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HOLOMORPHIC CONVEXITY OF SPACES OF ANALYTIC CYCLES

BY

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A la mémoire de
M. le Professeur W. Rothstein

RÉSUMÉ. — Utilisant la méthode d'approximation-L^2 de Hörmander pour l'opérateur d'', on généralise un résultat de R. NARASIMHAN sur le problème de Levi, en n'imposant qu'aux points réguliers le caractère plurisousharmonique de la fonction d'exhaustion. On utilise cette généralisation pour prouver que certains espaces de cycles analytiques sont de Stein.

SUMMARY. — Using Hörmander's L^2-estimation method for the $\bar{\partial}$-operator, we generalize a result of R. NARASIMHAN about the Levi problem, by requiring only to regular points the plurisubharmonicity of the exhausting function. We apply this result to prove the Steinness of some spaces of analytic cycles.

Let X be a complex manifold of dimension n.

X is said to be *strictly q-pseudoconvex* if X is a relatively compact open subset of another complex manifold and if locally the boundary of X is defined by the vanishing of a real-valued C^2 function φ such that $d\varphi \neq 0$ and the restriction of the Levi form of φ to the holomorphic tangent space of the boundary of X has $n-q-1$ positive eigenvalues and q negative eigenvalues.

X is said to be *strongly q-pseudoconvex* if there exists a C^2 exhaustion function φ on X such that for some compact subset K of X the Levi form of φ has at least $n-q$ positive eigenvalues at every point of $X-K$.

X is said to be *q-complete* if X is strongly q-pseudoconvex and the compact subset K in the preceding definition can be chosen to be the empty set.

All strictly q-pseudoconvex manifolds are strongly q-pseudoconvex.

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By a *positive analytic q-cycle* of X, we mean a formal finite linear combination of irreducible q-dimensional compact subvarieties of X with positive integers as coefficients. Denote by $C_q^+(X)$ the space of all positive analytic q-cycles of X. Suppose X is an open subset of the regular points of a subvariety of $\mathbf{P}_N$. The classical construction of the Chow variety gives $C_q^+(X)$ a complex structure ([2], § 2). As in [2], throughout this paper the complex structure of $C_q^+(X)$ is assumed to be so enriched that continuous function germs on $C_q^+(X)$ which are holomorphic at regular points are holomorphic. In [2], it is proved that, if X is strictly q-pseudoconvex, then $C_q^+(X)$ is holomorphically convex; and if, in addition, $H^{q+1}(X, \mathscr{F}) = 0$ for all coherent analytic sheaves $\mathscr{F}$ on X, then $C_q^+(X)$ is Stein. In this paper, we investigate the same problems for the case where X is only strongly q-pseudoconvex. We obtain the following theorem.

THEOREM 1. — *Suppose X is an open subset of the regular points of a subvariety of $\mathbf{P}_N$ and X is strongly q-pseudoconvex. Then $C_q^+(X)$ is holomorphically convex. If, in addition, $H^{q+1}(X, \mathscr{F}) = 0$ for all coherent analytic sheaves $\mathscr{F}$ on X (in particular, if X is q-complete), then $C_q^+(X)$ is Stein.*

Theorem 1 is proved by canonically associating to the exhaustion function on X a continuous exhaustion function on $C_q^+(X)$ which is plurisubharmonic on the set of regular points of $C_q^+(X)$ and then proving a criterion for Stein spaces involving the existence of such exhaustion functions. The canonical exhaustion function ψ on $C_q^+(X)$ is defined as follows. For $c = \sum_{i=1}^k m_i c_i$, $\psi(c)$ is the supremum of $\varphi(x)$ for $x \in \bigcup_{i=1}^k c_i$.

In [3], the analytic space $C_q^+(X)$ is constructed for any complex analytic space X, but important parts of our paper do not extend.

A complex space is said to be *K-separable* if, for every point x, the set of all points y, for which $f(x) = f(y)$ for all global holomorphic functions f, has dimension 0 at x. The criterion for Stein spaces we will prove is the following.

THEOREM 2. — *If X is a K-separable complex space and there is a continuous exhaustion function φ on X which is plurisubharmonic on the set of regular points of X, then X is Stein. Moreover, for $\lambda' < \lambda \leqslant \infty$, the restriction map $\Gamma(X_\lambda, \mathscr{O}_X) \to \Gamma(X_{\lambda'}, \mathscr{O}_X)$ has dense image, where $X_\mu = \{\varphi < \mu\}$ and $\mathscr{O}_X$ is the structure sheaf of X.*

Theorem 2 is proved by using Hörmander's L^2 estimates for the $\bar{\partial}$ operator [5]. It can be strengthened to the following.

THEOREM 2'. — *Suppose X is a complex space admitting a continuous exhaustion function φ which is plurisubharmonic on the set of regular points of X. Then X is Stein if there exist a continuous function σ on X with the following property: for every $x \in X$, there exist $\varepsilon > 0$ and a holomorphic map π with finite fibers from an open neighborhood U of x to an open subset G of some $\mathbf{C}^n$ such that, if ρ is a real-valued C^2 function on G with compact support whose partial derivatives of order $\leqslant 2$ have absolute values $\leqslant \varepsilon$ on G, then $\sigma + \rho \circ \pi$ is plurisubharmonic on the set of regular points of U.*

Theorem 2' generalizes NARASIMHAN's result on the Levi problem [6].

In the last part of this paper, we will investigate compact subvarieties of pure dimension $> q$ in a strongly q-pseudoconvex manifold. It is conjectured that the union of all compact subvarieties of pure dimension $> q$ in a strongly q-pseudoconvex manifold is a subvariety. We will prove this conjecture in some special cases.

The following notations will be used. The boundary of a set E is denoted by ∂E, and the closure of E is denoted by E^- or $\bar{E}$. If $c = \sum_{i=1}^k m_i c_i$ is a positive analytic q-cycle, then $| c |$ denotes $\bigcup_{i=1}^k c_i$. The coordinates of $\mathbf{C}^n$ are denoted by $z_1, \ldots, z_n$. Complex spaces are not necessarily reduced. $\mathcal{O}_X$ denotes the structure sheaf of X. A function on a subvariety of an open subset G of $\mathbf{C}^n$ is called C^k if it is the restriction of a C^k function on G. A function on a complex space is called C^k if locally it is C^k when the complex space is regarded as a subvariety of an open subset of $\mathbf{C}^n$.

1. Levi problem

(1.1) A real-valued function ψ on an open subset G of $\mathbf{C}^n$ is said to be *strongly plurisubharmonic* if, for every $x \in G$, there exists $\varepsilon > 0$ with the following property: if ρ is a real-valued C^2 function on G with compact support whose partial derivatives of order $\leqslant 2$ have absolute values $\leqslant \varepsilon$ on G, then $\psi + \rho$ is plurisubharmonic on some open neighborhood of x.

A real-valued function φ on a complex space X is said to be *plurisubharmonic* (respectively *strongly plurisubharmonic*) if, for every $x \in X$, there exists a biholomorphic map π from an open neighborhood U of x onto a subvariety of an open subset G of some $\mathbf{C}^n$ and there exists a plurisubharmonic (respectively strongly plurisubharmonic) function ψ on G such that $\varphi = \psi \circ \pi$ on U.

A real-valued (resp. complex-valued) continuous function on a complex space X is said to be *weakly plurisubharmonic* (resp. *weakly holomorphic*) if it is plurisubharmonic (resp. holomorphic) on the set of regular points of X.

If τ is a real-valued C^2 function on an open interval I with nonnegative first and second derivatives and if φ is a plurisubharmonic (respectively weakly plurisubharmonic) function on a complex space X whose range in contained in I, then $\tau \circ \varphi$ is a plurisubharmonic (respectively weakly plurisubharmonic) function on X.

We recall first two results of NARASIMHAN ([6], [7]) :

(i) a complex space is Stein if, and only if, it admits a continuous strongly plurisubharmonic exhaustion function.

(ii) a complex space is Stein if, and only if, the normalization of its reduction is Stein.

(1.2) A reduced complex space X of pure dimension n is called a *branched Riemann domain* over $\mathbf{C}^n$ if it is furnished with a holomorphic map π $X \longrightarrow \mathbf{C}^n$ whose fibers have dimension $\leqslant 0$. The set X_0 of points of X where π is locally biholomorphic is called the *unbranched portion* of X. $X' := X - X_0$ is called the *branching set* of X. When $X' = \varnothing$, X is called an (*unbranched*) *Riemann domain* over $\mathbf{C}^n$.

It is well-known ([4], V. D. 4) that a reduced complex space of pure dimension n is a branched Riemann domain over $\mathbf{C}^n$ if, and only if, it is K-separable.

For a branched Riemann domain $\pi : X \longrightarrow \mathbf{C}^n$ with unbranched portion X_0, we denote by dv the $2n$-form

$$\pi^*\left(\frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge \frac{\sqrt{-1}}{2} dz_n \wedge d\bar{z}_n\right) \quad \text{on } X_0 .$$

For a (0,1)-form $\omega = \sum_{i=1}^n \omega_i \, d\bar{z}_i$ on X_0, $|\omega|^2$ denotes $\sum_{i=1}^n |\omega_i|^2$.

(1.3) PROPOSITION. — *Suppose $\pi : X_0 \longrightarrow \mathbf{C}^n$ is an unbranched Riemann domain such that X_0 is Stein. Let φ be a plurisubharmonic function on X_0. Suppose ω is a C^∞ $\bar{\partial}$-closed* (0,1)-*form on X_0. Then there exists a C^∞ function η on X_0 such that $\bar{\partial}\eta = \omega$ and*

$$\int_{X_0} |\eta|^2 e^{-\varphi} dv \leqslant e\,\delta^2 \int_{X_0} |\omega|^2 e^{-\varphi} dv,$$

where δ is the diameter of $\pi(X_0)$.

The special case where $X_0 \subset \mathbf{C}^n$ is proved in [5] (Theorem 2.2.3), and the proof can be trivially modified to give the general case.

(1.4) LEMMA. — *Suppose* $\pi : X \to \mathbf{C}^n$ *is a branched Riemann domain and* X_0 *is the unbranched portion of* X. *Let* $\{ U_i \}$ *be an open covering of* X. *Then there exists a* C^∞ *partition of unity* $\{ \eta_i \}$ *subordinate to* $\{ U_i \}$ *such that all partial derivatives of* $\eta_i \mid X_0$ *with respect to the coordinates of* $\mathbf{C}^n$ *are locally bounded on* X.

Proof. — By taking a refinement, we can assume without loss of generality the following:

(i) $U_i \subset\subset X$,

(ii) $\pi \mid U_i : U_i \to \pi(U_i)$ is an analytic cover,

(iii) $\{ U_i \}$ is locally finite.

Choose a relatively compact open subset W_i of U_i such that $X \subset \bigcup_i W_i$. Take a nonnegative C^∞ function χ_i with compact support on $\pi(U_i)$ such that $\chi_i(x) > 0$ for $x \in \pi(W_i)$. Let $\tilde{\chi}_i$ be the C^∞ function on X which is the trivial extension of $\chi_i \circ (\pi \mid U_i)$. Define $\eta_i = \tilde{\chi}_i (\sum_j \tilde{\chi}_j)^{-1}$. Then $\{ \eta_i \}$ satisfies the requirement.

Q.E.D.

(1.5) LEMMA. — *Suppose* $\pi : X \to \mathbf{C}^n$ *is a branched Riemann domain with unbranched portion* X_0 *and branching set* X'. *Let* g *be a holomorphic function on* X *such that, for* $x \in X'$, $g = h(\sigma \circ \pi)$ *on some open neighbourhood* U *of* x, *where* h *is a holomorphic function on* U *and* σ *is a holomorphic function on* $\pi(U)$ *vanishing identically on* $\pi(U \cap X')$. *If* f *is a holomorphic function on* X_0 *and* $\int_{X_0} |f|^2 \, dv < \infty$, *then* fg *is weakly holomorphic on* X.

Proof. — Recall that, if G is an open subset of $\mathbf{C}^n$, and A is a subvariety of G of codimension $\geqslant 1$ then every square integrable holomorphic function on $G - A$ can be extended to a holomorphic function on G.

To prove the lemma, it is sufficient to verify it locally, in neighbourhoods of "generic" points, where coordinates can be chosen to realize the special situation:

(i) X is the open unit n-disc,

(ii) $\pi(z_1, \ldots, z_n) = (z_1^l, z_2, \ldots, z_n)$ with $l \geqslant 1$,

(iii) $g = z_1^l$.

Then:

$$dv = l^2 \, | z_1^{l-1} |^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge \frac{\sqrt{-1}}{2} dz_n \wedge d\bar{z}_n,$$

and

$$\int_{X_0} |fg|^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge \frac{\sqrt{-1}}{2} dz_n \wedge d\bar{z}_n \leqslant l^{-2} \int_{X_0} |f|^2 dv < \infty.$$

If follows that fg can be extended to a holomorphic function on X.

Q.E.D.

(1.6) LEMMA. — *Assume that Theorem 2 is true for complex spaces of dimension* $< n$. *Suppose* $\pi : X \to \mathbf{C}^n$ *is a branched Riemann domain such that* $\pi(X) \subset\subset \mathbf{C}^n$, *and* X *is normal and admits a continuous weakly plurisubharmonic exhaustion function* φ. *Suppose* g *is a holomorphic function on* $\pi(X)$ *which vanishes identically on* $\pi(X')$ *but does not vanish identically on any component of* $\pi(X)$. *Then* $H^1(X, \mathcal{O}_X) = 0$.

Proof. — Our first step is to show that X_0 is Stein. Before we can do it, we have to prove first the following.

($\star$) Suppose D is an open polydisc in $\mathbf{C}^n$, and W is a connected normal subvariety of $D \times \mathbf{C}^N$ such that the natural projection $p : D \times \mathbf{C}^N \to D$ makes W an analytic cover over D. Let W_0 be the unbranched portion of W. Then W_0 is Stein.

Let $p^* : W^* \to \mathbf{C}^n$ be the envelope of holomorphy of W_0. Since D is Stein, there exists a holomorphic function on D which can not be extended holomorphically across any boundary point of D and, as a consequence, $p^*(W^*) \subset D$. Let $w_1, \ldots, w_N$ be the coordinates of $\mathbf{C}^N$. The holomorphic function $w_i \mid W_0$ extends uniquely to a holomorphic function $\tilde{w}_i$ on W^* ($1 \leqslant i \leqslant N$). Let $\Phi : W^* \to D \times \mathbf{C}^N$ be defined by p^* and $\tilde{w}_1, \ldots, \tilde{w}_N$. $\Phi \mid W_0$ is the identity map of W_0. We have to show that $W_0 = W^*$. Suppose the contrary. Then there exists a sequence of points $\{ x_\nu \} \subset W_0$ such that $x_\nu \to x^* \in W^* - W_0$. $\Phi(x^*) \notin W_0$, otherwise $x_\nu = \Phi(x_\nu) \to \Phi(x^*)$ in W_0 and, by the uniqueness of the limit in W^*, $x^* = \Phi(x^*) \in W_0$. Because $D \times \mathbf{C}^N$ is Stein, W is the set of common zeros of a set of holomorphic functions $\{ u_j \}$ on $D \times \mathbf{C}^N$. $u_j \circ \Phi$ is a holomorphic function on W^* which is identically zero on W_0. Hence $u_j \circ \Phi \equiv 0$ on W^*. So $\Phi(W^*) \subset W$. Since p^* has Jacobian rank n everywhere on W^*, it follows that, for some open neighborhood U of x^* in W^*, $\Phi(U)$ is a locally closed complex submanifold of dimension n in $D \times \mathbf{C}^N$, and $\Phi(U)$ is a Riemann domain over D under the projection p. Since $\Phi(U) \subset W$ and W is locally irreductible and is of dimension n, $\Phi(U)$ is an open subset of W. Since $\Phi(U)$ is a Riemann domain over D under the projection p, it follows that $\Phi(U) \subset W_0$, contradicting $\Phi(x^*) \notin W_0$. Hence $W_0 = W^*$ is Stein, and $(\bigstar)$ is proved.

Now we are ready to prove that X_0 is Stein. Fix arbitrarily $c > 0$. Let $X' = X_0 \cap \{ \varphi < c \}$. By [4] (p. 283, Theorem 5), it suffices to show that X' is Stein. X' is relatively compact in X. Let x_0 be an arbitrary boundary point of X' in X. We can find an open polydisc D in $\mathbf{C}^n$ and a relatively compact open neighborhood W of x_0 in X such that π makes W an analytic cover over D. Let $\alpha : W \to \mathbf{C}^N$ be a proper holomorphic embedding. Then the map $W \to D \times \mathbf{C}^N$, defined by π and α, is a proper holomorphic embedding of W such that the natural projection $D \times \mathbf{C}^N \to D$ makes its image an analytic cover over D. By $(\bigstar)$, $W \cap X_0$ is Stein. For a point x of $W \cap X_0$ (respectively $W \cap X'$, X',) let $d_1(x)$ (respectively $d_2(x)$, $d_3(x)$) be the largest positive number such that π maps an open neighborhood of x in $W \cap X_0$ (respectively $W \cap X', X'$) biholomorphically onto the open ball centered at $\pi(x)$ with radius $d_1(x)$ (respectively $d_2(x)$, $d_3(x)$). Since $W \cap X_0$ is Stein, by [4] (p. 283, Theorem 4), $-\log d_1$ is a continuous plurisubharmonic function on $W \cap X_0$. Let θ be a C^2 function from $(-\infty, c)$ to $\mathbf{R}$ with nonnegative first and second derivatives such that

$$\lim_{\lambda \to c^-} \theta(\lambda) = \infty.$$

Then $-\log d_1 + \theta \circ \varphi + \sum_{i=1}^n | z_i \circ \pi |^2$ is a continuous strongly plurisubharmonic exhaustion function on $W \cap X'$. By NARASIMHAN's result quoted in (1.1), $W \cap X'$ is Stein. By [4] (p. 283, Theorem 4), $-\log d_2$ is a continuous plurisubharmonic function on $W \cap X'$. There exists an open neighborhood Q of the boundary of $W \cap X'$ in W such that d_2 agrees with d_3 on $Q \cap W \cap X'$. Hence $-\log d_3$ is plurisubharmonic on $Q \cap W \cap X'$. Since x_0 is an arbitrary boundary point of X' in X, as x_0 runs through all the boundary points of X' in X, for some compact subset K of X', the union of $Q \cap W \cap X'$ is $X' - K$. So $-\log d_3$ is plurisubharmonic on $X' - K$. Let A be a positive number greater than the supremum of $-\log d_3$ on K. Let

$$\psi = \max(A, -\log d_3).$$

Then $\psi + \sum_{i=1}^n | z_i \circ \pi |^2$ is a continuous strongly plurisubharmonic exhaustion function on X'. By NARASIMHAN's result quoted in (1.1), X' is Stein. Hence X_0 is Stein.

Let $\tilde{g} = g \circ \pi$. Since Theorem 2 is assumed true for complex spaces of dimension $< n$, the support of $\mathcal{O}_X/\tilde{g}\,\mathcal{O}_X$ is Stein. It follows that $H^1(X, \mathcal{O}_X/\tilde{g}\,\mathcal{O}_X) = 0$ and the map

$$\sigma : \quad H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X),$$

defined by multiplication by $\tilde{g}$, is surjective. To prove the Lemma, it suffices to show that the image of σ is 0.

Let $\tilde{\mathcal{U}} = \{ \tilde{U}_i \}$ and $\mathcal{U} = \{ U_i \}$ be two open coverings of X by Stein open subsets of X such that $U_i \subset\subset \tilde{U}_i$. Take $\tilde{f} = \{ \tilde{f}_{ij} \} \in Z^1(\tilde{\mathcal{U}}, \mathcal{O}_X)$. Let $f = \{ f_{ij} \} \in Z^1(\mathcal{U}, \mathcal{O}_X)$ be the restriction of $\tilde{f}$. It suffices to show that $\tilde{g}f$ belongs to $B^1(\mathcal{U}, \mathcal{O}_X)$. By (1.4), there exists a C^∞ partition of unity $\{ \eta_i \}$ subordinate to $\mathcal{U}$ such that all partial derivatives of $\eta_i \mid X_0$ with respect to the coordinates of $\mathbf{C}^n$ are locally uniformly bounded on X. Let ω be the (0,1)-form on X_0 which equals $\bar{\partial}(\sum_i \eta_i f_{ij})$ on $U_j \cap X_0$. Since $f_{ij} = \tilde{f}_{ij} \mid U_i \cap U_j$ and $U_i \cap U_j \subset\subset \tilde{U}_i \cap \tilde{U}_j$, it follows that f_{ij} is uniformly bounded on $U_i \cap U_j$, and the coefficients of ω are locally uniformly bounded on X. Since φ is an exhaustion function on X, there exists a C^2 function $\tau : \mathbf{R} \to \mathbf{R}$ with non negative first and second derivatives such that

$$\int_{X_0} |\omega|^2 e^{-\tau \circ \varphi} dv < \infty.$$

By (1.3), there exists a C^∞ function h on X_0 such that $\bar{\partial}h = \omega$ on X_0 and

$$\int_{X_0} |h|^2 e^{-\tau \circ \varphi} dv < \infty.$$

Let $f_j = \sum_i \eta_i f_{ij} - h$ on $U_j \cap X_0$. Then f_j is holomorphic on $U_j \cap X_0$. Since U_j is relatively compact in X, $\tau \circ \varphi$ is bounded by some positive number A_j on U_j. Hence

$$\int_{U_j \cap X_0} |h|^2 dv \leqslant e^{A_j} \int_{U_j \cap X_0} |h|^2 e^{-\tau \circ \varphi} dv < \infty.$$

Since f_{ij} is uniformly bounded on $U_i \cap U_j$, it follows that $\sum_i \eta_i f_{ij}$ is uniformly bounded on U_j and

$$\int_{U_j \cap X_0} |f_j|^2 dv < \infty.$$

By (1.5), $f_j \tilde{g}$ can be extended to a holomorphic function f_j^* on U_j. Let

$$f^* = \{f_j^*\} \in C^0(\mathcal{U}, \mathcal{O}_X).$$

Then $\delta f^* = \tilde{g}f$. Hence $\tilde{g}f \in B^1(\mathcal{U}, \mathcal{O}_X)$, and the image of σ is 0.

Q.E.D.

(1.7) LEMMA. — *Suppose X is a reduced complex space with a continuous weakly plurisubharmonic exhaustion function φ. Let S be the singular set of X, and $\mathcal{I}$ be the ideal-sheaf of S. Suppose S is finite-dimensional and Stein, and $H^1(X, \mathcal{I}) = 0$. If there exists a continuous strongly plurisubharmonic function σ on X, then X is Stein.*

Proof. — Since S is finite-dimensional and Stein, there exist holomorphic functions $g_1, \ldots, g_k$ on S such that $\psi := \sum_{i=1}^{k} |g_i|^2$ is an exhaustion function on S. We can choose a C^2 function $\tau : \mathbf{R} \to \mathbf{R}$ with nonnegative first and second derivatives such that $\tau \circ \psi > \varphi$ on S.

Since $H^1(X, \mathcal{I}) = 0$, g_i can be extended to a holomorphic function $\tilde{g}_i$ on $X (1 \leqslant i \leqslant k)$. Let $\tilde{\psi} = \sum_{i=1}^{k} |\tilde{g}_i|^2$, and let $\tilde{\varphi} = \max(\varphi, \tau \circ \tilde{\psi})$. Then $\tilde{\varphi}$ agrees with $\tau \circ \tilde{\psi}$ on an open neighbourhood of S and hence is plurisubharmonic on X. Since $\tilde{\varphi} + \sigma$ is a continuous strongly plurisubharmonic function on X, by NARASIMHAN's result quoted in (1.1), X is Stein.

Q.E.D.

(1.8) LEMMA. — *Assume that Theorem 2 is true for complex spaces of dimension $< n$. Suppose $\pi : X \to \mathbf{C}^n$ is a branched Riemann domain, and X is normal. Suppose φ^* is a continuous weakly plurisubharmonic exhaustion function on X, and σ is a continuous strongly plurisubharmonic function on $\pi(X)$. Let $\varphi = \varphi^* + \sigma \circ \pi$. Then, for $\lambda \in \mathbf{R}$, $X_\lambda := \{ \varphi < \lambda \}$ is Stein.*

Proof. — We can assume that $n \geqslant 1$. Fix arbitrarily $\lambda \in \mathbf{R}$. Let $\mathcal{I}$ be an arbitrary coherent ideal-sheaf on X whose zero-set has dimension $< n$. We are going to apply the "bumping" technique of ANDREOTTI-GRAUERT [1] to show that $H^1(X_\lambda, \mathcal{I})$ is finite-dimensional. Choose Stein open subsets $V_i \subset\subset U_i$ of $X (1 \leqslant i \leqslant k)$ such that

(i) the boundary of X_λ is contained in $\bigcup_{i=1}^{k} V_i$,

(ii) U_i is an analytic cover over $\pi(U_i)$.

Choose a nonnegative C^∞ function ρ_i on $\pi(U_i)$ with compact support such that $\rho_i(x) > 0$ for $x \in \pi(V_i) (1 \leqslant i \leqslant k)$. There exists $\varepsilon > 0$ such t at $\sigma - \varepsilon \sum_{j=1}^{k} \rho_j$ is plurisubharmonic on $\pi(X) (1 \leqslant i \leqslant k)$. Let $\tilde{\rho}_i$ be the C^∞ function on X which is the trivial extension of $\rho_i \circ (\pi \mid U_i)$ $(1 \leqslant i \leqslant k)$. Define $\varphi_i = \varphi - \varepsilon \sum_{j=1}^{i} \tilde{\rho}_j$ for $1 \leqslant i \leqslant k$. Let

$$Y_0 = X_\lambda \quad \text{and} \quad Y_i = \{ \varphi_i < \lambda \} \quad (1 \leqslant i \leqslant k).$$

Since U_i is Stein, there exists a continuous strongly plurisubharmonic exhaustion function ψ_i on U_i $(1 \leqslant i \leqslant k)$. Choose a C^2 function $\tau : (-\infty, \lambda) \to \mathbf{R}$ with nonnegative first and second derivatives such that $\lim_{\mu \to \lambda^-} \tau(\mu) = \infty$. Then $\max(\psi_i, \tau \circ \varphi_{i-1})$ is a continuous weakly plurisubharmonic exhaustion function on $Y_{i-1} \cap U_i$ $(1 \leqslant i \leqslant k)$. Since $\pi(U_i)$ is Stein, there exists a holomorphic function on $\pi(U_i)$ which vanishes identically on $\pi(U_i \cap X')$ but does not vanish identically on any component of $\pi(U_i)$ $(1 \leqslant i \leqslant k)$. It follows from (1.6) that $H^1(Y_{i-1} \cap U_i, \mathcal{O}_X) = 0$ for $1 \leqslant i \leqslant k$. Because U_i is Stein, there exists $g_i \in \Gamma(U_i, \mathcal{S})$ which does not vanish identically on any branch of U_i $(1 \leqslant i \leqslant k)$. Since Theorem 2 is assumed to be true for complex spaces of dimension $< n$, it follows that $Y_{i-1} \cap U_i \cap \operatorname{Supp}(\mathcal{S}/\mathcal{O}_X g_i)$ is Stein and

$$H^1\left(Y_{i-1} \cap U_i, \frac{\mathcal{S}}{\mathcal{O}_X} g_i \right) = 0 (1 \leqslant i \leqslant k).$$

From the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\alpha_i} \mathcal{S} \to \frac{\mathcal{S}}{\mathcal{O}_X} g_i \to 0,$$

on U_i (where α_i is defined by multiplication by g_i), we conclude that $H^1(Y_{i-1} \cap U_i, \mathcal{S}) = 0$ $(1 \leqslant i \leqslant k)$. Because $Y_i = Y_{i-1} \cap U_i$, it follows from the Mayer-Vietoris sequence

$$H^1(Y_i, \mathcal{S}) \to H^1(Y_{i-1}, \mathcal{S}) \oplus H^1(U_i, \mathcal{S}) \to H^1(Y_{i-1} \cap U_i, \mathcal{S}),$$

that the restriction map $\beta_i : H^1(Y_i, \mathcal{S}) \to H^1(Y_{i-1}, \mathcal{S})$ is surjective $(1 \leqslant i \leqslant k)$. Since $Y_0 \subset\subset Y_k$ and the restriction map

$$H^1(Y_k, \mathcal{S}) \to H^1(Y_0, \mathcal{S})$$

is surjective, we conclude that $H^1(Y_0, \mathcal{S})$ is finite-dimensional.

Next we are going to prove that $H^1(X_\lambda, \mathcal{S}) = 0$. Suppose the contrary. Let ω be a non-zero element of $H^1(X_\lambda, \mathcal{S})$. Let A be the set of all entire functions on $\mathbf{C}^n$. For $f \in A$, define $\Phi_f : \mathcal{S} \to \mathcal{S}$ by multiplication by $f \circ \pi$. For $f \in A - \{0\}$, both $\operatorname{Supp} \operatorname{Ker} \Phi_f$ and $\operatorname{Supp} \operatorname{Coker} \Phi_f$ have dimension $< n$. Since Theorem 2 is assumed to be true, and since $\tau \circ \varphi$ is a continuous weakly plurisubharmonic exhaustion function for X_λ, it follows that, for $f \in A - \{ 0 \}$, $X_\lambda \cap \operatorname{Supp} \operatorname{Ker} \Phi_f$ and $X_\lambda \cap \operatorname{Supp} \operatorname{Coker} \Phi_f$ are both Stein and

$$H^2(X_\lambda, \operatorname{Ker}\Phi_f) = 0,$$
$$H^1(X_\lambda, \operatorname{Coker}\Phi_f) = 0.$$

From the cohomology sequence of the short exact sequences

$$0 \to \operatorname{Ker}\Phi_f \to \mathscr{I} \to \operatorname{Im}\Phi_f \to 0,$$
$$0 \to \operatorname{Im}\Phi_f \to \mathscr{I} \to \operatorname{Coker}\Phi_f \to 0,$$

we conclude that the map

$$\Phi_f^* : \quad H^1(X_\lambda, \mathscr{I}) \to H^1(X_\lambda, \mathscr{I}),$$

induced by Φ_f is surjective for $f \in A - \{ 0 \}$. Since $H^1(X_\lambda, \mathscr{I})$ is finite-dimensional, Φ_f^* is also injective for $f \in A - \{ 0 \}$. Hence

$$f \mapsto \Phi_f^*(\omega),$$

defines a monomorphism from A to $H^1(X_\lambda, \mathscr{I})$, contradicting the finite-dimensionality of $H^1(X_\lambda, \mathscr{I})$. So $H^1(X_\lambda, \mathscr{I}) = 0$.

For $x \in X$ let m_x be the maximal sheaf of ideals for the subvariety $\{ x \}$. Choose $\lambda < \tilde{\lambda} < \infty$. Since $H^1(X_{\tilde{\lambda}}, m_x^2) = 0$ for $x \in X_{\tilde{\lambda}}$, it follows that holomorphic functions on $X_{\tilde{\lambda}}$ give local embedding of $X_{\tilde{\lambda}}$ at every point of $X_{\tilde{\lambda}}$. We can select a finite number of holomorphic functions $f_1, \ldots, f_l$ on $X_{\tilde{\lambda}}$ which give local embedding of X_λ at every point of X_λ. The function $\sum_{i=1}^l |f_i|^2$ is continuous and strongly plurisubharmonic on X_λ.

Since $H^1(X_\lambda, \mathscr{I}) = 0$ when $\mathscr{I}$ is the ideal-sheaf of X', it follows from (1.7) that X_λ is Stein.

Q.E.D.

(1.9) LEMMA. — *Suppose*

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r] \arrow[d, "r'"'] & E \arrow[r, "\beta"] \arrow[d, "r"'] & E'' \arrow[r] \arrow[d, "r''"'] & 0 \\
0 \arrow[r] & F' \arrow[r, "\sigma"] & F \arrow[r, "\tau"] & F'' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of Fréchet spaces and continuous linear maps whose rows are exact. Suppose one of the following two conditions is satisfied:

(i) *r' and r'' have dense image;*

(ii) *r'' has dense image and* $\operatorname{Im} \sigma \subset (\operatorname{Im} r)^-$.

Then r has dense image.

Proof. — Suppose the contrary. Then there exists a nonzero continuous linear functional f on F such that $f \circ r \equiv 0$ on E. Since $(\operatorname{Im} r')^{-} = F$ or $\operatorname{Im} \sigma \subset (\operatorname{Im} r)^{-}$, it follows that $f \circ \sigma = 0$ on F'. By the exactness of the last row of the diagram, there exists a continuous linear functional g on F'' such that $f = g \circ \tau$. Since β is surjective and $f \circ r \equiv 0$ on E, it follows that $g \circ r'' \equiv 0$ on E''. Since r'' has dense image, we have $g \equiv 0$ on F'', which contradicts that f is nonzero.

Q.E.D.

(1.10) LEMMA. — *Suppose X is a complex space, and φ is a continuous weakly plurisubharmonic exhaustion function on X such that $X_\lambda := \{ \varphi < \lambda \}$ is Stein fur $\lambda \in \mathbf{R}$. Then, for any coherent analytic sheaf $\mathscr{F}$ on X, the restriction map $\Gamma(X_\lambda, \mathscr{F}) \to \Gamma(X_{\lambda'}, \mathscr{F})$ has dense image for $\lambda' < \lambda < \infty$.*

Proof. — Since $X_\lambda \subset\subset X$ for $\lambda \in \mathbf{R}$, we can assume without loss of generality that $\dim X < \infty$. Let $n = \dim X$. We prove by induction on n. Suppose the Lemma is true for spaces of dimension $< n$. Fix $\lambda' < \lambda < \infty$.

First, we observe that for any given X, it suffices to prove the special case where $\mathscr{F} = \mathcal{O}_X$. For, by replacing X by $X_{\tilde{\lambda}}$ for some $\lambda < \tilde{\lambda} < \infty$, we can assume that there is a sheaf-epimorphism $\mathcal{O}_X^p \to \mathscr{F}$ on X and, since in the diagram

$$\begin{array}{ccc} \Gamma(X_\lambda, \mathcal{O}_X^p) & \to & \Gamma(X_\lambda, \mathscr{F}) \\ {\scriptstyle r_1}\downarrow & \eta & \downarrow{\scriptstyle r_2} \\ \Gamma(X_{\lambda'}, \mathcal{O}_X^p) & \to & \Gamma(X_{\lambda'}, \mathscr{F}) \end{array}$$

η is surjective and r_1 has dense image, it follows that r_2 has dense image.

Secondly, we reduce the general case to the case where X is reduced and normal. Suppose the Lemma is true when X is reduced and normal, and we want to prove it for the case of a general X. Let $\mathscr{K}$ be the subsheaf of nilpotent elements of $\mathcal{O}_X$. Let $\mathcal{O}'_X = \mathcal{O}_X / \mathscr{K}$, and let $\tilde{\mathcal{O}}_X$ be the sheaf of germs of weakly holomorphic functions on the reduction of X. By replacing X by $X_{\tilde{\lambda}}$ for some $\lambda < \tilde{\lambda} < \infty$, we can assume without loss of generality the following:

(i) $\mathscr{K}^k = 0$ for some nonnegative integer k;

(ii) there exists $u \in \Gamma(X, \mathcal{O}'_X)$ not vanishing identically on any branch of X such that $u \tilde{\mathcal{O}}_X \subset \mathcal{O}'_X$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to \Gamma(X_\lambda, \tilde{\mathcal{O}}_X) & \to & \Gamma(X_\lambda, \mathcal{O}'_X) & \to & \Gamma(X_\lambda, \mathcal{O}'_X/u\,\tilde{\mathcal{O}}_X) \to 0 \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a''} \\ 0 \to \Gamma(X_{\lambda'}, \tilde{\mathcal{O}}_X) & \to & \Gamma(X_{\lambda'}, \mathcal{O}'_X) & \to & \Gamma(X_{\lambda'}, \mathcal{O}'_X/u\,\tilde{\mathcal{O}}_X) \to 0 \end{array}$$

induced by the exact sequence

$$0 \to \tilde{\mathcal{O}}_X \xrightarrow{b} \mathcal{O}'_X \to \mathcal{O}'_X/u\,\tilde{\mathcal{O}}_X \to 0,$$

where b is defined by multiplication by u. Since we assume that the Lemma is true when X is reduced and normal, it follows that a' has dense image. By induction hypothesis, a'' has dense image. Hence by (1.9), a has dense image. Consider now the following commutative diagram with exact rows $(0 \leqslant \nu \leqslant k)$.

$$\begin{array}{ccccccc} 0 \to \Gamma(X_\lambda, \mathcal{K}^\nu/\mathcal{K}^{\nu+1}) & \to & \Gamma(X_\lambda, \mathcal{O}_X/\mathcal{K}^{\nu+1}) & \to & \Gamma(X_\lambda, \mathcal{O}_X/\mathcal{K}^\nu) \to 0 \\ \downarrow & & \downarrow{\scriptstyle \theta_{\nu+1}} & & \downarrow{\scriptstyle \theta_\nu} \\ 0 \to \Gamma(X_{\lambda'}, \mathcal{K}^\nu/\mathcal{K}^{\nu+1}) & \to & (X_{\lambda'}, \mathcal{O}_X/\mathcal{K}^{\nu+1}) & \to & \Gamma(X_{\lambda'}, \mathcal{O}_X/\mathcal{K}^\nu) \to 0 \end{array}$$

induced by the exact sequence

$$0 \to \mathcal{K}^\nu/\mathcal{K}^{\nu+1} \to \mathcal{O}_X/\mathcal{K}^{\nu+1} \to \mathcal{O}_X/\mathcal{K}^\nu \to 0.$$

Since $\mathcal{K}^\nu/\mathcal{K}^{\nu+1}$ is an $\mathcal{O}_X$-sheaf and a has dense image, we conclude from (1.9) and by induction on ν that $\theta_{\nu+1}$ has dense image for $0 \leqslant \nu < k$. In particular, the restriction map $\Gamma(X_\lambda, \mathcal{O}_X) \to \Gamma(X_{\lambda'}, \mathcal{O}_X)$ has dense image.

Now we assume that X is reduced and normal. By replacing X by $X_{\tilde{\lambda}}$ for some $\lambda < \tilde{\lambda} < \infty$, we can assume without loss of generality that

(i) X can be represented as a branched Riemann domain $\pi : X \to \mathbf{C}^n$,

(ii) there exists a holomorphic function h on X which vanishes identically on the branching set X' but does not vanish identically on any branch of X

By (1.5), there exists a positive integer l such that, if U is an open subset of X_λ and f is a holomorphic function on $U - X'$ with $\int_{U-X'} |f|^2\, dv < \infty$, then $h^l f$ can be extended to a holomorphic function on U. Let $g = h^l$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to \Gamma(X_\lambda, \mathcal{O}_X) & \to & \Gamma(X_\lambda, \mathcal{O}_X) & \to & \Gamma(X_\lambda, \mathcal{O}_X/g\,\mathcal{O}_X) \to 0 \\ \downarrow & \sigma & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r''} \\ 0 \to \Gamma(X_{\lambda'}, \mathcal{O}_X) & \to & \Gamma(X_{\lambda'}, \mathcal{O}_X) & \to & \Gamma(X_{\lambda'}, \mathcal{O}_X/g\,\mathcal{O}_X) \to 0 \end{array}$$

induced by the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\tau} \mathcal{O}_X \to \mathcal{O}_X/g\,\mathcal{O}_X \to 0$$

where τ is defined by multiplication by g. By induction hypothesis, r'' has dense image. By (1.9), to show that r has dense image, it suffices to show that $\operatorname{Im}\sigma \subset (\operatorname{Im} r)^{-}$.

Fix $\varepsilon > 0$ and $\lambda_1 < \lambda'$. There exist $\lambda_1 < \lambda_2 < \lambda'$ and an open neighbourhood W of X' in X such that

$$\|f\|_{X_{\lambda_1}^{-}} \leqslant \|f\|_{X_{\lambda_2}^{-}-W},$$

for $f \in \Gamma(X_{\lambda'}, \mathcal{O}_X)$, where $\| \;\|_E$ denotes the supremum norm on E. Fix $\lambda_2 < \lambda_3 < \lambda_4 < \lambda'$. There exists $C > 0$ such that

$$\|f\|_{X_{\lambda_2}^{-}-W} \leqslant C \int_{X_{\lambda_3}-X'} |f|^2\, dv \qquad \text{for } f \in \Gamma(X_{\lambda_3} - X', \mathcal{O}_X).$$

Take $f \in \Gamma(X_{\lambda'}, \mathcal{O}_X)$. By (1.4), there exists a nonnegative C^∞ function ρ on $X_{\lambda'}$ with compact support such that $\rho \equiv 1$ on X_{λ_4} and the partial derivatives of ρ on $X_{\lambda'} - X'$ with respect to the coordinates of $\mathbf{C}^n$ are locally uniformly bounded on $X_{\lambda'}$. Let $\omega = (\bar\partial \rho) f$ on $X_\lambda - X'$. Since $\omega \equiv 0$ on X_{λ_4}, we can choose a C^2 function $\tau : (-\infty, \lambda) \to \mathbf{R}$ with nonnegative first and second derivatives such that

(i) $\tau \equiv \lambda_3$ on $(-\infty, \lambda_3)$,

(ii) $\tau(\lambda_4)$ is large enough to give $\displaystyle\int_{X_\lambda - X'} |\omega|^2 e^{-\tau\circ\varphi}\, dv \leqslant \varepsilon$.

By (1.3), there exists a C^∞ function h on $X_\lambda - X'$ such that $\bar\partial h = \omega$ and

$$\int_{X_\lambda - X'} |h|^2 e^{-\tau\circ\varphi}\, dv \leqslant e\,\delta^2\,\varepsilon,$$

where δ is the diameter of $\pi(X_\lambda)$. Let $\tilde f = \rho f - h$. Then $\tilde f$ is holomorphic on $X_\lambda - X'$ and is locally square integrable on X_λ with respect to dv. Moreover,

$$\int_{X_{\lambda_3} - X'} |\tilde f - f|^2\, dv \leqslant e^{\lambda_3 + 1}\, \delta^2\, \varepsilon.$$

Hence:

$$\|\tilde f - f\|_{X_{\lambda_2}^{-} - W} \leqslant C\, e^{\lambda_3 + 1}\, \delta^2\, \varepsilon.$$

Let $C' = \| g \|_{X_{\lambda_2}^-}$. Since $g\tilde{f}$ is holomorphic on X_λ,

$$\| g f - g f \|_{X_{\lambda_1}^-} \leqslant \| g f - g f \|_{X_{\lambda_2}^- - W} \leqslant C' C e^{\lambda_3 + 1} \delta^2 \varepsilon.$$

Because $f \in \Gamma(X_{\lambda'}, \mathcal{O}_X)$, $\varepsilon > 0$, and $\lambda_1 \in (-\infty, \lambda')$ are arbitrarily chosen, it follows that $\operatorname{Im} \sigma \subset (\operatorname{Im} r)^-$.

Q.E.D.

(1.11) *Proof of Theorem* 2. – By (1.10), it suffices to show that X_λ is Stein for $\lambda \in \mathbf{R}$ (*cf.* [12], Satz 1.3, or [4], Cor. 9 and Th. 10, p. 214-215). We can assume that $\dim X$ is finite. We prove by induction on $\dim X$. We can also assume that X is reduced and normal. Since X is K-separable, we can represent X as a branched Riemann domain $\pi : X \to \mathbf{C}^n$.

Fix $\lambda_0 \in \mathbf{R}$. Choose a C^2 function $\tau : (-\infty, \lambda_0) \to \mathbf{R}$ with nonnegative first and second derivatives such that $\lim_{\mu \to \lambda_0^-} \tau(\mu) = \infty$. Let $\varphi^* = \tau \circ \varphi$. Then φ^* is a continuous weakly plurisubharmonic exhaustion function on X_{λ_0}. Let σ be a continuous strongly plurisubharmonic function on $\mathbf{C}^n$. Let $Y_\lambda = \{ \varphi^* + \sigma \circ \pi < \lambda \}$. By (1.8), Y_λ is Stein for every $\lambda \in \mathbf{R}$. By (1.10), the restriction map $\Gamma(Y_\lambda, \mathcal{O}_X) \to \Gamma(Y_{\lambda'}, \mathcal{O}_X)$ has dense image for $\lambda' < \lambda < \infty$. Hence $X_{\lambda_0} = \bigcup_{\lambda \in \mathbf{R}} Y_\lambda$ is Stein.

Q.E.D.

(1.12) *Proof of Theorem* 2'. – By Theorem 2, it suffices to show that $X_\lambda := \{ \varphi < \lambda \}$ is K-separable for every $\lambda \in \mathbf{R}$. Fix $\lambda_0 \in \mathbf{R}$. Let λ_1 be the supremum of σ on X_{λ_0}. Let $\lambda = \lambda_0 + \lambda_1$ and $Y = \{ \varphi + \alpha < \lambda \}$. Then $X_{\lambda_0} \subset Y$. ∂Y can be covered by a finite number of Stein open subsets U_i $(1 \leqslant i \leqslant k)$ with the following property: there exist $\varepsilon_i > 0$ and a proper holomorphic map π_i from U_i to an open subset G_i of $\mathbf{C}^{n_i}$ such that, if ρ is a real-valued C^2 function on G_i with compact support whose partial derivatives of order $\leqslant 2$ have absolute values $\leqslant \varepsilon_i$ on G_i, then $\sigma + \rho \circ \pi_i$ is weakly plurisubharmonic on U_i. Let $\varepsilon = \min_{1 \leqslant i \leqslant k} \varepsilon_i$ Choose open subsets

$$W_i \subset\subset Q_i \subset\subset G_i \quad (1 \leqslant i \leqslant k)$$

such that $\partial Y \subset \bigcup_{i=1}^k \pi^{-1}(W_i)$. Let α_i be a nonnegative C^2 function on Q_i with compact support $(1 \leqslant i \leqslant k)$ such that $\alpha_i(x) > 0$ for $x \in W_i$ and the partial derivatives of α_i of order $\leqslant 2$ have absolute values $\leqslant \varepsilon/2k$ on Q_i. Let β_i be a nonnegative C^2 function on G_i with compact support $(1 \leqslant i \leqslant k)$ such that $\beta_i(x) > 0$ for $x \in Q_i$ and the partial derivatives of β_i of order $\leqslant 2$ have absolute values $\leqslant \varepsilon/2k$ on G_i.

Define $\tilde{\alpha}_i$ (respectively $\tilde{\beta}_i$) as the C^2 function on X which is the trivial extension of $\alpha_i \circ \pi_i$ (respectively $\beta_i \circ \pi_i$) $(1 \leqslant i \leqslant k)$. Let

$$\varphi_0 = \varphi + \sigma,$$

$$\varphi_{i+1} = \varphi_i - \tilde{\alpha}_{i+1}, \qquad (0 \leqslant i < k),$$

$$\varphi_{i+1} = \varphi_i - \tilde{\beta}_{i-k+1}, \qquad (k \leqslant i < 2k).$$

Since $(\sigma/2k) - \tilde{\alpha}_i$ and $(\sigma/2k) - \tilde{\beta}_i$ are weakly plurisubharmonic on $X (1 \leqslant i \leqslant k)$, it follows that φ_i is weakly plurisubharmonic on X for $0 \leqslant i \leqslant 2k$. Let $Y_i = \{ \varphi_i < \lambda \} (0 \leqslant i \leqslant 2k)$ and let $U_{k+i} = U_i$. for $1 \leqslant i \leqslant k$. Since U_i is Stein, there exists a continuous strongly plurisubharmonic exhaustion function ψ_i on $U_i (1 \leqslant i \leqslant 2k)$. Choose a C^2 function $\tau : (-\infty, \lambda) \to \mathbf{R}$ with nonnegative first and second derivatives such that $\lim_{\mu \to \lambda^-} \tau(\mu) = \infty$. Since $\max(\psi_i, \tau \circ \varphi_{i-1})$ is a continuous weakly plurisubharmonic function on $Y_{i-1} \cap U_i (1 \leqslant i \leqslant 2k)$, by Theorem 2, $Y_{i-1} \cap U_i$ is Stein.

Let $\mathcal{I}$ be an arbitrary coherent ideal-sheaf on X whose zero-set is disjoint from Y_0. We are going to show that, for $1 \leqslant i \leqslant k$, the restriction map

$$f_i : \quad \Gamma(U_i, \mathcal{I}) \to \Gamma(Y_{i-1} \cap U_i, \mathcal{I}),$$

has dense image. Fix $1 \leqslant i \leqslant k$. Let K be an arbitrary compact subset of $Y_{i-1} \cap U_i$. Let A be the supremum of ψ_i on K. Define

$$\chi = \max[(\lambda/A)\psi_i, \varphi_{i-1}] \qquad \text{and} \qquad H = \{ \chi < \lambda \}.$$

Then: $K \subset H \subset Y_{i-1} \cap U_i$. Since χ is a continuous weakly plurisubharmonic exhaustion function on U_i, by Theorem 2 and (1.10), the restriction map

$$\Gamma(U_i, \mathcal{I}) \to \Gamma(H, \mathcal{I}),$$

has dense image. Since K is an arbitrary compact subset of $Y_{i-1} \cap U_i$, it follows that f_i has dense image.

Since $Y_i = Y_{i-1} \cup (Y_i \cap U_i)$, we have the following Mayer-Vietoris sequence $(1 \leqslant i \leqslant 2k)$:

$$\begin{aligned}\Gamma(Y_{i-1}, \mathcal{I}) \oplus \Gamma(Y_i \cap U_i, \mathcal{I}) &\xrightarrow{g_i} \Gamma(Y_{i-1} \cap U_i, \mathcal{I}) \\ &\xrightarrow{h_i} H^1(Y_i, \mathcal{I}) \to H^1(Y_{i-1}, \mathcal{I}) \oplus H^1(Y_i \cap U_i) \\ &\to H^1(Y_{i-1} \cap U_i, \mathcal{I}).\end{aligned}$$

Since $H^1(Y_{i-1} \cap U_i, \mathscr{I}) = 0$ by the fact that $Y_{i-1} \cap U_i$ is Stein, it follows that the restriction map:

$$\theta_i : \quad H^1(Y_i, \mathscr{I}) \to H^1(Y_{i-1}, \mathscr{I}),$$

is surjective for $1 \leqslant i \leqslant 2k$. In particular, the restriction map:

$$H^1(Y_{2k}, \mathscr{I}) \to H^1(Y_i, \mathscr{I}),$$

is surjective for $1 \leqslant i \leqslant k$. Since $Y_i \subset\subset Y_{2k}$ for $1 \leqslant i \leqslant k$, $H^1(Y_i, \mathscr{I})$ is finite-dimensional and is therefore Hausdorff for $1 \leqslant i \leqslant k$. It follows that $\operatorname{Im} g_i = \operatorname{Ker} h_i$ is closed for $1 \leqslant i \leqslant k$. Since g_i has dense image for $1 \leqslant i \leqslant k$, we conclude that, for $1 \leqslant i \leqslant k$, g_i is surjective and, due to the vanishing of $H^1(U_i, \mathscr{I})$, θ_i is bijective. In particular, the restriction map:

$$\xi : \quad H^1(Y_k, \mathscr{I}) \to H^1(Y_0, \mathscr{I}),$$

is an isomorphism. Consider the following diagram:

$$\begin{tikzcd}
\Gamma(Y_k, \mathcal{O}_X) \arrow[r, "\xi"] & \Gamma(Y_k, \mathcal{O}_X/\mathscr{I}) \arrow[r] & H^1(Y_k, \mathscr{I}) \arrow[r] \arrow[d, "\xi"'] & H^1(Y_k, \mathcal{O}_X) \arrow[d] \\
& & H^1(Y_0, \mathscr{I}) \arrow[r, "\eta"] & H^1(Y_0, \mathcal{O}_X)
\end{tikzcd}$$

where the rows come from

$$0 \to \mathscr{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathscr{I} \to 0,$$

and the vertical maps are restriction maps. Since $\mathscr{I} = \mathcal{O}_X$ on Y_0, η is an isomorphism. It follows that ξ is surjective.

For $x \in X$, let m_x be the maximum sheaf of ideals for the subvariety $\{x\}$. By setting $\mathscr{I}$ in turn equal to m_x^2 and $m_x \cap m_y$ for $x, y \in Y_k - Y_0$, we conclude from the surjectivity of ξ that holomorphic functions on Y_k separate points of $Y_k - Y_0$ and give local embedding of Y_k at every point of $Y_k - Y_0$. Let $\Omega \subset\subset Y_k$ be an open neighbourhood of $\bar{Y_0}$. We can select a finite number of holomorphic functions $F_1, \ldots, F_l$ on Y_k such that the map $F : \Omega \to \mathbf{C}^l$, defined by $F_1, \ldots, F_l$ maps $\Omega - Y_0$ injectively into $\mathbf{C}^l$. Let

$$Z = \{x \in \Omega ; \dim_x F^{-1} F(x) > 0\}.$$

We are going to show that Z is empty. Suppose the contrary. Then Z is a compact subvariety which is positive-dimensional at every point. Let Z_0 be a branch of Z. The supremum of σ on Z_0 is assumed at some point x^* of Z_0. There exist $\varepsilon^* > 0$ and a proper holomorphic map π^*

from an open neighbourhood U^* of x^* to an open neighbourhood G^* of some $\mathbf{C}^{n^*}$ such that, if ρ is a real-valued C^2 function on G^* with compact support whose partial derivatives of order $\leqslant 2$ have absolute values $\leqslant \varepsilon^*$, then $\sigma + \rho \circ \pi^*$ is weakly plurisubharmonic on U^*. Choose a real-valued C^2 function ρ on G^* with compact support such that $\rho(\pi^*(x^*)) > \rho(y)$ for all $y \in G$, and the partial derivatives of ρ of order $\leqslant 2$ have absolute values $\leqslant \varepsilon^*$ on G^*. The function $s := (\sigma + \rho \circ \pi^*) \mid Z_0 \cap U^*$ is weakly plurisubharmonic on $Z_0 \cap U^*$ and $s(x^*) > s(x)$ for $x \in Z_0 \cap U^*$, which is easily seen to be a contradiction when one considers a holomorphic map with finite fibers from the open unit disc to $Z_0 \cap U^*$ whose image contains x^* and considers the pullback of s under such a map. Hence $Z = \varnothing$ and Y_0 is K-separable.

Q.E.D.

2. Separation of cycles

If c is a compact subvariety of pure dimension q in a complex manifold X, and c_0 the set of regular points of c, then for a C^∞ (q, q)-form ω on X. we define $\int_c \omega$ as $\int_{c_0} \omega$, according to the well known result of P. LELONG,

(2.1) LEMMA. — *Suppose X is a complex manifold of dimension n, and $c_1, \ldots, c_k$ are distinct irreducible compact subvarieties of dimension q in X. Suppose one of the following two conditions is satisfied:*

(i) *$H^{q+1}(X, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ on X.*

(ii) *There exist $\lambda_0 \in \mathbf{R}$ and a C^2 exhaustion function φ on X such that the Levi form of φ has at least $n-q$ positive eigenvalues at every point of $\{ \varphi > \lambda_0 \}$, and c_1 is not contained in $\{ \varphi \leqslant \lambda_0 \}$.*

Then there exists a $\bar{\partial}$-closed C^∞ (q, q)-form ω on X such that

$$\int_{c_1} \omega = 1 \qquad \text{and} \qquad \int_{c_j} \omega = 0 \qquad \text{for} \quad 1 < j \leqslant k.$$

Proof. — Define K as follows. When (i) is satisfied, $K = \varnothing$. When (ii) is satisfied, $K = \{ \varphi \leqslant \lambda_0 \}$. Let Ω^q be the sheaf of germs of holomorphic q-forms on X, and let $\mathscr{I}$ be the ideal-sheaf of $c := c_1 \cup \ldots \cup c_k$. Define $\mathscr{F} = \Omega^q / \mathscr{I}\, \Omega^q$. Consider the commutative diagram

$$\begin{tikzcd} H^q(X, \Omega^q) \arrow[r, "\gamma"] \arrow[d] & H^q(X, \mathscr{F}) \arrow[r] \arrow[d, "\beta"] & H^{q+1}(X, \mathscr{I}\Omega^q) \arrow[d, "\alpha"] \\ H^q(X, \Omega^q) \arrow[r] & H^q(K, \mathscr{F}) \arrow[r] & H^{q+1}(K, \mathscr{I}\Omega^q) \end{tikzcd}$$

induced by

$$0 \to \mathscr{I}\Omega^q \to \Omega^q \to \mathscr{F} \to 0.$$

α is an isomorphism. For, when (i) is satisfied, both $H^{q+1}(X, \mathscr{I}\Omega^q)$ and $H^{q+1}(K, \mathscr{I}\Omega^q)$ are zero, and, when (ii) is satisfied,

$$H^{q+1}(K, \mathscr{I}\Omega^q) = \operatorname{ind}\lim_{\lambda > \lambda_0} H^{q+1}(X_\lambda, \mathscr{I}\Omega^q),$$

and, for $\lambda > \lambda_0$,

$$H^{q+1}(X, \mathscr{I}\Omega^q) \to H^{q+1}(X_\lambda, \mathscr{I}\Omega^q),$$

is an isomorphism [1], where $X_\lambda = \{ \varphi < \lambda \}$. It follows that $\operatorname{Ker}\beta \subset \operatorname{Im}\gamma$.

Let $\tilde{c}_j$ be the normalization of c_j $(1 \leqslant j \leqslant k)$. Then the disjoint union $\tilde{c}$ of $\tilde{c}_1, \ldots, \tilde{c}_k$ is the normalization of c. Let $\pi : \tilde{c} \to c$ be the normalization map. Let

$$\theta : \quad \mathscr{F} \to R^0 \pi_*(\pi^* \mathscr{F})$$

be the natural map (where $R^0 \pi_*$ denotes the zero-th direct image under π). Let $Y = X - K$. Since Supp Ker θ and Supp Coker θ have dimension $< q$, it follows from [9] that the map:

$$\tilde{\theta} : \quad H^q_*(Y \cap c, \mathscr{F}) \to H^q_*(Y \cap c, R^0 \pi_*(\pi^* \mathscr{F})),$$

induced by θ is surjective, where H^q_* denotes the q-dimensional cohomology group with compact supports. $H^q_*(Y \cap c, R^0 \pi_*(\pi^* \mathscr{F}))$ is naturally isomorphic to $\bigoplus_{j=1}^k H^q_*(\pi^{-1}(Y \cap c) \cap \tilde{c}_j, \pi^* \mathscr{F})$, and we identify these two groups. Let $\tilde{c}^0_j$ be the set of all regular points of $\tilde{c}_j$ $(1 \leqslant j \leqslant k)$. Since the singular set of $\tilde{c}_j$ has dimension $\leqslant q-2$, it follows from [9] that the map:

$$\sigma : \quad \bigoplus_{j=1}^k H^q_*(\pi^{-1}(Y \cap c) \cap \tilde{c}^0_j, \pi^* \mathscr{F}) \to \bigoplus_{j=1}^k H^q_*(\pi^{-1}(Y \cap c) \cap c_j, \pi^* \mathscr{F}),$$

induced by $\tilde{c}^0_j \subsetneq \tilde{c}_j$ is an isomorphism. Let Ω^q_j be the sheaf of germs of holomorphic q-forms on $\tilde{c}^0_j$ $(1 \leqslant j \leqslant k)$. Since $\dim \tilde{c}^0_j = q$, by [9], the sheaf-epimorphism $\pi^* \mathscr{F} \to \Omega^q_j$ on $\tilde{c}^0_j$ induces an epimorphism

$$\tau : \quad \bigoplus_{j=1}^k H^q_*(\pi^{-1}(Y \cap c) \cap \tilde{c}^0_j, \pi^* \mathscr{F}) \to \bigoplus_{j=1}^k H^q_*(\pi^{-1}(Y \cap c) \cap c^0_j, \Omega^q_j).$$

Because $\pi^{-1}(Y \cap c) \cap \tilde{c}^0_1 \neq \varnothing$, we can choose a C^∞ (q, q)-form ω_1 on $\pi^{-1}(Y \cap c) \cap \tilde{c}^0_1$ with compact support such that

$$\int_{\pi^{-1}(Y \cap c) \cap \tilde{c}^0_1} \omega_1 = 1.$$

For $1 < j \leqslant k$ let ω_j be the (q, q)-form on $\pi^{-1}(Y) \cap \tilde{c}_j^0$ which is identically zero. Denote by ω'_j the element of $H^q_*(\pi^{-1}(Y \cap c) \cap \tilde{c}_j^0, \Omega^q_j)$, defined by ω_j $(1 \leqslant j \leqslant k)$. There exists $f \in H^q_*(Y \cap c, \mathscr{F})$ such that $\tau\sigma^{-1}\tilde{\theta}(f) = \bigoplus_{j=1}^k \omega'_j$. Let

$$\xi : \quad H^q_*(Y \cap c, \mathscr{F}) \to H^q(X, \mathscr{F})$$

be induced by $Y \cap c \subsetneq X$. Since $\xi(f) \in \operatorname{Ker} \beta$, there exists $g \in H^q(X, \Omega^q)$ such that $\gamma(g) = \xi(f)$. Let ω be a $\bar{\partial}$-closed C^∞ (q, q)-form on X which defines g. It is easily verified that ω satisifies the requirement.

Q.E.D.

(2.2) PROPOSITION. — *Suppose X is an open subset of the regular points of a subvariety of $\mathbf{P}_N$.*

(i) *If $H^{q+1}(X, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ on X, then holomorphic functions on $C_q^+(X)$ separate points of $C_q^+(X)$.*

(ii) *If X is strongly q-pseudoconvex, then for every $c \in C_q^+(X)$ the components of $L_c := \{ c' \in C_q^+(X);\ f(c') = f(c)$ for all f holomorphic on $C_q^+(X) \}$ are compact.*

Proof. — Let c, c' be any two distinct positive analytic q-cycles in X. In this proof, for notational convenience, we write a positive analytic q-cycle as a formal finite linear combination of irreducible q-dimensional compact subvarieties of X with *nonnegative* integers as coefficients. With this convention, we can assume that

$$c = \textstyle\sum_{i=1}^k m_i c_i,$$
$$c' = \textstyle\sum_{i=1}^k m'_i c_i,$$

where $c_1, \ldots, c_k$ are distinct irreducible q-dimensional compact subvarieties of X and $m_1, \ldots, m_k$, $m'_1, \ldots, m'_k$ are nonnegative integers.

(i) Suppose that $H^{q+1}(X, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ on X. $m_i \neq m'_i$ for some $1 \leqslant i \leqslant k$. By (2.1), there exists a $\bar{\partial}$-closed C^∞ (q, q)-form on X such that

$$\int_{c_j} \omega = \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases}$$

By [2], (Theorem 5), the function f on $C_q^+(X)$ defined by

$$\textstyle\sum_{i=1}^l n_i e_i \mapsto \sum_{i=1}^l n_i \displaystyle\int_{e_i} \omega$$

is a holomorphic function on $C_q^+(X)$. f assumes different values at c and c'.

(ii) Suppose X is strongly q-pseudoconvex and $n = \dim X$. There exist $\lambda_0 \in \mathbf{R}$ and a C^2 exhaustion function on X whose Levi form at every point of $\{\varphi > \lambda_0\}$ has at least $n-q$ positive eigenvalues. For $e = \sum_{i=1}^{l} n_i e_i \in C_q^+(X)$ define $g(e) = \sum_{i \in I} n_i e_i$, where I is the set of all $1 \leqslant i \leqslant l$ such that $n_i \neq 0$, and e_i is not contained in $\{\varphi \leqslant \lambda_0\}$. We are going to show that, if $g(c) \neq g(c')$, then $f(c) \neq f(c')$ for some holomorphic function f on $C_q^+(X)$. Suppose $g(c) \neq g(c')$. Without loss of generality, we can assume that $m_1 \neq m'_1$ and c_1 is not contained in $\{\varphi \leqslant \lambda_0\}$. By (2.1), there exists a $\bar{\partial}$-closed C^∞ (q, q)-form ω on X such that

$$\int_{c_j} \omega = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j > 1. \end{cases}$$

By [2] (Theorem 5), the function f on $C_q^+(X)$ defined by

$$\textstyle\sum_{i=1}^{l} n_i e_i \mapsto \sum_{i=1}^{l} n_i \displaystyle\int_{e_i} \omega$$

is a holomorphic function on $C_q^+(X)$. f assumes different values at c and c'. It follows that every element of L_c is of the form $g(c) + \sum_{i=1}^{l} n_i e_i$ with $e_i \subset \{\varphi \leqslant \lambda_0\}$. Since the set of all $\sum_{i=1}^{l} n_i e_i \in C_q^+(X)$ with $e_i \subset \{\varphi \leqslant \lambda_0\}$ has compact components, we conclude that L_c has compact components.

Q.E.D.

(2.3) PROPOSITION. — *Suppose X is a reduced complex space and suppose, for $x \in X$, all the components of*

$$L_x := \{y \in X; f(y) = f(x) \text{ for all } f \in \Gamma(X, \mathcal{O}_X)\}$$

are compact. Let Y be the quotient topological space obtained by identifying points x, x' of X which belong to the same component of L_y for some $y \in X$. Let $\mathcal{O}_Y$ be the sheaf of germs of functions f on Y such that $f \circ \pi$ is a holomorphic function-germ on X, where $\pi : X \to Y$ is the natural projection. Then $(Y, \mathcal{O}_Y)$ is a K-separable complex space and $\pi : X \to Y$ is a proper holomorphic map.

Proof. — Fix arbitrarily $y \in Y$. Let $A = \pi^{-1}(y)$. Then A is a component of $L_{x'}$ for some $x' \in X$. Let U be a relatively compact open neighbourhood of A in $X - (L_{x'} - A)$. For $a \in A$ and $u \in \partial U$, there exists a holomorphic function $f_{a,u}$ on X which vanishes at a and does not vanish at u. For some $\varepsilon_{a,u} > 0$ and some open neighbourhood $W_{a,u}$ of u in X, $|f_{a,u}| \geqslant \varepsilon_{a,u}$

on $W_{a,u}$. Since we can cover ∂U by a finite number of such sets $W_{a,u}$, we obtain $\varepsilon > 0$ and holomorphic functions $f_1, \ldots, f_n$ on X such that $_i \mid A \equiv 0\,(1 \leqslant i \leqslant n)$ and $\max_{1 \leqslant i \leqslant n} \mid f_i(x) \mid \geqslant \varepsilon$ for $x \in \partial U$. Let $F : X \to \mathbf{C}^n$ be defined by $f_1, \ldots, f_n$. Let $P \subset \mathbf{C}^n$ be the open polydisc with center 0 and radius ε. Define $D = U \cap F^{-1}(P)$. The map $D \to P$ induced by F is proper, because $D = U^- \cap F^{-1}(P)$. Hence D is holomorphically convex. By [10], there exist a Stein complex space R and a proper surjective holomorphic map $\sigma : D \to R$ with connected fibers such that a function germ g on R is holomorphic if, and only if, $g \circ \sigma$ is holomorphic.

We are going to show that

$$(\star) \qquad \sigma^{-1}\sigma(x) = \pi^{-1}\pi(x) \qquad \text{for} \quad x \in D.$$

Fix $x \in D$. Since $\sigma^{-1}\sigma(x)$ is a connected compact subvariety of D, it is clear that $\sigma^{-1}\sigma(x) \subset \pi^{-1}\pi(x)$. On the other hand,

$$\pi^{-1}\pi(x) \subset L_x \subset F^{-1}F(x).$$

Since $\pi^{-1}\pi(x)$ is connected and intersects U and since $F^{-1}F(x)$ is disjoint from ∂U, it follows that $\pi^{-1}\pi(x) \subset F^{-1}F(x) \cap U \subset D$. Consequently $\pi^{-1}\pi(x)$ is a connected compact subvariety of D. Since R is Stein, $\pi^{-1}\pi(x) \subset \sigma^{-1}\sigma(x)$.

From $(\star)$, we conclude the following:

(i) $\pi^{-1}\pi(D) = D$;

(ii) these exists a canonical isomorphism τ of ringed spaces from $(R, \mathcal{O}_R)$ onto an open subset of $(Y, \mathcal{O}_Y)$;

(iii) $\pi = \tau \circ \sigma$ on D.

Since $y \in \tau(R)$ and y is an arbitrary point of Y, it follows that Y is a complex space and π is proper. It is clear from the definition of Y that Y is K-separable.

Q.E.D.

3. Weakly plurisubharmonic exhaustion of the cycle space

(3.1) PROPOSITION. — *Suppose X is an open subset of the regular points of a subvariety of $\mathbf{P}_N$ of pure dimension n. Suppose $\lambda_0 \in \mathbf{R}$ and φ is a C^2 exhaustion function on X whose Levi form at every point of $\{\varphi > \lambda_0\}$ has at least $n-q$ positive eigenvalues. For $c \in C_q^+(X)$ let $\psi(c)$ be the supremum of $\varphi(x)$ for $x \in |c|$. Then ψ is a continuous exhaustion function on every component of $C_q^+(X)$ and ψ is weakly plurisubharmonic on $\{\psi > \lambda_0\}$.*

Proof. — Suppose $c_\nu \to c$ in $C_q^+(X)$. Take arbitrarily $\varepsilon > 0$. Let x_0 be a point of $|c|$ where the supremum of φ on $|c|$ is achieved, i. e. $\psi(c) = \varphi(x_0)$. There exists an open neighbourhood U of x_0 in X such that $\varphi(x) > \varphi(x_0) - \varepsilon$ for $x \in U$. Since $c_\nu \to c$, for ν sufficiently large, $|c_v| \cap U \neq \varnothing$. Hence $\psi(c_\nu) > \varphi(x_0) - \varepsilon$ for ν sufficiently large. On the other hand, since $|c|$ is compact, there exists an open neighbourhood W of $|c|$ in X such that $\varphi(x) < \psi(c) + \varepsilon$ for $x \in W$. For ν sufficiently large, $|c_\nu| \subset W$. Hence $\psi(c_\nu) < \psi(c) + \varepsilon$ for ν sufficiently large. So, $\lim_{\nu \to \infty} \psi(c_\nu) = \psi(c)$ and ψ is continuous on $C_q^+(X)$. Let Y be the set of all $(x, c) \in X \times C_q^+(X)$ such that $x \in |c|$. By [2] (p. 44, Remarque 1), Y is a subvariety of $X \times C_q^+(X)$. Let $\pi : Y \to X$ and $p : Y \to C_q^+(X)$ be the natural projections. Take $c_0 \in C_q^+(X)$ such that $\psi(c_0) > \lambda_0$. We are going to show that ψ is weakly plurisubharmonic on an open neighbourhood of c_0. Choose $\lambda_0 < \lambda_1 < \psi(c_0)$. Let

$$B = |c_0| \cap \{\varphi \geqslant \lambda_1\}.$$

Take arbitrarily $x \in B$. There exists a biholomorphic map τ from an open neighbourhood U of x in X onto an open neighbourhood Ω of 0 in $\mathbf{C}^n$ such that

(i) $\tau(x) = 0$,

(ii) $\Delta^n \subset\subset \Omega$ (where Δ is the open unit disc in $\mathbf{C}$),

(iii) $((\partial \Delta^{n-q}) \times \bar{\Delta}^q) \cap \tau(|c_0| \cap U) = \varnothing$,

(iv) $(\varphi|_U \circ \tau^{-1}) \mid \Delta^{n-q} \times \{\xi\}$, is strongly plurisubharmonic for every $\xi \in \Delta^q$.

Let $D_x = \tau^{-1}(\Delta^n)$. Choose an open neighbourhood W_x of c_0 in $C_q^+(X)$ such that

(i) $\tau(|c| \cap U) \cap ((\partial \Delta^{n-q}) \times \bar{\Delta}^q) = \varnothing$ for $c \in W_x$,

(ii) $\tau(|c| \cap U) \cap \Delta^n \neq \varnothing$ for $c \in W_x$.

For any $\xi \in \Delta^q$,

$$p_\xi : \quad (\tau \circ \pi | \pi^{-1}(U))^{-1} (\Delta^{n-q} \times \{\xi\}) \cap p^{-1}(W_x) \to W_x,$$

is an analytic cover, where p_ξ is induced by p. It follows that, for $\xi \in \Delta^q$, the function θ_ξ on W_x, defined by

$$\theta_\xi(c) = \sup\{(\varphi \circ \pi)(y);\ y \in p_\xi^{-1}(c)\},$$

is continuous and weakly plurisubharmonic on W_x. Let ψ_x be the function on W_x, defined by

$$\psi_x(c) = \sup_{\xi \in \Delta^q} \theta_\xi(c).$$

Since B is compact, there exist $x_1, \ldots, x_k \in B$ such that $B \subset \bigcup_{i=1}^{k} D_{x_i}$. Since $\psi(c_0) > \lambda_1$ and the supremum of φ on $|c_0| - \bigcup_{i=1}^{k} D_{x_i}$ is $< \lambda_1$, there exists an open neighbourhood W of c_0 in $\bigcap_{i=1}^{k} W_{x_i}$ such that

(i) the infimum of ψ on W is $> \lambda_1$;

(ii) the supremum of φ on $p^{-1}(W) - \bigcup_{i=1}^{k} D_{x_i}$ is $< \lambda_1$.

Then $\psi = \max_{1 \leqslant i \leqslant k} \psi_{x_i}$ on W. Since $\psi_{x_i} \mid W$ is the supremum function of a family of continuous weakly plurisubharmonic functions on W, $\psi \mid W$ is the supremum function of a family of continuous weakly plurisubharmonic functions on W. It follows from the continuity of ψ that ψ is weakly plurisubharmonic on W.

Q.E.D.

(3.2) *Proof of Theorem* 1. — Let $n = \dim X$. There exist $\lambda_0 \in \mathbf{R}$ and a C^2 exhaustion function φ on X whose Levi form at every point of $\{ \varphi > \lambda_0 \}$ has at least $n-q$ positive eigenvalues. By (3.1) the function ψ on $C_q^+(X)$, defined by

$$\psi(c) = \sup_{x \in |c|} \varphi(x),$$

is a continuous exhaustion function on every component of $C_q^+(X)$ and is weakly plurisubharmonic on $\{ \psi > \lambda_0 \}$. Choose $\lambda_0 < \lambda_1 < \infty$. Let $\tilde{\psi} = \max(\lambda_1, \psi)$. Then $\tilde{\psi}$ is a continuous weakly plurisubharmonic exhaustion function on every component of $C_q^+(X)$.

Consider first the special case where $H^{q+1}(X, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ on X. By (2.2), holomorphic functions on $C_q^+(X)$ separate points. It follows from Theorem 2 that $C_q^+(X)$ is Stein for this special case.

For the general case, by (2.2) and (2.3) there exist a K-separable complex space Y and a proper surjective holomorphic map $\pi : C_q^+(X) \to Y$ with connected fibers. Since each fiber of π is a compact connected subvariety, $\tilde{\psi}$ is constant on each fiber of π. It follows that there exists a continuous function ψ^* on Y such that $\tilde{\psi} = \psi^* \circ \pi$. Clearly ψ^* is an exhaustion function on every component of Y. Let Y_1 be a branch of Y of dimension k. There exists a branch Z of $\pi^{-1}(Y_1)$ such that $\pi(Z) = Y_1$. Let $l = \dim Z$. Let A be the set of all $x \in Z$ such that

$$\dim_x \pi^{-1} \pi(x) \cap Z \geqslant l - k + 1.$$

$\pi(A)$ is a subvariety of Y of dimension $< k$. Let B be the singular set of Y_1. For every $y_0 \in Y_1 - (B \cup \pi(A))$ there exist a subvariety of V

of an open subset of $Z-A$ and an open neighbourhood W of y_0 in $Y_1-(B \cup \pi(A))$ such that $(\pi \mid V): V \to W$ is an analytic cover. Since

$$\psi^*(y) = \sup_{x \in V \cap \pi^{-1}(y)} \psi(x) \qquad \text{for} \quad y \in W,$$

it follows that ψ^* is plurisubharmonic on W. Hence ψ^* is weakly plurisubharmonic on Y. By Theorem 2, Y is Stein. Consequently, $C_q^+(X)$ is holomorphically convex.

Q.E.D.

(3.3) *Remark.* – The proof of (3.1) can easily be modified to show that the conclusions of (3.1) remain valid if the Levi form of φ is only assumed to have at least $n-q$ nonnegative (instead of positive) eigenvalues at every point of $\{\varphi > \lambda_0\}$. Hence, in Theorem 1, if $H^{q+1}(X, \mathscr{F}) = 0$ for all coherent analytic sheaves $\mathscr{F}$ on X, then the conclusion that $C_q^+(X)$ is Stein remains valid if the C^2 exhaustion function on X is only assumed to have a Levi form with at least $n-q$ nonnegative (instead of positive) eigenvalues at every point of the complement of a compact subset of X.

4. Compact subvarieties and $\partial\bar{\partial}$-cohomology

(4.1) Suppose X is a complex manifold of dimension n and $1 \leqslant k \leqslant n$. Define $\Lambda_*^{k,k}(X)$ as:

$$\frac{\{\text{all closed } (k, k)-\text{currents on } X \text{ with compact supports}\}}{\partial\bar{\partial}\{\text{all}(k-1, k-1)-\text{currents on } X \text{ with compact supports}\}}.$$

A compact irreducible subvariety of X of dimension k is called *maximal* if it is not contained in a compact irreductible subvariety of X of dimension $> k$. Let $M_k(X)$ be the group of all formal finite linear combinations of maximal compact irreducible subvarieties of X of dimension k with coefficients in $\mathbf{C}$. Since every compact subvariety of X of pure dimension k defines an associated closed $(n-k, n-k)$-current on X with compact support by integration over its regular points, there is a natural map:

$$\Phi_k(X): \quad M_k(X) \to \Lambda_*^{n-k,n-k}(X),$$

defined by mapping a maximal compact irreducible subvariety of X of dimension k to its associated closed $(n-k, n-k)$-current with compact support.

Suppose X is strongly q-pseudoconvex. In the remaining portion of this paper, we will discuss the conjecture that the union of all compact subvarieties of X of pure dimension $> q$ is a subvariety. The conjecture is clearly equivalent to the statement that $M_k(X)$ is finite-dimensional for $k > q$. We will adopt the following approach and carry it out for some special cases. We will first show that $\Phi_k(X)$ is injective for $k > q$ and then we will show that $\Lambda_*^{n-k,n-k}(X)$ is finite-dimensional for $k > q$.

(4.2) PROPOSITION. — *If X is a connected noncompact complex manifold of dimension n, then $\Phi_{n-1}(X)$ is injective.*

Proof. — Suppose the contrary. Then there exists a nonzero element $\sum_{i=1}^k \alpha_i c_i$ of $M_{n-1}(X)$ with distinct $c_1, \ldots, c_k$ such that the associated (1,1)-current $\sum_{i=1}^k \alpha_i [c_i]$ of $\sum_{i=1}^k \alpha_i c_i$ equals $\sqrt{-1}\,\partial\bar{\partial} f$ for some (0,0)-current f on X with compact support. We can assume without loss of generality that α_1 is a positive real number. Since f is pluriharmonic on $X - \bigcup_{i=1}^k c_i$ and f has compact support, it follows that $f \equiv 0$ on $X - \bigcup_{i=1}^k c_i$. Because $\alpha_1 > 0$, f is a plurisubharmonic function on $X - \bigcup_{i=2}^k c_i$. Since $f \equiv 0$ on $X - \bigcup_{i=1}^k c_i$, it follows that the plurisubharmonic function f on $X - \bigcup_{i=2}^k c_i$ must be identically zero, which contradicts the fact that $\sqrt{-1}\,\partial\bar{\partial} f = \alpha_1 [c_1]$ on $X - \bigcup_{i=2}^k c_i$.

Q.E.D.

(4.3) PROPOSITION. — *Suppose $\tilde{X}$ is an n-dimensional complex submanifold of $\mathbf{P}_N$. Suppose $0 \leqslant q < n$ and X is an open subset of $\tilde{X}$ such that $\mathbf{P}_N - X$ contains a linear subspace of $\mathbf{P}_N$ of dimension $N-q-1$. Then $\Phi_q(X)$ is injective.*

Proof. — Let $\mathbf{G}$ be the Grassmannian of all $(N-q-1)$-dimensional linear subspaces of $\mathbf{P}_N$. Define $\tilde{R}$ to be the set of all $(T, x) \in G \times \mathbf{P}_N$ such that $x \in T$. Let $\tilde{p} : \tilde{R} \to \mathbf{G}$ and $\tilde{\pi} : \tilde{R} \to \mathbf{P}_N$ be the natural projections. Let $R = \tilde{\pi}^{-1}(\tilde{X})$ and let $p : R \to \mathbf{G}$ (respectively $\pi : R \to \tilde{X}$) be induced by $\tilde{p}$ (respectively $\tilde{\pi}$).

Suppose the proposition is not true. Then there exists a nonzero element $\sum_{i=1}^k \alpha_i c_i$ of $M_q(X)$ with distinct $c_1, \ldots, c_k$ such that the associated $(n-q, n-q)$-current $\sum_{i=1}^k \alpha_i [c_i]$ of $\sum_{i=1}^k \alpha_i c_i$ equals $\sqrt{-1}\,\partial\bar{\partial} f$ for some $(n-q-1, n-q-1)$-current f on X with compact support. We can assume without loss of generality that $\alpha_1 > 0$. Both $\sum_{i=1}^k \alpha_i [c_i]$ and f can be naturally regarded as currents on $\tilde{X}$. Since $\pi : R \to \tilde{X}$ is a fiber bundle, the pull-backs $\pi^* [c_i]$ and $\pi^* f$ are well-defined, and

$$\sum_{i=1}^k \alpha_i \pi^* [c_i] = \sqrt{-1}\,\partial\bar{\partial}\, \pi^* f.$$

It follows that

$$\sum_{i=1}^{k} \alpha_i \, p_* \pi^* [c_i] = \sqrt{-1}\, \partial\bar{\partial}\, p_* \pi^* f,$$

where $p_* \pi^* [c_i]$ and $p_* \pi^* f$ are the push-forwards. It is easy to see that $\dim R - \dim \mathbf{G} = n - q - 1$. Hence $p_* \pi^* f$ is a (0,0)-current and $\sum_{i=1}^{k} \alpha_i p_* \pi^* [c_i]$ is a (1,1)-current. The support of $\sum_{i=1}^{k} \alpha_i p_* \pi^* [c_i]$ is contained in the subvariety $p(\pi^{-1}(\bigcup_{i=1}^{k} c_i))$ of $\mathbf{G}$. $p_* \pi^* f$ is a pluriharmonic function on $\mathbf{G} - p(\pi^{-1}(\bigcup_{i=1}^{k} c_i))$. Since $\mathbf{P}_N - X$ contains a linear subspace of $\mathbf{P}_N$ of dimension $N - q - 1$ and f has compact support in X, it follows that the support of $p_* \pi^* f$ is a proper subset of $\mathbf{G}$. Consequently $p_* \pi^* f \equiv 0$ on $\mathbf{G} - p(\pi^{-1}(\bigcup_{i=1}^{k} c_i))$. Choose $T_0 \in \mathbf{G}$ such that

(i) $T_0 \cap c_1 \neq \varnothing$,

(ii) $T_0 \cap (\bigcup_{i=2}^{k} c_i) = \varnothing$,

(iii) T_0 intersects c_1 normally at some regular point of c_1 (which condition makes sure that $p_* \pi^* [c_1] \neq 0$ on the following W).

There exists an open neighbourhood W of T_0 in $\mathbf{G}$ such that, on W,

$$\sum_{i=1}^{k} \alpha_i \, p_* \pi^* [c_i] = \alpha_1 \, p_* \pi^* [c_1] \neq 0.$$

It follows that $p_* \pi^* f \mid W$ is plurisubharmonic and not identically zero,contradicting that the support of $p_* \pi^* f \mid W$ is contained in the subvariety $W \cap p(\pi^{-1}(\bigcup_{i=1}^{k} c_i))$ of W of codimension $\geqslant 1$.

Q.E.D.

(4.4) *Remark.* — In the proofs of (4.2) and (4.3), the maximality of compact irreducible subvarieties of the dimension under consideration is not used, but the maximality is always satisfied under the given assumptions.

5. Compact subvarieties and the finiteness of $\partial\bar{\partial}$-cohomology

(5.1) We will consider the finite-dimensionality of $\Lambda_*^{k,k}(X)$, and we will use the method of [8] together with duality. First, we recall an exact sequence given in [8]. Suppose X is a complex manifold. Let $A^{k,l}(X)$ be the Fréchet space of all C^∞ (k, l)-forms on X. Let

$$A_r^{k,l}(X) = \bigoplus_{i+j=r,\, i,\, j \geqslant 0} A^{k+i,\, l+j}(X).$$

Define:

$$H_r^{k,l}(X) = \frac{\operatorname{Ker}(A_r^{k,l}(X) \xrightarrow{d} A_{r+1}^{k,l}(X))}{d A_{r-1}^{k,l}(X)}.$$

Let Ω^l be the sheaf of germs of holomorphic l-forms on X. The following short exact sequences of complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_r^{k,l}(X) & \xrightarrow{\alpha} & A_{r+1}^{k-1,l}(X) & \xrightarrow{\beta} & A^{k-1,l+r+1}(X) & \to & 0 \\
 & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & \\
0 & \to & A_{r+1}^{k,l}(X) & \to & A_{r+2}^{k-1,l}(X) & \to & A^{k-1,l+r+2}(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(where α is the inclusion map and β is the projection map) gives rise to the following exact sequence:

$$\ldots \to H^{l+r}(X, \Omega^{k-1}) \to H_r^{k,l}(X) \to H_{r+1}^{k-1,l}(X) \to H^{l+r+1}(X, \Omega^{k-1}) \to \ldots$$

which is the "first exact sequence" in [8].

(5.2) PROPOSITION. — *If X is a strongly q-pseudoconvex manifold, $s, t \geqslant q$, and $H^{s+t+1}(X, \mathbf{C})$ is finite-dimensional, then $H_1^{s,t}(X)$ is finite-dimensional.*

Proof. — Since $H^l(X, \Omega^{k-1})$ is finite-dimensional for $l > q$, it follows from the exact sequence of (5.1) that

$$H_r^{k,l}(X) \to H_{r+1}^{k-1,l}(X)$$

has finite-dimensional kernel and cokernel for $l+r > q$. Consequently,

$$H_1^{s,t}(X) \to H_{s+1}^{0,t}(X)$$

has finite-dimensional kernel and cokernel. Since $H_r^{k,l}(X)$ is isomorphic to $H_r^{l,k}(X)$ by conjugation,

$$H_r^{k,l}(X) \to H_{r+1}^{k,l-1}(X)$$

has finite-dimensional kernel and cokernel for $k+r > q$. Consequently,

$$H_{s+1}^{0,t}(X) \to H_{s+t+1}^{0,0}(X)$$

has finite-dimensional kernel and cokernel. The proposition now follows from

$$H_{s+t+1}^{0,0}(X) = H^{s+t+1}(X, \mathbf{C})$$

Q.E.D.

(5.2′) *Remark.* — This result is a particular case of the following: *If X is a strongly q-pseudoconvex manifold, $r > 0$, $s+r > q$, $t+r > q$, and $H^{s+t+r}(X, \mathbf{C})$ is finite-dimensional, then $H_r^{s,t}(X)$ is finite-dimensional* which results from Theorem 1 (iii) of [8].

We define

$$V^{k,k}(X) = \frac{\operatorname{Ker}(A^{k,k}(X) \xrightarrow{\partial\bar{\partial}} A^{k+1,k+1}(X))}{\partial A^{k-1,k}(X) + \bar{\partial} A^{k,k-1}(X)}.$$

(5.3) Proposition. — *If X is a strongly q-pseudoconvex manifold and $H^{2q+3}(X, \mathbf{C})$ is finite-dimensional, then $V^{q+1,q+1}(X)$ is finite-dimensional.*

Proof. — Consider the following sequence

$$H^{q+1}(X, \Omega^{q+1}) \oplus H^{q+1}(X, \Omega^{q+1}) \xrightarrow{\sigma} V^{q+1,q+1}(X) \xrightarrow{\tau} H_1^{q+1,q+1}(X),$$

where σ and τ are defined as follows:

(i) if $(\xi^*, \eta^*) \in H^{q+1}(X, \Omega^{q+1}) \oplus H^{q+1}(X, \Omega^{q+1})$ is represented by a couple (ξ, η) of $\bar{\partial}$-closed C^∞ $(q+1, q+1)$-forms on X, then $\sigma(\xi^*, \eta^*)$ is represented by $\xi + \bar{\eta}$ ($\bar{\eta}$ being the conjugate of η);

(ii) if $\omega^* \in V^{q+1,q+1}(X)$ is represented by a $\partial\bar{\partial}$-closed C^∞ $(q+1, q+1)$-form ω on X, then $\tau(\omega^*)$ is represented by $(\partial - \bar{\partial})\,\omega$.

We are going to prove that the sequence is exact. If ξ, η are $\bar{\partial}$-closed C^∞ $(q+1, q+1)$-forms on X then:

$$(\partial - \bar{\partial})(\xi + \bar{\eta}) = d(\xi - \bar{\eta}).$$

Hence $\tau\sigma = 0$. Suppose ω and ζ are C^∞ $(q+1, q+1)$-forms on X and $(\partial - \bar{\partial})\,\omega = d\zeta$. If we set $\xi = 1/2\,(\omega + \zeta)$ and $\eta = 1/2\,(\bar{\omega} - \bar{\zeta})$, then ξ and η are $\bar{\partial}$-closed and $\omega = \xi + \bar{\eta}$. Hence $\operatorname{Ker}\sigma \subset \operatorname{Im}\tau$. Since $H^{q+1}(X, \Omega^{q+1})$ is finite-dimensional, the proposition follows from the exactness of the sequence and (5.2).

Q.E.D.

(5.3′) *Remark.* — This result is a particular case of the following: *If X is a strongly q-pseudoconvex manifold, $s > q$, $t > q$ and $H^{s+t+1}(X, \mathbf{C})$ is finite-dimensional, then $V^{s,t}(X)$ is finite-dimensional,* which results from Theorem 1, (vi), in [8].

We define

$$\Lambda^{k,k}(X) = \frac{\operatorname{Ker}(A^{k,k}(X) \xrightarrow{d} A_1^{k,k}(X))}{\partial\bar{\partial} A^{k-1,k-1}(X)}$$

(5.4) PROPOSITION. — *If X is a strongly q-pseudoconvex manifold, $t > q$, and $H^{2t}(X, \mathbf{C})$ is finite-dimensional, then $\Lambda^{t,t}(X)$ is finite-dimensional.*

Proof. — Consider the following sequence

$$H^t(X, \Omega^{t-1}) \xrightarrow{\sigma} \Lambda^{t,t}(X) \xrightarrow{\tau} H_1^{t-1,t}(X),$$

where σ and τ are defined as follows:

(i) if $\xi^* \in H^t(X, \Omega^{t-1})$ is represented by a $\bar{\partial}$-closed C^∞ $(t-1, t)$-form ξ on X, then $\sigma(\xi^*)$ is represented by $\partial\xi$.

(ii) if $\omega^* \in \Lambda^{t,t}(X)$ is represented by a closed C^∞ (t, t)-form ω on X, then $\tau(\omega^*)$ is represented by ω.

We are going to prove that the sequence is exact. $\tau\sigma = 0$, because, if ξ is a $\bar{\partial}$-closed C^∞ $(t-1, t)$-form on X, then $\partial\xi = d\xi$. Suppose ω is a closed C^∞ (t, t)-form on X and $\omega = d\eta$ for some $C^\infty(t-1, t)$-form η on X. Then $\omega = \partial\eta$ and $\bar{\partial}\eta = 0$. The element of $\Lambda^{t,t}(X)$ represented by ω is the image under σ of the element of $H^t(X, \Omega^{t-1})$ represented by η. Hence $\operatorname{Ker}\tau \subset \operatorname{Im}\sigma$. Since $H^t(X, \Omega^{t-1})$ is finite-dimensional, the proposition follows from the exactness of the sequence and (5.3).

Q.E.D.

(5.4′) *Remark.* — This result is a particular case of the following: *If X is a strongly q-pseudoconvex manifold, $s > q$, $t > q$, and $H^{s+t}(X, \mathbf{C})$ is finite-dimensional, then $\Lambda^{s,t}(X)$ is finite-dimensional,* which results from Theorem 1 (v), in [8].

(5.5) PROPOSITION. — *Suppose X is a strongly q-pseudoconvex manifold of dimension n and $H^k(X, \mathbf{C})$ is finite-dimensional for $k = 2q+3$, $2q+4$. Then $\Lambda_*^{n-q-1,n-q-1}(X)$ is finite-dimensional.*

Q.E.D.

Proof. — Let $D^{k,l}(X)$ be the set of all (k, l)-currents on X with compact supports. The following two sequences are transposes of each other:

$$A^{q,q+1}(X) \oplus A^{q+1,q}(X) \xrightarrow{\partial\oplus\bar{\partial}} A^{q+1,q+1}(X) \xrightarrow{\partial\bar{\partial}} A^{q+2,q+2}(X),$$

$$D^{n-q,n-q-1}(X) \oplus D^{n-q-1,n-q}(X)$$
$$\xleftarrow{d} D^{n-q-1,n-q-1}(X) \xleftarrow{\partial\bar{\partial}} D^{n-q-2,n-q-2}(X).$$

By (5.4), $\Lambda^{q+2,q+2}(X)$ is finite-dimensional. By (5.3), $V^{q+1,q+1}(X)$ is finite-dimensional. It follows that the maps in the first sequence have closed images. From the two sequences, we conclude that $\Lambda_*^{n-q-1,n-q-1}(X)$ is dual to $V^{q+1,q+1}(X)$, and is therefore finite-dimensional.

Q.E.D.

(5.5′) *Remark.* — More generally, considering the two exact sequences:

$$A^{s,t+1}(X) \oplus A^{s+1,t}(X) \xrightarrow{\partial \oplus \bar{\partial}} A^{s+1,t+1}(X) \xrightarrow{\partial\bar{\partial}} A^{s+2,t+2}(X),$$

$$D^{n-s,n-t-1}(X) \oplus D^{n-s-1,n-t}(X)$$

$$\xleftarrow{d} D^{n-s-1,n-t-1}(X) \xleftarrow{\partial\bar{\partial}} D^{n-s-2,n-t-2}(X),$$

and using remarks (5.3′) and (5.4′), one gets:

Suppose X is a strongly q-pseudoconvex manifold of dimension n, $s \geqslant q$, $t \geqslant q$ and $H^k(X, \mathbf{C})$ is finite-dimensional for $k = s+t+3$, $k = s+t+4$. Then $\Lambda_*^{n-s-1,n-t-1}(X)$ is finite-dimensional.

(5.6) COROLLARY. — *Let X be a complex manifold of dimension n and φ be a C^2 function on X. Suppose $\{ \varphi \leqslant 0 \}$ is compact, $d\varphi$ is nowhere zero on $\{ \varphi = 0 \}$, and the Levi form of φ has at least $n-q$ positive eigenvalues at every point of $\{ \varphi = 0 \}$. Let $D = \{ \varphi < 0 \}$. Then $\Lambda_*^{n-q-1,n-q-1}(D)$ is finite-dimensional.*

Proof. — Since $d\varphi$ is nowhere zero on the compact set $\{ \varphi = 0 \}$, there exists $\lambda < 0$ such that

(i) $d\varphi$ is nowhere zero on $\{ \varphi = \lambda \}$,

(ii) $\{ \varphi \leqslant \lambda \}$ is a strong deformation retract of D.

It follows that $H^k(D, \mathbf{C})$ is finite-dimensional for $k \geqslant 0$. Since D is strongly q-pseudoconvex, by (5.5) $\Lambda_*^{n-q-1,n-q-1}(D)$ is finite-dimensional.

Q. E. D.

(5.6′) *Remark.* — As a Corollary of Remark (5.5′), one gets more generally:

Under the hypotheses of Corollary (5.6), $\Lambda_*^{n-s-1,n-t-1}(X)$ *is finite-dimensional for s, $t \geqslant q$.*

(5.7) THEOREM. — *If X is a strongly $(n-2)$-pseudoconvex manifold of dimension n, then the union Z of all compact subvarieties of X of pure dimension $> n-2$ is a subvariety.*

Proof. — There exist $\lambda_0 \in \mathbf{R}$ and a C^2 exhaustion function φ on X whose Levi form at every point of $\{ \varphi > \lambda_0 \}$ has at least 2 positive eigenvalues. By Sard's theorem, there exists $\lambda_0 < \lambda < \infty$ such that $d\varphi$ is nowhere zero on $\{ \varphi = \lambda \}$. Let $D = \{ \varphi < \lambda \}$. Since $Z \subset D$, the theorem follows from (5.6) and (4.2).

Q. E. D.

(5.8) *Remark.* – Theorem (5.7) can also be proved in the following way. Clearly one needs only consider the noncompact components of X. By [11], $H^n(X, F) = 0$ for every coherent analytic sheaf F on X (because we now assume that X is noncompact). Hence condition (i) of Lemma (2.1) is satisfied for $q = n-1$. Now we want to show that

$$\dim M_{n-1}(X) \leqslant \dim H^{n-1}(X, \Omega^{n-1}).$$

Suppose $c_1, \ldots, c_k$ are distinct elements of $C^+_{n-1}(X)$. By Lemma (2.1), there exist C^∞ $\bar{\partial}$-closed $(n-1, n-1)$-forms ω_j such that

$$\int_{c_i} \omega_j = \delta_{ij} \text{(Kronecker delta)};$$

$\omega_1, \ldots, \omega_k$ are linearly independent in $H^{n-1}(X, \Omega^{n-1})$, because, if $\sum_{i=1}^k \alpha_i \omega_i = \bar{\partial}\eta$ for some $c_1, \ldots, c_k \in \mathbf{C}$ and some C^∞ $(n-1, n-2)$-form η on X, then:

$$\alpha_j = \int_{c_j} \sum_{i=1}^k \alpha_i \omega_i = \int_{c_j} \bar{\partial}\eta = \int_{c_j} d\eta = \int_{\partial c_j} \eta = 0.$$

Since X is strongly $(n-2)$-pseudoconvex, $H^{n-1}(X, \Omega^{n-1})$ is finite-dimensional. Hence there are only a finite number of irreducible compact subvarieties of dimension $\geqslant n-1$ in X.

(5.9) THEOREM. – *Suppose $\tilde{X}$ is an n-dimensional complex submanifold of $\mathbf{P}_N$. Suppose X is an open subset of $\tilde{X}$ such that X is strongly q-pseudoconvex and $\mathbf{P}_N - X$ contains a linear subspace of $\mathbf{P}_N$ of dimension $N-q-2$. Then the union Z of all compact subvarieties of X of pure dimension $> q$ is a subvariety.*

Proof. – There exist $\lambda_0 \in \mathbf{R}$ and a C^2 exhaustion function φ on X whose Levi form at every point of $\{ \varphi > \lambda_0 \}$ has at least $n-q$ positive eigenvalues. By Sard's theorem, there exists $\lambda_0 < \lambda < \infty$ such that $d\varphi$ is nowhere zero on $\{ \varphi = \lambda \}$. Let $D = \{ \varphi < \lambda \}$. Since $Z \subset D$, the theorem follows from (5.6) and (4.3).

Q. E. D.

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