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RIESZ SPACES VALUED MEASURES AND PROCESSES

BY

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ABSTRACT. — We give necessary and sufficient conditions for the weak convergence (resp. strong convergence, resp. order convergence) of L^1-bounded (resp. uniformly bounded, resp. order bounded) supermartingales and, more generally, order asymptotic martingales valued in Banach lattices.

RÉSUMÉ. — On donne des conditions nécessaires et suffisantes pour la convergence faible (resp. forte, resp. pour l'ordre) des surmartingales et plus généralement des martingales asymptotiques pour l'ordre à valeurs dans un treillis de Banach et qui sont bornées dans L^1 (resp. uniformément bornées, resp. bornées pour l'ordre).

0. Introduction

This paper is mainly concerned with Riesz spaces valued measures and processes. We first study the lattice properties of processes of vector measures valued in an ordered vector space, but the main goal is to analyze those Banach lattice-valued processes of random variables, which include martingales, submartingales and supermartingales, that is an extension of the notion of asymptotic martingales to the infinite dimensional setting. Different extensions of this notion were studied by A. BELLOW (uniform amarts), by R. V. CHACON, L. SUCHESTON and G. EDGAR (strong amarts), by A. BRUNEL and L. SUCHESTON (weak amarts). In this paper (section II.3) we show that these notions do not, in general, preserve the lattice properties of the real asymptotic martingales and we study another notion (orderamart) which is stable under the lattice operation and shares most of the properties of the other extensions.

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In section II.2 we prove the Riesz decomposition for o-amarts, while in sections II.4, II.5 and II.6 we give necessary and sufficient conditions on the Banach lattice and the "right" boundedness conditions on the processes to ensure the weak convergence, the strong convergence and the order convergence of these processes.

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1. Riesz spaces valued measures

1.1. Generalities

Let E be an order complete Riesz space (i. e. every majorized set of E has a supremum). We will say that a family (e_α) in E decreases to zero (and we will write $e_\alpha \downarrow 0$) iff (e_α) is directed downward and $\inf_\alpha e_\alpha = 0$.

A net (x_α) in E is said to be order convergent to x, if there exists $e_\beta \downarrow 0$ such that: $|x_\alpha - x| \leqslant e_\beta$ where $\alpha \geqslant P(\beta)$ which is equivalent to: $\sup_\alpha |x_\alpha|$ exists and $\lim\sup x_\alpha = \lim\inf x_\alpha = x$.

A net (x_γ) in E is called o-Cauchy, if there exists $e_\beta \downarrow 0$ such that: $|x_\alpha - x_\gamma| \leqslant e_\beta$ for all α, $\gamma \geqslant P(\beta)$. It is easy to show that a net(x_α) is o-convergent if and only if it is o-Cauchy.

For complete studies of Riesz spaces we refer to the books ([24], [27]).

Let now $(\Omega, \mathscr{F}, P)$ be a probability space and let E_+ denote the positive cone of E.

Definition 1.1.1. — A set function $\mu : \mathscr{F} \to E$ is said to be a *positive measure* iff μ is E_+-valued, finitely additive and order-countably additive, that is: for every disjoint sequence (A_n) in $\mathscr{F}$ we have $\mu(\bigcup_n A_n)$ = order limit of $\sum_1^k \mu(A_n)$. It can be easily shown that the latter condition is equivalent to the following property:

★ If (A_n) is a sequence of elements of $\mathscr{F}$, decreasing to $\emptyset$, then $\mu(A_n) \downarrow 0$.

Definition 1.1.2. — A set function $\mu : \mathscr{F} \to E$ is said to be a *signed measure* iff μ is the difference of two positive measures.

Suppose μ is a signed measure which is the difference of the two positive measures μ_1 and μ_2. Since $\mu \leqslant \mu_1$, we can define for every $A \in \mathscr{F}$, $\mu^+(A) = \sup_{B \in \mathscr{F}} \mu(A \cap B)$ which is a positive measure since $0 \leqslant \mu^+(A) \leqslant \mu_1(A)$ for every $A \in \mathscr{F}$. Thus, μ^+ is the smallest positive measure majorizing μ.

Also, if λ and μ are two signed measures then $\mu \vee \lambda = \lambda + (\mu - \lambda)^+$. We deduce easily the following:

PROPOSITION 1.1.3. — *The class $\mathcal{M}_s(\mathcal{F}, E)$ of the E-valued signed measures is an order complete Riesz space.*

1.2. ORDER ASYMPTOTIC MARTINGALES OF MEASURES

Let now $(\mathcal{F}_n)_n$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = \sigma\ (\bigcup_n \mathcal{F}_n)$. Denote by T the set of bounded stopping times. A sequence (μ_n) of E-valued signed measures is said to be $(\mathcal{F}_n)$-adapted if for every $n \in \mathbb{N}$, μ_n is a signed measure on $\mathcal{F}_n$. If σ is a stopping time in T, we define the σ-field:

$$\mathcal{F}_\sigma = \{ A \in \mathcal{F};\ A \cap \{\sigma = n\} \in \mathcal{F}_n \text{ for every } n \in \mathbb{N} \}.$$

For $\sigma \in T$, we also define μ_σ on $\mathcal{F}_\sigma$ by:

$$\mu_\sigma(A) = \textstyle\sum_n \mu_n(A \cap \{\sigma = n\}) \quad \text{for each } A \in \mathcal{F}_\sigma.$$

It is easily seen that μ_σ is then a signed measure and that $(\mu_\sigma)_{\sigma \in T}$ is adapted to $(\mathcal{F}_\sigma)_{\sigma \in T}$.

DEFINITIONS 1.2.1. — An $\mathcal{F}_n$-adaped sequence of signed measures (μ_n) is said to be an *order asymptotic martingale* (o-amart) if the net $\{\mu_\sigma(\Omega)\}_{\sigma \in T}$ is o-convergent. The sequence (μ_n) is said to be a *martingale* (resp. *sub* or *supermartingale*) if for every $n \in \mathbb{N}$ and every $A \in \mathcal{F}_n$, $\mu_{n+1}(A) = \mu_n(A)$ (resp. $\geqslant$ or $\leqslant$). It is clear that every submartingale or supermartingale such that $\sup_n |\mu_n|(\Omega)$ exists is an o-amart.

The sequence is said to be an *order potential* (o-potential) if the net $|\mu_\sigma|(\Omega)$ o-converges to zero.

LEMMA 1.2.2. — *Let (μ_n) be an E-valued o-amart, then the net $\mu_\sigma(A)$ o-converges to a limit $\mu(A)$ for each $A \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n = \bigcup_{\sigma \in T} \mathcal{F}_\sigma$ and the convergence is uniform, that is: there exists $e_\alpha \downarrow 0$ such that:*

$$\sigma \in T \quad \textit{and} \quad \sigma \geqslant N(\alpha) \Rightarrow \sup_{A \in \mathcal{F}_\sigma} |\mu_\sigma(A) - \mu(A)| \leqslant e_\alpha.$$

Proof. — There exists $e_\alpha \downarrow 0$ and $N(\alpha)$ such that if σ, $\rho \geqslant N(\alpha)$ then:

$$|\mu_\rho(\Omega) - \mu_\sigma(\Omega)| \leqslant e_\alpha.$$

Let $\sigma \geqslant \tau \geqslant N(\alpha)$. For any $A \in \mathcal{F}_\tau$, define the following stopping time:

$$\rho = \begin{cases} \tau & \text{on } A, \\ \sigma & \text{on } A^c. \end{cases}$$

It follows that:

$$|\mu_\tau(A)-\mu_\sigma(A)|=|\mu_\rho(\Omega)-\mu_\sigma(\Omega)|\leqslant e_\alpha,$$

if $\sigma\geqslant\tau\geqslant N(\alpha)$ and the net $\mu_\sigma(A)$ is o-Cauchy and hence o-convergent to $\mu(A)$. Letting σ go to infinity we get:

$$\sup_{A\in\mathscr{F}_\tau}|\mu_\tau(A)-\mu(A)|\leqslant e_\alpha \quad \text{whebever } \tau\geqslant N(\alpha).$$

1.3. RIESZ DECOMPOSITION OF ORDER AMARTS OF MEASURES:

THEOREM 1.3.1. – *Every o-amart* (μ_n) *can be written uniquely as the sum of a martingale* m_n *and an o-potential* ρ_n.

Proof. – For every $A\in\mathscr{F}_n$, let $m_n(A)$ be the o-limit of $\mu_k(A)$ given by Lemma 1.2.2. It is easily seen that m_n is a signed measure for every n and that (m_n) is a martingale since $m_{n+1}(A)=\lim\mu_k(A)=m_n(A)$ for $A\in\mathscr{F}_n$. To prove that $\rho_n=\mu_n-m_n$ is an o-potential, we use again Lemma 1.2.2 to get:

$$\sup_{A\in\mathscr{F}_\sigma}|\rho_\sigma(A)|=\sup_{A\in\mathscr{F}_\sigma}|\mu_\sigma(A)-m_\sigma(A)|=\sup_{A\in\mathscr{F}_\sigma}|\mu_\sigma(A)-\mu(A)|\leqslant e_\alpha,$$

whenever $\sigma\in T$ and $\sigma\geqslant N(\alpha)$. It follows that $\rho_\sigma^+(\Omega)\leqslant e_\alpha$ if $\sigma\geqslant N(\alpha)$. The same holds for ρ_σ^-, hence $|\rho_\sigma|(\Omega)$ o-converges to zero.

If $\mu_n=m'_n+\rho'_n$ is another Riesz decomposition of (μ_n) then, $|m_n-m'_n|$ is a submartingale which is also an o-potential, hence $m_n=m'_n$ and $\rho_n=\rho'_n$ and the decomposition is unique.

COROLLARY 1.3.2. – *If* (μ_n) *is an o-amart such that* $\liminf|\mu_n|(\Omega)$ *exists then* $\sup_{\sigma\in T}|\mu_\sigma|(\Omega)$ *exists.*

Proof. – By Lemma 1.2.2, there exists $e_\alpha\downarrow 0$ such that:

$$|\mu_n(A)-\mu_m(A)|\leqslant e_\alpha \qquad \text{if} \quad m\geqslant n\geqslant N(\alpha) \quad \text{for every } A\in\mathscr{F}_n.$$

Thus, for every $A\in\mathscr{F}_n$, $\mu_n(A)\leqslant e_\alpha+\liminf|\mu_m|(\Omega)$ and:

$$m_n(A)=o\text{-}\lim\mu_n(A)\leqslant e_\alpha+\liminf|\mu_m|(\Omega).$$

Since it holds for every $A\in\mathscr{F}_n$, we get that $m_n^+(\Omega)\leqslant e_\alpha+\liminf|\mu_m|(\Omega)$ and consequently $|m_n|(\Omega)\leqslant e_\alpha+\liminf|\mu_m|(\Omega)$. Now $(|m_n|)_n$ is a submartingale such that $\sup_n|m_n|(\Omega)=\sup_{\sigma\in T}|m_\sigma|(\Omega)$ exists, but clearly $\sup_\sigma|\rho_\sigma|(\Omega)$ exists, hence $\sup_{\sigma\in T}|\mu_\sigma|(\Omega)$ exists.

COROLLARY 1.3.3. – *The class* $\mathscr{A}_b^0$ *of o-amarts* (μ_n) *such that* $\liminf|\mu_n|(\Omega)$ *exists is a Riesz space.*

Proof. – For that it is enough to notice that for every $\sigma \in T$, $-|\rho_\sigma| \leqslant \mu_\sigma^+ - m_\sigma^+ \leqslant |\rho_\sigma|$. If $\lim \inf |\mu_n|(\Omega)$ exists, then $\sup_\sigma m_\sigma^+(\Omega)$ exists, hence $m_\sigma^+(\Omega)$ o-converges and consequently $\mu_\sigma^+(\Omega)$ o-converges.

2. Banach lattices valued random variables

2.1. Generalities

Let E be a Banach space. A function $\mu : \mathscr{F} \rightarrow E$ is said to be a vector measure if μ is finitely additive and countably additive for the norm topology in E. An application $X : \Omega \rightarrow E$ is said to be *strongly measurable* if X is almost everywhere norm limit of E-valued simple functions. A strongly measurable function X is said to be Bochner integrable iff $\int \| X \| \, dP < \infty$. For more details on vector measures we refer to the book [12].

Let now E be a Banach lattice. We recall that E has an *order continuous norm* if and only if each order convergent net in E is norm convergent. A well-known characterizations of *weak sequential complete* Banach lattices is that every norm bounded increasing sequence is norm convergent.

An E-valued vector measure μ is said to be of bounded variation if $\sup_\pi \sum_{A \in \pi} \| \mu(A) \| < \infty$ where the supremum is taken over all finite partitions π of Ω. We denote by:

$\mathscr{M}(\mathscr{F}, E)$ the space of E-valued norm-countably additive measures;

$\mathscr{M}_b(\mathscr{F}, E)$ the subspace of $\mathscr{M}(\mathscr{F}, E)$ consisting of the measures of bounded variation;

$\mathscr{M}_d(\mathscr{F}, E)$ the subspace of $\mathscr{M}_b(\mathscr{F}, E)$ consisting of the differentiable measures ($\mu(A) = \int_A X \, dP$ where X is Bochner integrable).

$\mathscr{M}_s(\mathscr{F}, E)$ the space of signed E-valued measures.

The following relations between the different sets of measures defined above are easily verifiable.

(i) In any Banach lattice E we have $\mathscr{M}_d(\mathscr{F}, E) \subseteq \mathscr{M}_s(\mathscr{F}, E)$.

(ii) E has an order continuous norm if and only if $\mathscr{M}_s(\mathscr{F}, E) \subseteq \mathscr{M}(\mathscr{F}, E)$.

(iii) E is weakly sequentially complete if and only if $\mathscr{M}_b(\mathscr{F}, E) \subseteq \mathscr{M}_s(\mathscr{F}, E)$.

(iv) We will say that E has the Radon-Nikodym property if and only if $\mathscr{M}_b(\mathscr{F}, E) = \mathscr{M}_d(\mathscr{F}, E)$.

Note also that if E has an order continuous norm then $\mathscr{M}_d(\mathscr{F}, E)$ is an order ideal in $\mathscr{M}_s(\mathscr{F}, E)$ and that the space $\mathscr{M}_s(\mathscr{F}, E)$ equipped with the norm $N(\mu)=\| |\mu|(\Omega)\|$ is a Banach lattice which has an order continuous norm.

2.2. Decomposition of o-amarts of random variables

Let (X_n) be an $\mathscr{F}_n$-adapted sequence of E-valued Bochner integrable random variables.

Definitions 2.2.1. — The sequence (X_n) is said to be an o-amart (resp. an o-potential) if $\left(\int X_\sigma\right)_{\sigma\in T}$ o-converges in E $\left(\text{resp. } \left(\int |X_\sigma|\right) o\text{-onverges to zero}\right)$.

The sequence (X_n) is said to be a submartingale (resp. a supermartingale) if $\left(\int X_\sigma\right)_{\sigma\in T}$ is increasing (resp. decreasing).

It is clear that (X_n) is an o-amart (resp. submartingale or supermartingale) if the measures $(X_n P)$ form an o-amart (resp. a submartingale or supermartingale) in the sense of section I. If the space E has also an order continuous norm, then for every $n\in\mathbb{N}$, $|X_n P|=|X_n|\,.\,P$, and the two notions of o-potentials coincide.

We denote by $\mathscr{A}_b^0$ (resp. $\mathscr{A}_B^0$) the spaces of o-amarts (X_n) such that $\liminf \int |X_n|$ exists $\left(\text{resp. } \liminf \int \|X_n\| < \infty\right)$.

Proposition 2.2.2. — (1) *If E has an order continuous norm then $\mathscr{A}_b^0$ is a Riesz space.*

(2) *If E is weakly sequentially complete then $\mathscr{A}_B^0$ is a Riesz space and $\mathscr{A}_B^0 \subseteq \mathscr{A}_b^0$.*

Proof. — (1) Let $\mu_n = X_n . P$. If $(X_n)\in\mathscr{A}_b^0$ then $\liminf |\mu_n|(\Omega)$ exists, hence (μ_n^+) is also an o-amart. But $\mu_n^+ = X_n^+ P$ since E has an order continuous norm.

(2) Again let $\mu_n = X_n . P$ and let $\mu_n = m_n + \rho_n$ be its Riesz decomposition. Following [14], we observe that:

$$\text{variation } m_n \leqslant \liminf \int \|X_n\|.$$

To see this, given $\varepsilon>0$ choose disjoint sets A_i, $i=1, \ldots, k$ so that variation $(m_n)-\sum_{i=1}^k \|m_n(A_i)\|<\varepsilon$. Next, find N so large that $n>N$ implies for all i :

$$\|m_n(A_i)-\int_{A_i} X_n\|<\varepsilon/k.$$

Now:

$$\liminf \sum_{i=1}^k \left\|\int_{A_i} X_n\right\| \leqslant \liminf \sum_{i=1}^k \int_{A_i} \|X_n\| \leqslant \liminf \int \|X_n\|,$$

which implies that:

$$\text{variation } m_n \leqslant \liminf \int \|X_n\|+2\varepsilon.$$

Note now that $\| |m_n|(\Omega)\| \leqslant \|m_n\|(\Omega) \leqslant \liminf \int \|X_n\|$. Hence $(|m_n|(\Omega))$ is increasing and norm bounded in the weakly sequentially complete space E. Therefore, $\sup_n |m_n|(\Omega)$ exists. Since $\sup_{\sigma\in T} |\rho_\sigma|(\Omega)$ *exists, we get that* $\sup_n \int |X_n|$ exists and the result follows from the first part of the proposition.

THEOREM 2.2.3. — *For a Banach lattice E, the following properties are equivalent:*

(1) *E is weakly sequentially complete;*

(2) *every o-amart* (X_n) *in* $\mathscr{A}_b^0$ *can be written uniquely as* $X_n=Y_n+Z_n$ *where* (Y_n) *is a martingale and* (Z_n) *is an o-potential;*

(3) *every submartingale* (X_n) *in* $\mathscr{A}_B^0$ *can be written uniquely as* $X_n=Y_n-Z_n$ *where* (Y_n) *is a martingale and* (Z_n) *is a positive supermartingale and an o-potential.*

Proof. — (1) $\Rightarrow$ (3) follows easily from the fact that if (X_n) is a submartingale in $\mathscr{A}_B^0$, then, for every $n\in\mathbb{N}$, the sequence of random variables $\{E^{\mathscr{F}_n}[X_m]; m\geqslant n\}$ is increasing and norm bounded in $L^1[E]$, hence convergent to Y_n in $L^1[E]$. The sequence (Y_n) is a martingale and (Y_n-X_n) is a supermartingale and an o-potential.

(3) $\Rightarrow$ (1) is straightforward by considering norm bounded increasing sequences in E.

(1) $\Rightarrow$ (2) By the preceding proposition, every o-amart in $\mathscr{A}_B^0$ is the difference of two positive elements of $\mathscr{A}_B^0$, thus it is enough to prove the decomposition for positive o-amarts.

Let $\mu_n = X_n . P$ the positive o-amart. Write $\mu_n = m_n + \rho_n$ its Riesz decomposition and recall that for every n, $m_n(A) = o\text{-}\lim \mu_\sigma(A)$ uniformly in $A \in \mathscr{F}_n$. It follows that μ_p converges to m_n in the Banach lattice $\mathscr{M}_s(\mathscr{F}, E)$ equipped with the order continuous norm $N(\mu) = \| |\mu|(\Omega) \|$. Since the lattice operations are continuous in this space, we get that $\mu_p \wedge keP \to m_n \wedge keP$ for every k. Clearly, $\mu_p \wedge keP = (X_p \wedge ke).P$ and from the remarks of section 2.1 we get a $Y_k^n \in L^1[E]$, $0 \leqslant Y_k^n \leqslant ke$ a. e. and $m_n \wedge ke.P = Y_k^n.P$. The same reasoning as in the preceding proposition gives that:

$$\int \| Y_k^n \| = \text{variation}(m_n \wedge ke) \leqslant \liminf \int \| X_p \wedge ke \| \leqslant \liminf \int \| X_p \| < \infty.$$

Thus the sequence $(Y_k^n)_k$ is increasing and norm bounded in $L^1[E]$, hence convergent to Y_n in $L^1[E]$. Now, it is enough to show that $m_n = Y_n . P$. Letting $\nu_n = X_n . P$, we have:

$$|\nu_n - m_n| \leqslant |\nu_n - \nu_n \wedge keP| + |\nu_n \wedge keP - \mu_p \wedge keP| + |\mu_p \wedge keP - \mu_p| + |\mu_p - m_n|,$$

which clearly implies that $\nu_n = m_n$.

2.3. The other notions of amarts

An $\mathscr{F}_n$-adapted sequence (X_n) in $L^1[E]$ is said to be a *strong amart* (resp. *weak amart*) if the net $\left(\int X_\sigma\right)_{\sigma \in T}$ converges strongly (resp. weakly). It is known [18] that (X_n) is a *strong amart* if:

$$\lim_{\sigma \geqslant n \to \infty} (E^{\mathscr{F}_n}[X_\sigma] - X_n) = 0 \quad \text{in Pettis norm.}$$

The sequence (X_n) is said to be a *uniform amart* [3] if:

$$\lim_{\tau \geqslant \sigma \geqslant n;\, \sigma, \tau \in T;\, n \to \infty} \int \| E^{\mathscr{F}_\sigma}[X_\tau] - X_\sigma \| = 0.$$

We now give the analogous characterisation of o-amarts.

THEOREM 2.3.1. — *If E has an order continuous norm then the following are equivalent:*

(1) (X_n) *is an o-amart.*

(2) $o\text{-}\lim_{\sigma \geqslant \tau;\, \sigma, \tau \in T} \int | X_\tau - E^{\mathscr{F}_\tau}[X_\sigma] | = 0.$

Proof. — If (X_n) is an o-amart, there exists $e_\alpha \downarrow 0$ such that:

$$\left| \int X_\rho - \int X_\sigma \right| \leqslant e_\alpha \qquad \text{if} \quad \rho, \sigma \geqslant P(\alpha).$$

Let $\sigma \geqslant \tau \geqslant P(\alpha)$. For any $A \in \mathscr{F}_\tau$, define the following stopping time:

$$\rho = \begin{cases} \tau & \text{on } A, \\ \sigma & \text{on } A^c. \end{cases}$$

Thus,

$$\int_A (X_\tau - E^{\mathscr{F}_\tau}[X_\sigma]) = \int_A X_\tau - \int_A X_\sigma = \int X_\rho - \int X_\sigma.$$

Since:

$$\rho \geqslant P(\alpha) : \int_A X_\tau - E^{\mathscr{F}_\tau}[X_\sigma] \leqslant e_\alpha \qquad \text{if} \quad \sigma \geqslant \tau \geqslant P(\alpha)$$

and:

$$\sup_{A \in \mathscr{F}_\tau} \int_A X_\tau - E^{\mathscr{F}_\tau}[X_\sigma] \leqslant e_\alpha \qquad \text{if} \quad \sigma \geqslant \tau \geqslant P(\alpha).$$

Hence:

$$\int (X_\tau - E^{\mathscr{F}_\tau}[X_\sigma])^+ \leqslant e_\alpha \qquad \text{if} \quad \sigma \geqslant \tau \geqslant P(\alpha).$$

The same holds for $\int [X_\tau - E^{\mathscr{F}_\tau}[X_\sigma]]^-$ and finally we get:

$$\int | X_\tau - E^{\mathscr{F}_\tau}[X_\sigma] | \leqslant e_\alpha \qquad \text{if} \quad \sigma \geqslant \tau \geqslant P(\alpha).$$

We now compare the notion of o-amarts to the other notions. We note first that:

$$\{\text{uniform amarts}\} \subseteq \{\text{strong amarts}\} \subseteq \{\text{weak amarts}\}.$$

THEOREM 2.3.2. — (1) *E has an order continuous norm if and only if* $\{$*o-amarts*$\} \subseteq \{$*strong amarts*$\}$.

(2) *E is isomorphic (as a topological vector lattice) to an $A-M$ space if and only if $\{$strong amarts$\} \subseteq \{$o-amarts$\}$.*

(3) *E is isomorphic (as a topological vector lattice) to an $A-L$ space if and only if $\{$o-amarts$\} \subseteq \{$uniform amarts$\}$.*

Proof. — (1) *is obvious.*

(2) If E is an $A-M$ space and $\left(\int X_\sigma\right)$ is norm convergent hence it is norm convergent in E'' which is an $A-M$ space with unit hence isomorphic as a topological vector lattice to a $C(K)$ where K is compact stonian. The order convergence follows from [26]. The reverse implication follows from (p. 243, [27]) since every null sequence is then order bounded.

(3) Let (X_n) be an o-amart and E be an $A-L$ space. Then by (p. 244, [27]) there exists $f \in E'_+$ such that $\|x\| \leqslant f(|x|)$ for all $x \in E$. Using Theorem 2.4.1, we get that:

$$\int \| X_\tau - E^{\mathcal{F}_\tau}[X_\sigma] \| \leqslant \int f(| X_\tau - E^{\mathcal{F}_\tau}[X_\sigma] |)$$

and hence goes to zero when $\sigma \geqslant \tau \to \infty$.

Suppose now E is not an $A-L$ space, thus by (p. 242, [27]) there exists a positive summable sequence (x_n) in E_+ such that $\sum_n \|x_n\| = \infty$. Hence, we may find an increasing sequence of integers $(m_k)_k$ such that $\sum_{n=m_k+1}^{m_{k+1}} \|x_n\| \geqslant 1$. Without loss of generality, and by multiplying some x_n's by coefficients smaller than 1, we can assume $\sum_{n=m_k+1}^{m_{k+1}} \|x_n\| = 1$.

For every $k \in \mathbb{N}$, divide the interval $[0, 1]$ into $(m_{k+1} - m_k)$ sub intervals $A_{k,n}$ such that the length of each $A_{k,n}$ is $\|x_n\|$. Let $\Omega_i = [0, 1]$, λ_i is the Lebesgue measure, $\sum_i$ the Borel sets, $\Omega = \prod_{i \in \mathbb{N}} \Omega_i$, $\mathcal{F} = \prod_{i \in \mathbb{N}} \sum_i$ and $P = \prod_{i \in \mathbb{N}} \lambda_i$. Let ω_k denote the k-th coordinate of $\omega \in \Omega$. Define the sequence of E-valued random variables $X_k : \Omega \to E$ by:

$$X_k(\omega) = \sum_{n=m_k+1}^{m_{k+1}} \frac{x_n}{\|x_n\|} 1_{A_{k,n}}(\omega_k) + \sum_{j > m_{k+1}} x_j.$$

The sequence (X_n) is an o-amart. In fact it is a positive supermartingale. Since the X_k's are independent, it is enough to show that $E[X_k] \geqslant X_{k-1}$.

$$E[X_k](\omega) = \sum_{n=m_k+1}^{m_{k+1}} \|x_n\| \left[\frac{x_n}{\|x_n\|} + \sum_{j > m_{k+1}} x_j \right]$$
$$= \sum_{n=m_k+1}^{m_{k+1}} x_j + \sum_{j > m_{k+1}} x_j \leqslant X_{k-1}(\omega).$$

The sequence $\int X_k = \sum_{i \geqslant m_k+1} x_i$ decreases to zero since (x_n) is summable, so is the net $\left(\int X_\sigma \right)_{\sigma \in T}$.

Now, (X_k) is not a uniform amart since if it was $(\| X_k \|)$ must converge to zero, which obviously is not the case.

COROLLARY 2.3.3. — *The notions of uniform amarts and o-amarts coincide if and only if E is finite dimensional.*

Proof. — E will then be an $A-L$ and an $A-M$ space.

COROLLARY 2.3.4. — *The notions of strong amarts and o-amarts coincide if and only if E is* $c_0(\Gamma)$.

Proof. — It follows from the fact that an $A-M$ space with an order continuous norm is isomorphic to $c_0(\Gamma)$ [24].

Now, we show that generally the space of strong amarts is not a Riesz space. Recall that (X_n) is said to be a *strong potential* if (X_σ) converges to zero in Pettis norm.

THEOREM 2.3.5. — (1) *The Banach lattice E is isomorphic (as a topological vector lattice) to an $A-M$ space if and only if the absolute value of every strong potential is a strong potential.*

(2) *E is weakly sequentially complete if and only if the absolute value of every L^1-bounded martingale is a strong amart.*

(3) *E is finite dimensional if and only if the L^1-bounded strong amarts form a Riesz space.*

Proof. — (1) Let E be an $A-M$ space, E'' is then a $C(K)$ where K is a stonian compact. There is no loss of generality if we suppose $E=C(K)$. Let now (X_n) be a strong potential, that is $\sup_{A \in \mathcal{F}_\sigma} \| \int_A X_\sigma \| \to 0$.

We claim that $\int | X_\sigma |$ norm converges to zero, thus that $| X_n |$ is a strong potential.

Suppose not, then there exists $\varepsilon > 0$, such that for every $n \in \mathbb{N}$, there exists $\sigma_n \in T$, $n \leqslant \sigma_n \leqslant m_n$ and $t_n \in K$ and:

$$\int | X_{\sigma_n}(\omega)(t_n) | \, dP(\omega) \geqslant \varepsilon.$$

Clearly $A=\{\omega;\ X_{\sigma_n}(\omega)(t_n)\geqslant 0\}\in\mathscr{F}_{m_n}$ and:

$$\int_A X_{\sigma_n}(\omega)(t_n)\,dP(\omega)\geqslant\varepsilon/2.$$

Define the stopping time:

$$\tau_{p,n}=\begin{cases}\sigma_n & \text{on } A,\\ p\geqslant m_n & \text{on } A^c.\end{cases}$$

We have:

$$\left|\int_\Omega X_{\tau_{p,n}}(\omega)(t_n)\right|=\left|\int_A X_{\sigma_n}(\omega)(t_n)+\int_{A^c} X_p(\omega)(t_n)\right|.$$

Since the third term goes to zero when $p\to\infty$, we may then find p_n large enough such that:

$$\left|\int_\Omega X_{\tau_{n,p_n}}(\omega)(t_n)\right|\geqslant\varepsilon/4,$$

which is a contradiction.

Suppose now that E is not an $A-M$ space, hence by [27], there exists a summable sequence (x_n) in E such that $(|x_n|)$ is not summable. By Orlisz-Pettis theorem, there exists $f\in E'_+$, such that $f(|x_n|)$ is not summable (absolutely summable in $\mathbb{R}$). We may construct an increasing sequence of integers (m_k) such that:

$$\sum_{n=m_k+1}^{m_{k+1}} f(|x_n|)=1.$$

For every $k\in\mathbb{N}$, divide the interval $[0,1]$, into $(m_{k+1}-m_k)$ sub-intervals $A_{k,n}$ such that the length of each $A_{k,n}$ is $f(|x_n|)$. Let $\Omega=[0,1]$ and P be the Lebesgue measure. Define:

$$X_k:\ (\Omega,\mathscr{F},P)\to E,$$

$$X_k(\omega)=\sum_{n=m_k+1}^{m_{k+1}}\frac{x_n}{f(|x_n|)}1_{A_{k,n}}(\omega).$$

For every k, $\mathscr{F}_k$ will be the σ-algebra generated by $\{X_1, X_2, \ldots, X_k\}$. Let σ be a bounded stopping time $\geqslant N$. For each $k\geqslant N$, let $B_{k,n}=A_{k,n}\cap\{\sigma=k\}$. We have for every $D\in\mathscr{F}_\sigma$,

$$\left\|\int_D X_\sigma\right\|=\left\|\sum_{k\geqslant N}\int_{D\cap\{\sigma=R\}}X_k\right\|=\left\|\sum_{k\geqslant N}\sum_{n=m_k+1}^{m_{k+1}}P(D\cap B_{k,m})\frac{x_n}{f(|x_n|)}\right\|.$$

Since $D \cap B_{k,n} \subseteq A_{k,n}$,

$$\frac{P(D \cap B_{k,n})}{f(|x_n|)} = \alpha_{k,n} \leqslant 1,$$

thus:

$$\sup_{A \in \mathcal{F}_\sigma} \left\| \int_A X_\sigma \right\| \leqslant \left\| \sum_{k \geqslant N} \sum_{n=m_k+1}^{m_{k+1}} x_n \right\|,$$

which goes to zero since (x_n) is summable. However $(|X_n|$ is not a potential, since if it was, $f(|X_n|)$ must converge to zero a. e. But, it is easy to chech that $f(|X_n|(\omega)) = 1$ for each n and each $\omega \in \Omega$.

(2) To prove that c_0 does not embed in E, consider (Y_n) an independent sequence of real valued random variables taking the values ± 1 with probabilities $1/2$, $1/2$. Clearly, the sequence:

$$X_n: \ (\Omega, \mathcal{F}, P) \to c_0,$$
$$X_n(\omega) = (Y_0, Y_1, \ldots, Y_n, 0, 0, \ldots),$$

is an L^1-bounded martingale ($\|X_n\|_{c_0} = 1$).

On the other hand, $\int |X_n| = (1, 1, \ldots, 1, 0, 0, \ldots)$ is not norm convergent to an element of c_0.

(3) Follows from the fact that a weakly sequentially complete $A - M$ space is finite dimensional.

2.4. WEAK CONVERGENCE OF o-AMARTS. — An adapted sequence (X_n) in $L^1[E]$ is said to be a *weak potential* if for every A in $\bigcup_n \mathcal{F}_n$, $\left(\int_A X_n\right)$ converges weakly to zero. We say that (X_n) is of class (B) if $\sup_{\sigma \in T} \int \|X_\sigma\| < \infty$.

In [8], it is shown that every separable subspace of E has a separable dual (E' has R.N.P.) if and only if every E-valued weak potential of class (B) converges weakly a. e. We now show that if we restrict ourselves to positive potentials, then the R.N.P. in the dual is not needed.

For every subspace F of E, we denote by dens (F) the density character of F and for every subset A of E, denote by H_A the smallest closed ideal generated by A. An element u in E_+ is said to be a quasi-interior point if $H_{\{u\}} = E$.

THEOREM 2.4.1. — *For a Banach lattice E the following properties are equivalent:*

(1) *Every closed separable sublattice of E has a quasi-interior point in the dual;*

(2) *every E-valued positive weak potential of class (B) converges weakly a. e.;*

(3) *for every closed sublattice F of E, there exists A in F′*, card $(A) \leq$ dens (F) *such that* $H_A = F'$.

Proof. — (1)$\Rightarrow$(2). Let (Z_n) be a positive weak potential of class (B). Using the maximal inequality of [10], we get that $\sup_n \| Z_n \| < \infty$ outside a set Ω_0 with $P(\Omega_0) = 0$. Since (Z_n) is almost separately valued, there exists a separable sublattice F such that $P[Z_n \in F] = 1$ for every n.

Let u be a quasi-interior element in F'_+. Clearly, $(u(Z_n))$ is an L^1-bounded real potential, hence $u(Z_n) \to 0$ outside Ω_u with $P(\Omega_u) = 0$. Let f be any element in F'_+, we have:

$$f(Z_n(\omega)) = f \wedge mu(Z_n(\omega)) + (f - f \wedge mu)(Z_n(\omega)).$$

Clearly, $f \wedge mu(Z_n(\omega))$ converges to zero outside Ω_u when $n \to \infty$ and $\lim_m (f - f \wedge mu)(Z_n(\omega)) = 0$ uniformly in n, outside Ω_0. Thus $f(Z_n)$ converges to zero outside $\Omega_0 \cup \Omega_u$.

(2) $\Rightarrow$(3). We can suppose $F = E$. Let K denote the positive part of the unit ball of E' equipped with the weak-star topology. It admits a base of open sets of cardinality $\alpha =$ dens (E). Suppose (3) is not satisfied, that is if $A \subset E'$ is of cardinality $\leq \alpha$, then $H_A \neq E'$.

We will need the following lemma.

LEMMA 2.4.2. — *Let* $W = \cup \{ V$ *weak-star open set of K such that there exists* $A \subset E'$, card $A \leq \alpha$ *and* $V \subset H_A + (1/2) K \}$. *Then:* $L = K \setminus W$ *is non empty and for every open set* $V \neq \emptyset$ *of L and all v in E′ we have:*

$$V \setminus \left(H_{\{v\}} + \frac{1}{2} K \right) \neq \emptyset.$$

Proof of the lemma. — Since $\bigcup_i H_{A_i} \subset H_{\bigcup_i A_i}$ and W has a base of open sets of cardinality$=\alpha$, we have $W \subset H_A + (1/2) K$ for one A with card $A \leq \alpha$. If $W = K$, we have $K \subset H_A + (1/2) K$ and since $H_A + H_A = H_A$, by induction we get that $K \subset H_A + (1/2^n) K$, thus $K \subset H_A$ and H_A being closed, $E' = H_A$ which is a contradiction. Thus $L \neq \emptyset$.

Set now $I = [0, 1]$ and λ the Lebesgue measure on I. We shall construct an increasing family of finite algebras $(\mathscr{F}_n)$ on I, and a sequence of E_+ valued,

$\mathscr{F}_n$-measurable random variables, and for every t in I, closed subsets $V_n(t)$ of L verifying the following conditions:

(a) $s_n=\inf\{\lambda(B);\ B\in\mathscr{F}_n\}>0$;

(b) $\|E^{\mathscr{F}_n}[X_{n+1}](t)\|\leqslant 2^{-n}s_n,\ t\in I$;

(c) $\|X_n(t)\|\leqslant 1,\ t\in I;\ n\in\mathbb{N}$;

(d) $\mathring{V}_n(t)\neq\emptyset$ and $V_{n+1}(t)\subset V_n(t),\ t\in I$;

(e) For $h\in V_n(t)$, $h(X_n(t))\geqslant 1/3$;

(f) $V_n(t)$ is constant on each atom of $\mathscr{F}_n$.

Denote by B the positive unit ball of E.

Suppose the construction is made till the rank n. Let Z be an atom of $\mathscr{F}_n$. Let V be the only representative of the $(V_n(t))$ for t in Z. Set $D=\{x\in B;\ h\in\mathring{V},\ h(x)\geqslant 1/2\}$ and $C=\overline{\text{conv}}\ D$. We first prove that $d(0,\ C)=0$. If not, the norm being increasing on E_+, we have $d(0,\ \overline{C+E_+})=d(0,\ C)=\beta>0$. By Hahn-Banach theorem, there exists f in E' with $f\geqslant\beta$ on $C+E_+$, that is an $f\geqslant 0$ with $f\geqslant\beta$ on D. By Lemma 2.4.2, there exists $h\in\mathring{V}$ such that $h\notin H_{\{f\}}+(1/2)K$. Let $n\geqslant 2\beta^{-1}$. Then, $h\notin[0,f]+(1/2)\,K-E'_+$ which is weak-star closed. Again, by Hahn-Banach theorem, there exists x in E, $\|x\|=1$ such that for every $k\in K$, $0\leqslant g\leqslant nf$ and $l\in E'_+$, we have $g(x)+(1/2)\,k(x)-l(x)\leqslant h(x)$. Thus $x\in E_+$ and since $(1/2)k(x)\leqslant h(x)$ for $k\in K$, we have $h(x)\geqslant 1/2$. Finally $nf(x)\leqslant h(x)\leqslant 1$ and $f(x)=0$. This contradiction shows that $d(0,\ C)=0$. Hence, there exists a finite family $(x_i)_{i\leqslant k}$ of D and real numbers $\alpha_i>0$ such that $\sum_i\alpha_i=1$ and $\|\sum\alpha_i x_i\|\leqslant s_n 2^{-n}$.

Divide now Z into k disjoint subsets Z_i of measure $\alpha_i\lambda(Z)$ and define X_{n+1} on Z by $X_{n+1}(t)=x_i$ if $t\in Z_i$. Also set:

$$V_{n+1}(t)=\left\{h\in\mathring{V}_n(t);\ h(X_{n+1}(t))\geqslant\frac{1}{3}\right\}.$$

For each n, let $a_n=\int X_n$. If $\mathscr{F}_{n+1}$ is the smallest σ-field generated by X_{n+1}, we get $a_n=\int E^{\mathscr{F}_{n-1}}[X_n]$, hence $\|a_n\|\leqslant 2^{-n+1}\ s_{n-1}$. Set $b_n=\sum_{p\geqslant n}a_p/s_{p-1}$, then $\|b_n\|\leqslant 2^{-n+2}$. Let now $Y_n(t)=X_n(t)+b_n$ for t in I.

We get that $\| Y_n(t) \| \leqslant 3$. Also, (Y_n) is a supermartingale. Indeed, for $t \in Z$. we have:

$$E^{\mathscr{F}_n}[Y_{n+1}](t) = b_{n+1} + E^{\mathscr{F}_n}[X_{n+1}](t) = b_{n+1} + \frac{1}{m(Z)} \int_Z X_{n+1}$$

$$\leqslant b_{n+1} + \frac{1}{m(Z)} \int X_{n+1} = b_{n+1} + \frac{a_{n+1}}{s_n}.$$

Thus, $E^{\mathscr{F}_n}[Y_{n+1}] \leqslant b_n \leqslant Y_n$.

Moreover, $\left\| \int Y_n \right\| = \| a_n + b_n \| \leqslant 2^{-n+3}$ norm converges to zero.

On the other hand, for every h in $V_n(t)$, $h(Y_n(t)) \geqslant h(X_n(t)) \geqslant 1/3$, hence if $h \in \bigcap_n V_n(t)$ which exists from (*d*), $h(Y_n(t)) \geqslant 1/3$ for every n. Thus if (Y_n) converges weakly, the limit cannot be zero which is absurd.

(4) $\Rightarrow$ (1) follows from the fact that if $A = \{a_n, \ n \in \mathbb{N}\}$ and $a = \sum_n 2^{-n}(a_n / \| a_n \|)$ then, $H_{\{a\}} = H_A$.

COROLLARY 2.4.3. — *For a Banach lattice E, the following properties are equivalent:*

(1) *E has the R.N.P. and every separable sublattice of E has a quasi-interior point in the dual;*

(2) *every E-valued o-amart of class (B) converges weakly a. e.*

Proof. — Follows from the Riesz decomposition and the preceding theorem.

Now, we prove that we cannot weaken the (B) boundedness in order to still get the weak convergence of positive potentials.

THEOREM 2.4.4. — *If E is a Banach lattice, the following properties are equivalent:*

(1) *E is isomorphic (as a Banach lattice) to an $A-L$ space;*

(2) *every E-valued o-potential (or positive weak potential) converges weakly a. e.;*

(3) *every L^1-bounded o-potential (or positive weak potential) converges weakly a. e.*

Proof. — (1) $\rightarrow$ (2), (3) follows from Theorem 2.4.1.

(3) $\Rightarrow$ (1) As in Theorem 2.3.2, suppose there exists (x_n) in E_+ summable, such that $\sum_{n=m_k+1}^{m_{k+1}} \| x_n \| = 1$ for an increasing sequence of integers (m_k). For each k, divide the interval $[0, 1/k]$ into $(m_{k+1} - m_k)$ disjoint subintervals $(A_{n,k})$

such that the length of each $A_{n,k}$ is $\|x_n\|/k$. Set $A_k = \bigcup_{n=m_k+1}^{m_{k+1}} A_{n,k}$. Let $(\Omega, \mathscr{F}, P)$ be as in Theorem 2.3.2 and define:

$$X_k:\ \Omega \to E,$$

by:

$$X_k(\omega) = \begin{cases} \sum_{n=m_k+1}^{m_{k+1}} k \dfrac{x_n}{\|x_n\|} 1_{A_{n,k}}(\omega_k) + \sum_{j>m_k+1} x_j & \text{if } \omega_k \in A_k, \\ \sum_{j>m_k+1} x_j & \text{if } \omega_k \notin A_k. \end{cases}$$

It is easily seen that (X_k) is an L^1-bounded positive supermartingale which is an o-potential (also a weak potential).

To show that (X_k) is not weakly convergent, let $B_k = \{\omega \in \Omega;\ \omega_k \in A_k\}$. The sets (B_k) are independant and $\sum_k P(B_k) = \sum_k 1/k = \infty$, hence from the Borel-Cantelli theorem, almost all ω belongs to an infinite number of B_k. But if $\omega \in B_k$, $\|X_k(\omega)\| \geqslant k$ and (X_k) cannot converge weakly.

COROLLARY 2.4.4. — *If E is a Banach lattice, the following properties are equivalent:*

(1) *E is isomorphic (as a Banach lattice) to an $l^1(\Gamma)$;*

(2) *each E-valued o-amart in $\mathscr{A}_b^0$ is weakly convergent a.e.;*

(3) *each E-valued o-amart in $\mathscr{A}_B^0$ is weakly convergent a.e.*

Proof. — (1) $\Rightarrow$ (2) follows from the Riesz decomposition and the fact that in an $A-L$ space if $\sup_n \int |X_n|$ exists then $\sup_n \int \|X_n\| < \infty$.

(3) $\Rightarrow$ (1) An $A-L$ space with the Radon-Nikodym property is an $l^1(\Gamma)$.

The following corollary shows that if E is infinite dimensional, then the real valued submartingales and supermartingales convergence theorems hold if and only if E is $l^1(\Gamma)$.

COROLLARY 2.4.5. — *The following are equivalent:*

(1) *E is an $l^1(\Gamma)$ for some set Γ;*

(2) *each E-valued positive supermartingale converges weakly a.e.;*

(3) *each E-valued submartingale such that $\sup_n \int X_n^+$ exists converges weakly a.e.*

2.5. STRONG CONVERGENCE OF o-AMARTS

We first show that the strong convergence of (B) bounded o-amarts generally fails.

THEOREM 2.5.1. — *The following properties are equivalent:*

(1) *E is isomorphic (as a Banach lattice) to an $A-L$ space;*

(2) *every E-valued o-potential (or positive weak potential) converges strongly a. e.;*

(3) *every E-valued o-potential (or positive weak potential) of class (B) converges strongly a. e.*

Proof. — The same as in Theorem 2.3.2, since we exhibit a positive supermartingale of class (B) which is not a uniform potential, hence not strongly convergent to zero.

COROLLARY 2.5.2. — *The following properties are equivalent:*

(1) *E is isomorphic to a $l^1(\Gamma)$;*

(2) *each E-valued o-amart of class (B) converges strongly a. e.*

COROLLARY 2.5.3. — *The following properties are equivalent:*

(1) *E is isomorphic to $l^1(\Gamma)$;*

(2) *E has the shur property and a quasi-interior point in the dual.*

Proof. — If E has the shur property, and a quasi-interior point in the dual, then every E-valued o-potential of class (B) converges weakly, thus strongly. By the preceding theorem, E is an $A-L$ space. The Kakutani's representation theorem gives that E is an $L^1[\mu]$ for some measure μ. If μ is not purely atomic, $L^1[\mu]$ will contain a sublattice isomorphic to $L^1[0, 1]$ which contradicts the shur property. Hence E is an $l^1(\Gamma)$.

We now show that o-amarts valued in a weakly compact set do not necessarily converge strongly like martingales do. The following example is of a supermartingale valued in the unit ball of l^2 which is not strongly convergent.

Example. — Let $(\Omega, \mathscr{F}, P)$ be as in theorem. Define $X_m:(\Omega, \mathscr{F}, P) \to l_2$ by:

$$[X_m(\omega)]_{2^n+k} = \begin{cases} \dfrac{1}{2^n} & \text{if } m<n; \\ 1 & \text{if } m=n \text{ and } \dfrac{k-1}{2^m} \leqslant \omega_m \leqslant \dfrac{k}{2^m}; \\ 0 & \text{otherwise,} \end{cases}$$

for each m, n and $1 \leqslant k \leqslant 2^n$.

It is easily seen that (X_m) is a positive supermartingale which converges weakly to 0. It is uniformly bounded since:

$$\| X_n(\omega) \|_2^2 = 1 + \sum_{m>n} \left(\frac{1}{2^m} \right)^2 . 2^m \leqslant 2.$$

It is not convergent in norm to zero since $\| X_n(\omega) \|_2 \geqslant 1$. However, if one considers weakly compact sets which are "close" to order intervals, we can get strong convergence. The reason is the following theorem which shows that weakly compact order intervals "behave" like compact sets, namely that their extreme points are denting.

THEOREM 2.5.4. — *A Banach lattice E has an order continuous norm if and only if whenever (x_n) is a sequence in E, $0 \leqslant x_n \leqslant x$ and x_n converges weakly to zero then it converges strongly to zero.*

Proof. — (1) Let $0 \leqslant x_n \leqslant x$ and $x_n \to x$ weakly. For every denting point z of $[0, x]$, $x_n \vee z \to x$ weakly, hence $x_n \wedge z$ converges weakly to z and consequently $x_n \wedge z \to z$ strongly. From [25], we get that $x_n \wedge z \to z$ strongly for every z in the convex hull of the denting points. Now, write $x - x_n \leqslant 2(x-z) + (z - z \wedge x_n)$ and use that a weakly compact set is the closed convex hull of its denting points to get the result.

(2) If E does not have an order continuous norm then there exists an increasing sequence (x_n) order bounded by x and not norm Cauchy, that is there exists $\alpha > 0$ and a subsequence (x_{n_j}) of (x_n) so that the vectors $u_j = x_{n_{j+1}} - x_{n_j}$ satisfies $\| u_j \| \geqslant \alpha$ and $u_j \leqslant x$ for all j. Clearly $u_j \to 0$ weakly and not strongly which is a contradiction.

COROLLARY 2.5.5. — *If E has an order continuous norm and (x_n), (y_n) are two sequences verifying:*

(i) $0 \leqslant x_n \leqslant y_n$, $\forall n$;

(ii) $x_n \to x$ *weakly and* $y_n \to x$ *strongly then* $x_n \to x$ *strongly.*

Proof. — Clearly, $x_n \leqslant x_n \vee x \leqslant y_n \vee x$, hence $x_n \vee x \to x$ weakly and $x_n \wedge x \to x$ weakly thus strongly. But:

$$0 \leqslant (y_n - x_n) \leqslant y_n - x_n \wedge x \leqslant | y_n - x | + | x - x_n \wedge x |,$$

which gives the strong convergence of (x_n).

We now give a general form of Theorem 2.2.1. It will allow us to prove the strong convergence of two subclasses of L^1-bounded o-amarts.

THEOREM 2.5.6. — *If E is weakly sequentially complete and (X_n) is an o-amart such that $0 \leqslant |X_n| \leqslant Y_n$ where (Y_n) is an L^1-bounded norm convergent sequence in $L^1[E]$, then (X_n) converges strongly a.e.*

Proof. — The sequence (X_n) is L^1-bounded, hence we can suppose (X_n) positive and $0 \leqslant X_n \leqslant Y_n$. Now, $Y \leqslant X_n \vee Y \leqslant Y_n \vee Y$ where Y is the limit of (Y_n). Thus $X_n \vee Y$ converges weakly to Y and therefore strongly. For $f \in E'_+$, $f(X_n)$ converges a.e., hence $f(X_n \wedge Y)$ converges a.e. But $(X_n \wedge Y)$ is valued in the weakly compact set $[0, Y(\omega)]$, thus it converges weakly to a Bochner integrable X_∞. Therefore, X_n converges weakly a.e. to X_∞. Let $Z_n = X_n - E^{\mathscr{F}_n}[X_\infty]$. It is an o-potential. Now, $0 \leqslant Z_n^+ \leqslant X_n \leqslant Y_n$. The sequence $(Z_n^+ \wedge Y)$ converges weakly to zero, therefore strongly. Also, $(Z_n^+ \vee Y)$ converges weakly to Y and therefore strongly since:

$$0 \leqslant Z_n^+ \vee Y \leqslant Y_n \vee Y.$$

It follows that (Z_n^+) converges strongly a.e. Again, $0 \leqslant Z_n^- \leqslant E^{\mathscr{F}_n}[X_\infty]$ and the same proof gives that (Z_n^-) converges strongly a.e.

2.6. POSITIVE SUBMARTINGALES AND HYPOMARTINGALES

DEFINITION 2.6.1. — An o-amart (X_n) is said to be an *hypomartingale* if for every $A \in \bigcup_n \mathscr{F}_n$, the sequence $\left(\int_A X_n\right)$ converges while being below its limit. If E is weakly sequentially complete and (X_n) is an L^1-bounded hypomartingale, then $X_n = M_n - Z_n$ where (M_n) is a martingale and (Z_n) is a positive weak potential.

We can now state the following.

THEOREM 2.6.2. — *The following properties are equivalent:*

(1) *E has the Radon-Nikodym property;*

(2) *every L^1-bounded positive submartingale converges strongly a.e.;*

(3) *every L^1-bounded positive hypomartingale converges strongly a.e.*

Proof. — Since $X_n = M_n - Z_n$, (M_n) is norm convergent and $0 \leqslant Z_n \leqslant M_n$, the theorem follows from Theorem 2.5.6.

This theorem shows the surprising difference between positive submartingales and positive supermartingales in infinite dimensional spaces. On the other hand, it shows how the order structure blends with the Radon-Nikodym property and that the R.N.P. in Banach lattices appears like the random analog of weak sequential completeness since submartingales can be considered as "randomly increasing sequences".

2.7. ORDER BOUNDED o-AMARTS

The results of section 2.6 clearly imply that order bounded o-amarts norm converges a. e. In this section, we show that they are even order convergent.

The key of the proof is to show an inequality extending the one of R. V. CHACON in the real case [3]. That is if (X_n) is an L^1-bounded $(\mathscr{F}_n)$-adapted sequence of real random variables such that $\left(\int X_\sigma\right)_{\sigma \in T}$ is bounded then the following holds:

$$\int (\limsup X_n - \liminf X_n)\, dP \leqslant \lim \sup_{\sigma, \tau \in T} \int (X_\sigma - X_\tau)\, dP.$$

In the finite dimensional case, the L^1-boundedness implies—*via* the maximal inequality—that the process is finite a.e. In the infinite dimensional case, such a property is not satisfied, so we must assume that for almost all ω in Ω, $\sup_n |X_n(\omega)|$ exists in the Banach lattice, at least to assure the existence of $\limsup X_n$ and $\liminf X_n$.

Clearly, the inequality is of interest in ordered spaces where the order convergence is stronger than the norm convergence, that is in spaces which have an order continuous norm.

We shall start recalling some well-known facts.

LEMMA 2.7.1. — *Let E be an order continuous Banach lattice which has a weak unit, then there exists a probability space (X, Σ, μ) such that E is order isometric to an ideal of $L^1[X, \Sigma, \mu]$.*

For a proof, *see* LINDENSTRAUSS and TZAFRIRI [24].

The following lemma is also standard.

LEMMA 2.7.2. — *For every Bochner integrable random variable $X:(\Omega, \mathscr{F}, P) \to L^1[X, \Sigma, \mu]$, there exists $Y:\Omega \times X \to \mathbb{R}$, $\mathscr{F} \otimes \Sigma$ measurable such that P a.e. $Y(\omega, .)$ represents $X(\omega)$. Moreover, Y is unique modulo a $P \otimes \mu$ negligible function. The following also holds:*

(i) $X \in L^1_E[P] \Leftrightarrow Y \in L^1_{\mathbb{R}}(P \otimes \mu]$ *where* $E = L^1[X, \Sigma, \mu]$;

(ii) *if* $X \in L^1_E[P]$ *then* $\int X\, dP$ *is represented by* $t \to \int Y(\omega, t)\, dP(\omega)$;

(iii) *if* (Y_n) *is associated to a sequence* (X_n) *such that for almost all* ω, *the sequence* $|X_n|(\omega)$ *is order bounded in E, then* $\limsup Y_n$ *and* $\liminf Y_n$ *represent the functions* $\omega \to \limsup X_n(\omega)$ *and* $\omega \to \liminf X_n(\omega)$.

Consider now the filtration $(\mathscr{F}_n \otimes \Sigma)$ on $\Omega \times X$. The next lemma describes an approximation process for the stopping times relative to this filtration. Since the arguments are standard, we give only a sketch of the proof.

LEMMA 2.7.3. — *Let τ be a stopping time for the filtration $(\mathscr{F}_n \otimes \Sigma)$ such that $p \leqslant \tau \leqslant q$. Then, for all $\varepsilon > 0$, there exists a finite partition (A_l) of X by measurable sets and stopping times (σ_l) on Ω such that $p \leqslant \sigma_l \leqslant q$ and that if the stopping time σ is defined by:*

$$(\star) \qquad \sigma(\omega, t) = \textstyle\sum_l 1_{A_l}(t)\, \sigma_l(\omega),$$

then:

$$P \otimes \mu \{ \tau \neq \sigma \} \leqslant \varepsilon.$$

Proof. — Let $k \in \mathbb{N}$ and $p \leqslant k \leqslant q$. The set $\{\tau = k\}$ belongs to $\mathscr{F}_k \otimes \Sigma$, hence there exists a set B_k which is a union of rectangles $C \times D$ with $C \in \mathscr{F}_k$ and $D \in \Sigma$ and such that:

$$P \otimes \mu(\{\tau = k\}) \Delta B_k) \leqslant \frac{\varepsilon}{q^2}.$$

Let σ be such that $\{\sigma = k\} = B_k \setminus \bigcup_{p \leqslant s < k} B_s$ for each $k, p \leqslant k \leqslant q$. Clearly, σ is a stopping time and $P \otimes \mu \{ \tau \neq \sigma \} \leqslant \varepsilon$. If we denote by A_l the atoms of the finite algebra of X generated by the projections on X of all the rectangles involved, then σ can be written as in $(\star)$.

THEOREM 2.7.4. — *If E is a Banach lattice with an order continuous norm and (X_n) is a sequence of E-valued Bochner integrable random variables verifying:*

(i) $\sup_n |X_n(\omega)|$ *exists for almost all $\omega \in \Omega$;*

(ii) $\left(\int X_\sigma \right)_{\sigma \in T}$ *is order bounded in E,*

then the following inequality is satisfied:

$$\int (\limsup X_n - \liminf X_n)\, dP \leqslant \limsup_{\sigma, \tau \in T} \int (X_\sigma - X_\tau)\, dP.$$

Proof. — Since the X_n's are almost separably valued, we can assume that E is separable, hence with a weak unit. By Lemma 2.7.1, it is enough to show the inequality in $L^1[X, \Sigma, \mu]$. Denote by T' the set of bounded $(\mathscr{F}_n \otimes \Sigma)$ stopping times and by T'_s the set of those which can be written as in the formula $(\star)$.

For every n, let Y_n be an $\mathcal{F}_n\otimes\Sigma$ representation of X_n. Let $\tau\in T'$ with $p\leqslant\tau\leqslant q$. It follows from Lemma 2.7.3 that for every $\varepsilon>0$, there exists $\tau'\in T'_s$ with $p\leqslant\tau'\leqslant q$ and $\left|\int Y_\tau-\int Y_{\tau'}\right|\leqslant\varepsilon$. Hence we have:

$$\sup_{\tau\in T',\,\tau\geqslant p}\int Y_\tau=\sup_{\tau\in T'_s,\,\tau\geqslant p}\int Y_\tau.$$

Write now, $\tau(\omega,t)=\sum_l 1_{A_l}(t)\,\sigma_l(\omega)$ for $\tau\in T'_s$ where $p\leqslant\sigma_l\leqslant\sup\tau$.

$$\int Y_\tau\,d(P\otimes\mu)=\sum_l\int_{A_l}\int_\Omega Y_{\sigma_{l(\omega)}}(\omega,t)\,dP(\omega)\,d\mu(t)$$

$$=\sum_l 1_{A_l}\int_\Omega X_{\sigma_l}\,dP\leqslant\sup_{\sigma\geqslant p,\,\sigma\in T}\int_\Omega X_\sigma\,dP.$$

Since $t\to\int_\Omega Y_{\sigma_{l(\omega)}}(\omega,t)\,dP(\omega)$ represents $\int_\Omega X_{\sigma_l}\,dP(\omega)$. This shows that:

$$\iint Y_\tau\leqslant\int_X\left(\sup_{\sigma\geqslant p,\,\sigma\in T}\int_\Omega X_\sigma\,dP\right)d\mu(t).$$

The same proof shows that:

$$\iint Y_\tau\geqslant\int_X\left(\inf_{\sigma\geqslant p,\,\sigma\in T}\int_\Omega X_\sigma\,dP\right)d\mu(t).$$

Using now Chacon's inequality of the real line, we get:

$$\iint(\limsup Y_n(\omega,t)-\liminf Y_n(\omega,t))$$

$$\leqslant\limsup_{\tau,\,\rho\in T}\iint(Y_\tau(\omega,t)-Y_\rho(\omega,t))\,d(P\otimes\mu)$$

$$\leqslant\int_X\left[\limsup_{\sigma\in T}\int_\Omega X_\sigma\,dP-\liminf_\sigma\int_\Omega X_\sigma\,dP\right]d\mu.$$

By Lemma 2.7.2, we get:

$$\int_X\left[\int_\Omega(\limsup X_n(\omega)-\liminf X_n(\omega))\,dP(\omega)\right]d\mu$$

$$\leqslant\int_X\left(\limsup_{\tau,\,\sigma\in T}\int_\Omega(X_\sigma-X_\tau)\,dP\right)d\mu.$$

Replacing X by any measurable subset A, the inequality will still hold hence:

$$\int_{\Omega} (\limsup X_n - \liminf X_n) \leqslant \limsup_{\sigma, \tau} \int (X_\sigma - X_\tau) \, \mu \text{ a. e.},$$

which concludes the proof.

COROLLARY 2.7.5. — *(a) If E has an order continuous norm and (X_n) is an o-amart such that* $\sup_n |X_n(\omega)|$ *exists for almost all $\omega \in \Omega$, then (X_n) order converges a. e.*

(b) The same holds when we replace the hypothesis (X_n) o-amart by (X_n) supermartingale.

Proof. — (*a*) Follows immediately from the inequality. For (*b*), we notice that for almost all ω, $(X_n(\omega))$ is valued in a weakly compact order interval, hence (X_n) converges weakly to $X \in L^1[E]$. Consider now $Z_n = X_n - E^{\mathscr{F}_n}[X]$. We have $\inf_\sigma \left(\int Z_\sigma \right) = 0$, thus (X_n) is an o-amart which is order convergent.

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