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THE STANDARD FORM OF A REPRESENTATION-FINITE ALGEBRA

BY

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RÉSUMÉ. — On connaît depuis [RI 1] et [BG] l'importance de la technique des revêtements pour l'étude des représentations des algèbres de dimension finie. Dans la définition du groupe fondamental Φ d'une algèbre A, qui est donnée dans [BG] lorsque A a un nombre fini de représentations indécomposables à isomorphisme près, les modules indécomposables sont supposés connus. Dans ce qui suit, nous donnons une description directe de Φ, du revêtement universel et de la forme standard de A.

ABSTRACT. — The importance of covering techniques for the investigation of the representations of finite-dimensional algebras is well-known since [RI 1] and [BG]. When the algebra A admits only finitely many isomorphism classes of indecomposable representations, the definition of the fundamental group Φ of A given in [BG] requires the knowledge of all indecomposables. In the sequel, we describe Φ, the universal cover and the standard form of A directly in terms of A.

A very convenient way to determine whether a given algebra A is representation-finite (and to compute its representations) consists in finding a simply connected cover of a suitable degeneration of A ([BG], [G 2], [G0]). In theorem 3.1. we describe the standard form $\bar{A}$ of A, which is the best possible degeneration, and we determine the universal Galois covering of $\bar{A}$ in terms of A (Theorem 1.5 and Remark 3.3 *a*). Our results heavily rest on the existence of a multiplicative basis in $\bar{A}$, i. e. of a basis such that the product of two basis vectors is again a basis vector or else is zero. The existence of such bases in representation-finite algebras was first stated by KUPISCH in the symmetric case [K] (*see* also [KS]). It was then proved by BONGARTZ for algebras whose quiver contains no oriented cycle [B 1]. BONGARTZ' proof is easily extended to simply connected *locally* representation-finite "algebras" [BRL], a case to which our existence proof

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can be reduced. Our own researches were started by ROITER 's publication [RO], in which the existence of a multiplicative basis for general representation-finite algebras is stated. Clearly, ROITER 's work lies on a higher level of difficulty, and we have not been in a position to check all the points in his developments. Paraphrasing Picard's comment on Poincaré 's Duality Theorem (Picard et Simart, Théorie des Fonctions Algébriques, tome I, chap. II, 1897), we might say: « ROITER a donné une démonstration générale de l'existence d'une base multiplicative. Sa démonstration repose sur des considérations entièrement différentes, mais peut-être plusieurs points auraient-ils besoin d'être complétés. Aussi, avons-nous suivi une autre voie, mais avons dû nous limiter au cas des algèbres standard. »

Well-read mathematicians will easily interpret our proofs as statements on singular homology of triangulated topological spaces associated with the considered algebras. In fact, our intuition is geometrical, and so were our original proofs. The translation from geometry to algebra has been worked out during the spring vacation 1982 in order to take care of pure algebraists. For them we replaced short references to algebraic topology (to [ML], IV, 11.5 or [GZ], Ap. II, 3.6 for instance) by longer elementary variations on the snake-lemma. We hope that these variations will not be dismissed as mere exercises for a first-year course in homological algebra, since they come from, rest on and elucidate a mass of examples in representation-theory ([NR], [BGP], [L], [SZ], [BR], [BG], [BRL], [B 2], [CG]).

The method used here was presented in June 1981 in Oberwolfach, where corollary 2.2, 2.6, 2.11 and 3.3 *b* were stated. The remaining results were presented in november 1981 in a lecture at the university of Trondheim. Since then, we received from MARTINEZ and DE LA PEÑA the proof [MP 1] of an older conjecture of ours which simplifies the demonstration of Theorem 1.5. Nevertheless, we maintained the first proof in paragraph 2 because of its own virtues (2.6-2.8). As they informed us at the beginning of March, MARTINEZ and DE LA PEÑA also noticed that our Oberwolfach-proof for the existence of a multiplicative basis in schurian algebras extends to standard algebras [MP 2].

Our investigations commenced with BONGARTZ and ROITER as a joint discussion which diverged too rapidly. We like to express our thanks to both of them.

The notations are those introduced in [G 1], [BG] and [G 2]. In particular, k always denotes an algebraically closed field.

1. The group of constraints of a representation-finite algebra

1.1. We first fix the *notations*: Let Λ be a locally bounded k-category ([BG], 2.1). By $\Lambda(a, b)$ we denote the space of morphisms from a to b in Λ, by $\mathscr{R}^0 \Lambda(a, b) = \Lambda(a, b) \supset \mathscr{R}^1 \Lambda(a, b) \supset \mathscr{R}^2 \Lambda(a, b) \ldots$ the radical series of $\Lambda(a, b)$ considered as a bimodule over $\Lambda(b, b)$ and $\Lambda(a, a)$. We say that a morphism $\mu \in \Lambda(a, b)$ has *level* $n \in \mathbb{N}$ if $\mu \in \mathscr{R}^n \Lambda(a, b) \setminus \mathscr{R}^{n+1} \Lambda(a, b)$; the zero-morphism has level ∞ by definition.

In the sequel, we assume that Λ is *locally representation-finite* ([BG], 2.2). We denote by $I = \operatorname{ind} \Lambda$ the category formed by chosen representatives of the isoclasses of indecomposable finite-dimensional Λ-modules, and we agree that $a^* = \Lambda(?, a)$ is chosen as a representative for each $a \in \Lambda$. Given two objects $a, b \in \Lambda$, we set $g(a, b, n) = \sup \{ p : \mathscr{R}^p I(a^*, b^*) = \mathscr{R}^n \Lambda\ (a, b)^* \}$ for each $n \in \mathbb{N} \cup \{ \infty \}$, where $I \supset \mathscr{R} I \supset \mathscr{R}^2 I \ldots$ is the radical series of I ([BG], 2.1). We say that a morphism $\mu \in \Lambda\ (a, b)$ of level n has *grade* $g(\mu) = g\ (a, b, n)$ in Λ. In case $g(\mu) = g < \infty$, this implies the existence of irreducible morphisms ([AR], § 1; [G 1], 1.6)

$$a^* \xrightarrow{\mu_1} m_1 \xrightarrow{\mu_2} m_2 \ldots m_{g-1} \xrightarrow{\mu_g} b^*$$

of I such that $\mu^* - \mu_g \ldots \mu_2 \mu_1 \in \mathscr{R}^{n+1} \Lambda(a, b)^*$ (remember that $\mathscr{R}^{n+1} \Lambda(a, b)$ has codimension $\leqslant 1$ in $\mathscr{R}^n \Lambda(a, b)$; *see* [G 2], 2.4).

In the following lemma, we denote by $m \xrightarrow{\bar{v}} n$ the arrow of Γ_Λ (= the Auslander-Reiten quiver of Λ) which is associated with an irreducible morphism $v : m \to n$.

LEMMA. — *With the above notations, the homotopy-class* ([BG], 1.2) $S(a, b, n)$ *of the path*

$$a^* \xrightarrow{\bar{\mu}_1} m_1 \xrightarrow{\bar{\mu}_2} m_2 \ldots m_{g-1} \xrightarrow{\bar{\mu}_g} b^*$$

of Γ_Λ *depends only on* a, b *and* n.

Proof. — Let $\pi : \tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$ be a universal covering and $F : k(\tilde{\Gamma}_\Lambda) \to I$ a well-behaved functor ([RI 2], 2.5; [BG], 3.1). Choose a sequence of morphisms

$$x \xrightarrow{v_1} z_1 \xrightarrow{v_2} z_2 \ldots z_{g-1} \xrightarrow{v_g} y$$

of $k(\tilde{\Gamma}_\Lambda)$ such that $F\nu_i - \mu_i \in \mathcal{R}^2 I$ for each i. We then get $F(\nu_g \ldots \nu_1) - \mu_g \ldots \mu_1 \in \mathcal{R}^{g+1} I$. Since we have

$$\mathcal{R}^g I/\mathcal{R}^{g+1} I(a^*, b^*) : k] = 1$$

and

$$\bigoplus_{\pi z = b} (\mathcal{R}^g k(\tilde{\Gamma}_\Lambda)/\mathcal{R}^{g+1} k(\tilde{\Gamma}_\Lambda))(x, z) \xrightarrow{\sim} (\mathcal{R}^g I/\mathcal{R}^{g+1} I)(a^*, b^*),$$

y is the unique point of $\pi^{-1}(b^*)$ such that

$$(\mathcal{R}^g k(\tilde{\Gamma}_\Lambda)/\mathcal{R}^{g+1} k(\tilde{\Gamma}_\Lambda))(x, y) \xrightarrow{\sim} (\mathcal{R}^g I/\mathcal{R}^{g+1} I)\,(a^*, b^*).$$

As a consequence, if we lift the path $\mu_g \ldots \bar{\mu}_1$ of Γ_Λ to a path of $\tilde{\Gamma}_\Lambda$ with prescribed origin x, the terminus y of the lifted path depends only on a, b and g.

Remark. — Consider an arrow $a \xrightarrow{\bar{\mu}} b$ of the quiver Q_Λ of Λ and assume that μ is a *representative* of $\bar{\mu}$, i. e. that $\mu \in \mathcal{R}\Lambda(a, b) \setminus \mathcal{R}^2 \Lambda(a, b)$. Then we simply write $S\bar{\mu}$ instead of $S(a, b, 1)$, and we have:

$$\nu = \nu_g \ldots \nu_1 \in \mathcal{R}\tilde{\Lambda}(x, y) \setminus \mathcal{R}^2 \tilde{\Lambda}(x, y),$$

since F induces a covering $\tilde{\Lambda} \to \Lambda$ ($\tilde{\Lambda}$ denotes the universal cover of Λ which is defined in [G 2], 2.1). In other words, if we lift $\bar{\mu}$ to an arrow $\bar{\nu}$ of $Q_{\tilde{\Lambda}}$ with tail x, the head of $\bar{\nu}$ is the terminus of the lifted path $\bar{\nu}_g \ldots \bar{\nu}_1$.

1.2. Denote by GQ_Λ the *free groupoid generated by* Q_Λ ([GZ], II, § 6), by $G\Gamma_\Lambda$ the *fundamental groupoid of* Γ_Λ: the objects of GQ_Λ are the points of the quiver Q_Λ of Λ; the (invertible) morphisms of GQ_Λ are formal compositions of arrows and of formal inverses of arrows of Q_Λ. Similarly, the objects of $G\Gamma_\Lambda$ are the vertices of Γ_Λ, i. e. the objects of I; the morphisms of $G\Gamma_\Lambda$ are the homotopy classes of paths of Γ_Λ.

The map $\bar{\mu} \mapsto S\bar{\mu}$ described in 1.1 defines a functor $S : GQ_\Lambda \to G\Gamma_\Lambda$ such that $Sa = a^* = \Lambda(?, a)$ if $a \in \Lambda$. We denote by $S_a = S(a, a)$ the induced group homomorphism from

$$\Pi(Q_\Lambda, a) = (GQ_\Lambda)(a, a) \text{ to } \Pi(\Gamma_\Lambda, a^*) = (G\Gamma_\Lambda)\,(a^*, a^*).$$

By definition, $\Pi(Q_\Lambda, a)$ is the *fundamental group* of the quiver Q_Λ at a, and $\Pi(\Gamma_\Lambda, a^*)$ is the *fundamental group* of the translation-quiver Γ_Λ at a^* (or equivalently of Λ at a).

PROPOSITION. — *Let Λ be locally representation-finite and connected, a a point of Λ, $\tilde{a}$ a point of the universal cover $\tilde{\Lambda}$ which lies over a. Then S_a gives rise to an exact group sequence:*

$$1 \to \Pi(Q_{\tilde{\Lambda}}, \tilde{a}) \to \Pi(Q_\Lambda, a) \xrightarrow{S_a} \Pi(\Gamma_\Lambda, a^*) \to 1.$$

In particular, $\Pi(Q_{\tilde{\Lambda}}, \tilde{a})$ *is identified with* $\operatorname{Ker} S_a$.

Proof. — Let $\pi : Q \to Q_\Lambda$ be a covering of quivers and Π a group acting on Q from the right. Assume that Q is connected and that Π acts freely and transitively on $\pi^{-1}(a)$. Then each $b \in \pi^{-1}(a)$ determines an exact sequence:

$$1 \to \Pi(Q, b) \xrightarrow{\varkappa} \Pi(Q_\Lambda, a) \xrightarrow{\tau} \Pi \to 1,$$

where $\varkappa$ is induced by π and τ is defined by $b\tau(w) = wb$ (= terminus of the lifting of w to Q with origin b).

This works in particular if $Q = Q_{\tilde{\Lambda}}$, $b = \tilde{a}$ and $\Pi = \Pi(\Gamma_\Lambda, a^*)$. In this case, π is induced by the universal covering $\tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$, and we define the action of Π on $\tilde{\Gamma}_\Lambda$ and $Q_{\tilde{\Lambda}}$ in such a way that $\tilde{a}v = v\tilde{a}$ (= terminus of the lifting of v to $\tilde{\Gamma}_\Lambda$ with origin $\tilde{a}$). It then remains to show that $\tau(w) = S_a(w)$ for each $w \in \Pi(Q_\Lambda, a)$, or equivalently that $\tau(w)\tilde{a} = S_a(w)\tilde{a}$. Now $\tau(w)\tilde{a}$ is the terminus of the lifting of w to $Q_{\tilde{\Lambda}}$ with origin $\tilde{a}$. Similarly, $S_a(w)\tilde{a}$ is obtained by lifting $S_a(w)$. Since w is a composition of arrows and of inverses of arrows of $Q_{\tilde{\Lambda}}$, our statement follows from the remark at the end of 1.1.

1.3. In the sequel, the kernel of S_a (1.2) will be called *the group of constraints of* Λ *at* a. It coincides with the group defined in [G 2], 2.2, and our purpose here is to produce a set of generators. For this sake, we first recall that a *schurian* category in the terminology of Roiter is a locally finite-dimensional k-category M such that $[M(a, b) : k] \leqslant 1$ for all $a, b \in M$. The terminology is justified by the fact that M is schurian if it is locally representation-finite and if $M(a, a) \xrightarrow{\sim} k$ for each $a \in M$. This last condition is satisfied in particular if M is locally bounded and *directed* (i.e. the quiver Q_M contains no oriented cycle).

Assume M to be locally bounded and schurian. We say that a path

$$x = x_0 \xrightarrow{v_1} x_1 \xrightarrow{v_2} x_2 \dots x_{p-1} \xrightarrow{v_p} x_p = y$$

of Q_M is a *zero-path* of M if the composition-map

$$M(x_0, x_1) \times \dots \times M(x_{p-1}, x_p) \to M(x_0, x_p)$$

is zero. We call *non-zero contour* of M a pair (v, w) of non-zero paths v and w with the same origin x and the same terminus y (compare with [RO]). We call the non-zero contour *simple* if x and y are the only common vertices of v and w. Finally, we denote by $C_a(v, w)$ the conjugacy class of the element $u^{-1} w^{-1} vu \in \Pi(Q_M, a)$ associated with a walk u from $a \in M$ to the origin of the contour (v, w). This conjugacy class is independent of u.

LEMMA. — *Let* Λ *be locally representation-finite and simply connected and* $a \in \Lambda$. *The fundamental group* $\Pi(Q_\Lambda, a)$ *is generated by the conjugacy classes* $C_a(v, w)$ *associated with the simple non-zero contours of* Q_Λ.

Our original proof is given in § 2.10 below. It uses Galois coverings whose groups are torsion-free and mainly rests on [G 2], 3.6. The following simpler proof rests on a conjecture recently proved by MARTINEZ and DE LA PEÑA.

Proof. — First consider the more general case where Λ is schurian. Denote by P the subgroup of $\Pi = \Pi(Q_\Lambda, a)$ generated by the conjugacy classes $C_a(v, w)$. Let $\tilde{Q}_\Lambda$ be the universal cover of Q_Λ, Q the quotient $\tilde{Q}_\Lambda/P$ and $\pi : Q \to Q_\Lambda$ the canonical projection. Then, two paths of Q with the same origin must have the same terminus, provided their projections on Q_Λ make up a non-zero contour.

Q is the quiver of a locally bounded k-category Λ' whose morphism-spaces are defined as follows: $\Lambda'(x, y) = 0$ if the projections of all paths from x to y are zero-paths of Λ; otherwise, $\Lambda'(x, y)$ is identified with $\Lambda(\pi x, \pi y)$. The composition $\Lambda'(x, y) \times \Lambda'(y, z) \to \Lambda'(x, z)$ is identified with $\Lambda(\pi x, \pi y) \times \Lambda(\pi y, \pi z) \to \Lambda(\pi x, \pi z)$, whenever $\Lambda'(x, y) \neq 0$, $\Lambda'(y, z) \neq 0$ and $\Lambda'(x, z) \neq 0$; it is zero otherwise. By construction, the action of Π/P on Q extends to a free action of Π/P on Λ', and the quiver-morphism π extends to a (Π/P)-invariant functor $F : \Lambda' \to \Lambda$. A path of Q is non-zero in Λ' iff its projection is non-zero in Λ. We infer that F is a Galois covering with group Π/P. By [MP 1], Π/P acts freely on ind Λ'. By [G 2], 3.6 F induces a Galois covering ind $\Lambda' \to$ ind Λ.

In the particular case where Λ is simply connected (hence directed and schurian), the covering ind $\Lambda' \to$ ind Λ must be trivial. Hence $\Pi = P$.

1.4. Let us now turn to the general case of a *connected locally representation-finite* Λ.

LEMMA. — *Let* $a_0 \xrightarrow{\bar{\alpha}_1} a_1 \to \xrightarrow{\bar{\alpha}_2} a_2 \ldots a_{n-1} \xrightarrow{\bar{\alpha}_n} a_n$ *be a path of* Q_Λ *and* $\alpha_i \in \mathscr{R}\Lambda(a_{i-1}, a_i)$ *a representative of* $\bar{\alpha}_i$ (1.1). *The following statements are equivalent:*

(i) $g(\alpha_n \ldots \alpha_2 \alpha_1) = \sum_{i=1}^n g(\alpha_i)$;
(ii) $\alpha'_n \ldots \alpha'_2 \alpha'_1 \neq 0$ *for all representatives* α'_i *of the* $\bar{\alpha}_i$;
(iii) $\bar{\alpha}_n \ldots \bar{\alpha}_2 \bar{\alpha}_1$ *is the projection of a non-zero path of* $\tilde{\Lambda}$.

Proof. — Let $x_0 \xrightarrow{\xi_1} x_1 \xrightarrow{\xi_2} x_2 \ldots x_{n-1} \xrightarrow{\xi_n} x_n$ be a lifting of $\bar{\alpha}_n \ldots \bar{\alpha}_2 \bar{\alpha}_1$ to $Q_{\tilde{\Lambda}}$ and $F : k(\tilde{\Gamma}_\Lambda) = \tilde{I} \to I = \operatorname{ind} \Lambda$ a well-behaved functor. The formulae:

$$\coprod_{Fz=Fy} (\mathscr{R}^p \tilde{I}/\mathscr{R}^{p+1} \tilde{I})(x, z) \xrightarrow{\sim} (\mathscr{R}^p I/\mathscr{R}^{p+1} I)(F x, F y)$$

(see [BG], 3.2 and 3.3) imply that $\alpha_i = F \xi_i + \sum_j F \rho_{ij}$ for some representative ξ_i of $\bar{\xi}_i$ and some morphisms ρ_{ij} with domain x_{i-1} and grade $> g(\alpha_i) = g(\xi_i)$. Hence $\alpha_n \ldots \alpha_2 \alpha_1 = F(\xi_n \ldots \xi_2 \xi_1) + \rho$, where $g(\rho) > \sum_{i=1}^n g(\xi_i)$ (F preserves the grade). Accordingly, statement (i) means that $g(\xi_n \ldots \xi_2 \xi_1) = \sum_{i=1}^n g(\xi_i)$, or equivalently that $\xi_n \ldots \xi_2 \xi_1 \neq 0$. (Remember that $\tilde{I} \xrightarrow{\sim} \operatorname{Gr} \tilde{I}$ with the notations of [BG], 5.1). We infer that (i) $\Leftrightarrow$ (iii).

Now statement (iii) is independent of the choice of the representatives α_i. Therefore, (iii) also implies $g(\alpha'_n \ldots \alpha'_1) = \sum_{i=1}^n g(\alpha'_i) < \infty$ and $\alpha'_n \ldots \alpha'_1 \neq 0$. Finally, choosing $\alpha'_i = F \xi_i$, we see that (ii) implies $\xi_n \ldots \xi_1 \neq 0$. Hence (ii) $\Rightarrow$ (iii).

1.5. We call a path $a_0 \xrightarrow{\bar{\alpha}_1} a_1 \xrightarrow{\bar{\alpha}_2} a_2 \ldots a_{n-1} \xrightarrow{\bar{\alpha}_n} a_n$ of Q_Λ *stable* if it satisfies statement (ii) of lemma 1.4. If this is so, the level l of $\alpha_n \ldots \alpha_2 \alpha_1$ stays constant whatever α_i we choose as representative of $\bar{\alpha}_i$. This constant level l is by definition the *level of the path* $\bar{\alpha}_n \ldots \bar{\alpha}_2 \bar{\alpha}_1$ (compare with [G 2], 2.4). The associated number $g(a_0, a_n, l)$ (1.1) is called the *grade* of $\bar{\alpha}_n \ldots \bar{\alpha}_2 \bar{\alpha}_1$.

A *stable contour* of Λ is a pair (v, w) of two stable paths of Q_Λ which have the same origin, the same terminus and the same level. With each stable contour (v, w) and each $a \in \Lambda$ we associate a conjugacy class $C_a(v, w)$ of $\Pi(Q_\Lambda, a)$ as we did in 1.3.

THEOREM. — *The group of constraints K_Λ (1.3) of a locally representation-finite connected k-category Λ is generated by the conjugacy classes $C_a(v, w)$ associated with the stable contours (v, w) of Λ.*

This immediately follows from 1.2, 1.3 and 1.4. Notice that the stable contours coincide with the non-zero contours if Λ is *schurian*.

1.6. COROLLARY. — *Under the assumptions of theorem 1.5, let v, w be two paths from x to y and u a path from a to x in Q_Λ. Assume v to be stable with level n. Then w is stable with level n iff $u^{-1} w^{-1} vu \in K_\Lambda$.*

Proof. – The condition is necessary by theorem 1.5. Conversely, suppose that $u^{-1}w^{-1}vu \in K_\Lambda$. Let $F : \tilde{\Lambda} \to \Lambda$ be a universal covering. If $\tilde{x} \in \tilde{\Lambda}$ lies over x, the liftings $\tilde{v}$ and $\tilde{w}$ of v and w to $Q_{\tilde{\Lambda}}$ with origin $\tilde{x}$ have a common terminus $\tilde{y}$ by assumption. Since $\tilde{v}$ is non-zero, so is $\tilde{w}$ ([BRL], 5.1). Therefore, w is stable (1.4). Finally, $\tilde{v}$ and $\tilde{w}$ have the same grade. So do v and w, since F preserves the grade.

1.7. For non-specialists we recall a celebrated *example* of Riedtmann: Let Λ be the k-category defined by the quiver $b \underset{\delta}{\overset{\sigma}{\leftrightarrows}} a \circlearrowleft \rho$ and the relations $\delta\sigma = \rho^2$, $\sigma\delta = \sigma\rho\delta$, $\rho^4 = 0$. The maximal stable paths are $\sigma\rho\delta$, ρ^3, $\delta\sigma\rho$ and $\rho\delta\sigma$. So there is "essentially" one stable contour, namely $(\rho^2, \delta\sigma)$. The group $\Pi(Q_\Lambda, a)$ is freely generated by ρ and $\delta\sigma$. The group K_Λ is generated by the conjugates of $\sigma^{-1}\delta^{-1}\rho^2$.

For a generalization of this example *see* [W 1], [RI 3].

2. Directed and simply connected algebras

2.1. Let M be a *schurian* category (1.3) and $S_n M$ the set of its *non-degenerate n-simplices*, i.e. of the sequences $x_0, \ldots, x_n$ of distinct objects of M such that the composition

$$M(x_0, x_1) \times \ldots \times M(x_{n-1}, x_n) \to M(x_0, x_n)$$

is not zero. By definition, $S_1 M$ is identified with the set of pairs (x, y) such that $M(x, y) \neq 0$; for $n = 0$, we agree that $S_0 M$ is the set of objects of M. The family $S_* M = (S_n M)_{n \in \mathbb{N}}$ is called the *simplicial frame* of M.

We denote by $C_n M$ the group of *n-chains* of M, i.e. the free abelian group generated by $S_n M$. The $C_n M$ give rise to a differential complex $\ldots C_2 M \xrightarrow{d} C_1 M \xrightarrow{d} C_0 M$ such that:

$$d(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

The homology groups of this complex $C_* M$ are denoted by $H_n M$ and are called the *simplicial homology groups* of M. If Z is an abelian group, the cohomology groups of the complex $\mathrm{Hom}(C_* M, Z)$ are denoted by $H^n(M, Z)$ and are called *simplicial cohomology groups* of M with coefficients in Z.

2.2. Two schurian categories M, N are said to be *equivalent* if they have the same simplicial frame and if there is an isomorphism $M \xrightarrow{\sim} N$ which is the identity on the objects.

LEMMA. — *Let M be schurian. Then there is a natural bijection between $H^2(M, k^*)$ and the set of equivalence classes of schurian categories N such that $S_* N = S_* M$.*

Proof. — Let N be schurian and such that $S_* N = S_* M$. Up to an equivalence we can assume that $N(x, y) = M(x, y)$ for all $x, y \in S_0 M$. For each $(x, y) \in S_1 M$ we choose a basis $(y \mid x)$ of $M(x, y)$. For each $(x, y, z) \in S_2 M$, the composition of N defines a non-zero scalar $c_N(x, y, z)$ such that:

$$(z \mid y)_{\dot{N}} (y \mid x) = c_N(x, y, z)(z \mid x).$$

The cochain $c_N : S_2 M \to k^*$, $(x, y, z) \mapsto c_N(x, y, z)$ obtained in this way is a 2-cocycle, because the composition is associative. The residue class $\bar{c}_N \in H^2(M, k^*)$ is independent of the choice of the basis vectors $(y \mid x)$. The map $N \mapsto \bar{c}_N$ yields the desired bijection.

2.3. On account of lemma 2.2, we have to determine second cohomology groups. This presupposes some information about $H_0 M$ and $H_1 M$.

Clearly, if we map the homology class $\bar{x} \in H_0 M$ of an object $x \in S_0 M$ onto the connected component of M in which x lies, we obtain an isomorphism between $H_0 M$ and the free abelian group which has the connected components of M as basis. From a dual point of view, $H^0(M, Z)$ is identified with the space of functions $S_0 M \to Z$ which are constant on the connected components.

For the interpretation of $H_1 M$, we focus on the relevant case in which M is *connected* and *locally bounded*. We denote by $\Pi = \Pi(Q_M, a)$ the fundamental group of the quiver Q_M of M at some vertex a, by P the normal subgroup of Π generated by the conjugacy classes $C_a(v, w)$ assigned to the non-zero contours (v, w) of M (1.3). We construct a homomorphism $\bar{\zeta} : \Pi/P \to H_1 M$ according to the following prescription: If w is a closed walk of Q_M from a to a, consider the 1-cycle $\zeta w = \sum_\varphi (w^+ \varphi - w^- \varphi) (t\varphi, h\varphi) \in C_1 M$, where $t\varphi \xrightarrow{\varphi} h\varphi$ is an arbitrary arrow of Q_M and $w^+ \varphi$ (resp. $w^- \varphi$) the number of steps from $t\varphi$ to $h\varphi$ (resp. from $h\varphi$ to $t\varphi$) in w. The homomorphism $\bar{\zeta}$ is obtained from $w \mapsto \zeta w$ by passing to the quotients.

LEMMA. — *Assume M to be connected, locally bounded and schurian. For each abelian group Z, the homomorphism $\bar{\zeta} : \Pi/P \to H_1 M$ induces isomorphisms*

$$H^1(M, Z) \xrightarrow{\sim} \operatorname{Hom}(H_1 M, Z) \xrightarrow{\sim} \operatorname{Hom}(\Pi/P, Z).$$

Proof. — The differential complex $C_* M$ yields exact sequences

$$0 \to Z_1 M \to C_1 M \xrightarrow{d} C_0 M \to H_0 M \to 0$$

and

$$C_2 M \to Z_1 M \to H_1 M \to 0,$$

where $Z_1 M = \operatorname{Ker} d$. Since $C_1 M$, $C_0 M$ and $H_0 M$ are free abelian groups, the first sequence splits and therefore remains exact when acted upon by an additive functor. Applying the functor $X \mapsto \operatorname{Hom}(X, Z) = X^*$, we thus get exact sequences

$$0 \leftarrow Z_1 M^* \leftarrow C_1 M^* \leftarrow C_0 M^* \leftarrow H_0 M^* \leftarrow 0$$

and

$$C_2 M^* \leftarrow Z_1 M^* \leftarrow H_1 M^* \leftarrow 0,$$

which induce the first canonical isomorphism $H^1(M, Z) \xrightarrow{\sim} \operatorname{Hom}(H_1 M, Z)$.

The composition $H^1(M, Z) \to \operatorname{Hom}(\Pi/P, Z)$ of this isomorphism with $\operatorname{Hom}(\zeta, Z)$ is induced by the map which assigns the value $\sum_\varphi (w^+ \varphi - w^- \varphi) f(t\varphi, h\varphi) \in Z$ to each 1-cocycle f of $\operatorname{Hom}(C_* M, Z)$ and each closed walk w from a to a in Q_M. In order to prove that this composition is bijective, we just produce the reciprocal map: Choose a walk ${}_x w$ from a to each vertex $x \in Q_M$; for each $(x, y) \in S_1 M$ choose a non-zero path ${}_y p_x$ from x to y; denote by ${}_y \bar{p}_x$ the image of the closed walk ${}_y w^{-1} {}_y p_{xx} w$ in Π/P. The reciprocal map sends $g \in \operatorname{Hom}(\Pi/P, Z)$ onto the cohomology class of the 1-cocycle $(x, y) \mapsto g({}_y \bar{p}_x)$.

2.4. For practical purposes, the following interpretation of $H^1(M, Z)$ turns out to be useful: Denote by Q_M^1 the set of arrows of Q_M. A function $f : Q_M^1 \to Z$ is called *closed* if $\sum_i f(v_i) = \sum_j f(w_j)$ for each non-zero contour (v, w) of M (the v_i and w_j denote the arrows occurring in v and w respectively). The function f is called *exact* if there is a function $e : S_0 M \to Z$ such that $f(\varphi) = e(h\varphi) - e(t\varphi)$ for all $\varphi \in Q_M^1$.

LEMMA. — *Let M be schurian and locally bounded. The canonical injection $Q_M^1 \to S_1 M$, $\varphi \mapsto (t\varphi, h\varphi)$ induces an isomorphism from $H^1(M, Z)$ onto the quotient of the space of Z-valued closed functions by the subspace of exact functions.*

Clear.

2.5. LEMMA. — *Assume* Λ *to be connected, locally representation-finite. Then* Λ *is simply connected iff* Λ *is schurian and* $H_1 \Lambda$ *is a torsion-group.*

Proof. — First assume $\Lambda = M$ to be simply connected. Then Λ is directed, hence schurian. Denote by $t H_1 \Lambda$ the torsion subgroup of $H_1 \Lambda$. By 2.3 $H_1 \Lambda$ is identified with the largest abelian factor-group of Π/P. We infer that $H_1 \Lambda / t H_1 \Lambda \xrightarrow{\sim} \Pi/K$ for some invariant subgroup K of Π which contains P. By [G 2], 3.1, the connected Galois covering $\Lambda' \to \Lambda$ with group Π/P constructed in the proof of lemma 1.3 yields a connected Galois covering $\Lambda'/(K/P) \to \Lambda$ with group $\Pi/K = H_1 \Lambda / t H_1 \Lambda$. Since Π/K is torsion-free, the Galois covering $\Lambda'/(K/P) \to \Lambda$ yields a Galois covering of $\operatorname{ind} \Lambda$ ([G 2], 3.6) with group Π/K. Since Λ is simply connected, Π/K is trivial and $H_1 \Lambda = t H_1 \Lambda$.

Conversely, assume $\Lambda = M$ schurian. By 2.3, $H_1 \Lambda / t H_1 \Lambda$ is the largest factor-group of Π/P which is both abelian and torsion-free. So, if $H_1 \Lambda$ is torsion, $\{1\}$ is the only torsion-free abelian factor-group of Π/P. On the other hand, the group of constraints K_Λ contains P by 1.2 and 1.4 (or equivalently by [G 2], 2.4). Therefore, the fundamental group $\Pi(\Gamma_\Lambda, a^*)$ of Λ is a factor-group of Π/P; it is free non-commutative by [BG], 4.2. If it was not trivial, $\Pi(\Gamma_\Lambda, a^*)$ and Π/P would admit a non-trivial torsion-free abelian factor-group.

2.6. Our main theorem in this section rests on some definitions, which we introduce now: Let Q be a *directed* quiver (i.e. a quiver without oriented cycles). Given two vertices x, y of Q, the inequality $x \leqslant y$ means that Q contains a path from x to y. We denote by $[x, y]$ the *interval* $\{z \in Q : x \leqslant z \leqslant y\}$ and call Q *interval-finite* if all intervals $[x, y]$ are finite. We call a set C of vertices of Q *convex* if it contains $[x, y]$ whenever $x, y \in C$.

THEOREM. — *Assume* Λ *locally representation-finite and* Q_Λ *directed and interval-finite. Then* $H_1 \Lambda$ *is a free abelian group, and* $H_i \Lambda = 0$ *for* $i \geqslant 2$. *If* Λ' *is a full subcategory of* Λ *whose set of objects is convex in* Q_Λ, *the inclusion* $\Lambda' \to \Lambda$ *induces an isamorphism of* $H_1 \Lambda'$ *onto a pure subgroup of* $H_1 \Lambda$.

Proof. — First we consider a *partially ordered set* Σ and the associated k-category Σ_k : Set $\Sigma_k(x, y) = k(y \mid x) =$ 1-dimensional space with basis $(y \mid x)$ or $\Sigma_k(x, y) = 0$ according as $x \leqslant y$ or $x \nleqslant y$; set $(z \mid y) \circ (y \mid x) = (z \mid x)$ if $x \leqslant y \leqslant z$. Assume that s is the *largest* element of Σ and denote by $h_n : C_n \Sigma_k \to C_{n+1} \Sigma_k$ the map sending $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, s)$ if $x_n \neq s$

or else to 0. Then we have $dh_n c = h_{n-1} dc - (-1)^n c$ for each $n \geqslant 1$ and each $c \in C_n \Sigma_k$. We infer that $H_n \Sigma_k = 0$ if $n \geqslant 1$.

Now consider the case where $\Sigma_k = M$ is *representation-finite*. Then we have $H_1 \Sigma_k = 0$: Indeed, assume M to be connected. In [G 2], 2.5 we have shown that the fundamental group $\Pi = \Pi(Q_M, a)$ is generated by the conjugacy classes attached to the contours of Q_M. With the notations of 2.3, this means that $\Pi/P = \{1\}$. Accordingly, the equality $H_1 \Sigma_k = 0$ follows from lemma 2.3.

In a third step of our proof, we suppose that Λ contains a *maximal element* s and denote by Λ' the full subcategory formed by the objects of Λ other than s. We set $\Sigma_s = \{ t \in \Lambda : \Lambda(t, s) \neq 0 \}$ and $\Sigma'_s = \Sigma_s \setminus \{ s \}$, and we partially order these sets by setting $t \leqslant t'$ iff $\Lambda(t', s) \circ \Lambda(t, t') \neq 0$. The associated k-categories Σ_{sk} and Σ'_{sk} are clearly identified with subcategories of Λ, and we have a short exact sequence of differential complexes:

$$0 \to C_* \Sigma'_{sk} \xrightarrow{\begin{bmatrix} u \\ -v \end{bmatrix}} C_* \Sigma_{sk} \oplus C_* \Lambda' \xrightarrow{[i\, j]} C_* \Lambda \to 0,$$

where u, v, i and j are induced by the inclusion-functors. Passing to homology, we obtain the celebrated *Mayer-Vietoris sequence:*

$$\ldots H_n \Sigma'_{sk} \to H_n \Sigma_{sk} \oplus H_n \Lambda' \to H_n \Lambda \xrightarrow{\partial} H_{n-1} \Sigma'_{sk} \ldots$$

In a fourth step, we assume that Λ is associated with a finite *partially ordered set*. Then, using induction on the cardinality, we suppose that $H_n \Sigma'_{sk} = H_n \Lambda' = 0$ for $n \geqslant 1$. Since Σ_s has only one maximal element, we have $H_n \Sigma_{sk} = 0$ for $n \geqslant 1$ by our first step. As a consequence, the Mayer-Vietoris sequence tells us that $H_n \Lambda = 0$ for $n \geqslant 2$. Together with step two, this proves that $H_n \Lambda = 0$ for $n \geqslant 1$.

In *step five*, we turn to the case where Λ has finitely many objects. The categories Σ'_{sk} and Σ_{sk} are representation-finite because the extension by 0 yields full embeddings from their module-categories into that of Λ. Therefore we have $H_n \Sigma'_{sk} = H_n \Sigma_{sk} = 0$ for $n \geqslant 1$ by step four, and the Mayer-Vietoris sequence reduces to $H_n \Lambda' \xrightarrow{\sim} H_n \Lambda$ for $n \geqslant 2$ and to the exact sequence:

$$0 \to H_1 \Lambda' \to H_1 \Lambda \xrightarrow{\partial} H_0 \Sigma'_{sk} \xrightarrow{\iota} H_0 \Sigma_{sk} \oplus H_0 \Lambda' \to H_0 \Lambda \to 0.$$

By induction, we infer that $H_n \Lambda = 0$ for $n \geqslant 2$. Moreover, $H_0 \Sigma'_{sk}$ is free abelian by 2.3 and so is $\operatorname{Ker} \iota$. By induction we get that $H_1 \Lambda \xrightarrow{\sim} H_1 \Lambda' \oplus \mathbb{Z}^r$ is free abelian.

Step six: Let Λ have infinitely many objects. We may suppose that Λ is connected. By lemma 2.7 below, Λ is the "union" of an increasing sequence of full finite subcategories Λ_n such that Λ_{n+1} is obtained by adding to Λ_n an extremal element of $Q_{\Lambda_{n+1}}$. Therefore, $H_1 \Lambda \xrightarrow{\sim} \varinjlim_n H_1 \Lambda_n$ can be constructed step by step by successively adding new basis vectors (step 5). Hence $H_1 \Lambda$ is free. Moreover, $H_i \Lambda \xrightarrow{\sim} \varinjlim_n H_i \Lambda_n$ is zero for $i \geqq 2$.

Step seven: Let us now turn to the last statement. Suppose that Λ' has finitely many objects. By lemma 2.7 below, Λ is the "union" of an increasing sequence of full finite subcategories Λ_n such that $\Lambda_0 = \Lambda'$ and that Λ_{n+1} is obtained by adding to Λ_n an extremal element of $Q_{\Lambda_{n+1}}$. Therefore, $H_1 \Lambda$ is obtained by successively adding new basis vectors to $H_1 \Lambda' = H_1 \Lambda_0$. Accordingly, $H_1 \Lambda'$ is identified with a direct summand of $H_1 \Lambda$.

Last step: Let Λ' have infinitely many objects. By lemma 2.7 below, Λ' is the "union" of an increasing sequence of full finite subcategories Λ'_n such that... We infer that $H_1 \Lambda'$ is identified with the union of an increasing sequence of direct summands $H_1 \Lambda'_n$ of $H_1 \Lambda$. Therefore, $H_1 \Lambda'$ is pure in $H_1 \Lambda$.

2.7. LEMMA. — *Let C be a finite convex set of vertices of a directed, interval-finite, connected and locally finite* ([BG], 2.1) *quiver Q. Then, the set of vertices of Q is the union of an increasing sequence $C = C_0 \subset C_1 \subset C_2 \subset \ldots$ of finite convex sets such that, for each $n \in \mathbb{N}$, the difference $C_{n+1} \setminus C_n$ is empty or consists of one extremal ($=$ maximal or minimal) element of C_{n+1}.*

Proof. — Let $q_1, \ldots, q_m, \ldots$ be the vertices of Q (a countable set). Let D_m be the union of the intervals $[x, y]$, where $x, y \in C \cup \{ q_i : i \leqslant m \}$. Then D_m is convex and finite, it contains C, and the set of vertices of Q equals $\bigcup_{m \in \mathbb{N}} D_m$. It is therefore sufficient to construct for each m a finite sequence

$$D_m = D_m^0 \subset D_m^1 \subset \ldots \subset D_m^r = D_{m+1}$$

of convex sets such that each D_m^{s+1} is obtained from D_m^s by addition of a maximal or of a minimal point. In fact, we construct D_m^{s+1} from D_m^s as follows:

If one of the sets

$$\{ x \in D_{m+1} \setminus D_m^s : x \geqslant y \text{ for some } y \in D_m^s \}$$

or

$$\{ x \in D_{m+1} \setminus D_m^s : x \leqslant y \text{ for some } y \in D_m^s \},$$

is not empty, we adjoin to D^s_m a minimal point of the first set or a maximal point of the second. If both are empty, we can add any point of $D_{m+1} \setminus D^s_m$ to D^s_m, or else we have $D^s_m = D_{m+1}$.

2.8. COROLLARY. — *Let Λ be locally representation-finite and simply connected. A full subcategory Λ' of Λ is simply connected provided its set of objects is convex in Q_Λ.*

Proof. — Q_Λ is interval-finite by [BG], 1.6. By 2.5 and 2.6 we have $H_1 \Lambda = 0$. This implies $H_1 \Lambda' = 0$ by 2.6 and our statement by 2.5.

2.9. Assume Λ to be *locally bounded and directed.* For each $s \in \Lambda$, denote by Q_s the full subquiver of Q_Λ formed by the vertices $t \not\geqslant s$, by Σ'_s the partially ordered set $\{t < s : \Lambda(t, s) \neq 0\}$ (where $t \leqslant t'$ iff $\Lambda(t', s) \circ \Lambda(t, t') \neq 0$; see the proof of theorem 2.6). We call s *separating* if each connected component of Q_s contains at most one connected component of Σ'_s.

COROLLARY (Separation-criterion of Bautista-Larrion). — *Let Λ be representation-finite and directed. Then Λ is simply connected iff each of its objects is separating.*

Proof. — First we assume Λ simply connected. By 2.5 and 2.6 we have $H_1 \Lambda = 0$. We denote by Λ'_s and Λ_s the full subcategories of Λ supported by Q_s and $Q_s \cup \{s\}$ respectively, by Σ_s the partially ordered set $\{t \leqslant s : \Lambda(t, s) \neq 0\}$. Since the set of objects of Λ_s is convex in Q_Λ, we have $H_1 \Lambda_s \subset H_1 \Lambda = 0$ by 2.6. The Mayer-Vietoris sequence (*see* 2.6)

$$H_1 \Lambda_s \xrightarrow{\partial} H_0 \Sigma'_{sk} \xrightarrow{\iota} H_0 \Sigma_{sk} \bigoplus H_0 \Lambda'_s$$

tells us that ι is injective. By 2.3, we have $H_0 \Sigma'_{sk} \xrightarrow{\sim} \bigoplus \mathbb{Z} C$, $H_0 \Lambda'_s \xrightarrow{\sim} \bigoplus \mathbb{Z} D$ and $H_0 \Sigma_{sk} \xrightarrow{\sim} \mathbb{Z} \Sigma_s$, where C and D range over the connected components of Σ'_s and Q_s respectively. Furthermore, ι maps C onto $\Sigma_s + \overline{C} \in \mathbb{Z} \Sigma_s \bigoplus (\bigoplus \mathbb{Z} D)$, where $\overline{C}$ denotes the connected component of Q_s containing C. Therefore, ι *is injective iff the map $C \mapsto \overline{C}$ is injective.* We infer that s is separating.

Conversely, suppose that the objects $x_1, \ldots, x_n$ of Λ are separating. We may assume that $i < j$ implies $x_i \not\geqslant x_j$. Denoting then by Λ_p the full subcategory of Λ formed by $x_1, \ldots, x_p$, we shall prove by induction on p that $H_1 \Lambda_p = 0$ and then apply 2.5. Actually, assume Λ_{p-1} simply connected and set $s = x_p$. On account of the Mayer-Vietoris sequence:

$$0 = H_1 \Lambda_{p-1} \to H_1 \Lambda_p \xrightarrow{\partial} H_0 \Sigma'_{sk} \xrightarrow{\iota} H_0 \Sigma_{sk} \oplus H_0 \Lambda_{p-1},$$

we have to show that ι is injective, or equivalently that the map $C \mapsto C'$ is injective, where C' is the connected component of Λ_{p-1} containing C. But this is true, because s is separating and $\Lambda_{p-1} \subset \Lambda'_s$.

2.10. *An alternative proof for lemma* 1.3. Let Λ be locally representation-finite and simply connected, and let $a_0 - a_1 - \ldots - a_n - a_{n+1} = a_0$, $n \geqslant 2$, be a simple closed walk in Q_Λ. We have to show that it can be "decomposed" into non-zero simple contours of Λ. Now, the union of the intervals $[a_i, a_j]$ is a finite convex set of objects of Q_Λ. Replacing Λ if necessary by the full subcategory formed by this union and applying 2.8, we may assume that Λ has a finite number of objects and proceed by induction on this number.

Let s be a maximal point of Q_Λ. If $a_i \neq s$ for all i, we apply our induction hypothesis to the full subcategory $\Lambda' = \Lambda'_s$ of Λ formed by the points other than s. Otherwise, we may assume that $a_0 = s$ and set $\Sigma' = \Sigma'_s = \{ t \in \Lambda' : \Lambda(t, s) \neq 0 \}$. Since $a_1 - \ldots - a_n$ is a walk of Q_Λ, a_1 and a_n lie in the same connected component of Λ'. Since s is separating, a_1 and a_n are in the same connected component of Σ', and of course they are maximal in Σ'. The following cases only can occur:

(1) Σ' contains a point b such that $a_1 \geqslant b \leqslant a_n$. If v and w are paths from b to a_1 and a_n, the given closed walk decomposes into the simple "contour" $b \overset{v}{-} a_1 \to a_0 \leftarrow a_n \overset{w}{-} b$ and the closed walk $b \overset{v}{-} a_1 - \ldots - a_n \overset{w}{-} b$ in Q_Λ. Induction applies.

(2) Σ' is connected and has a third maximal point m besides a_1 and a_n. (Remember that Q_Λ has at most 3 arrows heading for $a_0 = s$). Furthermore, Σ' contains two points c and d such that $a_1 \geqslant c \leqslant m \geqslant d \leqslant a_n$. In this case, we choose two paths v and v' from c to a_1 and m on one hand, and two paths w' and w from d to m and a_n on the other. The given closed walk then decomposes into two simple "contours" $c \overset{v}{-} a_1 \to a_0 \leftarrow m \overset{v'}{-} c$, $d \overset{w'}{-} m \to a_0 \leftarrow a_n \overset{w}{-} d$ and the closed walk $a_1 - \ldots - a_n \overset{w}{-} d \overset{w'}{-} m \overset{v'}{-} c - a$. Again induction applies.

2.11. Corollary. — *Assume Λ to be schurian, connected and locally representation-finite. Let $\Phi = \Pi(\Gamma_\Lambda, a^*)$ be the fundamental group of Γ_Λ at some projective vertex a^* and Z an abelian group. Then $H^1(\Lambda, Z)$ is canonically isomorphic to* $\mathrm{Hom}(\Phi, Z)$.

Proof. — By Theorem 1.5, Φ is identified with the group Π/P of 2.3. Our statement therefore follows from lemma 2.3.

2.12. COROLLARY. — *Assume* Λ *to be schurian, connected and locally representation-finite. Let* Z *be a non-zero abelian group. Then* Λ *is simply connected iff* $H_1 \Lambda = 0$, *or equivalently iff* $H^1(\Lambda, Z) = 0$.

Proof. — Being free non-commutative ([BG], 4.2), Φ is trivial iff $\mathrm{Hom}(\Phi, Z) \xrightarrow{\sim} H^1(\Lambda, Z)$ is zero (2.11), or equivalently iff the largest abelian factor group $H_1 \Lambda$ of $\Phi = \Pi/P$ is trivial (2.3).

3. The standard form of a locally representation-finite algebra

Let Λ be locally representation-finite.

3.1. The *standard form* $\bar{\Lambda}$ of Λ is by definition the full subcategory formed by the projective points of the mesh-category $k(\Gamma_\Lambda)$ ([BG], 2.2 and 5.1). We know by [BG] that $\bar{\Lambda}$ is locally representation-finite, has the same Auslander-Reiten quiver as Λ and is a degeneration of Λ in the sense of algebraic geometry. The description of $\bar{\Lambda}$ is important from a practical point of view, because some concrete algebras are proved to be representation-finite by first exhibiting a representation-finite degeneration and then applying [G0], 4.2.

Let x, y be two objects of Λ. A morphism from x to y in the quiver-category $k\,Q_\Lambda$ is by definition a formal linear combination of paths from x to y in the quiver Q_Λ. We consider the subspace $I_\Lambda(x, y)$ of $(k\,Q_\Lambda)(x, y)$ which is generated by the non-stable paths and by the differences $v - w$, where (v, w) ranges over the stable contours with origin x and terminus y (1.5). In view of lemma 1.4, it is clear that the family $(I_\Lambda(x, y))_{x, y \in \Lambda}$ is an ideal of $k\,Q_\Lambda$: we call it the *standard ideal*.

THEOREM. — *The standard form* $\bar{\Lambda}$ *of* Λ *is isomorphic to* $k\,Q_\Lambda / I_\Lambda$.

Proof. — We first consider the case where Λ is simply connected and therefore isomorphic to $\bar{\Lambda}$. In this case, we have $H_n \Lambda = 0$ for $n \geqslant 1$ (2.5 and 2.6); equivalently, the sequence of free abelian groups

$$\ldots C_2 \Lambda \xrightarrow{d} C_1 \Lambda \xrightarrow{d} C_0 \Lambda \to H_0 \Lambda \to 0$$

is exact and splits. If we let the functor $\mathrm{Hom}(?, Z)$ act on it, the induced sequence remains exact and yields $H^n(\Lambda, Z) = 0$ for $n \geqslant 1$. In particular, we have $H^2(\Lambda, k^*) = 1$. By 2.2 this means that Λ admits a *multiplicative basis* (in the terminology of [RO]), i.e. that we can choose a basic-vector $(y \mid x)$ in each non-zero morphism space $\Lambda(x, y)$ in such a way that $(z \mid y) \circ (y \mid x) = (z \mid x)$ if $\Lambda(y, z) \circ \Lambda(x, y) \neq 0$. This clearly proves our theorem in the considered case (for historical comments, *see* the introduction).

In the general case, we may assume Λ connected. We then denote by Φ the fundamental group of the Auslander-Reiten quiver Γ_Λ at some projective point a^*. The universal covering $\pi : \tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$ induces Galois coverings

$$\begin{array}{ccc} \tilde{\Lambda} & \subset & k(\tilde{\Gamma}_\Lambda) \\ \downarrow & & \downarrow \\ \bar{\Lambda} & \subset & k(\Gamma_\Lambda) \end{array}$$

with group Φ. We infer that $\bar{\Lambda}$ is identified with $\tilde{\Lambda}/\Phi$ ([G 2], 3.1). On the other hand, π induces a Galois covering $k\,Q_{\tilde{\Lambda}}/I_{\tilde{\Lambda}} \to k\,Q_\Lambda/I_\Lambda$ (apply lemma 1.4). Accordingly, $k\,Q_\Lambda/I_\Lambda$ is identified with $(k\,Q_{\tilde{\Lambda}}/I_{\tilde{\Lambda}})/\Phi$. Now we already know that $k\,Q_{\tilde{\Lambda}}/I_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}$. The trouble is that we lack a Φ-equivariant isomorphism, which would induce an isomorphism between the quotients of $k\,Q_{\tilde{\Lambda}}/I_{\tilde{\Lambda}}$ and $\tilde{\Lambda}$ by Φ.

Let $\varepsilon : k\,Q_{\tilde{\Lambda}}/I_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}$ be an isomorphism. If $x \neq y$ and $\tilde{\Lambda}(x, y) \neq 0$, the paths from x to y have a common image ${}_y\varepsilon_x$ in $\tilde{\Lambda}(x, y)$. The basis vectors ${}_y\varepsilon_x$ produced by ε satisfy ${}_z\varepsilon_y \circ {}_y\varepsilon_x = {}_z\varepsilon_x$ whenever $(x, y, z) \in S_2\tilde{\Lambda}$, i. e. they form a multiplicative basis $({}_y\varepsilon_x)$ of $\tilde{\Lambda}$. As a matter of fact, the map $\varepsilon \mapsto ({}_y\varepsilon_x)$ yields a bijection between the isomorphisms and the multiplicative bases. In this bijection the Φ-equivariant isomorphisms are associated with the Φ-invariant multiplicative bases, i. e. with the multiplicative bases $({}_y\varepsilon_x)$ such that ${}_{y\varphi}\varepsilon_{x\varphi} = {}_y\varepsilon_x\varphi$ for all x, y and each $\varphi \in \Phi$. The existence of a Φ-invariant multiplicative basis is proved in 3.2 below.

3.2. *Construction of a Φ-invariant multiplicative basis of $\tilde{\Lambda}$.* First we choose a vector ${}_y\eta_x \neq 0$ in each $\tilde{\Lambda}(x, y) \neq 0$ such that $x \neq y$. We assume that ${}_{y\varphi}\eta_{x\varphi} = {}_y\eta_x\varphi$ for all $\varphi \in \Phi$. This can be done because the action of Φ on $\tilde{\Lambda}$ is free. In case $(x, y, z) \in S_2\tilde{\Lambda}$, we then have ${}_z\eta_y \circ {}_y\eta_x = c(x, y, z)\,{}_z\eta_x$, where $c \in \mathrm{Hom}(C_2\tilde{\Lambda}, k^*)$ is a Φ-invariant 2-cocycle, i. e. a 2-cocycle such that $c(x\varphi, y\varphi, z\varphi) = c(x, y, z)$ for all $\varphi \in \Phi$ and all $(x, y, z) \in S_2\tilde{\Lambda}$. From the lemma farther on, we infer the existence of scalars $b(x, y) \in k^*$ defined whenever $(x, y) \in S_1\tilde{\Lambda}$ and satisfying $b(x\varphi, y\varphi) = b(x, y)$ and $c(x, y, z) = b(y, z)\,b(x, z)^{-1}\,b(x, y)$. The required Φ-invariant multiplicative basis is determined by ${}_y\varepsilon_x = b(x, y)^{-1}\,{}_y\eta_x$.

LEMMA. — *Let Z be an abelian group. For each $n \geqslant 2$ and each Φ-invariant n-cocycle $c \in \mathrm{Hom}(C_n\tilde{\Lambda}, Z)$, there exists a Φ-invariant $b \in \mathrm{Hom}(C_{n-1}\tilde{\Lambda}, Z)$ whose coboundary is c.*

Proof. — Since Φ acts freely on $S_n \tilde{\Lambda}$ and $H_n \tilde{\Lambda}$ is 0 for $n \geqslant 1$ (2.5 and 2.6), the differential complex $C_* \tilde{\Lambda}$ yields a free resolution of the trivial Φ-module $\mathbb{Z} \xrightarrow{\sim} H_0 \tilde{\Lambda}$. Accordingly, the n-th cohomology group of the differential complex $\mathrm{Hom}_\Phi(C_* \tilde{\Lambda}, Z)$ is identified with the n-th cohomology group $H^n(\Phi, \mathrm{Z})$ of Φ with coefficients in Z. This group vanishes for $n \geqslant 2$ because Φ is free ([ML], IV, 7.3). Our lemma follows from the fact that $\mathrm{Hom}_\Phi(C_* \tilde{\Lambda}, Z)$ is identified with the complex of Φ-invariant cochains.

3.3. *Remarks.* (*a*) Among other things, theorem 3.1 yields a description of the *universal cover* $\tilde{\Lambda}$ ([G2], 2.1) of Λ: Assume Λ to be connected. Let $\tilde{Q}_\Lambda$ be the universal cover of Q_Λ, K_Λ the group of constraints of Λ, which is determined by the stable contours of Λ (1.5). Then $Q_{\tilde{\Lambda}}$ equals $\tilde{Q}_\Lambda / K_\Lambda$. The non-zero paths of $\tilde{\Lambda}$ are the paths of $Q_{\tilde{\Lambda}}$ whose projections on Q_Λ are stable (1.4). By 3.1 we obtain $\tilde{\Lambda}$ from the quiver-category $k Q_{\tilde{\Lambda}}$ by annihilating zero-paths and equalizing non-zero paths with the same ends.

(*b*) With the notations of 3.1, assume Λ to be *schurian*. Then $S_n \Lambda$ is clearly identified with $S_n \tilde{\Lambda}/\Phi$ and $\mathrm{Hom}(C_n \Lambda, Z)$ with $\mathrm{Hom}_\Phi(C_n \tilde{\Lambda}, Z)$. As a result, lemma 3.2 tells us that $H^2(\Lambda, k^*) = 1$, i. e. Λ has a multiplicative basis and is identified with its standard form $\bar{\Lambda}$.

We leave it to the reader to interpret $S_n \tilde{\Lambda}/\Phi$ and $\mathrm{Hom}_\Phi(C_n \tilde{\Lambda}, Z)$ in terms of Λ in the general case.

(*c*) With the notations of 3.1, *assume that there exists a Galois covering* $F : M \to \Lambda$ *with group* G *such that* M *is standard. Then* Λ *is standard:* Indeed, by restriction to the connected components of Λ and M, we can reduce the problem to the connected case. If $\tilde{M}$ is the universal cover of M, the stable paths of Λ are the projections of the non-zero paths of $\tilde{M}$ by lemma 1.4 (indeed, F induces a Galois covering $\mathrm{Ind}\, M \to \mathrm{Ind}\, \Lambda$ by [MP 1] and [G 2], 3.6; therefore, $\tilde{\Gamma}_\Lambda$ is identified with $\tilde{\Gamma}_M$, and $\tilde{\Lambda}$ with $\tilde{M}$). We infer that the stable paths of Λ are the projections of the stable paths of M. This implies isomorphism (2) in the series:

$$\bar{\Lambda} \underset{(1)}{\xrightarrow{\sim}} k Q_\Lambda / I_\Lambda \underset{(2)}{\xrightarrow{\sim}} (k Q_M / I_M)/G \underset{(3)}{\xrightarrow{\sim}} \bar{M}/G \xrightarrow{\sim} M/G \underset{(4)}{\xrightarrow{\sim}} \Lambda,$$

where (1) and (3) follow from 3.1, and (4) from [G 2], 3.1.

(*d*) Assume that Λ is representation-finite and has a directed quiver. In 1979 Bongartz and Riedtmann constructed a polyhedron which is in fact the geometric realization of the 2-dimensional skeleton of $S_* \Lambda$ (2.1). They conjectured that Λ is simply-connected iff their polyhedron is so (private communication). Theorem 1.5 shows that Γ_Λ and the polyhedron have the same fundamental group.

On this occasion, we like to repair a lack of precision in a reference to BONGARTZ' contribution to [G 2]. Knowing about the theory developed in [G 2], he had the idea of truncating the universal cover in order to get simple projectives and to start the efficient construction of the AUSLANDER-REITEN quiver which goes back to BAUTISTA and was exploited and transmitted to us by RINGEL ([G2], 4.2).

(*e*) Suppose that Λ *is* finite and *self-injective*. The universal cover $\tilde{\Lambda}$ is described in [RI 2] when Λ is of tree-class A_n. In case D_n, Riedtmann presented a description of $\tilde{\Lambda}$ using her classification of the translation quivers $\tilde{\Gamma}_\Lambda$ (Ottawa, August 1979). In Puebla (August 1980), Waschbüsch proposed a direct description of $\tilde{\Lambda}$ in the "regular" selfinjective case (*see* [W 2]). Some of his proofs seem defective, but it remains that Waschbüsch was the first to propose the use of contours – considered independently by Roiter [RO] – for a description of $\tilde{\Lambda}$.

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