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AN EXAMPLE CONCERNING A QUESTION OF ZARISKI

BY

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RÉSUMÉ. — Dans ce papier, nous donnons un exemple d'une famille de singularités de surfaces qui est équisingulière à la Zariski pour une projection transversale mais qui n'est pas équisingulière à la Zariski pour une projection générique.

ABSTRACT. — In this paper we present an example of a family of surface singularities which is Zariski-equisingular for a transversal projection, but it is not Zariski-equisingular for a generic projection.

In this paper we present an example of a family of surfaces $p : X \to Y$ having a transversal direction L such that the discriminant corresponding to the projection parallel to L is equisingular along Y at P_0, but X does not have dimensional type 2, that is, the discriminant of a generic projection is not equisingular along Y at P_0.

This example answers to a question which goes back to ZARISKI (*cf.* [5], p. 490, question 1). In [2] BRIANÇON and SPEDER give an example showing that equisingularity of the discriminant corresponding to a non-transversal projection does not imply generic equisingularity. Subsequently, ZARISKI develops in [6] and [7] a theory of equisingularity based on the discriminant of a generic projection. Nevertheless it remained to know whether generic equisingularity is or not stronger than equisingularity of the discriminant of a transversal projection.

In this paper the show the answer to this question is a affirmative.

Consider the family of surfaces $p : X \to Y$ with equation

$$x^{10} + tyz^3 x^7 + y^{10} + y^6 z^4 + z^{16},$$

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where

$$Y=\text{Sing } X=\{(x, y, z, t)\in\mathbb{C}^4 \mid x=y=z=0\}$$

and

$$p(x, y, z, t)=t.$$

Let $L=(a, b, c, d)$ be a non tangent direction to X in $P_0=(0, 0, 0, 0)$. We will denote by $\pi_L : X\to\mathbb{C}^3$ the projection of X in the direction parallel to L, B_L the ramification locus of π_L, and $D_L=\pi_L(B_L)$ the discriminant of π_L. We will see that if $L=(1, 0, c, 0)$ and c if generic, then D_L is equisingular along Y at P_0, and if $L=(a, b, c, d)$ is a generic direction, D_L is not equisingular along Y at P_0.

In [4] (*and* [1], *if* X_t is an isolated singularity) it is shown that D_L is equisingular at P_0 for a linear generic projection π_L if and only if the discriminant of a non-linear generic projection is equisingular at P_0, i. e., X has dimensional type two at P_0 (*cf.* [7]). Hence, in our example X is not generically equisingular along Y at P_0. Essentially, to prove the above statements for X one calculates in each case the parametric equations of the ramification locus.

Let $L=(1, 0, c, 0)$. To find the discriminant we make the change:

$$x=x,\quad y=y,\quad z=z'+cx,\quad t=t.$$

In this new system of coordinates $L=(1, 0, 0, 0)$ and D_L results by eliminating x out of the new equation of X and its derivative with respect to x, namely

$$(1)\qquad x^{10}+ty(z'+cx)^3x^7+y^{10}+y^6(z'+cx)^4+(z'+cx)^{16},$$

and

$$(2)\qquad 10x^9+7ty(z'+cx)^3x^6+3cty(z'+cx)^2x^7 +4cy^6(z'+cx)^3+16c(z'+cx)^{15}.$$

As we have mentioned before, we have to find parametric equations of B_L. It is known (*see* [3]) that if X_0 has an isolated singular point (as in our case), then for the generic direction L, D_L is reduced and also

$$m(D_L)=\mu^2(X_0)+m(X_0)-1 \qquad \text{where} \quad \mu^2(W_0)$$

is the Milnor number of a generic plane section of X_0 and $m(X_0)$ is its multiplicity. In the example, it is easy to see that $\mu^2(X_0)=81$, and so, for every t, the multiplicity of a generic projection is 90. Observe that the direction $L=(1, 0, c, 0)$ is not necessarily generic, but if $c\neq 0$, the multiplicity of $(D_L)_t$ is also 90, and as a consequence of the parametrization we will have as a matter of fact that $(D_L)_t$ is reduced too.

The fact that $m((D_L)_t)=90$ follows easily since if

$$c=0 \quad \text{and} \quad t=0, \qquad D_L=(y^{10}+y^6 z^4+z^{16})^9$$

so in general the term of $(D_L)_t$ with degree 90 is not null; on the other hand, from the form of (1) and as the discriminant of a polynomial is quasihomogeneous in the coefficients of the polynomial, $(D_L)_t$ has no term with degree <90, resulting $m((D_L)_t)=90$.

Let us show that the parametric equations of some branches of D_L have the form

$$x=u^5 h(t, u),$$
$$(3) \qquad y=u^6 g(t, u),$$
$$z'=u^3.$$

Putting (3) in (1) and (2), and taking out common factors u^{48} and u^{45} respectively, we get

$$(4) \quad M(g, h, u, t)=u^2 h^{10}+tu^2 g(1+cu^2 h)^3 h^7 + u^{12} g^{10}+g^6(1+cu^2 h)^4+(1+cu^2 h)^{16}=0,$$

$$(5) \quad N(g, h, u, t)=10\, h^9+7\, tg(1+cu^2 h)^3 h^6+3\, ctu^2 g(1+cu^2 h)^2 h^7 + 4\, cg^6(1+cu^2 h)^3+16\, c(1+cu^2 h)^{15}=0.$$

The solution of (4) and (5) with $t=0$, $u=0$ are such that

$$(6) \quad \begin{cases} g^6+1=0, \\ 10\, h^9+4cg^6+16c=10\, h^9+12c=0. \end{cases}$$

Let (g_0, h_0) be one of the 54 pairs of solutions of (6). Easily,

$$\begin{vmatrix} \dfrac{\delta M}{\delta g}(g_0, h_0, 0, 0) & \dfrac{\delta M}{\delta h}(g_0, h_0, 0, 0) \\ \dfrac{\delta N}{\delta g}(g_0, h_0, 0, 0) & \dfrac{\delta N}{\delta h}(g_0, h_0, 0, 0) \end{vmatrix} = 540\, g_0^5 h_0^8 \neq 0.$$

The Implicit Function Theorem guanantees the existence of power series $h(t, u)$ and $g(t, u)$, solutions of (4) and (5) such that

$$h(0, 0)=h_0 \qquad \text{and} \qquad g(0, 0)=g_0.$$

Then the corresponding series obtained by means of (3) parametrize a part of the ramification locus B_L and hence,

$$y=u^6 g(t, u) \qquad \text{and} \qquad z'=u^3$$

parametrize a part of the discriminant D_L. To check that the image of this parametrization is a family of curves equisingular along Y in P_0, put

$$g(t, u)=g_0(t)+g_1(t)u+g_2(t)u^2+\ldots$$

From (4),

$$g_0(t)=g_0 \qquad \text{and} \qquad g_1(t)=0.$$

On the other hand, from (3), (4) and (5) it follows that

$$g_2(t)=h_0(t)(9h_0^9(t)+6tg_0h_0^6(t))(6g_0^5)^{-1},$$

where

$$h_0(t)=h(t, 0).$$

Now the resulting 54 parametrizations can be grouped into 18 sets, each one with three elements, in such a way that the parametrizations of each set are conjugated and parametrize a branch of B_L with multiplicity 3, So, these equations represent for each t, 18 branches of $(D_L)_t$ each one with multiplicity 3. But from (3), (4) and (5) it is immediate to obtain the multiplicity of intersection of these branches pairwise, proving that they form an equisingular family of curves.

So we have obtained a parametrization of a subvarity of D_L with multiplicity 54. This subvariety cannot be the whole D_L since $m(D_L)=90$. In fact, of we conside a parametrization like the following

$$(7) \qquad \begin{aligned} x&=ug(t, u),\\ y&=u,\\ z&=uh(t, u), \end{aligned}$$

similarly, we see that the solutions of (1) and (2) of the form (7) with $t=u=0$, are such that

$$g^{10}+1+(h+cg)^4=0, \tag{8}$$

$$10g^9+4c(h+cg)^3=0. \tag{9}$$

Now then it is not difficult to show that two affine curves with equations (8) and (9) have an intersection point in the line of infinity and the multiplicity of intersection of both curves in this same point is 54. Moreover, if c is generic, the 36 intersection points lying on the affine part of both curves are different. The Jacobian determinant correspon-ding to the point gives

$$4(h_0+cg_0)^3(90g_0^8+12c^2(h_0+cg_0)^2) \tag{10}$$

and if c is generic, (10) has no solution in common with (8) and (9). Now the Implicit Function Theorem gives 36 nonsingular surfaces included into B_L and as $m(B_L)=90$, these branches and the ones calculated before form the whole B_L.

On the other hand by computation of the solutions of (8) and (9) in a neighborhood of. $(h_0, 0, 0)$ with $h_0\neq 1$, one can deduce that for c generic the solutions of (8) and (9) verify that if

$$(g_0, h_0)\neq(g_0', h_0') \quad \text{then} \quad h_0\neq h_0'\neq 0.$$

This implies that the parametrizations $y=u$, $z=uh(t, u)$ give 36 non singu-lar branches with different tangents and transverslas to the ones found before. But as $h_0\neq 0$, these branches are transversals to the ones found before. Therefore, we have calculated all branches of $(D_L)_t$, resulting an equisingular family of curves in a neighborhood of $t=0$, i.e., D_L is equisingular along Y at P_0.

Notice that if $t=0$, $(B_L)_0$ and $(D_L)_0$ are not equisaturated, that is $(D_L)_0=\pi_L((B_L)_0)$ is not generic plane projection of $(B_L)_0$. However, Briançon and Henry show in [1] that if L' is a generic direction, $(B_{L'})_0$ and $(D_{L'})_0$ are equisaturated, i.e., $(D_{L'})_0$ is the generic projection of $(B_{L'})_0$. Newertheless, one can show that the family $(B_L)_t$ is equisaturated, that is a generic projection of $(B_L)_t$ is an equisingular family of curves, this is a stronger condition than the single projection $(D_L(B_L)_t)=(D_L)_t$ being equisingular at P_0.

In order to find the discriminant of a generic projection, suppose $L=(a,\ 1,\ c,\ d)$ and make the following change of coordinates

$$x=x'+ay',$$
$$y=y',$$
$$z=z'+cy',$$
$$t=t'+dy'.$$

Now, the equation of X in the coordinate system $\{x', y', z', t'\}$ is

$$(11)\quad (x'+ay')^{10}+(t'+dy')\,y'(z'+cy')^3(x'+ay')^7+y'^{10} +y'^6(z'+cy')^4+(z'+cy')^{16}=0,$$

$L=(0,\ 1,\ 0,\ 0)$ and the equations of the ramification locus B_L are (11) and its derivative with respect to y'

$$\begin{aligned}(12)\quad & 10\,a\,(x'+ay')^9+dy'(z'+cy')^3(x'+ay'')^7\\ &+(t'+dy')(z'+cy')^3(x'+ay')^7\\ &+3\,c\,(t'+dy')\,y'(z'+cy')^2(x'+ay')^7\\ &+7\,a\,(t'+dy')\,y'(z'+cy')^3(x'+ay')^6\\ &+10\,y'^9+6\,y'^5(z'+cy')^{4+4cy'6}(z'+cy')^3\\ &+16\,c\,(z'+cy')^{15}=0.\end{aligned}$$

If we want to find the parametric equations of B_L, we have a fundamental difference with the above case: for certain branches of B_L it is not possible to find any parametrization

$$x'=x'(u,\ t'),\qquad y'=y'(u,\ t'),\qquad z'=z'(u,\ t')$$

with

$$x'(0,\ 0)=y'(0,\ 0)=z'(0,\ 0)=0$$

as before. In fact it is only possible to find a parametrization for the curve $(B_L)_0$ and for $(B_L)_{t'}$, $t'\neq 0$ indepently, as we will show now.

Let $t_0\neq 0$, sufficiently near to 0, and consider (11) and (12) as equations in $\{x', y', z'\}$ defining $(D_L)_{t_0}=B_L\cap\{t'=t_0\}$. Proceeding as before,

$$(13)\qquad \begin{aligned}x'&=u^{40}(h_0+h_1u^6+h(u)\,u^7),\\ y'&=u^{51}(g_0+g(u)\,u),\\ z'&=u^{25},\end{aligned}$$

are parametric equations of a part of $(D_L)_{t_0}$, where

$$\text{(14)} \qquad \begin{cases} h_0^{10}+1=0, \\ t_0 h_0^7+6g_0^5=0 \end{cases}$$

and $h(u)$, $g(u)$ are convenient series. Furthermore, $h_1=1/2 g_0^6 h_0^{-9} \neq 0$. In this way we see that if we consider all the solutions of (14), the corresponding equations as (13) parametrize two branches of $(D_L)_{t_0}$ with multiplicity 25. Now then as $L=(0, 1, 0, 0)$ the parametric equations of the image by π_L of both branches are

$$x'=h_0 u^{40}+h_1 u^{46}+h(u) u^{47},$$
$$z'=u^{25}$$

and these represent two branches of $(D_L)_{t_0}$ with characteristic exponents (40/25, 46/25). By the same method we can calculate the remaining branches of $(D_L)_{t_0}$, obtaining 40 non singular branches, but we do not go into the details since to show that $(D_L)_{t_0}$ and $(D_L)_0$ are not equisingular it is enough to find the characteristic exponents of both branches of $(D_L)_0$ corresponding to the above calculated ones. In fact if $t'=0$,

$$x'=u^{40}(h_0+h_1 u^{12}+h(u) u^{13}),$$
$$y'=u^{52}(g_0+g(u) u),$$
$$z'=u^{25},$$

are parametric equations of two branches of $(D_L)_0$, each one having multiplicity 25. So $h_0 \neq 0$, $g_0 \neq 0$, and if $a \neq 0$, $h_1=-5/6\ a g_0 \neq 0$. Now the projections of these branches of $(B_L)_0$ are two branches of $(D_L)_0$ with characteristic exponents (40/25, 52/25) and hence they are not equisingular with the corresponding branches of $(D_L)_{t_0}$. So then $(D_L)_0$ and $(D_L)_{t_0}$ are not equisingular if $t_0 \neq 0$ and consequently, D_L is not equisingular along Y at $P_0=(0, 0, 0, 0)$ since it is possible to determine the equisingularity for families of plane curves by comparing the sections. Finally, notice that in this case, for every t, $(B_L)_t$ and $(D_L)_t$ are equisaturated, as the mentioned Briançon-Henry theorem asserts.

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