathscr{C}(K)$ is a Banach space (b) implies that $\mathscr{C}(X)$ is an open surjective limit. $\mathscr{C}(X)$ may not be a complete locally convex space [43] but its completion is also an open surjective limit of $\mathscr{C}(K)$, K compact in X. The Tietze extension theorem implies that $\mathscr{C}(X)$ is a bounded surjective limit. Now let K denote a compact subset of X and suppose B is a compact subset of $\mathscr{C}(K)$. There exists a sequence in $\mathscr{C}(K)$, $(x_n)_{n=1}^{\infty}$, which converges to 0 and B is contained in the closed convex hull of this sequence. By the Tietze extension theorem we can extend each x_n to a continuous function on X, $\tilde{x}_n$, such that $\| x_n \|_K = \| \tilde{x}_n \|_X$. Hence $\tilde{x}_n$ converges to 0 in $\mathscr{C}(X)$ and its closed convex hull is compact. This shows that $\mathscr{C}(X)$ is a compact and sequential surjective limit.

(d) Let X denote a locally compact space, μ a Radon measure on X and $\mathscr{L}_{\text{loc}}^p(X, \mu)$ the space of all locally $p-\mu$-summable functions on X with its natural topology, $1 \leqslant p \leqslant \infty$. The associated Hausdorff space $L_{\text{loc}}^p(X, \mu)$ is an open, bounded, compact and sequential surjective limit of Banach spaces.

PROPOSITION 2.8. — Let $E = \lim_{n} E_n$ denote a strict inductive limit then E_{β}' is a directed surjective limit of $((E_n)_{\beta}')_{n=1}^{\infty}$ ([22], 2.12).

Proof. — We may suppose that $E = \bigcup_{n=1}^{\infty} E_n$. Since E induces on E_n its original topology we see, via the Hahn-Banach theorem, that the transpose of the canonical injection of E_n into E is a surjective mapping from E_{β}' onto $(E_n)_{\beta}'$. The strong topology on E' is the topology of

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([6]) The Tietze extension theorem shows that the restrictions mappings are surjections.

uniform convergence on bounded subsets of E and since each bounded subset of E is contained and bounded in some E_n it follows that the topology of E'_β is the weakest topology for which all the transpose mappings are continuous. Hence E'_β is a surjective limit of $((E_n)'_\beta)_{n=1}^\infty$ and it is obviously a directed surjective limit. This completes the proof.

It now follows that the strong dual of the strict inductive limit of $\mathscr{RF}$ spaces is an open surjective limit of $\mathscr{DRF}$ spaces and the strong dual of the strict inductive limit of $D\mathscr{F}$ spaces is an open surjective limit of Fréchet spaces.

EXAMPLE 2.9. — Let Ω denote an open subset of $\mathbf{R}^n$ and let $\mathscr{D}(\Omega)$ denote the set of all compex valued C^∞-functions on Ω with compact support. It is well known (see [22]) that $\mathscr{D}(\Omega)$ with its natural topology is the strict inductive limit of Fréchet-Schwartz spaces. Hence $\mathscr{D}'(\Omega)$ is an open surjective limit of $\mathscr{DFS}$ spaces.

EXAMPLE 2.10. — The strong dual of the strict inductive limit of Fréchet-Montel spaces is an open and compact surjective limit of $\mathscr{DFM}$ spaces.

Proof. — Let $E = \lim_n \overrightarrow{E_n}$. We have already noted that $E'_\beta = \lim_n \overleftarrow{(E_n)'_\beta}$ is an open surjective limit. Let K_n denote a compact subset of $(E_n)'_\beta$. There exists a convex balanced neighbourhood of 0 in E_n, V, such that $V^0 \supset K_n$ (V^0 is the polar of V in $(E_n)'_\beta$). Since E is a strict inductive limit there exists a neighbourhood of 0 in E, W, such that $V \supset W \cap E_n$. Now if $\varphi \in V^0$ then $|\varphi(W \cap E_n)| \leqslant 1$ and, by the Hahn-Banach theorem, there exists $\tilde{\varphi} \in E'$ such that $|\tilde{\varphi}(W)| \leqslant 1$ and $\tilde{\varphi}|_{E_n} = \varphi$. E is a Montel space and hence W^0 is a compact subset of E'_β. This completes the proof.

Remark. — GROTHENDIECK ([15], p. 95) gives an example of a Fréchet-Montel space which has a quotient space isomorphic to l_1. Since l_1 is non-reflexive and every quotient space of a Fréchet-Schwartz space is itself a Fréchet-Schwartz space and hence reflexive it follows that there exist Fréchet-Montel spaces which are not Fréchet-Schwartz spaces (see also [22], p. 279).

We now define complete surjective representations. If u is a continuous linear mapping from the locally convex space E into the locally convex space F then $\hat{u}$ will denote the unique extension of u to a continuous linear mapping from $\hat{E}$ into $\hat{F}$. Let $\alpha = (E_i, \pi_i)_{i \in A}$ denote a surjective representation of the locally convex space E. *The α-completion of E is*

the subspace of $\hat{E}$ containing E which equals $\bigcap_{i \in A} (\hat{\pi}_i)^{-1} (E_i)$. We let E_α denote the α or $(E_i, \pi_i)_{i \in A}$-completion of E. $(E_i, \hat{\pi}_i|_{E_\alpha})_{i \in A}$ is a surjective representation of E_α. If $E = E_\alpha$ then we say that E is an $(E_i, \pi_i)_{i \in A}$-complete space and $(E_i, \pi_i)_{i \in A}$ is a complete surjective representation of E.

PROPOSITION 2.11. — If $\alpha = (E_i, \pi_j^i, \pi_i)_{i \in A, i \geqslant j}$ is a directed surjective representation of E then E_α is isomorphic to

$$\tilde{E} = \left\{ (x_i)_{i \in A} \in \prod_{i \in A} E_i \, \middle| \, \pi_j^i(x_i) = x_j \text{ for all } i, j \in A, i \geqslant j \right\}.$$

Proof. — Let π, $\hat{E} \to \prod_{i \in A} \hat{E}_i$ denote the mapping which takes $x \in \hat{E}$ onto $(\hat{\pi}_i(x))_{i \in A}$. By the definition of surjective limit E is isomorphic to $\pi(E)$ and E_α is isomorphic to $\pi(E_\alpha)$. To complete the proof it suffices to show that the sets $\pi(E_\alpha)$ and $\tilde{E}$ coincide. If $x \in E_\alpha$ then $\hat{\pi}_i(x) \in E_i$ for all $i \in A$. Since $\pi_j^i \circ \pi_i = \pi_j$ for all $i, j \in A$, $i \geqslant j$, it follows that $\pi_j^i \circ \hat{\pi}_i|_{E_\alpha} = \hat{\pi}_j|_{E_\alpha}$. Hence $\pi_j^i(\hat{\pi}_i(x)) = \hat{\pi}_j(x)$ for all $x \in E_\alpha$ and $\pi(E_\alpha) \subset \tilde{E}$. Conversely let $(x_i) \in \tilde{E}$. Let $\alpha_1, \ldots, \alpha_n \in A$ and let $\beta \in A$, $\beta \geqslant \alpha_i$ for $i = 1, \ldots, n$. Choose $x \in E$ such that $\pi_\beta(x) = x_\beta$. Hence

$$\pi_{\alpha_i}^\beta \circ \pi_\beta(x) = \pi_{\alpha_i}^\beta(x_\beta) = x_{\alpha_i} \qquad \text{for} \quad i = 1, \ldots, n.$$

It now follows that $(x_i) \in \overline{\pi(E)}$ where the closure is taken in $\prod_{i \in A} E_i$. Since $\pi(E_\alpha)$ is a closed subspace of $\prod_{i \in A} E_i$ we see that

$$\pi(E_\alpha) \subset \tilde{E} \subset \overline{\pi(E)} \subset \pi(E_\alpha).$$

Hence $\pi(E_\alpha) = \tilde{E}$ and this completes the proof.

Remark. — There may exist more than one surjective limit associated with the same surjective system (e. g. if $(E_i, \pi_i)_{i \in A}$ is a directed surjective representation of E and E is not $(E_i, \pi_i)_{i \in A}$-complete). The complete surjective limit associated with a surjective system is, when it exists, the classical projective limit associated with a given projective system. A surjective limit $E = \lim_i (E_i, \pi_i)$ is said to be complete if E is $(E_i, \pi_i)_{i \in A}$-complete.

Our next proposition is frequently useful and explains our terminology.

PROPOSITION 2.12. — A surjective limit of complete locally convex spaces is complete if and only if it is a complete surjective limit.

Proof. — If E is complete then $\hat{\pi}_i = \pi_i$ and hence

$$E = \bigcap_{i \in A} (\pi_i)^{-1}(E_i) = \bigcap_{i \in A} (\hat{\pi}_i)^{-1}(E_i) = E_\alpha.$$

Conversly let $E = E_\alpha$. Since E_i is complete

$$(\hat{\pi}_i)^{-1}(E_i) = \hat{E} \quad \text{and} \quad E_\alpha = \bigcap_{i \in A}(\hat{\pi}_i)^{-1}(E_i) = \hat{E}.$$

This completes the proof.

Proposition 2.12 is equivalent to the statement that the collection of all complete locally convex spaces is closed under the operation of complete surjective limits. We do not, in general, know if we can associate a surjective limit with a given surjective system. However, if the indexing set A contains a smallest element then we can always construct an associated surjective limit by using the mapping π of Proposition 2.11. Moreover, if $(E_i, \pi_j^i)_{i, j \in A, i \geqslant j}$ is a surjective system then

$$E(j_0) = \left\{ (x_i)_{i \in A, i \geqslant j} \in \prod_{i \geqslant j_0} E_i \,\middle|\, \pi_j^i(x_i) = x_j \right\}$$

is the complete surjective limit associated with $(E_i, \pi_j^i)_{i, j \in A, i \geqslant j \geqslant j_0}$. It is not difficult to show that $E(j_0)$ and $E(j_1)$ are isomorphic as locally convex spaces for any $j_0, j_1 \in A$. Moreover if $(E_i, \pi_j^i, \pi_i)_{i, j \in A, i \geqslant j}$ is a representation of the locally convex space E then the $(E_i, \pi_i)_{i \in A}$-completion of E is isomorphic to $E(j_0)$ for any $j_0 \in A$. Since we are primarily interested in the topological vector space structure of surjective limits and not in the surjective system used to generate them we thus find that we can always "associate" a surjective limit with a given surjective system.

3. Vector valued holomorphic functions

In this section we give the various definitions of holomorphic functions between locally convex spaces that we shall need in this work ([7]). We found it necessary to define continuous and locally bounded holomorphic functions on a collection of sets which, in general, properly contain all open sets. This we did by using a Taylor series expansion which is valid for G-holomorphic vector valued functions defined on finitely open subsets of a vector space. We also define hypoanalytic, Silva holomorphic and weak holomorphic mappings and prove a number of results showing the relationship between these definitions. In the

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([7]) For further definitions of holomorphic mappings between locally convex spaces, we refer to [27], [35] and [37].

latter part of this section we discuss definitions which involve functions defined on surjective limits.

U, a subset of the vector space E, is finitely open if and only if $U \cap F$ is an open subset of the Euclidean space F for each finite dimensional subspace F of E. The finitely open subsets define a translation invariant topology on E, t_f, which is a vector topology if and only if the (algebraic) dimension of E is countable. We shall frequently use the facts that linear surjections between vector spaces are t_f open mappings and the balanced t_f neighbourhoods of zero form a basis for the t_f neighbourhoods of zero.

A function f defined on a finitely open subset, U, of a vector space E with values in a locally convex space F is said to be Gateaux or G-holomorphic if it satisfies the following condition.

For each $a \in U$, $b \in E$, $\varphi \in F'$ the complex valued function of one complex variable, $\lambda \to \varphi \circ f(a + \lambda b)$, is holomorphic in some neighbourhood of $0 \in \mathbf{C}$.

We let $\mathcal{H}_G(U; F)$ denote the set of all G-holomorphic functions from U into F. Each f in $\mathcal{H}_G(U; F)$ has a (unique) Taylor series expansion about each point of U consisting of polynomials from E into $\hat{F}$, i. e. for each ξ in U there exists a sequence of polynomials from E into $\hat{F}$, $(\hat{d}^n f(\xi)/n!)_{n=1}^\infty$, such that

$$(3.1) \qquad\qquad f(\xi + y) = \sum_{n=0}^\infty \frac{\hat{d}^n f(\xi)}{n!}(y)$$

for all y in some t_f neighbourhood of 0. For $\xi \in U$ we let

$$T_{(f, \xi)}(y) = \lim_{m \to \infty} \sum_{n=0}^m \frac{\hat{d}^n f(\xi)}{n!}(y) = \sum_{n=0}^\infty \frac{\hat{d}^n f(\xi)}{n!}(y)$$

whenever this limit exists in $\hat{F}$.

Now suppose E and F are locally convex spaces. *An F-valued function defined on a finitely open subset of E, U, is a continuously holomorphic (resp. locally bounded holomorphic) function if it is G-holomorphic and for each $\xi \in U$, $y \to T_{(f, \xi)}(y)$ defines a continuous (resp. locally bounded) function on some neighbourhood of zero* [8]. When U is open $f \in \mathcal{H}_G(U; F)$

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[8] The t_f topology is finer than any locally convex topology on E and hence every open subset of E is finitely open.

is continuously holomorphic (resp. locally bounded holomorphic) if and only if f is continuous (resp. locally bounded) on U. We let $\mathscr{H}(U; F)$ (resp. $\mathscr{H}_{LB}(U; F)$) denote the set of all continuously holomorphic (resp. locally bounded holomorphic) functions from U into F.

The following is a useful result and the proof is immediate. We use the above notation.

LEMMA 3.1.

(a) If $F = \lim_{i \in A}(F_i, \pi_i)$ is a surjective limit then $f \in \mathscr{H}(U; F)$ if and only if $\pi_i \circ f \in \mathscr{H}(U; F_i)$ for all $i \in A$.

(b) If F is a normed linear space then $\mathscr{H}(U; F) = \mathscr{H}_{LB}(U; F)$ where U and E are arbitrary.

(c) If $\mathscr{H}(U; F) = \mathscr{H}_{LB}(U; F)$ for any locally convex space F and any finitely open subset U of E then E is a normed linear space.

A G-holomorphic F valued function defined on an open subset of a l. c. space is hypoanalytic if its restriction to compact sets is continuous. HIRSCHOWITZ [17] defines a hypoanalytic function as a G-holomorphic function which is bounded on compact sets. Our definition is obviously more restrictive and is not equivalent since the identity mapping from an infinite dimensional Hilbert space with the weak topology into itself with the norm topology maps compact sets onto bounded sets but is not continuous on compact sets. We let $\mathscr{H}_{HY}(U; F)$ denote the set of all hypoanalytic functions from the open set U into F.

An F valued G-holomorphic function, f, defined on a finitely open subset, U, of a locally convex space E is said to be scalarly holomorphic at $x \in U$ if for each $\mathbf{C}$-valued continuously holomorphic function, g, defined on a neighbourhood of $f(x)$ in F the function $g \circ f$ is continuous in some neighbourhood of x. f is said to be scalarly holomorphic on U if it is scalarly holomorphic at all points of U. We let $\mathscr{H}_\omega(U; F)$ denote the set of all scalarly holomorphic functions from U into F.

If F is a locally convex space and F_ω denotes F with the $\sigma(F, F')$ topology then the elements of $\mathscr{H}(U; F_\omega)$ are the weakly holomorphic functions from U, a finitely open subset of the locally convex space E, into F.

The proof of the following proposition, which shows the relationship between the various definitions of holomorphic function, is immediate.

PROPOSITION 3.2. — *Let E and F denote locally convex spaces with U a finitely open subset of E. We have the following inclusions;*

$$\mathcal{H}_{LB}(U,F) \subset \mathcal{H}(U;F) \begin{array}{c} \subset \mathcal{H}_{\omega}(U;F) \subset \mathcal{H}(U;F_{\omega}) \\ \subset \mathcal{H}_{HY}(U;F) \subset \end{array} \mathcal{H}_G(U;F)$$

The next proposition gives conditions under which *G*-holomorphic functions are hypoanalytic.

PROPOSITION 3.3. — *A G-holomorphic function from $U \subset E$ into F, f, is hypoanalytic if any of the following conditions are satisfied :*

(a) *f is bounded on compact sets and each polynomial from E into F which is bounded on compact sets is hypoanalytic.*

(b) *$f \in \mathcal{H}_{\omega}(U ; F)$ and $F = \lim_{i \in A} F_i$ where each F_i is a normed linear space and the closed unit ball of $(F_i)'$ is weak* sequentially compact* ([9]).

(c) *E is separable and f is scalarly holomorphic.*

(d) *$f \in \mathcal{H}(U; F_{\omega})$ and each closed bounded subset of F is compact.*

(e) *f is bounded on compact sets, $(E', \sigma(E', E))$ is separable and E satisfies the Mackey convergence criterion ([22] p. 285)* ([10]).

(f) *f is bounded on compact sets and E is metrizable.*

Proof. — (a) Since *E* is locally convex we may assume that *U* is convex and balanced. Let *K* denote a balanced compact subset of *U*. Now *f* is bounded on *K* and hence, since $\lambda K \subset U$ for some $\lambda > 1$, we may use the Cauchy integral formula to show

$$\lim_{n \to \infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_K^{1/n} < 1.$$

A further application of Cauchy's integral formula shows that $\hat{d}^n f(0)/n!$ is bounded on compact sets for each *n* and hence is hypoanalytic by our hypothesis. It is now trivial to complete the proof.

([9]) Separable locally convex spaces, Schwartz spaces and reflexive Banach spaces all have representations of this form.

([10]) i. e. if $(x_n)_{n=1}^{\infty}$ is a null sequence in *E* then there exists a sequence of scalars, $(\lambda_n)_{n=1}^{\infty}$, which diverges to $+\infty$ such that $(\lambda_n x_n)_{n=1}^{\infty}$ is a null sequence in *E*.

(b) The bounding subsets of F_i are precompact for all $i \in A$ [8]. Hence if K is compact in U and $\pi_i(f(K))$ is not a precompact subset of F_i then there exists a $g \in \mathscr{H}(F_i; \mathbf{C})$ such that $\|g \circ \pi_i \circ f\|_K = \infty$. This contradicts the fact that $f \in \mathscr{H}_\omega(U; F)$. Hence f maps compact subsets onto precompact subsets of F. Now if $(x_\alpha)_{\alpha \in \Gamma}$ is a convergent net in K then $(f(x_\alpha))_{\alpha \in \Gamma}$ is a precompact net in F. If $(f(x_\alpha))_{\alpha \in \Gamma_1}$ and $(f(x_\alpha))_{\alpha \in \Gamma_2}$ are two subnets of $(f(x_\alpha))_{\alpha \in \Gamma}$ then

$$\lim_{\alpha \to \infty, \, \alpha \in \Gamma_1} \varphi(f(x_\alpha)) = \lim_{\alpha \to \infty, \, \alpha \in \Gamma_2} \varphi(f(x_\alpha))$$

for any $\varphi \in \mathscr{H}(F_\omega; \mathbf{C})$. By the Hahn-Banach theorem it follows that $(f(x_\alpha))_{\alpha \in \Gamma}$ is a convergent net and hence f is hypoanalytic.

(c) Suppose $\{x_n\}_{n=1}^\infty$ denotes a dense sequence in U and F is a Banach space. Let F_1 denote the closure of the Banach subspace of F generated by $(f(x_n))_{n=1}^\infty$. If $\varphi \in F'$, $\varphi|_{F_1} = 0$ then $\varphi \circ f(x_n) = 0$ for all n and since $f \in \mathscr{H}_\omega(U; F)$ it follows that $\varphi \circ f = 0$. Hence $f(U) \subset F_1$. Now let K denote a compact subset of U. If $f(K)$ is not a precompact subset of F_1 then we can find (see [8]) $(\varphi_n)_{n=1}^\infty$, a sequence of elements of F_1', such that $g = \sum_{n=1}^\infty \varphi_n^n \in \mathscr{H}(F_1)$ and $\|g \circ f\|_K = \infty$. Now $\sup_n \|\varphi_n\|_{F_1} < \infty$ and hence we may use the Hahn-Banach theorem to find $(\tilde{\varphi}_n)_{n=1}^\infty$, a sequence in F', such that $\tilde{\varphi}_n|_{F_1} = \varphi_n$ for all n and $\sup_n \|\tilde{\varphi}_n\|_F < \infty$. It is now easy to see that $g = \sum_{n=1}^\infty \varphi_n^n \in \mathscr{H}(V)$ for some open subset V of F which contains F_1. Hence $g \circ f \in \mathscr{H}(U)$ and $\|g \circ f\|_K = \infty$. This contradicts the fact that $f \in \mathscr{H}_\omega(U; F)$. Hence $f(K)$ is a precompact subset of U and the F and the proof may be completed as in (b) ([11]).

(d) If $f \in \mathscr{H}(U; F_\omega)$ then $\varphi(f(K))$ is a bounded subset of $\mathbf{C}$ for each compact subset K of U and each $\varphi \in F'$. Hence f is bounded on compact subsets of U. By our hypothesis on F it follows that $f(K)$ is a precompact subset of F for each compact subset K of U. The proof may now be completed as in (b).

(e) If $(E', \sigma(E', E))$ is separable then each compact subset of E is metrizable and hence it suffices to consider sequential convergence. By (a), we may suppose that f is an m-homogeneous polynomial. If $(x_n)_{n=1}^\infty$ is a sequence in E which convergence to 0 then, by hypothesis,

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([11]) In particular if E is a $\mathscr{DFM}$ space then $\mathscr{H}(U; F) = \mathscr{H}_\omega(U; F) = \mathscr{H}_{HY}(U; F)$ for any domain U spread over E.

there exists a sequence of scalars, $(\lambda_n)_{n=1}^\infty$ which diverges to $+\infty$ and $\lambda_n x_n \to 0$ as $n \to \infty$. Since f is bounded on compact sets

$$\bigcup_n f(\lambda_n x_n) = \bigcup_n \lambda_n^m f(x_n)$$

is a bounded subset of F. Hence $f(x_n) \to 0$ as $n \to \infty$ and this completes the proof.

(f) Trivial.

We now briefly discuss one further definition of a holomorphic function ([6], [27]). If B is a closed convex balanced bounded subset of a locally convex space E we let E_B denote the vector space spanned by the set B endowed with the norm $\| x \|_B = \inf \{ \lambda > 0 \mid x \in \lambda B \}$. τ_M (cf. [20]) is the finest topology on E for which the injections $E_B \to E$ are continuous for all such B. *A locally convex space (E, τ) for which $\tau = \tau_M$ is called superinductive* (note that (E, τ_M) will not, in general, be a locally convex space). Fréchet spaces and $\mathscr{DFS}$ spaces are examples of superinductive locally convex spaces.

Let E and F denote locally convex spaces. *A G-holomorphic F valued function defined on an open subset of E, U, is S-holomorphic or Silva holomorphic if it is continuous when U is given the induced τ_M topology.* We let $\mathscr{H}_S(U; F)$ denote the space of all S-holomorphic mappings from U into F.

PROPOSITION 3.4. — *Hypoanalytic and weakly holomorphic functions are Silva holomorphic.*

Proof. — Let f; $U \subset E \to F$ denote a G-holomorphic function and let B denote a closed convex balanced bounded subset of E. To show f is Silva holomorphic it suffices to prove that f is bounded on each $\| \ \|_B$ convergent sequence in U. Now if $\{ x_n \}_{n=1}^\infty$ is a $\| \ \|_B$ convergent sequence then it is also a convergent sequence in E. Hence if $f \in \mathscr{H}(U; F_\omega)$ or $f \in \mathscr{H}_{HY}(U; F)$ then f is bounded on $\{ x_n \}_{n=1}^\infty$ and this completes the proof.

We now give an example in which the converse is true. *E, a locally convex space, has property (S)* [21] *if for each compact subset K of E there exists a balanced convex closed bounded subset of E, B, such that K is contained and compact in E_B.* Strict inductive limits of Fréchet spaces and strong duals of infrabarrelled Schwartz spaces have property (S) [21].

PROPOSITION 3.5. — *If E has property (S) then $\mathscr{H}_{HY}(U; F) = \mathscr{H}_S(U; F)$ for all open subsets U of E and any locally convex space F.*

Proof. — Let τ denote the locally convex topology on E. If K is a compact subset of (E, τ) and B is a closed convex balanced subset of E such that K is compact subset of E_B, then it follows that τ, τ_M and $\| \quad \|_B$ induce the same topology on K. If $f \in \mathcal{H}_S(U; F)$ then the restriction of f to (K, τ) is continuous and f is hypoanalytic.

Notation. — When $F = \mathbf{C}$ we shall write $\mathcal{H}(U)$, etc., in place of $\mathcal{H}(U; \mathbf{C})$, etc.

A manifold spread over E, a locally convex space, is a pair (Ω, p) where Ω is a connected Hausdorff space and p is a function from Ω into E which is a local homeomorphism. Holomorphic functions on Ω are defined by means of the function p and the holomorphic functions on E (*see* [4] and [39] for further details).

We now consider holomorphic functions defined on surjective limits.

DEFINITION 3.6. — *Let U denote a connected finitely open subset of the locally convex space E, let $(E_i, \pi_i)_{i \in A}$ denote a surjective representation of E and let F denote a locally convex space.*

(1) *U has the weak (resp. strong) local F factorization property with respect to the surjective representation $(E_i, \pi_i)_{i \in A}$ if for each $f \in \mathcal{H}(U; F)$ (resp. $\mathcal{H}_{LB}(U; F)$) there exists an $i \in A$ with the following properties:*

(a) *There is a neighbourhood of each x in U, V_x, and a neighbourhood of $\pi_i(x)$ in E_i, W_x, such that $\pi_i(V_x) \subset W_x$.*

(b) *There is an $f_x \in \mathcal{H}(W_x; F)$ (resp. $\mathcal{H}_{LB}(W_x; F)$ such that*

$$f\big|_{V_x} = f_x \circ \pi_i \big|_{V_x}.$$

(2) *U has the weak (resp. strong) global F factorization property with respect to the surjective representation $(E_i, \pi_i)_{i \in A}$ if it has the weak (resp. strong) local F factorization property with respect to $(E_i, \pi_i)_{i \in A}$ and*

$$f_x\big|_{\pi_i(U) \cap W_x \cap W_y} = f_y\big|_{\pi_i(U) \cap W_x \cap W_y}$$

for all x, y in U.

HIRSCHOWITZ [16] proves the existence of open subsets of $\prod_{n=1}^{\infty} \mathbf{C}$ which have the local ([12]) factorization property with respect to $(\mathbf{C}^n, \pi_n)_{n=1}^{\infty}$

([12]) We define various holomorphic properties of locally convex spaces using the prefixes weak and strong to distinguish between properties which refer to continuous and locally bounded holomorphic functions respectively. We use no prefix when the prefixes are interchangable (as is the case here).

but which do not have the global factorization property with respect to the same representation ([13]). NACHBIN [32] proves that E has the global factorization property with respect to any open surjective representation of E by normed linear spaces and also proves, that $\mathscr{H}$ (C) (with the compact open topology) does not have the local C-factorization property with respect to the canonical normed surjective representation of $\mathscr{H}$ (C). A balanced open subset which has the weak (resp. strong) local F factorization property for some surjective representation can easily be shown (*via* Taylor series expansion and analytic continuation) to have the weak (resp. strong) global F factorization property with respect to the same surjective representation.

If π is a linear mapping from the vector space E_1 onto the vector space E_2 and F is a locally convex space we let $^t\pi$ denote the transpose function which maps $f \in \mathscr{H}_G (\pi (U); F)$ onto $f \circ \pi \in \mathscr{H}_G (U; F)$. It is immediate that

$$^t\pi (\mathscr{H} (\pi(U); F)) \subset \mathscr{H} (U; F) \quad \text{and} \quad ^t\pi (\mathscr{H}_{LB}(\pi (U); F)) \subset \mathscr{H}_{LB}(U; F).$$

U, a finitely open subset of E, has the weak (resp. strong) global F factorization property if and only if

$$\mathscr{H} (U; F) = \bigcup_{i \in A} {}^t\pi_i (\mathscr{H} (\pi_i(U); F))$$
$$(\text{resp. } \mathscr{H}_{LB}(U; F) = \bigcup_{i \in A} {}^t\pi_i (\mathscr{H}_{LB}(\pi_i(U); F))).$$

In discussing holomorphically complete locally convex spaces [10] we introduced the concept of ω-space. We rephrase and slightly generalise this definition so that it is now a concept involving factorizing locally bounded holomorphic functions on locally convex spaces. Our new definition was also motivated by our use of hereditary Lindelöf spaces in the study of the Levi problem [13].

DEFINITION 3.7. — *Let E and F denote locally convex spaces. E is an $F-\omega$-space if the following conditions are satisfied.*

(1) *each open subset of E has the strong global F factorization property with respect to the canonical metrizable surjective representation of E.*

(2) *for each U open in E and $f \in \mathscr{H}_{LB} (U; F)$ there exists a sequence of bounded subsets of F, $(B_n)_{n=1}^\infty$, such that $U = \bigcup_{n=1}^\infty (f^{-1} (B_n))^\circ$ ([14]).*


([13]) π_n is the usual projection $\prod_{n=1}^\infty$ C onto the space spanned by the first n coordinates.

([14]) A° denotes the interior of A.

Now let $f \in \mathscr{H}_{LB}(U; F)$ where U is an open subset of the $F - \omega$-space E. Let $(B_n)_{n=1}^{\infty}$ denote an increasing sequence of convex balanced bounded subsets of F such that

$$U = \bigcup_{n=1}^{\infty} (f^{-1}(B_n))^0.$$

For each n let F_n denote the vector subspace of F spanned by B_n and normed by the Minkowski functional of B_n. $G = \lim_{\longrightarrow n} F_n$ is a (DF)-space in the sense of GROTHENDIECK [15] and the canonical inclusion of G in F, j, is continuous. By condition (1) it is now possible to find a locally convex metrizable space, E_1, a continuous mapping π from E onto E_1 and $f \in \mathscr{H}_{LB}(\pi(U); G)$ such that the following diagram commutes
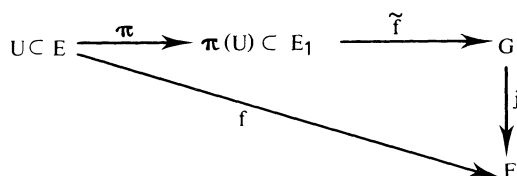
This shows that each F-valued locally bounded mapping from an $F - \omega$-space is, modulo two linear mappings, a locally bounded mapping from a metrizable space into a DF-space.

Now suppose that every open subset of the locally convex space E is a Lindelöf space. Let $f \in \mathscr{H}_{LB}(U; F)$ where U is open in E and F is arbitrary. For each x in U choose a continuous seminorm on E, p_x, such that $V_x = \bigcup_y \{ f(x+y); p_x(y) < 1 \text{ and } x+y \in U \}$ is a bounded subset of F. Choose a sequence of points of U, $(x_n)_{n=1}^{\infty}$, such that $\bigcup_{n=1}^{\infty} \{ x_n + y; p_{x_n}(y) < 1 \} = U$. f is a locally bounded function from U, endowed with the topology induced by $(p_{x_n})_{n=1}^{\infty}$, into F and since $\bigcup_{n=1}^{\infty} V_{x_n}$ is a countable union of bounded subsets of F we have shown that E is an $F - \omega$-space.

If each open subset of E_n is Lindelöf and $E = \lim_{\longrightarrow n} E_n$ then each open subset of E has the same property and thus a countable inductive limit of separable metrizable locally convex spaces is an $F - \omega$-space for any locally convex space F. A $\mathscr{DFM}$ space is a Suslin space (i. e. a continuous image of a complete separable metric space) and hence it is a hereditary Lindelöf space and an $F - \omega$-space for any locally convex space F.

If F is a $D\mathscr{F}$ space, condition (2) of Definition 3.7 is always satisfied (e. g. a locally convex metrizable space is a $D\mathscr{F} - \omega$-space).

E. GRUSELL and M. SCHOTTENLOHER have given, independantly, examples of locally convex spaces which are not $C-\omega$-spaces.

4. Pseudo-Convex domains ([15])

An open subset, U, of a locally convex space, E, is finitely polynomially convex if $U \cap F$ is polynomially convex for each finite dimensional subspace F of E.

PROPOSITION 4.1. — *The collection of locally convex spaces in which the finitely polynomially convex open subsets are polynomially convex is closed under arbitrary surjective limits.*

Proof. — As in Proposition 1.7 of [13].

EXAMPLE 4.2. — The following are examples of spaces in which the finitely polynomially convex open subsets are polynomially convex;

(*a*) Locally convex spaces with an equi-Schauder basis (*see* example 2.4, and [13]).

(*b*) Nuclear spaces (*see* Example 2.2, and [13]).

(*c*) Locally convex spaces with the strong approximation property ([35], p. 69).

PROPOSITION 4.3. — *The collection of locally convex spaces in which the finitely polynomially convex open subsets are domains of holomorphy (resp. domains of existence of holomorphic functions) is closed under open surjective limits.*

Proof. — (*see* Proposition 1.6 of [13]). — If U is a finitely polynomially convex open subset of $\lim_{i \in A} (E_i, \pi_i)$ then there exists an $i \in A$ such that $U = \pi_i^{-1} (\pi_i (U))$ and $\pi_i (U)$ is a finitely polynomially convex open subset of E_i. Now suppose $\pi_i (U)$ is a domain of holomorphy. If U_1, U_2 are open subsets of E such that $U_2 \subset U$, $U \cap U_1 \supset U_2$ and for each $f \in \mathscr{H} (U)$ there exists an $f_1 \in \mathscr{H} (U_1)$ with $f|_{U_2} = f_1|_{U_2}$ then, since π_i is an open mapping, this implies that $\pi_i (U_1) \subset \pi_i (U)$. Hence $U_1 \subset \pi_i^{-1} (\pi_i (U)) = U$ and U is a domain of holomorphy. Now suppose $\pi_i (U)$ is a domain of existence for the function f. Now if U_1 and U_2 are two convex subsets

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([15]) A number of the results in this section were announced in [9], and proved in [13].

of E, $U_2 \subset U \cap U_1$ and there exists $g \in \mathscr{H}(U_1)$ such that $g\big|_{U_2} = f \circ \pi_i\big|_{U_2}$ then it follows that $\pi_i(U_1) \subset \pi_i(U)$ and hence U is a domain of existence for the function $f \circ \pi_i$.

EXAMPLE 4.4.

(a) Every finitely polynomially convex open subset of a Fréchet space with a basis is the domain of existence of a holomorphic function. We proved this result in [13] in a rather laborious fashion but noted that the proof could be considerably simplified if we knew that there existed a continuous norm on E. It is now possible to use Proposition 4.3 and the result in Example 2.4 to simplify the proof.

(b) (see [35] p. 81). Every finitely polynomially convex open subset of $\mathscr{C}(X)$ is the domain of existence of a holomorphic function.

(c) If E is a hereditary Lindelöf, barrelled space and possesses a basis (e. g. a $\mathscr{DFM}$ space with a basis or a countable direct sum of Fréchet spaces each of which has a basis) then the finitely polynomially convex open subsets of E are domains of holomorphy.

PROPOSITION 4.5. — *The collection of locally convex spaces in which the pseudo convex open subsets are holomorphically convex (resp. domains of holomorphy, domains of existence) is closed under open surjective limits.*

Proof. — As in proposition 1.6 of [13] and proposition 4.3 above.

EXAMPLE 4.6. — The pseudo convex open subsets of $\mathscr{C}(X)$ are domains of existence of holomorphic functions for any completely regular space X in which the compact subsets are metrizable ([35], p. 99).

Before discussing locally convex spaces in which the holomorphically convex open subsets are domains of existence we obtain a result concerning the metrizability of $\mathscr{H}(U)$ which is of independent interest (*see* also [21]).

PROPOSITION 4.7. — *Let E denote an infrabarrelled locally convex space in which the convex hull of each compact set is compact then $\mathscr{H}(U)$, endowed with the compact open topology, is a Fréchet space for all U open in E if and only if E is a $\mathscr{DFM}$ space.*

Proof. — First let U denote an open subset of the $\mathscr{DFM}$ space E. We have already seen that U is a Lindelöf space and hence U may be written as a countable union of translates of convex balanced open subsets of E.

Now E is also a DF space and a Montel space and hence there exists an exhaustive sequence of convex, balanced compact subsets of E, $(K_n)_{n=1}^{\infty}$. Let $(\lambda_n)_{n=1}^{\infty}$ denote a sequence of positive numbers which strictly increases to 1. If V is a convex balanced open subset of E then $(\lambda_n V \cap K_m)_{n,\ m=1}^{\infty}$ is an exhaustive sequence of compact subsets of V. It now follows that U is hemicompact and hence $\mathscr{H}(U)$ with the compact open topology is metrizable. Since E is a k-space $\mathscr{H}(U)$ is complete and we have shown that $(\mathscr{H}(U), \mathscr{T}_0)$ is a Fréchet space.

Conversely if $(\mathscr{H}(U), \mathscr{T}_0)$ is a Fréchet space for all U open in E then E'_c (the dual of E with the compact open topology) is also a Fréchet space. Since the closed convex hull of each compact set is compact it follows that E contains a countable fundamental family of compact sets. Now let B denote an arbitrary bounded subset of E and let $\mathscr{T}_{c_1}$ denote the topology, on E', of uniform convergence on compact subsets of E together with uniform convergence on B. Since E'_c is a Fréchet space it follows that $(E', \mathscr{T}_{c_1})$ is also a Fréchet space. By the open mapping Theorem and the Hahn-Banach Theorem it follows that B is relatively compact. Hence E is a semi-Montel space and since we assume E is infrabarrelled it follows that E is a Montel space. Since E is Montel $E'_c = E'_\beta$ is a Fréchet space and hence E' is a Fréchet-Montel space. Montel spaces are reflexive and hence $E \cong (E'_\beta)'_\beta$ is a $\mathscr{DFM}$ space.

PROPOSITION 4.8. — *A holomorphically convex open subset of a $\mathscr{DFM}$ space is the domain of existence of a holomorphic function.*

Proof. — Let U denote a holomorphically convex open subset of the $\mathscr{DFM}$ space E. Since each compact subset of E is a separable metrizable space and U is hemicompact there exists a sequence of points in δU [16], $(x_n)_{n=1}^{\infty}$, and for each n a sequence of points of U, $(x_n(m))_{m=1}^{\infty}$, such that $x_n(m) \to x_n$ as $m \to \infty$ for each n and the set $\{x_n\}_{n=1}^{\infty}$ is dense in δU. Since U is holomorphically convex, we may choose an increasing sequence of holomorphically convex compact subsets of U, $(K_n)_{n=1}^{\infty}$, such that each compact subset of U is contained in some K_n. We may now construct, as in the finite dimensional case, a G-holomorphic function on U, f, which is bounded on compact subsets of U and is such that $\sup_m |f(x_n(m))| = \infty$ for all n. Since U is a k-space it follows that $f \in \mathscr{H}(U)$ and hence U is the domain of existence of a holomorphic function.

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[16] The boundary of U.

5. Zorn spaces ([17])

Let E and F denote locally convex spaces. *E is called a weak (resp. strong) F-Zorn space if each F-valued G-holomorphic function defined on a domain spread over E has an open and closed set of points of continuity (resp. points of local boundedness).*

Since local boundedness and continuity are local properties we need only consider domains which are open convex balanced subsets of E in the definition of Zorn spaces. We collect in the next two propositions some elementary stability properties of Zorn spaces.

PROPOSITION 5.1.

(a) *If E is a weak F_i-Zorn space for each $i \in A$ and $F = \lim_{i \in A} (F_i, \pi_i)$ then E is a weak F-Zorn space.*

(b) *Quotients of weak (resp. strong) F-Zorn spaces are themselves weak (resp. strong) F-Zorn spaces.*

(c) *If E is a weak (resp. strong) F-Zorn space and $E \subset E_1 \subset \hat{E}$ then E_1 is a weak (resp. strong) F-Zorn space.*

Proof.

(a) Trivial.

(b) Immediate since quotient mappings are open.

(c) Let U denote a convex balanced open subset of E_1 and suppose f is a G-holomorphic mapping from U into F which is continuous (resp. locally bounded) at 0. It suffices to show that for each $x_1 \in U$ there exists a dense subspace of E_1, $E_1(x_1)$, such that $x_1 \in E(x_1)$ and $f|_{E_1(x_1) \cap U}$ is continuous (resp. locally bounded). Now if $x_1 \in U$, $x_2 \in E$, $x_2 \neq 0$, we let E_2 denote a closed subspace of E_1 such that $E_1 = \{x_1\} + \{x_2\} + E_2$ where $\{x_i\}$ is the subspace generated by x_i for $i = 1,2$.

We define an endomorphism of E_1, π, by

$$\pi(\lambda_1 x_1 + \lambda_2 x_2 + \omega) = \lambda_2 x_1 + \lambda_1 x_2 + \omega$$

for all $\lambda_i \in \mathbf{C}$, $i = 1,2$ and $\omega \in E_2$. π is a linear homeomorphism. Hence $\pi(E)$ is a weak (resp. strong) F-Zorn space, which is dense in E_1 and

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([17]) A number of the results proved in this section were announced in [11] and [14].

$\pi(x_2) = x_1 \in E(x_1) = \pi(E)$. Now $f|_{U \cap E(x_1)}$ is G-holomorphic and continuous (resp. locally bounded) at 0. Hence

$$f|_{U \cap E(x_1)} \in \mathscr{H}(U \cap E(x_1); F) \quad (\text{resp. } \mathscr{H}_{LB}(U \cap E(x_1); F)$$

and this completes the proof.

Now suppose F_1 and G_1 are locally convex spaces and that G_1 may be identified with a space of linear functionals on F_1. A subset B of G_1 is F_1-equicontinuous if there exists a neighbourhood of 0 in F_1, U, such that

$$B \subset U^0 = \{ \varphi \in G_1 ; \, |\varphi(x)| \leqslant 1 \text{ for all } x \in U \}.$$

PROPOSITION 5.2.

(a) If $E \times F_1$ is a **C**-Zorn space and the bounded subsets of G_1 are precisely the F_1-equicontinuous sets then E is a strong G_1-Zorn space.

(b) If $E \times F_\beta'$ is a **C**-Zorn space then E is a strong F-Zorn space.

(c) If $E \times F$ is a **C**-Zorn space and F is infrabarrelled then E is a strong F_β'-Zorn space.

Proof.

(a) Let U denote an open subset of E and suppose $f \in \mathscr{H}_G(U; F_1)$ is locally bounded at $x_0 \in U$. We define $\hat{f} \in \mathscr{H}_G(U \times G_1)$ by the equation $\hat{f}(x, y) = y(f(x))$. By hypothesis there exists a neighbourhood of x_0 in E, U_1, such that $f(U_1)$ is a bounded subset of G_1. By our assumptions on F_1 and G_1 there exists a neighbourhood of 0 in F_1, V, such that $f(U_1) \subset V^0$. Hence $\hat{f}(U_1 \times V)$ is a bounded subset of **C**. Since $E \times F_1$ is a **C**-Zorn space f is locally bounded at all points of $U \times F_1$ and hence $f \in \mathscr{H}_{LB}(U; G_1)$.

(b) The strong topology on F_β' is the topology of uniform convergence on the bounded subsets of F. Hence the F_β'-equicontinuous subsets of F are precisely the bounded subsets of F_β' and we may apply (a) to complete the proof.

(c) When F is infrabarrelled the strongly bounded subsets of F_β' are equicontinuous [22], and we may apply (a) to complete the proof.

Our next result, the main result of this section, describes surjective limits of Zorn spaces and other related results.

THEOREM 5.3. — *Let* $(E_i, \pi_i)_{i \in A}$ *denote a surjective representation of the locally convex space E and let U denote a connected domain spread over E. Let F denote a locally convex space and let f denote an F-valued G-holomorphic function on U which is locally bounded at some point of U. f is locally bounded at all points of U if any of the following conditions are satisfied.*

(a) *Each* E_i *is a strong F-Zorn space.*

(b) $(E_i, \pi_i)_{i \in A}$ *is a compact surjective representation of E*, $f \in \mathcal{H}_{HY}(U; F)$, *and each* $g \in \mathcal{H}_{HY}(V; F)$, *V a connected domain spread over some* E_i, *which is locally bounded at some point of V is locally bounded at all points of V.*

(c) $(E_i, \pi_i)_{i \in A}$ *is a bounded surjective representation of E*, $f \in \mathcal{H}_S(U; F)$, *and each* $g \in \mathcal{H}_S(V; F)$, *V a connected domain spread over some* E_i, *which is locally bounded at one point of V is locally bounded at all points of V.*

(d) $(E_i, \pi_i)_{i \in A}$ *is an open surjective representation of E*, $f \in \mathcal{H}_\omega(U; F)$, *and each* $g \in \mathcal{H}_\omega(V; F)$, *V a connected domain spread over some* F_i, *which is locally bounded at one point of V is locally bounded at all points of V.*

(e) $(E_i, \pi_i)_{i \in A}$ *is an open surjective representation of E*, $f \in \mathcal{H}(U; F_\omega)$, *and each* $g \in \mathcal{H}(V; F_\omega)$, *V a connected domain spread over some* E_i, *which is locally bounded at one point of V is locally bounded at all points of V.*

Proof. — We may assume that U is a convex balanced open subset of E and that f is locally bounded at 0. Choose $i \in A$ and W a neighbourhood of 0 in E_i such that $\pi_i^{-1}(W) \subset U$ and f is bounded on $\pi_i^{-1}(W)$. Now if $x \in E$ and $\pi_i(x) = 0$ then $\{\lambda x \mid \lambda \in \mathbf{C}\} \subset U$. Hence if y is an arbitrary element of U there exists a positive δ such that

$$\sup_{\beta \in C}\{|\varphi \circ f(\alpha y + \beta x)|, |\alpha| \leqslant \delta\} < \infty$$

for all $\varphi \in F'$.

By Liouville's theorem it follows that

$$\varphi \circ f(\alpha y + \beta x) = \varphi \circ f(\alpha y)$$

for all $\alpha, |\alpha| \leqslant \delta$, and $\beta \in \mathbf{C}$. Hence

$$f(y + \beta x) = f(y)$$

for all β such that $y + \beta x \in U$. Now if $j \in A$ and there exists a convex balanced neighbourhood of 0 in E_j, W_j, such that $\pi_j^{-1}(W_j) \subset \pi_i^{-1}(W)$ then the same argument shows that $f(y) = f(y + \beta x)$ for any y in U, $x \in E$,

$\pi_j(x) = 0$ and $y + \beta\, x \in U$. Let U_j denote the largest balanced E_j-open (^{18}) subset of U. We define f_j on $\pi_j(U_j)$ by the formula $f_j(\omega) = f(\widetilde{\omega})$ if $\pi_j(\widetilde{\omega}) = \omega$. f_j is well defined, locally bounded at 0, G-holomorphic on $\pi_j(U_j)$ and $f_j \circ \pi_j\big|_{U_j} = f\big|_{U_j}$. We now consider each case separately.

(a) If E_j is a strong F-Zorn space then $f_j \in \mathscr{H}_{LB}(\pi_j(U_j); F)$.

(b) If $(E_i, \pi_i)_{i \in A}$ is a compact representation of E then, since $U_j = \pi_j^{-1}(\pi_j(U_j))$, for each compact subset K of $\pi_j(U_j)$ there exists a compact subset L of U_j such that $\pi_j(L) = K$. If $(x_\alpha)_{\alpha \in \Gamma}$ is a net in L such that $\pi_j(x_\alpha) \to \pi_j(x)$ as $\alpha \to \infty$ and $(y_\beta)_{\beta \in B}$ and $(z_\alpha)_{\alpha \in C}$ are subnets of $(x_\alpha)_{\alpha \in \Gamma}$ such that $y_\beta \to y$ as $\beta \to \infty$ and $z_\alpha \to z$ as $\alpha \to \infty$ then $\pi_j(y) = \pi_j(z)$ and $f_j(\pi_j(x)) = f(y) = f(z)$. Hence if $f \in \mathscr{H}_{HY}(U; F)$ then

$$f_j \in \mathscr{H}_{HY}(\pi_j(U_j); F)$$

and the remaining assumption in (b) shows that $f_j \in \mathscr{H}_{LB}(\pi_j(U_j); F)$ in this case.

(c) If $(E_i, \pi_i)_{i \in A}$ is a bounded representation of E and B is a bounded subset of E_j then there exists a bounded subset of E, C, such that $\pi_j(C) = B$. If $f \in \mathscr{H}_S(U; F)$ then for each $x_o \in U$ there exists $\alpha(x_o) > 0$ such that $f(x_o + \alpha(x_o)\, C)$ is a bounded subset of F. If $x_o \in U_j$ we see that $f_j(\pi_j(x_o) + \alpha(x_o)\, B)$ is a bounded subset of F and $f_j \in \mathscr{H}_S(\pi_j(U_j); F)$. The remaining assumption in (c) shows that $f_j \in \mathscr{H}_{LB}(\pi_j(U_j); F)$.

(d) If $f \in \mathscr{H}_\omega(U; F)$, $p \in U_j$ and g is a $\mathbf{C}$-valued function defined and holomorphic on some neighbourhood of $f(p)$ then $g \circ f = g \circ f_j \circ \pi_j$ is holomorphic on some neighbourhood of p. If π_j is an open mapping then $g \circ f_j$ is holomorphic on some neighbourhood of $\pi_j(p)$. Hence $f_j \in \mathscr{H}_\omega(U; F)$. The remaining assumption in (d) shows that $f_j \in \mathscr{H}_{LB}(\pi_j(U_j); F)$.

(e) Use the same method as in (d).

Thus we have shown that $f_j \in \mathscr{H}_{LB}(\pi_j(U_j); F)$ in all cases and hence $f\big|_{U_j} \in \mathscr{H}_{LB}(U_j; F)$. Since

$$U = \bigcup_j \{ U_j ; \ \exists\ W_j \text{ a neighbourhood of } 0 \text{ in } E_j \text{ and } \pi_j^{-1}(W_j) \subset \pi_i^{-1}(W) \}$$

we have completed the proof.

(^{18}) $U \subset E = \lim_{i \in A} (E_i, \pi_i)$ is E_j-open if there exists an open subset of E_j, V, such that $(\pi_j)^{-1}(V) = U$. Hence U is E_j-open if and only if $\pi_j(U)$ is an open subset of E_j and $U = \pi_j^{-1}(\pi_j(U))$.

It is possible to prove analogous results for continuously holomorphic functions by imitating the proof of Theorem 5.3. However, in this case, we must assume that there exists a continuous norm on F in order to construct f_j and we must assume that $(E_i, \pi_i)_{i \in A}$ is an open surjective representation of E in order to get f_j continuous at 0.

The following result may be proved in this manner.

PROPOSITION 5.4. — Let $(E_i, \pi_i)_{i \in A}$ denote an open surjective representation of E and let F denote a locally convex space on which there exists a continuous norm. If each E_i is a weak F-Zorn space then E is also a weak F-Zorn space.

On the other hand we may use Proposition 5.1 (a) and Theorem 5.3 directly to obtain results about weak Zorn spaces.

PROPOSITION 5.5. — If $(E_i, \pi_i)_{i \in A}$ is a surjective representation of E and E_i is a weak F-Zorn space for any locally convex space F and any $i \in A$ then E is a weak F-Zorn space for any locally space F.

Proof. — By Proposition 5.1 (a), we way assume that F is a normed linear space. The locally bounded and the continuous holomorphic mappings into F coincide and hence we may apply theorem 5.3 (a) to complete the proof.

EXAMPLE 5.6.

(a) $\prod_\omega \mathbf{C}$, ω arbitrary, is an open surjective limit of finite dimensional (and hence locally compact) vector spaces. Since each finite dimensional vector space is a strong and weak F-Zorn space for any locally convex space F it follows that $\prod_\omega \mathbf{C}$ is a weak (use Proposition 5.5) and strong (use Theorem 5.3 (a)) F-Zorn space for any locally convex space F.

(b) HIRSCHOWITZ [18] gives an example of a locally convex nuclear space which is not a $\mathbf{C}$-Zorn space. Since every nuclear space is the surjective limit of inner product spaces we may conclude that there exist inner product spaces which are not $\mathbf{C}$-Zorn spaces. NACHBIN [32] gives an example of a holomorphic function on $\mathscr{H}(\mathbf{C})$ which is not uniformly holomorphic. If $(p_n)_{n=1}^\infty$ is an increasing family of seminorms on $\mathscr{H}(\mathbf{C})$ which defines its topology it then follows that $(\mathscr{H}(\mathbf{C}), p_n)$ is not a $\mathbf{C}$-Zorn space for all n sufficiently large.

(c) The product of **C**-Zorn spaces is not necessarily a **C**-Zorn space [18] however if $E = \prod_{i \in A} E_i$ and $\prod_{i \in A_1} E_i$ is a **C**-Zorn space for each finite subset A_1 of A then E is a **C**-Zorn space. In particular, an arbitrary product of Fréchet spaces is a weak F-Zorn space for any space F and an arbitrary product of $\mathscr{DFS}$ spaces is a weak F-Zorn space for any space F.

(d) Fréchet spaces are weak F-Zorn spaces for any locally convex space F [34]. Thus $\mathscr{C}(X)$ is a weak F-Zorn space for any completely regular space X.

(e) If $\mathscr{C}$ is a collection of locally convex **C**-Zorn spaces stable under the formation of finite products then $E \in \mathscr{C}$ is a strong F-Zorn space for any $F \in D \mathscr{C}$. In particular, we note that the strong dual of the strict inductive limit of Fréchet-Schwartz spaces is a strong F-Zorn space for any $\mathscr{FS}$ space F and the surjective limit of Fréchet spaces is a strong F-Zorn space for any $D \mathscr{F}$ space F.

(f) Let E denote a Fréchet space which does not admit a continuous norm. There exists a continuous open linear mapping of E'_β onto $\sum_{n=1}^\infty \mathbf{C}$ (*see* section 7) hence $\sum_{n=1}^\infty \mathbf{C}$ is isomorphic to a quotient space of E'_β. Since $\mathscr{H}(\mathbf{C}) \times \sum_{n=1}^\infty \mathbf{C}$ [18] and $\mathscr{C}[0, 1] \times \sum_{n=1}^\infty \mathbf{C}$ ([5] and Proposition 5.2 (c)) are not **C**-Zorn spaces it follows by Proposition 5.1 (b) that $\mathscr{H}(\mathbf{C}) \times E'_\beta$ and $\mathscr{C}[0, 1] \times E'_\beta$ are not **C**-Zorn spaces. Since $\mathscr{C}[0, 1]$ is a quotient space of l_1 a further application of Proposition 5.1 (b) shows that $l_1 \times \sum_{n=1}^\infty \mathbf{C}$ is not a **C**-Zorn space.

(g) Let $E = \prod_{i \in \omega} E_i$ where each E_i is a metrizable locally convex space and suppose F is an arbitrary locally convex space, then

$$\mathscr{H}(U; F) = \mathscr{H}(U; F_\omega)$$

for any domain spread over E (there exists an E satisfying the conditions of this example which is neither a **C**-Zorn space nor a k-space).

(h) Let E denote the strong dual of the strict inductive limit of Fréchet-Montel spaces. If U is a domain spread over E and F is any locally convex space then the set of points of continuity of hypoanalytic and scalarly holomorphic F-valued functions on U is open and closed. The proof consists in applying Proposition 3.3 (b), the result of Example 2.10 and Theorem 5.3 (b).

We complete this section by using the method of Theorem 5.3 to prove a number of results.

PROPOSITION 5.7. — *Let F denote a locally convex space on which there exists a continuous norm and let* $E = \lim_{i \in A} (E_i, \pi_i)$ *denote an open surjective limit in which* $\mathscr{H}_{LB}(U_i; F) = \mathscr{H}(U_i; F)$ *for all domains spread over* E_i, $i \in A$, *then* $\mathscr{H}_{LB}(U; F) = \mathscr{H}(U; F)$ *for all domains U spread over E.*

Proof. — Let $f \in \mathscr{H}(U; F)$ where U is an open subset of E. Since the definition of $\mathscr{H}_{LB}(U; F)$ is purely local it suffices to show f is locally bounded at any preassigned point $x_o \in U$. Without loss of generality we may assume $x_o = 0$ and $f(x_o) = 0$. Let p denote a continuous norm on F. Since f is continuous at 0 there exists an element i of A and W a convex balanced neighbourhood of 0 in E_i such that $f^{-1}(V_p) \supset \pi_i^{-1}(W)$ where V_p denotes the open unit p-ball of F.

We define $\tilde{f}: W \to F$ by the equation $\tilde{f}(x) = f(y)$ if $\pi_i(y) = x$. Now if $\pi_i(y) = \pi_i(z) = x \in W$ and $f(y) \neq f(z)$ then $p(f(y)-f(z)) \neq 0$. Hence there exists a p-continuous linear functional on F, φ, such that $\varphi(f(y)) \neq \varphi(f(z))$. The function $\varphi \circ f$ is continuous and holomorphic on U. Since φ is p-continuous there exists a positive number ε such that $(\varphi \circ f)^{-1}(Z, |Z| < \varepsilon) \supset \pi_i^{-1}(W)$. By Liouville's theorem this implies that $f(x) = f(y)$. Hence $\tilde{f}$ is well defined. Since E is an open surjective limit $\tilde{f}$ is continuous on W and hence, by hypothesis, it is locally bounded on W. Hence f is locally bounded at zero and this completes the proof.

REMARK. — The requirement, in the above proposition, that there exists a continuous norm on F is non trivial. Indeed any continuously holomorphic function from $\mathbf{C}^n$, n a positive integer, into an arbitrary locally convex space is locally bounded. $\prod_{n=1}^{\infty} \mathbf{C}$ is an open surjective limit of finite dimensional spaces and hence every continuously holomorphic function from $\prod_{n=1}^{\infty} \mathbf{C}$ into a locally convex space on which there exists a continuous norm is locally bounded. However, since $\prod_{n=1}^{\infty} \mathbf{C}$ is not a normed linear space the identity mapping from $\prod_{n=1}^{\infty} \mathbf{C}$ into itself is continuous but is not locally bounded. We now prove a result concerning local factorization of holomorphic functions.

PROPOSITION 5.8. — *A connected open subset, U, of the locally convex space E has the strong local F-factorization property with respect to the surjective representation of E,* $(E_i, \pi_i)_{i \in A}$, *if either of the following conditions are satisfied.*

(1) $(E_i, \pi_i)_{i \in A}$ *is an open surjective representation of E.*

(2) $(E_i, \pi_i)_{i \in A}$ *is a directed surjective representation of* E, U *is* E_i-*open for some* $i \in A$ *and each* E_i *is a strong F-Zorn space.*

Proof. — We may assume that 0 lies in U. In both cases we may choose an $i \in A$ and W a convex balanced open subset of E_i such that $\pi_i^{-1}(W) \subset U$, $f(\pi_i^{-1}(W))$ is a bounded subset of F and $\pi_i(U)$ is an open subset of E_i. We show that f factors through E_i at each point of U. We may assume that U is a convex balanced open subset of E. Now if $x \in U$, $y \in E$ and $\pi_i(y) = 0$ then there exists $\lambda_0 > 0$ such that

$$\{\lambda x + \beta y \mid |\lambda| \leqslant \lambda_0, \beta \in \mathbf{C}\} \subset W.$$

Hence $\varphi \circ f|_{\{\lambda x + \beta y \mid |\lambda| \leqslant \lambda_0, \beta \in \mathbf{C}\}}$ is a bounded holomorphic function for all $\varphi \in F'$. This implies that $f(\alpha x + \beta y) = f(\alpha x)$ for all α, β such that $\alpha x + \beta y \in W$ and $\alpha x \in W$. Hence we may define $\tilde{f}$ on $\pi_i(U)$ by the formula $\tilde{f}(\theta) = f(z)$ if $\pi_i(z) = \theta$. $\tilde{f} \in \mathscr{H}_G(\pi_i(U); F)$ and $f = \tilde{f} \circ \pi_i$.

If $(E_i, \pi_i)_{i \in A}$ is an open surjective representation of E then $\tilde{f} \in \mathscr{H}_{LB}(\pi_i(U); F)$. $\tilde{f}(W)$ is a bounded subset of F and hence if E_i is a strong F-Zorn space then f is locally bounded on U. This completes the proof.

The corresponding result for continuously holomorphic functions may be proved in an analogous fashion. We obtain the following result.

PROPOSITION 5.9. — *A connected open subset of a locally convex space* E *has the weak local F-factorization property with respect to the open surjective representation of* E, $(E_i, \pi_i)_{i \in A}$ *if there exists a continuous norm on* F.

6. Holomorphically complete and paracomplete locally convex spaces [19]

Let $f \in \mathscr{H}_{LB}(E; F)$ where E and F are locally convex spaces. Let

$$\tilde{\Omega}_f = \{x \in \hat{E} \mid \exists \ V_x \text{ a neighbourhood of } x \text{ in } \hat{E}$$

such that $f(V_x \cap E)$ is a bounded subset of $F\}$.

By using Taylor series expansions of f, it is possible to show that $\tilde{\Omega}_f$ is an open subset of $\hat{E}$ and that there exists an $\tilde{f}$ in $\mathscr{H}_{LB}(\tilde{\Omega}_f ; \hat{F})$ such that $\tilde{f}|_E = f$. We let $\Omega_f = (\tilde{f})^{-1}(F)$ and call Ω_f the natural strong domain of

---

[19] A number of the results in this section were announced in [11] and [14].

existence of f (note Ω_f is not necessarily an open subset of $\hat{E}$). By the same procedure as that used in [10], it is possible to show that

$$\bigcap_{f \in \mathscr{H}_{LB}(E, F)} \Omega_f$$

is a vector subspace of $\hat{E}$ containing E and is in fact the largest subspace of $\hat{E}$ to which all locally bounded holomorphic functions on E can be continued as F-valued locally bounded holomorphic functions.

DEFINITION 6.1.

(a) $\hat{E}_d [F] = \bigcap_{f \in \mathscr{H}_{LB}(E; F)} \Omega_f$ is called the strong F-holomorphic completion of E.

(b) If $\mathscr{C}$ is a collection of locally convex spaces $\hat{E}_d [\mathscr{C}] = \bigcap_{F \in \mathscr{C}} \hat{E}_d [F]$ is the strong $\mathscr{C}$-holomorphic completion of E. E is called a strong $\mathscr{C}$-holomorphically complete locally convex space if $\hat{E}_d [\mathscr{C}] = E$.

We now use the concept of strong holomorphic completion to define weak holomorphic completion.

Let $f \in \mathscr{H}(E; F)$ and let $q \in \mathscr{CS}(F)$. Let

$$\tilde{\theta}_{f,q} = \big\{ x \in \hat{E} \,\big|\, \exists\, V_x \text{ a neighbourhood of } x$$

$$\text{in } \hat{E} \text{ and } \pi_q \circ f (V_x \cap E) \text{ is a bounded subset of } F_{q/q^{-1}(0)} \big\}$$

$\tilde{\theta}_{f,q}$ is an open subset of $\hat{E}$. We let $\tilde{\theta}_f = \bigcap_{q \in \mathscr{CS}(F)} \tilde{\theta}_{f,q}$. By construction there exists $\tilde{f} \in \mathscr{H}(\tilde{\theta}_f; \hat{F})$ such that $\tilde{f}|_E = f$. We regard $\theta_f = (\tilde{f})^{-1}(F)$ as the natural domain of existence of f with values in F.

Again we find that $\bigcap_{f \in \mathscr{H}(E; F)} \theta_f$ is a vector subspace of $\hat{E}$ and is in fact the largest subspace of $\hat{E}$ to which all continuously holomorphic functions from E to F can be extended to continuously F-valued holomorphic functions.

If $\mathscr{C}$ is a collection of locally convex spaces

$$\hat{E}_d (\mathscr{C}) = \bigcap_{f \in \mathscr{H}(E; F); F \in \mathscr{C}} \theta_f$$

is called the weak $\mathscr{C}$-holomorphic completion of E. If $\hat{E}_d (\mathscr{C}) = E$ then we say that E is a weak $\mathscr{C}$-holomorphically complete locally convex space.

By allowing the collection $\mathscr{C}$ to become too large we do not achieve much more than an existence theorem (e. g. $\hat{E}_d (E) = E$ for any locally convex

space E) so we begin by defining a concept of completeness which is suitable for our purposes ([10], [11] and [12]). This is the concept of very strongly complete locally convex space which we may now interpret as a particular case of complete surjective limit (Proposition 6.3).

DEFINITION 6.2.

(a) *A net*, $(x_\beta)_{\beta \in B}$, *of elements of a locally convex space*, E, *is very strongly Cauchy if* $\{ \lambda_{\beta, \alpha} (x_\beta - x_\alpha) \}_{(\beta, \alpha) \in B \times B}$ *converges to zero for any set of scalars* $(\lambda_{\beta, \alpha})_{(\beta, \alpha) \in B \times B}$.

(b) E_*, *the very strong completion of* E, *is the subspace of* $\hat{E}$ *consisting of all limit points of very strongly Cauchy nets.*

(c) *If* $E = E_*$ *then* E *is said to be very strongly complete.*

It is immediate that a net $(x_\beta)_{\beta \in B}$ is very strongly Cauchy if and only if for each $p \in \mathscr{CS} (E)$ there exists $\beta_0 \in B$ such that $p (x_\beta - x_\alpha) = 0$ for all $\beta, \alpha \geqslant \beta_0$. E_* is a very strongly complete subspace of $\hat{E}$ which contains E.

PROPOSITION 6.3. — E *is very strongly complete if and only if it has a complete surjective representation by normed linear spaces.*

Proof. — Let $\theta = (E_i, \pi_i)_{i \in A}$ denote a surjective representation of E by normed linear spaces and let E_θ denote the θ-completion of E. Since each locally convex space can be representend by normed linear spaces we may complete the proof by showing $E_\theta = E_*$. Each E_i defines a continuous seminorm on E in the following manner $\tilde{p}_i (x) = p_i (\pi_i (x))$ where p_i is a norm on E_i which defines the topology of E_i. It also follows by the definition of surjective limit that $(\tilde{p}_i)_{i \in A}$ is a directed family of seminorms on E which defines the topology. Now if $x \in E_\theta = \bigcap_{i \in A} (\hat{\pi}_i)^{-1} (E_i)$ then for each $i \in A$ there exists $x_i \in E$ such that

$$\hat{\pi}_i(x) = \pi_i(x_i), \text{ i. e. } \hat{\pi}_i(x_i - x) = 0.$$

Hence if we let q_i denote the extension of p_i to $\hat{E}$ it follows that $q_i (x_i - x) = 0$. Hence $(x_i)_{i \in A}$ is a very strongly Cauchy net in $\hat{E}$ which implies that $E_\theta \subset E_*$.

Conversely, if $x \in E_*$ and $(x_\beta)_{\beta \in B}$ is a very strongly Cauchy net such that $x_\beta \to x$ as $\beta \to \infty$ then for any $i \in A$ it follows that $\pi_i (x_\beta - x_\alpha) = 0$ for all β, α sufficiently large. Hence $\hat{\pi}_i (x_\beta - x) = 0$ for all β sufficiently large and $x \in (\hat{\pi}_i)^{-1} (E_i)$ for all i. Thus $E_* \subset E_\theta$ and this completes the proof.

PROPOSITION 6.4. — *Let* $(F_i, \pi_i)_{i \in A}$ *denote a complete surjective repre-sentation of the locally convex space* F. *If* $E \subset E_1 \subset \hat{E}$ *are locally convex spaces and* $\hat{E}_d (F_i) \supset E_1$ *for all* i *then* $\hat{E}_d (F) \supset E_1$.

Proof. — Let $f \in \mathscr{H} (E; F)$. Now $\pi_i \circ f \in \mathscr{H} (E; F_i)$ and hence there exists $f_i \in \mathscr{H} (E_1; F_i)$ such that $f_i |_E = \pi_i \circ f$. If $(x_\alpha)_{\alpha \in B} \in E \to x \in E_1$ as $\alpha \to \infty$ then $\pi_i (f(x_\alpha))$ converges for all $i \in A$ as $\alpha \to \infty$. Hence $(f(x_\alpha))_{\alpha \in B}$ is a Cauchy net in $\hat{F}$ and $f(x_\alpha) \to \eta \in \hat{F}$ as $\alpha \to \infty$. Thus $\pi_i \circ f(x_\alpha) \to \pi_i (\eta) \in F_i$ as $\alpha \to \infty$ and

$$\eta \in \bigcap_{i \in A} (\hat{\pi}_i)^{-1} (F_i) = F$$

since F is a complete surjective limit. Hence there exists $\tilde{f} \in \mathscr{H} (E_1; F)$ such that $\tilde{f} |_E = f$ and this completes the proof.

PROPOSITION 6.5. — *Let* $(E_i, \pi_i)_{i \in A}$ *denote a surjective representation of* E. *The* $(E_i, \pi_i)_{i \in A}$-*completion of* E *lies in the strong (resp. weak)* F-*holomorphic completion of* E *for any locally convex space* F *(resp. any very strongly complete locally convex space).*

Proof. — Let E_C denote the $(E_i, \pi_i)_{i \in A}$-completion of E. We show $E_C \subset \hat{E}_d [F]$ and the remainder is proved by using this result, the canonical normed surjective representation of E and Proposition 6.4.

Let $f \in \mathscr{H}_{LB} (E; F)$. By the methods of the proceeding section we may find an $i \in A$ and $f_i \in \mathscr{H}_G (E_i; F)$ such that $f = f_i \circ \pi_i$. We let $\hat{\pi}_i$ denote the extension of π_i to a continuous linear mapping from E_C onto E_i and we define $\tilde{f}$ on E_C by the formula $\tilde{f} = f_i \circ \hat{\pi}_i$. Now if $x \in E_C$ then we can find $y \in E$ such that $\hat{\pi}_i (x) = \pi_i (y)$. Choose V a neighbourhood of 0 in E such that $f(y + V)$ is a bounded subset of F.

For any $z \in V$ we have

$$
\begin{aligned}
f(y + z) &= f_i \circ \pi_i (y + z) = f_i (\pi_i (y) + \pi_i (z)) \\
&= f_i (\hat{\pi}_i (x) + \pi_i (z)) = f_i \circ \hat{\pi}_i (x + z) \\
&= \tilde{f}(x + z).
\end{aligned}
$$

Hence f is locally bounded at x and this completes the proof.

COROLLARY 1. — *If* F *is a locally convex space (resp. a very strongly complete locally convex space) then* $E_* \subset \hat{E}_d [F]$ *(resp.* $E_* \subset \hat{E}_d (F)$*).*

COROLLARY 2. — *If $E = \lim_{i \in A} (E_i, \pi_i)$ and each E_i is a complete locally convex space then $\hat{E}_d [F] \overset{\longleftarrow}{=} \hat{E}$ (resp. $\hat{E} = \hat{E}_d (F)$) for any locally convex space F (resp. any very strongly complete locally convex space).*

We now give a number of examples in which the opposite inclusion is true.

PROPOSITION 6.6. — *Let E and F denote locally convex spaces and let $(E_i, \pi_i)_{i \in A}$ denote a surjective representation of E where each E_i is a strong (resp. weak) F-holomorphically complete locally convex space. The $(E_i, \pi_i)_{i \in A}$-completion of E is a strong (resp. weak) F-holomorphically complete locally convex space for any locally convex space (resp. any very strongly complete locally convex space) F if any of the following conditions are satisfied;*

(a) *$(E_i, \pi_i)_{i \in A}$ is an open surjective representation of E.*

(b) *Each E_i is a C-Zorn space.*

(c) *Each E_i is a C-ω-space.*

Proof. — By Proposition 6.5 we must show that $\hat{\pi}_i (\hat{E}_d [F]) \subset E_i$ for all $i \in A$. Let $x \in \hat{E}_d [F]$ (resp. $\hat{E}_d (F)$) and let $\tilde{E}$ denote the subspace of $\hat{E}$ spanned by E and x. Now if $f \in \mathcal{H}_{LB} (E_i; F)$ (resp. $\mathcal{H} (E_i; F)$) then there exists $\tilde{f} \in \mathcal{H}_{LB} (\tilde{E}; F)$ (resp. $\mathcal{H} (\tilde{E}; F)$) such that $\tilde{f}|_E = f \circ \pi_i$. We now obtain the following commutative diagram :
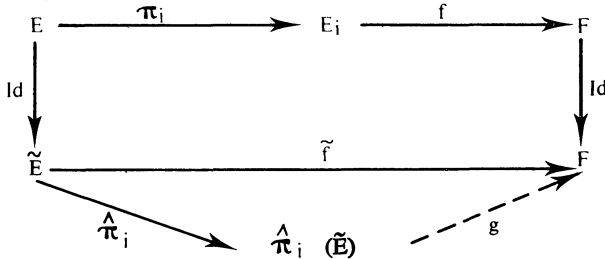
where $g(y) = \tilde{f}(\tilde{x})$ if $y = \hat{\pi}_i(\tilde{x})$. If $x_1, x_2 \in \tilde{E}$ and $\hat{\pi}_i(x_1) = \hat{\pi}_i(x_2)$ then $\hat{\pi}_i(x_1 - x_2) = 0$ and hence $x_1 - x_2 \in E$. Now if $y \in E$ then

$$\tilde{f}(y + x_1 - x_2) = f \circ \pi_i (y + x_1 - x_2)$$
$$= f \circ \pi_i (y) = \tilde{f}(y).$$

Hence if $y \in E$, $y_\alpha \in \tilde{E} \to y$ as $\alpha \to \infty$ then

$$\tilde{f}(y + x_1 - x_2) = \lim_{\alpha \to \infty} f(y_\alpha + x_1 - x_2) = \lim_{\alpha \to \infty} f(y_\alpha) = \tilde{f}(y).$$

In particular letting $y = x_2$ it follows that $\tilde{f}(x_1) = \tilde{f}(x_2)$ and hence g is well defined. g is a G-holomorphic function and $\tilde{f} = g \circ \hat{\pi}_i$.

(a) We complete the proof, in this case, by showing that $g \in \mathcal{H}_{LB}(\hat{\pi}_i(\tilde{E}); F)$ (resp. $\mathcal{H}(\hat{\pi}_i(\tilde{E}); F)$). If $p \in \mathcal{CS}(F)$ then $\tilde{f} \in \mathcal{H}_{LB}(\tilde{E}; F_p)$ and thus there exists a neighbourhood of 0 in E, V, such that

$$\sup_{y \in V; n = 0, 1, 2, \ldots} p\left(\frac{\hat{d}^n f(0)}{n!}(2x + 2y)\right) < \infty.$$

Hence

$$\sup_{y \in \overline{\pi_i(V)}, n = 0, 1, 2, \ldots} p\left(\frac{\hat{d}^n g(0)}{n!}(2\hat{\pi}_i(x) + 2y)\right) < \infty,$$

where $\overline{\pi_i(V)}$ is the closure of $\pi_i(V)$ in $\hat{\pi}_i(\tilde{E})$. (Note that any continuous polynomial from E into F can be extended to a continuous polynomial from $\hat{E}$ into $\hat{F}$.) Using the above inequalities it follows that

$$\sup_{y \in \overline{\pi_i(V)}; n = 0, 1, 2, \ldots} p\left(\sum_{n=0}^\infty \frac{\hat{d}^n g(0)}{n!}(\hat{\pi}_i(x) + y)\right) < \infty.$$

Since π_i is an open mapping it follows that $\overline{\pi_i(V)}$ is a neighbourhood of 0 in $\hat{\pi}_i(\tilde{E})$. If $f \in \mathcal{H}(E; F)$ then by letting p range over $\mathcal{CS}(F)$ it follows that $g \in \mathcal{H}(\hat{\pi}_i(\tilde{E}); F)$. If $f \in \mathcal{H}_{LB}(E; F)$ then we can choose V independent of p and it follows that $g \in \mathcal{H}_{LB}(\hat{\pi}_i(\tilde{E}); F)$.

(b) An application of proposition 5.1 (c) shows that if E_i is a C-Zorn space then

$$(\hat{E}_i)_d(\mathbf{C}) = \left\{ Z \in \hat{E}_i; \sum_{n=0}^\infty \frac{\hat{d}^n f(0)}{n!}(Z) \right.$$

$$\left. \text{converges for all } f = \sum_{n=0}^\infty \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(E_i) \right\}.$$

Using this fact and the same construction as in (a) we see that $\hat{\pi}_i(x) \in (\hat{E}_i)_d(\mathbf{C}) = E_i$. This completes the proof for (b).

(c) Now suppose that each E_i is a C-ω-space.

Then $\mathscr{H}(E_i) = \bigcup_{m \in \mathscr{M}(E_i)} {}^t\pi_m (\mathscr{H}(\pi_m(E_i))$ (π_m is the canonical mapping from E_i onto $(E_i)_m$) and

$$\left| \sum_{n=0}^{\infty} \frac{\hat{d}^n h(0)}{n!} (\pi_m(\hat{\pi}_i(x))) \right| < \infty \quad \text{for all } h \in \mathscr{H}(\pi_m(E_i)).$$

Since $\pi_m(E_i)$ is metrizable it follows (*see* [10]) that

$$\pi_m(\hat{\pi}_i(x)) \in ((\hat{E}_i)_m)_d(\mathbf{C})$$

for all $m \in \mathscr{M}(E_i)$ and hence $\hat{\pi}_i(x) \in (\hat{E}_i)_d(\mathbf{C}) = E_i$. This completes the proof of (*c*).

REMARK. — If

$$E = F = \{ (U_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbf{C}, \; U_n = 0 \text{ for all } n \text{ sufficiently large} \}$$

then $\prod_{n=1}^{\infty} \mathbf{C}$ is the $(\mathbf{C}^n, \pi_n)_{n=1}^{\infty}$-completion of E and $(\hat{\mathbf{C}^n})_d(F) = \mathbf{C}^n$ for all n. However the identity mapping from E into F cannot be extended to a continuously holomorphic function from $\prod_{n=1}^{\infty} \mathbf{C}$ into F. We thus see that it is necessary, in the case of continuously holomorphic functions, to place some conditions on the range space F in Proposition 6.7.

EXAMPLE 6.7. — If E is a locally convex space of countable algebraic dimension then $\hat{E}_d[F] = E_*$ for any locally convex space F and $\hat{E}_d(F) = E_*$ for any very strongly complete locally convex space F.

Proof. — Now $E = \lim_{p \in \mathscr{CS}(E)} (E_p, \pi_p)$ and each E_p is a normed linear space of countable dimension. In [10], we showed that $(\hat{E}_p)_d(\mathbf{C}) = E_p$ and since $\mathbf{C}$ can be isomorphically embedded as a closed subspace in any locally convex space of positive dimension it follows that

$$(\hat{E}_p)_d[F] = (\hat{E}_p)_d(F) = E_p$$

for any locally convex space F. Propositions 6.5 and 6.6 imply that the $(E_p, \pi_p)_{p \in \mathscr{CS}(E)}$-completion of E is the strong (resp. weak) F-holomorphic completion of E for any locally convex space F (resp. for any very strongly complete locally convex space F). An application of Proposition 6.3 now completes the proof.

EXAMPLE 6.8. — $\widehat{\mathscr{C}(X)}_d[E] = \widehat{\mathscr{C}(X)} = \mathscr{C}(X)_*$ for any locally convex space F and $\widehat{\mathscr{C}(X)}_d(F) = \widehat{\mathscr{C}(X)}$ for any very strongly complete locally convex space F.

The characterization of holomorphically complete locally convex spaces may be looked upon as the problem of finding the largest domain of definition of a family of holomorphic functions. We now look at the dual problem of finding the smallest range ([20]). Let U and V denote respectively a connected open subset of E, a locally convex space, and a connected domain spread over E and let F denote another locally convex space. *We say V is a weak (resp. strong) F-holomorphic extension of U if U can be identified with an open subset of V and each $f \in \mathscr{H}(U; F)$ (resp. $\mathscr{H}_{LB}(U; F)$) can be extended (uniquely) to an element of $\mathscr{H}(V; F)$ (resp. $\mathscr{H}_{LB}(V; F)$). (U, V) is then called a weak (resp. strong) F-extension pair over E.*

DEFINITION 6.9. — *F, a locally convex space, is a weak (resp. strong) E-paracomplete locally convex space if each $\mathbf{C}$-extension pair over E is also a weak (resp. strong) F-extension pair.*

Before investigating paracomplete spaces we look at G-holomorphic extensions.

DEFINITION 6.10.

(a) *A locally convex space F is R-complete if each $f \in \mathscr{H}(U; F)$, U an open subset of an arbitrary locally convex space, can be extended to an element of $\mathscr{H}_G(V; F)$ for any $\mathbf{C}$-extension pair (U, V).*

(b) *The R-completion of F, $\hat{F}_R$, is the intersection of all R-complete subspaces of $\hat{F}$ which contain F.*

PROPOSITION 6.11. — *A locally convex space F is R-complete if any of the following conditions holds :*

(1) *F is sequentially complete.*

(2) *F is a $\mathbf{C}$-holomorphically complete $\mathbf{C}$-ω-space.*

(3) *F is a $\mathbf{C}$-holomorphically complete $\mathbf{C}$-Zorn space.*

Proof. — If F is sequentially complete the proof follows immediately from results in [2] and [23]. Now suppose F is a $\mathbf{C}$-holomorphically complete locally convex space and that (U, V) is a $\mathbf{C}$-extension pair spread over some locally convex space. If $f \in \mathscr{H}(U; F)$ then there exists, since $\hat{F}$ is sequentially complete and hence R-complete, an $\tilde{f} \in \mathscr{H}_G(V; \hat{F})$ such that

$\tilde{f}|_U = f$.   To complete the proof we must show that $\tilde{f}(V) \subset F$.   Now $\varphi \circ \tilde{f} \in \mathscr{H}(V)$ for all $\varphi \in F'$ since V is a C-extension of U.   Hence the restriction of $\tilde{f}$ to finite dimensional submanifolds of V is continuous.   Since E is locally convex and V is connected it suffices to show the following to complete the proof; if $\varphi : [\,0, 1\,] \to V$ is a continuous curve such that $\varphi(0) = x \in U$, $\varphi(1) = y \in V$ and $\varphi([\,0, 1\,])$ is contained in a finite dimensional complex submanifold of V then

$$M = \sup\{\delta, \tilde{f}\circ\varphi([0, t]) \subset F \text{ all } t < \delta\} = 1.$$

Since $x \in U$ it follows that $0 < M \leqslant 1$.   If $M \neq 1$ then there exists $(t_n)_{n=1}^{\infty}$, $0 < t_n < 1$, $t_n \to t$ as $n \to \infty$, such that $\tilde{f}(\varphi(t_n)) \in F$ and $\tilde{f}(\varphi(t)) \in \hat{F}\backslash F$.   Since $\varphi([\,0, 1\,])$ is contained in a finite dimensional complex submanifold of V it follows that

$$\tilde{f}(\varphi(t_n)) \to \tilde{f}(\varphi(t))$$

as $n \to \infty$.   Now if F is a C-ω-space then we can find g in $\mathscr{H}(F)$ such that $g(\tilde{f}(\varphi(t_n))) \to \infty$ as $n \to \infty$ ([\,10\,], Proposition 5) and this contradicts the fact that (U, V) is a C-extension pair and completes the proof with condition (2).   Now if F is a C-Zorn space and

$$\sup_n |g \circ \tilde{f} \circ \varphi(t_n)| < \infty \quad \text{for all } g \in \mathscr{H}(F)$$

then

$$p(g) = \sup_n |g(\tilde{f}(\varphi(t_n)))| \in \mathscr{CS}(\mathscr{H}(F), \mathscr{T}_\delta) \quad [12].$$

Hence

$$\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n g(0)}{n!}\right) < \infty \quad \text{for all } g \in \mathscr{H}(F)$$

and this implies that

$$\sum_{n=0}^{\infty} \widetilde{\frac{\hat{d}^n g(0)}{n!}}(\tilde{f}(\varphi(t)) < \infty \quad \text{for all } g \in \mathscr{H}(F)$$

where $\widetilde{\hat{d}^n g(0)}/n!$ denotes the unique extension of $\hat{d}^n g(0)/n!$ to a continuous polynomial from $\hat{E}$ into C.   The proof of proposition 6.6 now implies that $\tilde{f}(\varphi(t)) \in \hat{F}_d(\mathbf{C}) = F$ and this completes the proof with condition (3).

Proposition 6.11 implies that the R-completion of a locally convex space, F, is an R-complete subspace of $\hat{F}$ which contains F.   Since the

intersection of R-complete spaces is R-complete, we may easily construct examples of R-complete spaces which do not satisfy any of the conditions (1), (2) or (3) of Proposition 6.11.

PROPOSITION 6.12. — *Let* $(F_i, \pi_i)_{i \in A}$ *denote a complete surjective representation of the locally convex space* F. *If each* F_i *is a weak* E-*paracomplete space* (*resp. an* R-*complete space*) *then the same is true of* F.

Proof. — Let (U, V) denote a C-extension over E (resp. over an arbitrary locally convex space) and suppose $f \in \mathscr{H}(U; F)$. Let $\tilde{f} \in \mathscr{H}_G(V; \hat{F}_R)$ denote the G-holomorphic extension of f to V. For each $i \in A$ there is an f_i in $\mathscr{H}_G(V; \hat{F}_i)$ such that the following diagram commutes
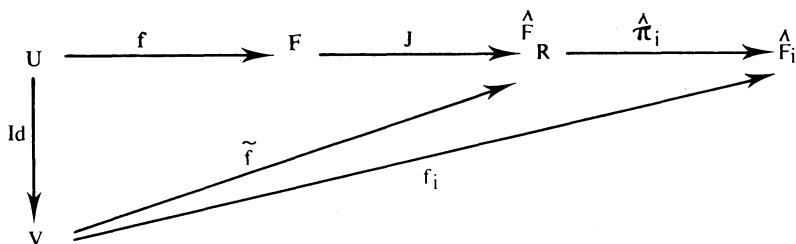
where J is the inclusion of F in $\hat{F}_R$ and $\hat{\pi}_i$ is the unique extension of π_i to a continuous linear mapping from $\hat{F}_R$ into $\hat{F}_i$. The diagram is commutative since $\hat{\pi}_i \circ \tilde{f}|_U = f_i$ and analytic continuation is unique. Now F_i is a weak E-paracomplete space (resp. an R-complete space) and hence $f_i \in \mathscr{H}(V; F_i)$ (resp. $\mathscr{H}_G(V; F_i)$) for all $i \in A$. It then follows that $(\hat{\pi}_i)^{-1}(f_i(V)) \subset (\pi_i)^{-1}(F_i)$ for all i. Hence $\tilde{f}(V) \subset \bigcap_{i \in A}(\hat{\pi}_i)^{-1}(F_i) = F$ and we obtain the following commutative diagram
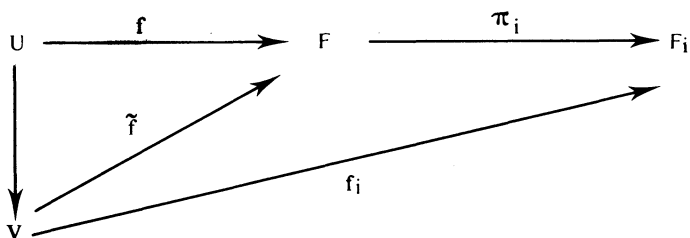
If F_i is R-complete for each $i \in A$ then F is also R-complete. If F_i is E-paracomplete then $\pi_i \circ \tilde{f} \in \mathscr{H}(V; F_i)$ for all $i \in A$ and $\tilde{f} \in \mathscr{H}(V; F)$. Hence F is E-paracomplete and this completes the proof of the proposition.

In the next proposition, we give a number of examples of strong E-para-complete locally convex spaces. Proposition 6.12 and 6.13 may be used to construct examples of weak E-paracomplete locally convex spaces.

PROPOSITION 6.13. — *Let E denote a locally convex space and let F denote an R-complete locally convex space. F is a strong E-paracomplete locally convex space if any of the following conditions hold.*

(a) *E is a strong F-Zorn space.*

(b) *E is an F-ω-space.*

(c) *$(\mathscr{H}(U), \mathscr{T}_0)$ is a barrelled locally convex space for any open subset U of E and F is arbitrary.*

(d) *$(E_i, \pi_i)_{i \in A}$ is a compact surjective representation of E, each E_i is a k-space and F is a separable normed linear space.*

(e) *$(E_i, \pi_i)_{i \in A}$ is a compact surjective representation of E, each E_i is a separable k-space and F is a normed linear space.*

(f) *$\mathscr{H}_{LB}(U; F) = \mathscr{H}(U; F_\omega)$ for all U open in E.*

Proof. — Let (U, V) be a **C**-extension pair and let $f \in \mathscr{H}_{LB}(U; F)$. Since F is R-complete there exists $\tilde{f} \in \mathscr{H}_G(V; F)$ such that $\tilde{f}|_U = f$. To complete the proof we must show that $\tilde{f} \in \mathscr{H}_{LB}(V; F)$ in all cases. Case (a) is trivial by the definition of strong F-Zorn spaces.

(b) By our hypothesis it suffices to prove the following; If U is a finitely open subset of a metrizable space $(E, p_n)_{n=1}^\infty$, F is a complete $D\mathscr{F}$ space and each $f \in \mathscr{H}_{LB}(U; F)$ can be extended to a G-holomorphic function, $\tilde{f}$, on a finitely open domain spread over E then $\tilde{f} \in \mathscr{H}_{LB}(V; F)$. Let $(B_n)_{n=1}^\infty$ denote a fundamental sequence of bounded subsets of F. For any countable cover of U, $\mathscr{V} = (V_n)_{n=1}^\infty$, we let $\mathscr{H}_{\mathscr{V}}(U)$ denote the subspace of $\mathscr{H}(U)$ consisting of all g such that

$$\sup \left\{ \xi \in V_m, P_m(y) \leqslant \frac{1}{m}; \left| \sum_{n=0}^\infty (\hat{d}^n g(\xi)/n!)(y) \right| < \infty \right\}$$

for all m.

$\mathscr{H}_{\mathscr{V}}(U)$, when endowed with its natural topology, is a Fréchet space and $\mathscr{H}(U)$ with the inductive limit topology, $\mathscr{T}_\delta$, generated by $\mathscr{H}_{\mathscr{V}}(U)$ as $\mathscr{V}$ ranges over all countable covers of U is a barrelled, bornological locally convex space. Let

$$W = \left\{ \xi \in V, g \in (\mathscr{H}(U), \mathscr{T}_\delta) \to \hat{d}^n g(\xi)(x) \in \mathbf{C} \right.$$
$$\left. \text{is continuous for all } x \in E \text{ and all } n \right\}.$$

Using Taylor series expansions about the points of W it follows that W is an open subset of V and hence $W = V$. For each n, let

$$W_n = \left\{ \xi \in U, \left| \varphi \left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(\xi)}{n!}(x) \right) \right| \leqslant 1 \text{ for all } \varphi \in B_n^0, p(x) \leqslant \frac{2}{n} \right\}.$$

Fix η in V and choose a positive integer N such that

$$\left| \tilde{h}(\eta) \right| \leqslant \sup_{m=1,\ldots,N} \sup_{\xi \in W_m, p_m(x) \leqslant 1/m} \left| \sum_{n=0}^{\infty} \frac{d^n h(\xi)}{n!}(x) \right|$$

for all $h \in \mathscr{H}(U)$ and $\tilde{h} \in \mathscr{H}(V)$ such that $\tilde{h}\big|_U = h$.

Hence $\left| \varphi \left(\hat{d}^n f(\eta)/n! \right)(x) \right| \leqslant 1$ whenever $\varphi \in B_N^0 \, p_N(x) \leqslant_N^0$, $1/N$ and n is arbitrary. This means that f is locally bounded at η and we have completed the proof in this case.

(c) Since $(\mathscr{H}(U), \mathscr{T}_0)$ is barrelled it follows that each point of V gives rise to a $\mathscr{T}_0$ continuous point evaluation on $\mathscr{H}(U)$. Hence if $\xi \in V$ then there exists a compact subset of U, K, such that

$$\left| h(\xi) \right| \leqslant \left\| h \right\|_K \quad \text{for all } h \in \mathscr{H}_G(V) \text{ such that } h\big|_U \in \mathscr{H}(U).$$

Now if $h \in \mathscr{H}_{LB}(U; F)$ then there exists a neighbourhood of 0 in E, W, such that $\bigcup_{x \in K, \, y \in W; \, n=0,1,2,\ldots} \left(\hat{d}^n f(x)/n! \right)(y) = B$ is a bounded subset of F. Hence $\bigcup_{y \in W; \, n=0,1,2,\ldots} \left(\hat{d}^n f(\xi)/n! \right)(y)$ lies in the closed convex balanced hull of B and f is locally bounded at ξ. This completes the proof in this case.

(d) Since (U, V) is a C-extension pair $f \in \mathscr{H}_\omega(V; F)$. By Proposition 3.3 (b), $f \in \mathscr{H}_{HY}(V; F)$ and an application of theorem 5.3 (b) completes the proof in this case.

(e) Choose $i \in A$ and W an open neighbourhood of x_0 in E_i such that $\pi_i^{-1}(W) \subset U$, and $f(\pi_i^{-1}(W))$ is a bounded subset of F. Now E_i is separable and hence $f(\pi_i^{-1}(W))$ is contained in a separable subspace of F. Since (U, V) is a C-extension pair we may use the Hahn-Banach theorem to show that $\tilde{f}(V)$ is contained in the same separable subspace of F. An application of (d) now completes the proof.

(f) Trivial.

Results related to the above may be found in [28], [33] and [41].

7. Hartogs' theorem ([21])

In this section, we extend theorem 5.3 (Zorn's Theorem) by replacing the local boundedness condition (theorem 7.12) and we extend Hartogs' theorem concerning separate holomorphicity (theorem 7.15). We begin by discussing very strongly and very weakly convergent sequences.

DEFINITION 7.1. — *A net of elements in a locally convex space* E, $(x_\alpha)_{\alpha \in A}$, *is very strongly convergent (resp. very weakly convergent) if* $(\lambda_\alpha x_\alpha)_{\alpha \in A}$ *converges to zero for any net of scalars* $(\lambda_\alpha)_{\alpha \in A}$ *(resp. for some net of non-zero scalars* $(\lambda_\alpha)_{\alpha \in A}$*).*

We note that $(x_\alpha)_{\alpha \in A}$ is a very strongly Cauchy net if and only if there exists an $x \in \hat{E}$, such that $(x_\alpha - x)_{\alpha \in A}$ is a very strongly convergent net. We shall primarily be concerned with very strongly convergent and very weakly convergent sequences in this section.

LEMMA 7.2. — *If* $(x_n)_{n=1}^\infty$ *is a very strongly convergent sequence, then*

$$\left\{ \sum_{n=1}^\infty \lambda_n x_n \,\middle|\, \lambda_n \in \mathbf{C} \text{ arbitrary} \right\} \subset E_*.$$

EXAMPLE 7.3.

(*a*) Every sequence in a metrizable locally convex space is very weakly convergent.

(*b*) If $E = \sum_{n=1}^\infty \mathbf{C}$ and $u_n = \underbrace{(0, \ldots 0, 1, 0 \ldots)}_{n-1}$ then $(u_n)_{n=1}^\infty$ is not a very weakly convergent sequence.

(*c*) If every sequence in (E, τ) is very weakly convergent and τ_1 is a weaker topology on E than τ then every sequence in (E, τ_1) is very weakly convergent.

It appears that the topics very strongly convergent sequences, very weakly convergent sequences and the existence of a continuous norm on a locally convex space are rarely discussed in the literature and for this reason we pause to give a number of elementary results concerning these concepts and properties. We find that the concepts of very strongly convergent sequence and very weakly convergent sequence are dual to one another in a certain sense.

([21]) Particular cases of some of the results of this section were announced in [14].

PROPOSITION 7.4.

(a) *If there exists a continuous norm on E then every very strongly convergent sequence is eventually zero.*

(b) *If E is separable and every sequence in E is very weakly convergent then there exists a continuous norm on E'_β.*

(c) *If E contains a very strongly convergent sequence which is not eventually zero then E'_β contains a sequence which is not very weakly convergent.*

Proof.

(a) Trivial. The converse is not true.

(b) Let $(x_n)_{n=1}^\infty$ denote a dense sequence in E. By hypothesis there exists a sequence of non-zero scalars, $(\lambda_n)_{n=1}^\infty$, such that $\lambda_n x_n \to 0$ as $n \to \infty$. Now $p(\varphi) = \sup_n |\varphi(\lambda_n x_n)|$, $\varphi \in E'$, is a continuous semi-norm on E'_β. If $p(\varphi) = 0$ then $\varphi(x_n) = 0$ for all n and hence $\varphi = 0$ since the sequence $(x_n)_{n=1}^\infty$ is dense in E. Hence p is a continuous norm on E'_β.

(c) Let $(x_n)_{n=1}^\infty$ denote a very strongly convergent sequence in E which is not eventually zero. For each integer n choose $\varphi_n \in E'$ such that $\varphi_n(x_n) = 1$. If $(\lambda_n)_{n=1}^\infty$ is any sequence of non-zero scalars then $p(\varphi) = \sup_n |\varphi(x_n/\lambda_n)|$ is a continuous seminorm on E'_β. Hence $p(\lambda_n \varphi_n) \geqslant |\varphi_n(x_n)| = 1$ and $\lambda_n \varphi_n \nrightarrow 0$ as $n \to \infty$. This completes the proof.

We now restrict ourselves to Fréchet spaces.

PROPOSITION 7.5.

(a) *If E is a Fréchet space then the following are equivalent.*

(1) *There does not exist any continuous norm on E.*

(2) *E contains a very strongly convergent sequence which is not eventually zero.*

(3) *E contains a subspace isomorphic to $\prod_{n=1}^\infty \mathbf{C}$.*

(4) *There exists a continuous linear mapping from E'_β onto $\sum_{n=1}^\infty \mathbf{C}$.*

(5) *$\sum_{n=1}^\infty \mathbf{C}$ is isomorphic to a quotient of E'_β.*

(b) *If E is a Fréchet space which is not a Banach space then E'_β contains a sequence which does not converge to 0 very weakly.*

(c) *If E is a Fréchet space which is not a Banach space then $\prod_{n=1}^{\infty} \mathbf{C}$ is isomorphic to a quotient of E.*

Proof.

(a) $(3) \Rightarrow (1) \Rightarrow (2)$ are trivial or well known. We now show $(3) \Rightarrow (4)$. If E contains a subspace isomorphic to $\prod_{n=1}^{\infty} \mathbf{C}$ then by the corollary to Proposition 17 of [22] (p. 308) there exists a continuous linear mapping of E'_β onto $(\prod_{n=1}^{\infty} \mathbf{C})'_\beta \cong \sum_{n=1}^{\infty} \mathbf{C}$. Hence $(3) \Rightarrow (4)$.

Now every convex balanced absorbing subset of $\sum_{n=1}^{\infty} \mathbf{C}$ is a neighbourhood of 0. Hence every linear mapping from a locally convex space onto $\sum_{n=1}^{\infty} \mathbf{C}$ is an open mapping. This shows $(4) \Leftrightarrow (5)$.

$(4) \Rightarrow (1)$. Let $T; E'_\beta \rightarrow \sum_{n=1}^{\infty} \mathbf{C}$ denote a continuous linear onto mapping. Let $T^0 = \{ x \in E, \varphi(x) = 0 \text{ all } \varphi \in \ker(T) \}$. T^0 is a closed subspace of E and hence is a Fréchet space. By the Corollary of [22] (p. 262) there exists a canonical bijective linear mapping from $E'_{\beta/\mathrm{Ker}(T)}$ onto $(T^0)'$. Hence $(T^0)'$ has a countably infinite basis. If there exists a continuous norm on T^0 then $(T^0)'$ is either finite dimensional or has uncountable dimension. Since both of these eventualities are impossible, we conclude that T^0 does not admit any continuous norm. Since T^0 is a subspace (algebraically and topologically) of E this means that E does not admit any continuous norm.

$(2) \Rightarrow (3)$. If (2) holds then we choose a sequence of continuous seminorms, $(p_n)_{n=1}^{\infty}$, which define the topology of E and a sequence of elements of E, $(x_n)_{n=1}^{\infty}$, such that $p_n(x_n) \neq 0$ and $p_m(x_n) = 0$ for all $n > m$. The set $\{ x_n \}_{n=1}^{\infty}$ consists of linearly independent vectors. We now consider the subspace of E generated by $(x_n)_{n=1}^{\infty}$ and call it E_1. For any integer n we have

$$p_n(\textstyle\sum_{j=1}^{m} \alpha_j x_j) \leqslant \sum_{j=1}^{n} |\alpha_j| \, p_n(x_j).$$

Hence the sequence $(x_j)_{j=1}^{\infty}$ is a Schauder basis for E_1, which has the topology of coordinatewize convergence. Since E is complete it follows that the completion of E_1 is a subspace of E isomorphic to $\prod_{n=1}^{\infty} \mathbf{C}$. Hence $(2) \Rightarrow (3)$ and this completes the proof of (a).

(b) If E is a Fréchet space which is not a Banach space then there exists on E an increasing sequence of non-equivalent semi-norms which define the topology, $(p_n)_{n=1}^{\infty}$. Since $E' \neq (E, p_n)'$ for any n we can choose a sequence of elements of E', $(\varphi_n)_{n=1}^{\infty}$, such that φ_{n+1} is p_{n+1} but not p_n

continuous for each $n \geqslant 1$. Now suppose $(\beta_n)_{n=1}^{\infty}$ is a sequence of non-zero scalars such that $\beta_n \varphi_n \to 0$ in E'_{β} as $n \to \infty$. We can choose a sequence of elements of E, $(x_n)_{n=1}^{\infty}$, such that $p_n(x_n) < 1/n$ and $|\varphi_n(x_{n-1})| > n/\beta_n$ and this is impossible. This completes the proof of (b).

The proof of (c) can be found in $[26]$ (p. 431).

EXAMPLE 7.6. — Let Ω denote the set of all ordinals less than or equal to the first uncountable ordinal with the order topology. The union of an increasing family of compact subsets of Ω is relatively compact. Hence $\mathscr{C}(\Omega)$ is a DF-space ($[43]$, Theorem 12). Now Ω is pseudocompact (i. e. every continuous function on Ω is bounded) by Theorem 11 of $[43]$. If $(f_n)_{n=1}^{\infty}$ is any sequence of elements of $\mathscr{C}(\Omega)$ then $f_n/n \, \|f_n\|_{\Omega}$ converges to zero uniformly on Ω (and hence on all compact subsets of Ω) as $n \to \infty$. Hence every sequence in $\mathscr{C}(\Omega)$ is very weakly convergent. This example shows that the result in (b) of the proceeding proposition is sharp.

E, a locally convex space, is said to be very weakly complete if for any sequence E, $(x_n)_{n=1}^{\infty}$ there exists a sequence in $\mathbf{C} \setminus \{0\}$, $(\lambda_n)_{n=1}^{\infty}$ such that $\lim_{n \to \infty} \sum_{m=1}^{n} \beta_m x_m = \sum_{n=1}^{\infty} \beta_n x_n \in E$ if $|\beta_n| \leqslant |\lambda_n|$ for all n.

If E is very weakly complete then every sequence in E converges to zero very weakly and a sequentially complete locally convex space in which every sequence converges to zero very weakly is very weakly complete.

E, a locally convex space, has the extension property if it contains a dense subspace, E_1, in which every sequence converges very weakly. If E_1 is very weakly complete we say that E has the complete extension property, if every point of E can be approached by a relatively compact net of elements of E_1 then we say that E has k-extension property and if E_1 is dense in (E, τ_M) then we say that E has the Silva extension property.

EXAMPLE 7.7.

(a) $\prod_{i \in \omega} E_i$ has the extension property (resp. complete extension property) if and only if each E_i has the extension property (resp. complete extension property).

(b) If X is a completely regular Hausdorff space then $\mathscr{C}_b(X)$ (the complex valued continuous bounded functions on X) is a dense subspace of $\mathscr{C}(X)$ which is very weakly complete. Hence $\mathscr{C}(X)$ has the complete extension property.

(c) If X is a normal Hausdorff space and $f \in \mathscr{C}(X)$ we let $V_n = \{ x \in X; |f(x)| \leqslant n \}$. By the Tietze extension theorem there exists f_n in $\mathscr{C}_b(X)$ such that $\|f_n\|_X \leqslant n$ and $f_n - f|_{V_n} = 0$. The sequence $(f_n - f)_{n=1}^{\infty}$ converges very strongly and hence in the τ_M topology on E. Thus $\mathscr{C}(X)$ has the complete Silva extension property whenever X is normal Hausdorff space.

(d) If there exists a continuous surjection from E onto $\sum_{n=1}^{\infty} \mathbf{C}$ then E does not possess the extension property.

(e) Now suppose E is a locally convex space with the complete extension property and $(f_n)_{n=1}^{\infty}$ is a sequence of non-zero elements of $\mathscr{H}(U)$, where U is a balanced convex open subset of E. We show that there exists an x in E such that $f_n(x) \neq 0$ for all n. By the complete extension property, since each f_n is continuous, we can choose a sequence of elements of E, $(x_n)_{n=1}^{\infty}$, such that $f_n(x_n) \neq 0$ and $\sum_{n=1}^{\infty} \lambda_n x_n \in U$ if $|\lambda_n| \leqslant 1$ for all n. Choose $\lambda_1^1 \in \mathbf{C}$, $0 \leqslant |\lambda_1^1| \leqslant 1/2$, and V_1 a neighbourhood of 0 in E such that $\lambda_1^1 x_1 + V_1 \subset U$ and $|f_1(\lambda_1^1 X_1)| > 0$.

Now suppose $(\lambda_i^j)_{1 \leqslant i \leqslant j \leqslant n}$ and V_n have been chosen so that;

(1) $|\lambda_i^i| \leqslant 1/2^{i+1}$ for $1 \leqslant i \leqslant n$.

(2) $|\lambda_i^j - \lambda_i^{j+1}| \leqslant 1/2^{j+1}$ for $1 \leqslant i \leqslant j \leqslant n$.

(3) $\sum_{i=1}^{n} \lambda_i^n x_i + V_n \subset U$

and

(4) $|f_i(\sum_{j=1}^{n} \lambda_j^n x_j + V_n)| > 1/2 |f_i(\sum_{j=1}^{i} \lambda_j^i x_j)| > 0$ for $i = 1, \ldots, n$.

Since non-zero holomorphic functions have nowhere dense zero sets we may choose $(\lambda_j^{n+1})_{j=1}^{n+1}$ and V_{n+1} to complete the next step in the induction process.

From (2), it follows that $\lim_{j \to \infty} \lambda_i^j = \lambda_i$ exists for all i and by (1) $|\lambda_i| \leqslant 1/2^i$.

By (1) and (2), and since $x_n \to 0$ as $n \to \infty$, it follows that

$$\sum_{i=1}^{n} \lambda_i^n x_i \to \sum_{i=1}^{\infty} \lambda_i x_i = x \in E.$$

By (4) it follows that $f_n(x) \neq 0$ for all n.

We now discuss the structure of the indexing set in surjective representations. Let $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ denote a surjective representation of the locally convex space E.

If B and C are subsets of A we let

$$B \wedge C = \{ \alpha \in A; \, \exists \, \alpha_1 \in B, \, \alpha_2 \in C \ \text{such that} \ \alpha \leqslant \alpha_1 \ \text{and} \ \alpha \leqslant \alpha_2 \}.$$

We write $\hat{B}$ in place of $B \wedge B$ and note that $B \wedge C = \hat{B} \wedge \hat{C} = \hat{B} \cap \hat{C}$. If $B = \hat{B}$, we say that B is a full (or an E-full) subset of A. A subset B of A is bounded (or E-bounded) if

$$B \subset \{ \alpha \}^{\hat{}} = \{ \beta \in A; \, \beta \leqslant \alpha \} \quad \text{for some} \ \alpha \ \text{in} \ A.$$

The E-complement of the subset B of A, B^c, is defined as follows

$$B^c = \{ \alpha \in A; \, \{ \alpha \} \wedge B = \varnothing \}.$$

It is immediate that $B \subset (B^c)^c$, $B^c = (B^{cc})^c = (B^c)^{cc}$, B^c is an E-full subset of A and $B \subset C$ implies $B^c \supset C^c$ for any subsets B and C of A. We now define the supports of elements of E. If $x \in E$,

$$N(x) = \{ \alpha \in A; \, \pi_\alpha(x) = 0 \}$$

is the $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ null set of x and $S(x) = N(x)^c$ is the $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support of x. We use the terms null set and support when there is no possibility of confusion. $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is called a symmetric surjective representation of E if $N(x) = S(x)^c$ for all x in E.

$B \subset A$ is called an *E-neighbourhood* of $\alpha \in A$ *if there exists an x in E such that $\alpha \in S(x)$ and $S(x) \subset B$. B is said to be E-open if it is an E-neighbourhood of each of its points (in particular $S(x)$ is an E-open subset of A).*

$(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *is an i (resp. j) surjective representation of E if for each unbounded subset of A (resp. each unbounded subset of A which is the union of a sequence of bounded subsets of A), B, there exists a sequence of E-open subsets of A, $(B_n)_{n=1}^\infty$, such that*

(7.1) $B_n \wedge B \neq \varnothing$ *for each n*

and

(7.2)   *if $\alpha \in A$ then $B_n \wedge \{ \alpha \} = \varnothing$ for all n sufficiently large.*

E *is said to admit $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partitions of unity if for each α in A and each E-neighbourhood of α, V, there exists an operator from E into E, P, such that the following hold for any x in E,*

(7.3) $S(P(x)) \subset V.$

(7.4) $N(x) \subset N(P(x)).$

(7.5) $N(x - P(x))$ *is an E-neighbourhood of α.*

P is called an $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *local partition of unity at* (α, V) *or at* α *when the E-neighbourhood is not specified.*

E is said to admit continuous (resp. hypocontinuous, Silva continuous) $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *local partitions of unity if for each* α *in A and each x in E,* $\pi_\alpha(x) = 0$, *there exists a net of* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *local partitions of unity at* α, $(P_\beta)_{\beta \in B}$, *such that* $P_\beta(x) \to 0$ *as* $\beta \to \infty$ *(resp.* $P_\beta(x) \to 0$ *as* $\beta \to \infty$ *and* $\bigcup_{\beta \in B} P_\beta(x)$ *is a relatively compact subset of E,* $P_\beta(x) \to 0$ *in* (E, τ_M) *as* $\beta \to \infty$).

A collection of E-open subsets of A, $(V_\beta)_{\beta \in B}$, *is an E-open cover of A if for each* α *in A there exists a* β *in B such that* $\{\alpha\} \wedge V_\beta \neq \varnothing$.

E admits continuous $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity if for each E-open cover of A,* $(V_\beta)_{\beta \in B}$, *and for each x in E there exists a family of mappings from E into E,* $(T_\gamma)_{\gamma \in C}$, *such that*

(7.6) *For each* γ *in C there exists a* β *in B such that* $S(T_\gamma(x)) \subset V_\beta$.

(7.7) $S(T_\gamma(x)) \subset S(x)$ *for all* γ *in C.*

(7.8) *The finite sums of* $\sum_{\gamma \in C} T_\gamma(x)$ *converge to x.*

Moreover if the finite sums of $\sum_{\gamma \in C} T_\gamma(x)$ *form a relatively compact subset of E then we say that E admits hypocontinuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity and if the finite sums of* $\sum_{\gamma \in C} T_\gamma$ *converge to x in* (E, τ_M) *then we say that E admits Silva continuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity.*

EXAMPLE 7.8. — $\mathscr{C}(X) = \lim_{K \in \mathscr{K}(X)} (\mathscr{C}(K), R_K)$ where $\mathscr{K}(X)$ is the set of compact subsets of X and R_K is the restriction (to K) mapping. $\mathscr{C}(X)$ induces on $\mathscr{K}(X)$ the usual set inclusion order.

$B \subset \mathscr{K}(X)$ is a $\mathscr{C}(X) =$ neighbourhood of $K \in \mathscr{K}(X)$ if there exists a neighbourhood of K in X, U such that $\mathscr{K}(U) \subset B$. If $f \in \mathscr{C}(X)$ and $n(f) = \{x \in X; f(x) = 0\}$ then $\mathscr{K}(n(f))$ is the $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ null set of f and $S(f) = \mathscr{K}(\{x; f(x) \neq 0\})$. $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ is a symmetric surjective representation of $\mathscr{C}(X)$. We obtain further properties by placing additional conditions on X.

(a) Let f denote a continuous function on a completely regular Hausdorff space X which is zero on the compact subset K of X. For each positive integer n let $V_n = \{n \in X; |f(n)| < 1/n\}$ and let g_n denote a local partition of unity at (K, V_n). $fg_n \to 0$ in $(\mathscr{C}(X), \tau_M)$ as $n \to \infty$ and hence $\mathscr{C}(X)$ admits continuous, hypocontinuous and Silva continuous $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ local partitions of unity.

(b) If X is a completely regular Hausdorff space then $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ is a i (resp. j) surjective representation of $\mathscr{C}(X)$ if and only if $\mathscr{C}(X)$ is infrabarrelled (resp. countably barrelled ([22])) [3].

Let Ω denote the set of all ordinals less than the first uncountable ordinal with the usual order topology. $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(\Omega)}$ is a j-surjective representation ([43], p. 276).

(c) If X is a paracompact space then $\mathscr{C}(X)$ admits hypocontinuous $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ partitions of unity by Ascoli's theorem and if X is a separated Lindelöf space then $\mathscr{C}(X)$ admits Silva continuous $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ partitions of unity.

EXAMPLE 7.9. — Let Γ denote an uncountable discrete set. $C_{0,\tau}(\Gamma)$ is the space $C_0(\Gamma)$ with the topology generated by the semi-norms $(P_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ where $\mathscr{K}_\sigma(\Gamma)$ is the set of all countable subsets of Γ and $P_B(Z) = \sup_{\alpha \in B}|Z_\alpha|$ for all $Z = (Z_\alpha)_{\alpha \in \Gamma} \in C_0(\Gamma)$ and all $B \in \mathscr{K}_\sigma(\Gamma)$. $(C_0(B), R_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ is a surjective representation of $C_{0,\tau}(\Gamma)$ where $R_B((Z_\alpha)_{\alpha \in \Gamma}) = (Z_\alpha)_{\alpha \in B}$. Since a countable union of countable sets is countable it follows that $(C_0(B), R_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ is a j-surjective representation of $C_{0,\tau}(\Gamma)$. $C_{0,\tau}(\Gamma)$ induces on $\mathscr{K}_\sigma(\Gamma)$ the usual set inclusion order. If $x = (x_\alpha)_{\alpha \in \Gamma} \in C_0(\Gamma)$ and $n(x) = \{\alpha \in \Gamma; x_\alpha = 0\}$ then $N(x) = \mathscr{K}_\sigma(n(x))$ and $S(x) = \mathscr{K}_\sigma(\Gamma \setminus n(x))$. Thus $A \subset \mathscr{K}_\sigma(\Gamma)$ is $C_{0,\tau}(\Gamma)$-open if and only if it is $C_{0,\tau}(\Gamma)$-full.

Since $\mathscr{K}_\sigma(B)^c = \mathscr{K}_\sigma(\Gamma \setminus B)$ for any subset B of Γ it follows that $(C_0(B), R_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ is a symmetric representation of $C_{0,\tau}(\Gamma)$. $\{R_{\{\alpha\}}\}_{\alpha \in \Gamma}$ is a continuous, hypocontinuous and Silva continuous $(C_0(B), R_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ partition of unity subordinate to any $C_{0,\tau}(\Gamma)$ open cover of $\mathscr{K}_\sigma(\Gamma)$. If $B \in \mathscr{K}_\sigma(\Gamma)$ then R_B is a local partition of unity at (B, V) for any $C_{0,\tau}(\Gamma)$-neighbourhood V of B. Hence $C_{0,\tau}(\Gamma)$ admits continuous, hypocontinuous and Silva continuous $(C_0(B), R_B)_{B \in \mathscr{K}_\sigma(\Gamma)}$ local partitions of unity.

EXAMPLE 7.10. — Let Γ denote an uncountable set. For each α in Γ let A_α denote a countable set and let $A = \bigcup_{\alpha \in \Gamma} A_\alpha$. Let $\mathscr{B}$ denote the set of all subsets of A whose intersection with each A_α is finite. For each $B \in \mathscr{B}$ we let $l_1(B) = \{ x = (x_i)_{i \in B}; \|x\|_B = \sum_{i \in B}|x_i| < \infty \}$.

We let $l_1(\mathscr{B}) = \{ (x_i)_{i \in A}; (x_i)_{i \in B} \in l_1(B) \text{ for all } B \in \mathscr{B} \}$, and we endow $l_1(\mathscr{B})$ with the topology generated by the semi-norms $\| \quad \|_B$ as B ranges over $\mathscr{B}$.

([22]) We refer to [24] for the definition, properties and examples of countably barrelled spaces.

$(l_1 (B), R_B)_{B \in \mathscr{B}}$ is a surjective representation of $l_1 (\mathscr{B})$.

An analysis similar to that undertaken in the proceeding example shows that $l_1 (\mathscr{B})$ admits Silva continuous $(l_1 (B), R_B)_{B \in \mathscr{B}}$ partitions of unity and Silva continuous $(l_1 (B), R_B)_{B \in \mathscr{B}}$ local partitions of unity. $l_1 (\mathscr{B})$ induces on B the usual set inclusion order and $(l_1 (B), R_B)_{B \in \mathscr{B}}$ is a symmetric i-surjective representation of $l_1 (\mathscr{B})$. A similar example is discussed in [10].

Now let U denote a convex balanced open subset of E and let f denote a function of U. If $(E_\alpha , \pi_\alpha)_{\alpha \in A}$ is a surjective representation of E then the $(E_\alpha , \pi_\alpha)_{\alpha \in A}$-null set of f, $N (f)$, is defined as follows:

$$N (f) = \{ \alpha \in A; \; \exists \; an \; E\text{-neighbourhood of } \alpha, \; N (\alpha), \; and$$

$$f(x + y) = f(x) \; for \; all \; (x, x+y) \in U \times U \; such \; that \; S(y) \subset N(\alpha) \}.$$

$$S (f) = N (f)^c \; is \; called \; the \; (E_\alpha, \pi_\alpha)_{\alpha \in A} \; support \; of \; f.$$

If B is an E-open subset of A and $B \wedge S (f) \neq \varnothing$ then there exists an x in U and a y in E such that $x+y \in U$, $S (y) \subset B$ and $f(x+y) \neq f(x)$.

The crucial part of many of our proofs consist in showing that certain functions have bounded support and factor through every bound of their support.

PROPOSITION 7.11. — *Let* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *denote a symmetric i-surjective representation of the locally convex space E. A Banach valued G-holomorphic function, f, defined on a convex balanced open subset of E has bounded* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *support and factors through every bound of its support if any of the following conditions are satisfied.*

(a) *E has the k-extension property, admits hypocontinuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity, admits hypocontinuous* $(E_\alpha, \pi_\alpha)_{n \in A}$ *local partitions of unity and f is hypoanalytic.*

(b) *E has the Silva extension property, admits Silva continuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity, admits Silva continuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *local partitions of unity and f is Silva holomorphic.*

(c) *E admits continuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *partitions of unity, admits continuous* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *local partitions of unity and f is continuous.*

Proof. — If $S (f)$ is not bounded then we can choose, since $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is an i-surjective representation of E, a sequence of E-open subsets of A, $(V_n)_{n=1}^\infty$, such that $S (f) \wedge V_n \neq \varnothing$ for all n and if $\alpha \in A$ then $\alpha \in V_n^c$ for

all n sufficently large. We now show that $S(f)$ is bounded when conditions (a) or (b) hold. By our continuity hypothesis on f and the extension hypothesis on E we may choose null sequences in U, $(x_n)_{n=1}^\infty$ and $(x_n+y_n)_{n=1}^\infty$, such that $S(y_n) \subset V_n$, $f(x_n+y_n) \neq f(x_n)$ for all n sufficently large (since $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is a symmetric surjective representation of E). Hence $(y_n)_{n=1}^\infty$ is a very strongly convergent sequence and $(x_n+\lambda_n y_n)_{n=1}^\infty$ is a null sequence in (E, τ_M) for any sequence of scalars $(\lambda_n)_{n=1}^\infty$. By Liouville's theorem we can choose $(\lambda_n)_{n=1}^\infty$ such that $\|f(x_n+\lambda_n y_n)\| \to \infty$ as $n \to \infty$. This is impossible and hence $S(f)$ is an E-bounded subset of A.

Now suppose condition (c) holds. We may choose two sequences in U, $(x_n)_{n=1}^\infty$ and $(x_n+y_n)_{n=1}^\infty$ such that $f(x_n+y_n) \neq f(x_n)$ and $S(y_n) \subset V_n$ for all n. Since $(y_n)_{n=1}^\infty$ is a very strongly convergent sequence and f is continuous we may choose a positive integer N and a neighbourhood of 0, V, such that $\{\lambda y_n \mid \lambda \in \mathbf{C}\} \subset V$ for all $n \geqslant N$ and $f(x+\lambda y_n)=f(x)$ for all x in V, all λ in $\mathbf{C}$ and all $n \geqslant N$. By Liouville's theorem, it follows that $f(x_n+y_n)=f(x_n)$ for all $n \geqslant N$, and this contradiction shows that $S(f)$ is an E-bounded subset of A.

Now let

$$\eta = \{W; \ W \text{ is an } E\text{-open subset of } A \text{ and } f(x+y)=f(x)$$
$$\text{for all } (x, x+y) \text{ in } U \times U \text{ such that } S(y) \subset W\}.$$

It is immediate that $N(f) = \bigcup_{W \in \eta} W$. Now if $S(f) \subset \{\alpha\}^\hat{}$ and V is any E-open subset of A that contains α then $\eta \cup V$ is an E-open cover of A. If $(x, x+y) \in U \times U$ and $S(y) \subset \mathrm{V}$ then we may choose, in case (c) (resp. (a), (b)), a continuous (resp. hypocontinuous, Silva continuous) $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ partition of unity, $(T_\beta)_{\beta \in B}$, such that

$$f(x+y) = \lim f(x+\textstyle\sum_\beta T_\beta(y)) = f(x).$$

Hence f factors through every E-neighbourhood of α. Since E admits continuous (resp. hypocontinuous, Silva continuous) $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partitions of unity it follows that $f(x+y)=f(x)$ for all $(x, x+y)$ in $U \times U$ such that $\pi_\alpha(y)=0$ if f is continuous (resp. hypoanalytic, Silva holomorphic). This completes the proof.

THEOREM 7.12. — *Let* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *denote a symmetric compact (resp. bounded) i-surjective representation of* E, *and let* X *denote a domain spread over* E. *Then* $\mathcal{H}(X; F) = \mathcal{H}_{HY}(X; F)$ *(resp.* $\mathcal{H}_S(X; F)$*) for any locally convex space* F *if* E *has the* k *(resp. Silva) extension property, admits hypo-*

continuous (resp. Silva continuous) $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ partitions of unity, admits hypocontinuous (resp. Silva continuous) $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partitions of unity and $\mathscr{H}(Y; F) = \mathscr{H}_{HY}(Y; F)$ (resp. $\mathscr{H}_S(Y; F)$) for any domain spread over E_α, α an arbitrary element of A, and any locally convex space F.

Proof. — Without loss of generality we may assume that X is a balanced convex open subset of E and that F is a normed linear space. By Proposition 7.11, there exists an α in A and f_α in $\mathscr{H}_G(\pi_\alpha(X); F)$ such that $f = f_\alpha \circ \pi_\alpha$. If f is hypoanalytic and $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is a compact surjective representation then $f_\alpha \in \mathscr{H}_{HY}(\pi_\alpha(X); F)$. By an appropriate choice of α we may assume that $\pi_\alpha(X)$ is a neighbourhood of 0 in E_α and hence, since $\mathscr{H}(Y, F) = \mathscr{H}_{HY}(Y; F)$ for all domains spread over E_α, f_α is continuous in some neighbourhood of zero. This shows that f is continuous at zero in E and completes the proof in the hypoanalytic case. The Silva holomorphic case is completed in a similar fashion.

EXAMPLE 7.13.

(a) $\mathscr{H}(U; F) = \mathscr{H}_{HY}(U; F)$ for any domain U spread over $\mathscr{C}(X)$ where X is a paracompact topological space and F is any locally convex space. $\mathscr{H}(U; F) = \mathscr{H}_S(U; F)$ for any domain U spread over $\mathscr{C}(X)$ where X is a Lindelöf space and F is any locally convex space.

(b) Since $\mathscr{H}(\prod_{n=1}^{\infty} \mathbf{C} \times \sum_{n=1}^{\infty} \mathbf{C}) \neq \mathscr{H}_{HY}(\prod_{n=1}^{\infty} \mathbf{C} \times \sum_{n=1}^{\infty} \mathbf{C})$ (otherwise $\mathscr{H}(\prod_{n=1}^{\infty} \mathbf{C} \times \sum_{n=1}^{\infty} \mathbf{C})$ endowed with the compact open topology would be complete [12]), we see that the k-extension property is necessary in theorem 7.13.

(c) Let X denote a discrete uncountable set and let E denote the subspace of $\mathscr{C}(X)$ spanned by the functions of compact support, $\mathscr{C}_0(X)$, and the constant functions. If $f \in E$, then there exists a unique $x(f) \in \mathscr{C}_0(E)$ and a unique $\beta(f) \in \mathbf{C}$ such that $f = x(f) + \beta(f) 1$ where 1 is the function identically equal to 1 on X. We let $\varphi(f) = \beta(f)$ for all $f \in E$. Since X is uncountable it follows that φ is bounded on bounded subsets of E and hence φ is Silva holomorphic. Since $\varphi|_{\mathscr{C}_0(X)} \equiv 0$ and $\varphi(1) = 1$ it follows that $\varphi \notin \mathscr{H}_{HY}(E)$. Since all other hypotheses are satisfied this shows that the Silva partition of unity requirement in Theroem 7.12 is necessary (every sequence in E converges very weakly).

(d) Let $C_0(\Gamma)$ denote the set of functions on Γ, an uncountable discreet set, which vanish at infinity endowed with the sup norm topology. The identity mapping belongs to $\mathscr{H}_{HY}(C_{0,\tau}(\Gamma); C_0(\Gamma))$. Since all other

hypotheses are satisfied this shows that the *i*-surjective representation requirement in Theorem 7.12 is necessary.

Let $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ denote a surjective representation of the locally convex space E. *A vector valued function, f, defined on a convex balanced open subset, U, of E has minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support if there exists an E-bounded subset B of A such that the following conditions are satisfied.*

(7.9) *If $B \subset \{\alpha\}^{\wedge}$, $(x, x+y) \in U \times U$ and $\pi_\alpha(y) = 0$ then $f(x+y) = f(x)$.*

(7.10) *If W is an E-open subset of A and $W \wedge B \neq \varnothing$ then there exists $(x, x+y) \in U \times U$ such that $S(y) \subset W$ and $f(x+y) \neq f(x)$.*

B is called a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support of f.

Since $S(f)$ always satisfies (7.10) it follows that $S(f)$ is a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support for f if and only if $S(f)$ is bounded and $f(x+y) = f(x)$ for all $(x, x+y) \in U \times U$ such that $\pi_\alpha(y) = 0$ for some α which is an E-bound for $S(f)$. Moreover if f has a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support B then $S(f) \supset B^{cc}$ and $S(f)$ is thus a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support for f if and only if it is E-bounded. In this case $S(f)$ is also the maximum ([23]) minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support for f.

Proposition 7.11 gives examples of functions with minimal support. We now give further examples.

PROPOSITION 7.14. — *Let $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ denote a symmetric representation of E and suppose E admits continuous $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partitions of unity. A continuous holomorphic Banach valued function, f, defined on a convex balanced open subset of E has minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support if the following conditions are satisfied.*

(7.11) $N(f) = S(f)^c$.

(7.12) $\{\alpha\}^{\wedge} = \bigcap \{V; V \text{ is an } E\text{-open subset of } A \text{ containing } \alpha\}$
             $= \bigcap_{x \in E, \, \alpha \in S(x)} S(x)$.

(7.13) *If α, $\beta \in A$ and V is an E-neighbourhood of β, then there exists a γ in A such that $\{\beta\} \wedge \{\gamma\} = \varnothing$ and $\alpha \in N(x)$ whenever $V \subset N(x)$ and $\gamma \in N(x)$.*

Proof. — Since f is continuous there exists a β in A such that $f(x+y) = f(x)$ for all x, $x+y$ in U with $\pi_\beta(y) = 0$. Let $\delta \in \{\beta\}^c$. By (7.12), we may

_____

([23]) Maximum with respect to set inclusion.

choose an E-neighbourhood of δ, V, such that $V \wedge \{\beta\} = \varnothing$. Now if $y \in E$ and $S(y) \subset V$ then $\beta \in V^c \subset S(y)^c = N(y)$ since $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is a symmetric surjective representation of E. Hence $\{\beta\}^c \subset N(f)$. Since $S(x) = S(x)^{cc}$ for all x in E and $\{\beta\}^\wedge = \bigcap_{\beta \in S(x)} S(x)$ it follows that $\{\beta\}^\wedge = \{\beta\}^{cc}$ and $S(f) \subset \{\beta\}^\wedge$. Hence f has bounded $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support. Now suppose $S(f) \subset \{\alpha\}^\wedge$ and V is an E-neighbourhood of α. By (7.13), we can choose γ in A such that $\gamma \in \{\alpha\}^c$ and $\pi_\beta(x) = 0$ whenever V and γ lie in $N(x)$. Now $\gamma \in \{\alpha\}^c \subset S(f)^c = N(f)$, hence we can choose an E-neighbourhood of γ, W, such that $W \subset N(f)$. Let P denote an $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partition of unity at (γ, W). Now suppose $y \in E$ and $V \subset N(y)$. Since $V \cup \{\gamma\} \subset N(y - P(y))$, it follows that $\pi_\beta(y) = \pi_\beta(P(y))$.

Hence $f(z + (y - P(y))) = f(z)$ for all $z \in U$, whenever $z + (y - P(y)) \in U$. By analytic continuation if follows that $f(z + y) = f(z)$ if z and $z + y$ lie in U. By using the continuous $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ local partition of unity, it is possible to complete the proof in an obvious manner.

If X is a completely regular Hausdorff space then each continuously holomorphic Banach valued function, F, defined on a convex balanced open subset of $\mathscr{C}(X)$ has minimal $(\mathscr{C}(K), R_K)_{K \in \mathscr{K}(X)}$ support. It is immediate that (7.12) is satisfied. Moreover, if

$$W = \{f \in \mathscr{C}(X); F(f_1 + f_2) = F(f_1) \text{ for all } f_1, f_2 \text{ such that } S(f_2) \subset S(f)\}$$

then $N(F) = \bigcup_{f \in W} S(f)$. Now if $K \in \mathscr{K}(\bigcup_{f \in W} S(f))^{cc}$ then there exists $f_1, \ldots, f_n$ in W such that $K \in \mathscr{K}(\bigcup_{i=1}^n S(f_i))^{cc}$. By using local partitions of unity it follows that $K \in N(F)$. Hence $N(F) = N(F)^{cc}$, and we may apply Proposition 7.14 to complete the proof.

We now extend Hartogs' theorem concerning separate analyticity.

THEOREM 7.15 (*Hartogs' theorem*).— *Let* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ *and* $(F_\beta, \rho_\beta)_{\beta \in B}$ *denote open i-surjective representations of the locally convex spaces E and F respectively and let G_1 denote a locally convex space. A G_1-valued separately locally bounded G-holomorphic function, f, defined on a domain spread over $E \times F$, X, is locally bounded if the following conditions are satisfied.*

(1) *For each α in A and each β in B, the G_1-valued separately locally bounded G-holomorphic functions defined on domains spread over $E_\alpha \times F_\beta$ are locally bounded.*

(2) *E and F satisfy the complete extension property.*

(3) *Each* **C**-*valued continuously holomorphic function defined on a balanced convex open subset of* E (resp. F) *has minimal* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ (resp. $(F_\beta, \rho_\beta)_{\beta \in B}$) *support.*

Proof. — We may assume without loss of generality that $X = U \times V$ where U and V are convex balanced open subsets of E and F respectively. For each x in U and y in V let $f(x, y) = f_x(y) = f^y(x)$ and let $B(\varphi \circ f^y)$ (resp. $B(\varphi \circ f_x)$) denote a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ (resp. $(F_\beta \rho_\beta)_{\beta \in B}$) support for $\varphi \circ f^y$ (resp. $\varphi \circ f_x$) where $\varphi \in (G_1)'$. Let $B_W(f^y) = \bigcup_{\varphi \in (G_1)'} B(\varphi \circ f^y)$. We now show that $B_1 = \bigcup_{x \in U} B_W(f_x)$ is an F-bounded subset of B. Suppose not, then, since $(F_\beta, \rho_\beta)_{\beta \in B}$ is an i-surjective representation of F we can choose a sequence of F-open subsets of B, $(W_n)_{n=1}^\infty$, such that $W_n \wedge B_1 \neq \varnothing$ for all n and if $\beta \in B$ then $\beta \in W_n^c$ for all sufficiently large n.

Hence we can find a sequence of elements in U, $(x_n)_{n=1}^\infty$, and two sequences in V, $(y_n)_{n=1}^\infty$, and $(y_n + z_n)_{n=1}^\infty$, such that $S(z_n) \subset W_n$ and $f(x_n, y_n + z_n) \neq f(x_n, y_n)$ for all n.

Since E satisfies the complete extension property we may choose x in U such that $f(x, y_n + z_n) \neq f(x, y_n)$ and since F satisfies the extension property we may assume that the sequence $(y_n)_{n=1}^\infty$ converges to zero in F. Since f is separately locally bounded we may choose a neighbourhood of zero in F, W, such that $W \subset V$ and $f(x, W)$ is a bounded subset of G_1. Since $(z_n)_{n=1}^\infty$ is a very strongly convergent sequence we may choose an integer N such that $\{ y_N + \lambda z_N \mid \lambda \in \mathbf{C} \} \subset W$. This contradicts the fact that we may choose, by Liouville's theorem and the Hahn Banach theorem, $\varphi \in (G_1)'$ such that $| \varphi \circ f(x, y_N + \lambda z_N) | \to \infty$ as $| \lambda | \to \infty$. Hence B_1 is an F-bounded subset of B. Similarly $A_1 = \bigcup_{y \in V} B_W(f^y)$ is an E-bounded subset of A.

Now choose α_0 in A and β_0 in B respectively such that $A_1 \subset \{ \alpha_0 \}^\wedge$ and $B_1 \subset \{ \beta_0 \}^\wedge$.

Suppose (x_1, y_1) and (x_2, y_2) are in $U \times V$ and

$$(\pi_{\alpha_0}(x_1), \rho_{\beta_0}(y_1)) = (\pi_{\alpha_0}(x_2), \rho_{\beta_0}(y_2)).$$

Let $\varphi \in (G_1)'$ be arbitrary. Since $B(\varphi \circ f^y) \subset A_1$ is a minimal $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ support for $\varphi \circ f^y$ and $B(\varphi \circ f_x) \subset B_1$ is a minimal $(F_\beta, \rho_\beta)_{\beta \in B}$ support for $\varphi \circ f_x$ it follows that

$$\varphi \circ f(x_1, y_1) = \varphi \circ f_{x_1}(y_1) = \varphi \circ f_{x_1}(y_2)$$
$$= \varphi \circ f^{y_2}(x_1) = \varphi \circ f^{y_2}(x_2) = \varphi \circ f(x_2, y_2).$$

Hence we may define the function $\tilde{f}$ on $\pi_{\alpha_0}(U) \times \rho_{\beta_0}(V)$ by the formula

$$\tilde{f}(x_2, y_2) = f(x_1, y_1)$$

if

$$(\pi_{\alpha_0}(x_1), \rho_{\beta_0}(y_1)) = (x_2, y_2).$$

Since π_{α_0} and ρ_{β_0} are open mappings if follows that $\tilde{f}$ is a separately locally bounded G_1 valued G-holomorphic function on $\pi_{\alpha_0}(U) \times \rho_{\beta_0}(V)$. By hypothesis (1) it follows that $\tilde{f} \in \mathscr{H}_{LB}(\pi_{\alpha_0}(U) \times \rho_{\beta_0}(V); G_1)$ and since $f = \tilde{f} \circ (\pi_{\alpha_0}, \rho_{\beta_0})$ it follows that $f \in \mathscr{H}_{LB}(U \times V; G_1)$. This completes the proof.

DEFINITION 7.16. — *A triple of locally convex spaces* (E, F, G_1) *is called a weak (resp. strong) Hartogs' triple if every* G_1*-valued* G*-holomorphic separately continuous (resp. locally bounded) function defined on a domain spread over* $E \times F$ *is continuous (resp. locally bounded).*

PROPOSITION 7.17.

(a) *If the locally convex spaces* F *and* G *form a duality such that every* F*-equicontinuous subset of* G *is bounded and* $(E, F, \mathbf{C})$ *is a Hartogs' triple then* $\mathscr{H}(U; G_\omega) = \mathscr{H}(U; G) = \mathscr{H}_{LB}(U; G)$ *for any domain spread over* E *where* G_ω *is the vector space* G *endowed with the* $\sigma(G, F)$ *topology.*

(b) *If* (E, F, G) *is a weak (resp. strong) Hartogs' triple and* E_1 *and* F_1 *are closed subspaces of* E *and* F *respectively then* $(E/E_1, F/F_1, G)$ *is a weak (resp. strong) Hartogs' triple.*

Proof.

(a) Since $\mathscr{H}(U; G_\omega) \supset \mathscr{H}(U; G) \supset \mathscr{H}_{LB}(U; G)$ it suffices to show $\mathscr{H}(U; G_\omega) \subset \mathscr{H}_{LB}(U; G_\omega)$. Let $f \in \mathscr{H}(U; G_\omega)$. Define $\tilde{f}(x, y) = y(f(x))$ for all x in U and all y in F. By hypothesis $\tilde{f}$ is separately continuous and, since $(E, F, \mathbf{C})$ is a Hartogs' triple, $\tilde{f} \in \mathscr{H}(U \times F)$. Hence we may choose, for each x in U, a neighbourhood of x in U, V, and a neighbourhood if 0 in F, W, such that $\tilde{f}(V \times W)$ is a bounded subset of G. Thus $f(V)$ is an F-equicontinuous subset of G and f is locally bounded. This completes the proof.

(b) Trivial.

As an immediate corollary, we obtain the following result.

COROLLARY.

(a) If $(E, F'_\beta, \mathbf{C})$ is a Hartogs' triple then

$$\mathscr{H}(U ; F_\omega) = \mathscr{H}(U ; F) = \mathscr{H}_{LB}(U ; F).$$

(b) If $(E, F, \mathbf{C})$ is a Hartogs' triple then

$$\mathscr{H}(U ; F'_\sigma) = \mathscr{H}(U ; F'_\beta) = \mathscr{H}_{LB}(U ; F'_\beta) \text{ where } F'_\sigma = (F', \sigma(F', F)).$$

We now use known extensions of Hartogs' theorem and theorem 7.15 to obtain new examples. We also show by various counterexamples that most of the rather technical conditions imposed on E and F in theorem 7.15 are necessary.

The following results are known;

(a) (E, F, G) is a strong Hartogs' triple if E and F are Fréchet spaces and G is a DF-space.

(b) (E, F, G) is a strong Hartogs' triple if E and F are $\mathscr{DFS}$ spaces and G is an $\mathscr{FS}$ space.

EXAMPLE 7.18.

(a) (See Examples 2.8 and 7.7.) $(\prod_{i \in w_1} E_i, \prod_{j \in w_2} F_j, G)$ is a Hartogs' triple if each E_i and F_j has the complete extension property and $(\prod_{i \in w_3} E_i, \prod_{j \in w_4} F_j, G)$ is a strong Hartogs' triple for any finite subset w_3 of w_1 and any finite subset w_4 of w_2. In particular $(\prod_{i \in w_1} E_i, \prod_{j \in w_2} F_j, G)$ is a strong Hartogs' triple if each E_i and F_j is a Fréchet space and G is a DF-space [34]. Since $(\sum_{n=1}^{\infty} \mathbf{C}, \prod_{n=1}^{\infty} \mathbf{C}, \mathbf{C})$ [24] is not a Hartogs' triple, we see that the complete extension property is necessary both in this example and in Theorem 7.15.

(b) If X and Y are completely regular Hausdorff spaces and $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ are infrabarrelled then $(\mathscr{C}(X), \mathscr{C}(Y), F)$ is a weak Hartogs' triple for any locally convex space F. Let Ω denote the set of all ordinals less than the first uncountable ordinal. $\mathscr{C}(\Omega)$ is not infrabarrelled but is countably barrelled (Example 7.8 (d)) and $\mathscr{C}(\Omega)'_\beta$ is a Fréchet space (Example 7.6). $(\mathscr{C}(\Omega), \mathscr{C}(\Omega)'_\beta, \mathbf{C})$ is not a Hartogs' triple and this shows that the i-surjective representation condition is necessary both in this Example and in Theorem 7.15.

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[24] If $(E, E'_\beta, \mathbf{C})$ is a Hartogs' triple then E is a normed linear space.

(c) Let X and Y denote completely regular topological spaces and suppose $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ are infrabarrelled. By corollary (b)

$$\mathscr{H}(\mathscr{C}(X); \mathscr{C}(Y)'_\beta) = \mathscr{H}_{LB}(\mathscr{C}(X), \mathscr{C}(Y)'_\beta).$$

Since $\mathscr{C}(X)$ is infrabarrelled the canonical mapping of $\mathscr{C}(X)$ into its second dual is an isomorphism onto its image. Hence $\mathscr{C}(Y)'_\beta$ is not a DF-space if Y is not hemicompact [43]. If X and Y are infinite dimensional Banach spaces then $\mathscr{H}(\mathscr{C}(X); \mathscr{C}(Y)) = \mathscr{H}_{LB}(\mathscr{C}(X); \mathscr{C}(Y))$ and $\mathscr{C}(Y)$ is not a DF-space.

(d) (See Example 7.9.) Since

$$(C_{0,\tau}(\Gamma))'_\beta = l_1(\Gamma)$$

it follows $(C_{0,\tau}(\Gamma), l_1(\Gamma), \mathbf{C})$ is not a Hartogs' triple if Γ is uncountable. This example shows that the i-surjective representation condition in theorem 7.15 may not be replaced by an equivalent j-surjective representation condition. It is also interesting to note that $\mathscr{H}(C_{0,\tau}\,\Gamma)) = \mathscr{H}(C_0(\Gamma))$ [25] but the above shows that $\mathscr{H}(C_{0,\tau}(\Gamma) \times F) \neq \mathscr{H}(C_0(\Gamma) \times F)$ for certain Banach spaces.

8. Locally convex topologies on $\mathscr{H}(U)$

In this section, we study locally convex topologies on spaces of holomorphic functions using the techniques developed in the previous section.

$\mathscr{T}_S$ will denote the topology of uniform convergence on Mackey convergent sequences. If E and F are locally convex spaces, U is a convex balanced open subset of E, and $\theta = (E_\alpha, \pi_\alpha)_{\alpha \in A}$ is a surjective representation of E we let $\mathscr{S}_\theta(U; F)$ denote the set of all F-valued functions on U which have minimal θ-support.

With the above notation, we obtain the following result.

PROPOSITION 8.1. — *Let θ denote a symmetric i-(resp. j) surjective representation of E.*

(a) *If F'_β has the extension property and E has the Silva extension property then every $\mathscr{T}_S$ bounded subset (resp. sequence), B, of $\mathscr{H}_S(U; F) \cap \mathscr{S}_\theta(U; F)$ factors through some α in A.*

(b) *If F'_β has the extension property, E has the complete Silva extension property and is very strongly complete then every $\mathscr{T}_p$ ([25]) bounded subset*

([25]) i. e. pointwise bounded.

(*resp. sequence*), *B*, of $\mathscr{H}_S(U;F) \cap \mathscr{S}_\theta(U;F)$ *factors through some* α *in* *A*.

(*c*) *If F is a Banach space then every* $\mathscr{T}_\delta$ *bounded subset* (*resp. sequence*), *B*, *of* $\mathscr{H}(U;F) \cap \mathscr{S}_\theta(U;F)$ *factors through some* α *in* *A*.

Proof. — For each f in B let $A(f)$ denote a minimal θ-support for f. If $A_1 = \bigcup_{f \in B} A(f)$ is an E-bounded subset of A then the required result follows immediately. Otherwise, using the fact that θ is an i (resp. j) surjective representation when B is a set (resp. a sequence), we can find a sequence of E-open subsets of A, $(W_n)_{n=1}^\infty$, with the following properties;

(8.1) $A_1 \wedge W_n \neq \varnothing$ for each n.

(8.2) If $\alpha \in A$ then there exists a positive integar $n(\alpha)$ such that $\alpha \in W_n^c$ for all $n \geqslant n(\alpha)$. Hence we may choose a sequence of elements in B, $(f_n)_{n=1}^\infty$ such that

(8.3) $f_n(x_n + y_n) \neq f_n(x_n)$ for all x

and

(8.4) $S(y_n) \subset W_n$.

(*a*) Since E has the Silva extension property we may assume that $x_n \to 0$ in (E, τ_M) as $n \to \infty$. Since F_β' has the extension property, we may choose a null sequence in F_β', $(\varphi_n)_{n=1}^\infty$, such that

$$\varphi_n \circ f_n(x_n + y_n) \neq \varphi_n \circ f_n(x_n).$$

Since $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is a symmetric surjective representation, (8.1), (8.2) and (8.4) imply that $(y_n)_{n=1}^\infty$ is a very strongly convergent sequence. By Liouville's theorem, we can choose a sequence of scalars, $(\lambda_n)_{n=1}^\infty$, such that $x_n + \lambda_n y_n \subset U$ for all n and $\left| \varphi_n \circ f_n(x_n + \lambda_n y_n) \right| \to \infty$ as $n \to \infty$. This contradicts the fact that B is a $\mathscr{T}_S$-bounded subset of $\mathscr{H}_S(U;F)$ and completes the proof of (*a*).

(*b*) If E has the complete Silva extension property then we may choose an x in U such that $f_n(x + y_n) \neq f_n(x)$ for all n. Since $(y_n)_{n=1}^\infty$ is a very strongly convergent sequence and E is a very strongly complete locally convex space there exists an integer N such that

$$x + \left\{ \textstyle\sum_{n=N}^\infty \lambda_n y_n \,\middle|\, \lambda_n \in \mathbf{C} \right\} \subset U.$$

By taking subsequences if necessary, we can find a sequence of scalars, $(\beta_n)_{n=1}^\infty$, and a null sequence in F_β', $(\varphi_n)_{n=1}^\infty$, such that

$$\sup_{n \geqslant N} \left| \varphi_n \circ f_n(x + \textstyle\sum_{m \geqslant N} \beta_m y_m) \right| = \infty.$$

This contradicts the fact that B is a $\mathcal{T}_p$-bounded subset of $\mathcal{H}_S(U; F)$ and completes the proof of (b).

(c) Since $(y_n)_{n=1}^\infty$ converges very strongly we can suppose

$$\|f_n(x_n+y_n) - f_n(x_n)\| \to \infty$$

as $n \to \infty$. Since each f in $\mathcal{H}(U; F)$ is locally bounded $f(x_n+y_n) = f(x_n)$ for all sufficiently large n. Now $(\mathcal{H}(U; F), \mathcal{T}_\delta)$ is barrelled hence

$$p(f) = \sup_n \|f(x_n+y_n) - f(x_n)\|$$

is a $\mathcal{T}_\delta$-continuous semi-norm on $(\mathcal{H}(U; F), \mathcal{T}_\delta)$. This contradiction completes the proof of (c).

REMARK 8.2.

(a) We may replace the hypothesis that E has the Silva extension property and $B \subset \mathcal{H}_S(U; F)$ by the hypothesis that E has the k-extension property and $B \subset \mathcal{H}_{HY}(U; F)$ in Proposition 8.1 (a) and (b).

(b) If θ is a symmetric i (resp. j) surjective representation of E, F_1 is a dense (resp. dense separable) subspace of F'_β in which every sequence converges very weakly, $\varphi \circ f$ has a minimal θ-support, $B(\varphi \circ f)$, for all φ in F_1 then $\bigcup_{\varphi \in F_1} B(\varphi \circ f)$ is a minimal θ-support for f if any of the following conditions hold.

$(b\,1)$ f is continuous.

$(b\,2)$ $f \in \mathcal{H}_S(U; F)$ and E has the Silva extension property.

$(b\,3)$ $f \in \mathcal{H}_{HY}(U; F)$ and E has the k-extension property.

(c) It is necessary to place some restriction on F in Proposition 8.2.

Let $E = F = \prod_{n=1}^\infty \mathbf{C}$ and let $\theta = (\mathbf{C}^n, \pi_n)_{n=1}^\infty$.

For each integer n let $f_n((x_m)_{m=1}^\infty) = (\sigma_{n,m} x_m)_{m=1}^\infty$ where $\sigma_{n,m} = 1$ if $n = m$ and $\sigma_{n,m} = 0$ if $n \neq m$. $F'_\beta = \sum_{n=1}^\infty \mathbf{C}$ does not have the extension property and $(f_n)_{n=1}^\infty$ does not factor through any n.

THEOREM 8.3. — Let $\theta = (E_\alpha, \pi_\alpha)_{\alpha \in A}$ denote a symmetric i (resp. j) surjective representation of the locally convex space E and let F denote a Banach space (resp. a Banach space with separable dual). If E has the extension property and $\mathcal{H}(U) \subset \mathcal{S}_\theta(U)$ for every balanced convex open subset U of E then the following results hold for any domain X spread over E.

(a) If θ is a compact surjective representation of E and

$$\mathcal{H}_{HY}(U_\alpha; F) = \mathcal{H}(U_\alpha; F)$$

for any convex balanced open subset of U_α of E_α, $\alpha \in A$, then

$$\mathcal{H}(X ; F_\omega) \cap \mathcal{H}_{HY}(X ; F) = \mathcal{H}(X ; F).$$

(b) *If θ is an open surjective representation of E and*

$$\mathcal{H}(U_\alpha ; F) = \mathcal{H}(U_\alpha ; F_\omega)$$

for any convex balanced open subset U_α of E_α, $\alpha \in A$, then

$$\mathcal{H}(X ; F) = \mathcal{H}(X ; F_\omega).$$

Proof. — We may assume, without loss of generality, that X is a convex balanced open subset of E. If θ is an i-surjective representation, we let B denote the closed unit ball of F'_β and if θ is a j-surjective representation we let B denote a dense sequence in the unit ball of F'_β. If $f \in \mathcal{H}(X ; F_\omega)$ then $\tilde{B} = (\varphi \circ f)_{\varphi \in B}$ is a bounded subset of $(\mathcal{H}(X), \mathcal{T}_0)$. By Proposition 8.1, $\tilde{B}$ factors through some α in A and thus, by the Hahn-Banach Theorem f factors through the same α. Hence there exists $\tilde{f} \in \mathcal{H}_G(\pi_\alpha(X) ; F)$ such that $\tilde{f} \circ \pi_\alpha = f$. If θ is a compact surjective representation of E and $f \in \mathcal{H}_{HY}(X ; F)$ then $\tilde{f} \in \mathcal{H}_{HY}(\pi_\alpha(X) ; F)$, and we may complete the proof of (a) in an obvious manner.

If θ is an open surjective representation of E then $\pi_\alpha(X)$ is a convex balanced open subset of E_α amd $\tilde{f} \in \mathcal{H}(\pi_\alpha(X) ; F_\omega)$. We may now complete the proof of (b) in an obvious manner.

EXAMPLE 8.4.

(a) If $\mathscr{C}(X)$ is infrabarrelled, then $\mathcal{H}(\mathscr{C}(X) ; F) = \mathcal{H}(\mathscr{C}(X) ; F_\omega)$ for any locally convex space F.

(b) We have already noted that

$$\mathcal{H}(C_{0,\tau}(\Gamma) ; C_0(\Gamma)) \neq \mathcal{H}(C_{0,\tau}(\Gamma) ; (C_0(\Gamma))_\omega)$$

if Γ is uncountable. Theorem 8.3 (b) shows that

$$\mathcal{H}(C_{0,\tau}(\Gamma) ; E) = \mathcal{H}(C_{0,\tau}(\Gamma) ; E_\omega)$$

for any *separable* Banach space.

Let $\mathcal{T}_{0,b}$ denote the bornological topology associated with $\mathcal{T}_0$ [12].

PROPOSITION 8.5. — *Let $\theta = (E_\alpha, \pi_\alpha)_{\alpha \in A}$ denote a compact, open, symmetric representation of E and let F denote a Banach space. If E has the extension*

property, U is a balanced open subset of E, and $\mathscr{H}(U; F) \subset \mathscr{S}_\theta(U; F)$ then the following are true.

(a) *If θ is an i-surjective representation of E and the $\mathscr{T}_0$-bounded subsets of $\mathscr{H}(\pi_\alpha(U); F)$ are equibounded for each α in A then the $\mathscr{T}_0$-bounded subsets of $\mathscr{H}(U; F)$ are equibounded* ([26]).

(b) *If θ is a j-surjective representation of E, and*

$$(\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_\delta) = (\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_{0,b})$$

for each α in A then

$$(\mathscr{H}(U; F), \mathscr{T}_\delta) = (\mathscr{H}(U; F), \mathscr{T}_{0,b}).$$

Proof.

(a) Let B denote a $\mathscr{T}_0$-bounded subset of $\mathscr{H}(U; F)$. By Proposition 8.1, B factors through some α in A. Since θ is a compact and an open surjective representation there exists a $\mathscr{T}_0$-bounded subset of $\mathscr{H}(\pi(U_\alpha); F)$, $\tilde{B}$, such that ${}^t\pi_\alpha(\tilde{B}) = B$. Since a $\mathscr{T}_0$-bounded subset of $\mathscr{H}(\pi(U_\alpha); F)$ is equibounded and π_α is an open mapping it follows that $\tilde{B}$ and hence ${}^t\pi_\alpha(\tilde{B})$ is an equibounded subset of $\mathscr{H}(U; F)$.

(b) Since $\mathscr{T}_\delta \geqslant \mathscr{T}_{0,b}$ and $\mathscr{T}_{0,b}$ is a bornological topology it suffices to show that every $\mathscr{T}_0$ bounded sequence, B, in $\mathscr{H}(U; F)$ is also $\mathscr{T}_\delta$-bounded. By using Proposition 8.1, we see, as in part (a), that there exists an α in A and a $\mathscr{T}_0$-bounded sequence, $\tilde{B}$, in $\mathscr{H}(\pi_\alpha(U; F))$ such that ${}^t\pi_\alpha(\tilde{B}) = B$. By our hypothesis, $\tilde{B}$ is a $\mathscr{T}_\delta$-bounded subset of $\mathscr{H}(\pi_\alpha(U); F)$. Using once more the fact that π_α is an open mapping we see that ${}^t\pi_\alpha$ is a bounded mapping and hence B is a $\mathscr{T}_\delta$-bounded subset of $\mathscr{H}(U; F)$. This completes the proof.

EXAMPLE 8.6.

(a) The $\mathscr{T}_0$-bounded subsets of $\mathscr{H}(\mathscr{C}(X))$ are equibounded if, and only if, $\mathscr{C}(X)$ is infrabarrelled.

Proof. — If $\mathscr{C}(X)$ is infrabarrelled then Proposition 8.5 implies that every $\mathscr{T}_0$-bounded subset of $\mathscr{H}(\mathscr{C}(X))$ is equibounded. Conversly, if every $\mathscr{T}_0$-bounded subset of $\mathscr{H}(\mathscr{C}(X))$ is equibounded then every strongly bounded subset of $\mathscr{C}(X)'_\beta$ is equicontinuous and hence $\mathscr{C}(X)$ is infrabarrelled ([22], p. 217).

([26]) i. e. locally uniformly bounded.

(b) Since

$$(\mathscr{H} \left(\sum_{n=1}^{\infty} \mathbf{C} \times \prod_{n=1}^{\infty} \mathbf{C} \right), \mathscr{T}_\delta) \neq (\mathscr{H} \sum_{n=1}^{\infty} \mathbf{C} \times \prod_{n=1}^{\infty} \mathbf{C}), \mathscr{T}_{0,b}) \, [12],$$

it follows that the extension condition is necessary in Proposition 8.5.

PROPOSITION 8.7. — *Let* $\theta = (E_\alpha, \pi_\alpha)_{\alpha \in A}$ *denote a compact, open, symmetric j-surjective representation of* E. *If* E *has the extension property,* U *is a convex balanced open subset of* E *and* $\mathscr{H}(U; F) \subset \mathscr{S}_\theta(U; F)$ *then* $(\mathscr{H}(U; F), \mathscr{T}_{0,b})$ *is barrelled if* $(\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_{0,b})$ *is barrelled for each* α *in* A;

Proof. — Let V denote a closed convex balanced absorbing subset os $(\mathscr{H}(U; F), \mathscr{T}_{0,b})$. Since $(\mathscr{H}(U; F), \mathscr{T}_{0,b})$ is bornological, it sufficef to show that V absorbs any $\mathscr{T}_0$-bounded sequence, B, in $\mathscr{H}(U; F)$. By Proposition 8.1, there exists an α in A and a $\mathscr{T}_0$-bounded sequence in $\pi_\alpha(U)$, $\tilde{B}$, such that ${}^t\pi_\alpha(\tilde{B}) = B$. Let $\tilde{V} = ({}^t\pi_\alpha)^{-1}(V)$. Since π_α is a compact mapping ${}^t\pi_\alpha$ is a continuous mapping from $(\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_{0,b})$ into $(\mathscr{H}(U; F), \mathscr{T}_{0,b})$ and hence $\tilde{V}$ is a closed convex balanced absorbing subset of $(\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_{0,b})$. Since $(\mathscr{H}(\pi_\alpha(U); F), \mathscr{T}_{0,b})$ is barrelled $\tilde{V}$ absorbs $\tilde{B}$ and hence V absorbs B. This completes the proof.

The above technique and Proposition 2.4 of [12] may also be used to show that certain spaces of holomorphic functions are complete.

PROPOSITION 8.8. — *Let* $\theta = (E_\alpha, \pi_\alpha)_{\alpha \in A}$ *denote a compact, open, i (resp. j) representation of* E *and let* U *denote a convex balanced open subset of* E. *If* $\mathscr{H}(U) \subset \mathscr{S}_\theta(U)$ *then the following are true.*

(a) *If* E *has the extension property then* $(\mathscr{H}(U), \mathscr{T}_0)$ *is quasi-complete* (*resp. sequentially complete*) *if* $(\mathscr{H}(\pi_\alpha(U)), \mathscr{T}_0)$ *is quasi-complete* (*resp. sequentially complete*) *for each* $\alpha \in A$.

(b) *If* E *has the extension property and each* F_α *is a normed linear space space then* $(\mathscr{H}(U), \mathscr{T}_{0,b})$ *is quasi-complete* (*resp. sequentially complete*).

(c) *If each* E_α *is a normed linear space then* $(\mathscr{H}(U), \mathscr{T}_\delta)$ *is quasi-complete* (*resp. sequentially complete*).

Proof. — Since θ is a compact and open surjective representation of E it follows that ${}^t\pi_\alpha$ is an isomorphism form $(\mathscr{H}(\pi_\alpha(U)), \mathscr{T}_0)$ onto its image in $(\mathscr{H}(U), \mathscr{T}_0)$ for any convex balanced open subset U of E. This fact and Proposition 8.1 may easily be combined to complete the proof of (a).

If B is a bounded subset of $(\mathscr{H}(\pi_\alpha(U)), \mathscr{T}_0)$ and E_α is a normed linear space then

$$\mathscr{T}_0(\mathscr{H}(\pi_\alpha(U)))|_B = \mathscr{T}_{0,b}(\mathscr{H}(\pi_\alpha(U)))|_B = \mathscr{T}_\delta(\mathscr{H}(\pi_\alpha(U)))|_B$$

(by [12] and [30]). Now using the identification between B and $'\pi_\alpha(B)$ to see that

$$\mathscr{T}_\delta(\mathscr{H}(\pi_\alpha(U)))|_B \geqslant \mathscr{T}_\delta(\mathscr{H}(U))|_{{}^t\pi_\alpha(B)}$$

since π_α is an open mapping.

Also :

$$\mathscr{T}_\delta(\mathscr{H}(U))|_{{}^t\pi_\alpha(B)} \geqslant \mathscr{T}_{0,b}(\mathscr{H}(U))|_{{}^t\pi_\alpha(B)}$$
$$\geqslant \mathscr{T}_0(\mathscr{H}(U))|_{{}^t\pi_\alpha(B)}$$
$$= \mathscr{T}_0(\mathscr{H}(\pi_\alpha(U)))|_B,$$

since θ is a compact surjective representation of E.

We may now complete the proof for (b) and (c) by using Proposition 8.1, our hypothesis and the fact that the different topologies coincide.

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