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SPECTRAL PROPERTIES OF G-SYMBOLIC MORSE SHIFTS

BY

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ABSTRACT. — A large class of G-symbolic Morse dynamical systems with simple spectrum is described, where G is a finite, abelian group. The problem of spectral multiplicity in case $G=Z_n$, n is a prime number and $x=b\times b\times \ldots$ is solved. Some examples of special substitutions having non-homogeneous spectra is presented.

Introduction

In the paper we study spectral properties of Morse shifts *over finite, abelian group*. This paper is a continuation of investigations from [8].

Let $x=b^0\times b^1\times \ldots$ be a continuous Morse sequence over G and let Ω_x be a G-symbolic minimal set defined by x (detailed definitions are given in paragraph 1). Let us denote by T the shift transformation on Ω_x and by Γ_0 the set of all eigenfunctions of T. T has a continuous spectrum on the orthogonal complement Γ of Γ_0. Many spectral properties of T on Γ are known in case $G=Z_2=\{0, 1\}$. KAKUTANI [5] and KEANE [7] described a mesure μ on R/Z that determines the maximal spectral type of T on Γ. Each two such measures are either orthogonal or equivalent. QUEFFELEC [14] gave more detailed description of such measures. DEL JUNCO [3] proved that T has a simple spectrum on Γ if

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$x=01\times 01\times\ldots$ (this is the well-known Morse sequence). It is shown in [8] that this result is true for every continuous Morse sequence over $\{0, 1\}$. In this connection the question of spectral multiplicity arises in the class of dynamical systems corresponding to continuous Morse sequences *over a finite, abelian group*. The value of multiplicity function of such systems is bounded by $|G|$ because it is not difficult to infer from [12] that the rank of each Morse dynamical system is less or equal than $|G|$ (*see* also [9]). The question of spectral multiplicity of Morse systems have a connection with ROBINSON'S and GOODSON'S papers [16], [4]. Using interval exchange transformations Robinson constructed ergodic automorphisms with arbitrary finite maximal spectral multiplicity. Goodson tried to simplify Robinson's construction by using the stacking method. He constructed an ergodic automorphism which admits simple approximation with multiplicity 3 having the maximal spectral multiplicity equal to 2. He wrote (without proof) that is method allow to construct ergodic automorphisms with rank $\leqslant M$ and with maximal spectral multiplicity equal to $M-1$ for each natural number M. Lemanczyk noticed that Goodson's example is exactly the shift transformation defined by a Morse sequence $x=010\times 010\times\ldots$ over Z_3. In this connection a suggestion arose that Goodson's construction is impossible in the class of stacking transformations obtained by Morse sequences of a form $x=b\times b\times\ldots$ for each natural number M. In our paper we confirm that suggestion. The other reason to investigate the multiplicity function of Morse dynamical systems over G, $|G|\geqslant 3$, is conected with considerations in [10] about centralizers of Morse shifts. The author proved that automorphisms arising from regular Morse sequences over Z_2 of the form $x=b^0\times b^1\times\ldots$, the set $\{b^i\}_{i\geqslant 0}$ is finite have countable but not trivial centralizers and noticed that his result is true for all regular Morse sequences over G satisfying the condition the set $\{b^i\}$ is finite and having simple spectra.

In this paper we give a necessary condition for Morse dynamical system induced by $x=b^0\times b^1\times\ldots$ over G to have multiple spectrum (*see* (6) in paragraph 2). In case $x=b\times b\times\ldots$ this condition become sufficient too. It is very probably that (6) is also sufficient if x is an arbitrary Morse sequence.

In paragraph 3 some applications of (6) are given. It allow to find a large class of Morse dynamical systems over G with simple spectra. It turns out that in some sense "most" of Morse sequences have simple spectra. We solve the problem of spectral multiplicity in case $G=Z_n$, n is a prime number and $x=b\times b\times\ldots$ We prove that T on Γ has a homogeneous spectrum and its multiplicity function is equal to two iff b is a symmetrical block and it is equal to one for the remaining blocks. We also place some examples of special substitutions having non-homogeneous spectrum on Γ. We finish the paper by proving that the spectral multiplicity T on Γ is less than $n-1$ for every substitution $i\rightarrow b+i$, $i\in Z_n$, where b is a block over Z_n and $n\geqslant 4$. This shows that the Goodson's construction is impossible in the class of stacking transformations arising from such substitutions. For spectral theory of unitary operators we refer to [15].

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1. Preliminaries

First we introduce notions, definitions and notations used in the paper. Let G be a finite, abelian group and let $X=\prod_{-\infty}^{+\infty}G$. Each finite sequence $b=(b[0]b[1]\ldots b[n-1])$, $b[i]\in G$, $i=0,\ldots,n-1$, $n\geqslant 1$ is called a block over G, the number n is called the length of b and denoted by $|b|$. If $y=\ldots y[-1]y[0]y[1]\ldots X$ and b is a block with $|b|=n$ then $y[i,j]$, $b[i,j]$, $0\leqslant i<j\leqslant n-1$, denote the blocks $(y[i]y[i+1]\ldots y[j])$ and $(b[i]b[i+1]\ldots b[j])$ respectively. Given $g\in G$ let $\sigma_g(b)$ be a block defined by $\sigma_g(b)[j]=b[j]+g$, $j=0,1,\ldots,n-1$. Similarly we can define $\sigma_g(y)$ if y is a two-sided sequence over G. If $c=(c[0]\ldots c[m-1])$ is another block over G then we denote by bc the block

$$(b[0]\ldots b[n-1]c[0]\ldots c[m-1]).$$

Further we define

$$b\times c=\sigma_{c[0]}(b)\,\sigma_{c[1]}(b)\ldots\sigma_{c[m-1]}(b).$$

In the sequel all blocks considered are over G and we will write "blocks" if no confusion can arise.

Assume $b^0, b^1, \ldots$ are finite blocks with $|b^t|\geqslant 2$, $t\geqslant 0$, starting with 0 (here $0\in G$). Then we may define an one-sided sequence as follows

(1) $$x=b^0\times b^1\times\ldots$$

We will assume that each block b^t, $t \geqslant 0$, contains every symbol from G and that x is nonperiodic. Such a sequence x is called *a generalized Morse sequence over G* (*see* [11], [12]). It is known that there exist an almost periodic two-sided sequence ω such that $\omega[l]=x[l]$ for $l=0, 1, \ldots$ and its orbit-closure Ω_x in X under the shift transformation T is a symbolic minimal set. The detailed description of Ω_x is given in [11].

Set

$$\lambda_t = |b^t|, \qquad n_t = \lambda_0 \ldots \lambda_t, \qquad c_t = b^0 \times \ldots \times b^t, \qquad t \geqslant 0.$$

If b, c are blocks with $|b| \leqslant |c|$ then by $fr(b, c)$ we denote the frequency b in c i.e.

$$fr(b, c) = \operatorname{card}\{j,\ 0 \leqslant j \leqslant |c| - |b|,\ c[j, j+|b|-1] = b\}.$$

If y is an one-sided sequence and b is a block then the number

$$\mu_y(b) = \lim \frac{1}{n} fr(b, y[0, n-1])$$

is called *the relative average frequency of b in y*, whenever the limit exists.

It is known [12] that (Ω_x, T) *is strictly ergodic* iff

$$\mu_{x_t}(g) = \frac{1}{|G|}$$

for every $g \in G$ and $t \geqslant 0$, where

$$x_t = b^t \times b^{t+1} \times \ldots$$

The unique T-invariant (ergodic) measure μ_x on Ω_x can be obtained as the relative average frequency of the blocks in x. For the remainder of the paper we assume that the sequence $x = b^0 + b^1 \times \ldots$ *is strictly transitive* i.e. (Ω_x, T) is a strictly ergodic system. It is not difficult to show that the conditions $fr(g, b^t) \geqslant \rho > 0$ for every $g \in G$ and $t = 0, 1, \ldots$ imply x is a strictly transitive sequence.

Now we recall some spectral properties of Morse dynamical system (Ω_x, T, μ_x) contained in [12]. Let $\hat{G}$ be the dual group of G. $\hat{G}$ is isomorphic to G and we denote by $\hat{g}$ the element of $\hat{G}$ corresponding to $g \in G$. We define

$$\Gamma_g=\{f\in L^2\Omega_x, \mu_x); \text{ for every } h\in G,\ y\in\Omega_x,$$
$$f(\sigma_h y)=\hat{g}(h)f(y).$$

It is clear that Γ_g, $g\in G$, are T-invariant subspaces of $L^2(\Omega_x, \mu_x)$ because $\sigma_h T=T\sigma_h$, $h\in G$. It is not difficult to verify that $L^2(\Omega_x, \mu_x)=\oplus_{g\in G}\Gamma_g$. The subspace Γ_0 consists of eigenfunctions of T corresponding to all n_t-roots of unity. A Morse sequence x is called continuous if Γ_0 contains all eigenfunctions of T. MARTIN [12] presented *a necessary and sufficient condition of continuouity of* x which is a generalization of a condition given by KEANE [6] when $G=Z_2$.

2. Spectral investigations

In the sequel we will assume that (1) is a continuous Morse sequence. Therefore T has a continuous spectrum on $\Gamma=\oplus_{g\neq 0}\Gamma_g$.

In this section we will examine a measure μ_g belonging to the maximal spectral type of T on Γ_g, $g\in G$, $g\neq 0$, for a given continuous Morse sequence x.

THEOREM 1. — *T has a simple spectrum on each Γ_g, $g\neq 0$, and the function $h_g:\Omega_x\to C$ defined by $h_g(\omega)=\hat{g}(\omega[0])$ determines the maximal spectral type of T on Γ_g.*

Proof. — The theorem can be shown in a similar way as in [8], [14], [4]. According to BAXTER lemma [1] it is sufficient to indicate a sequence $\{V_t^g\}_{t\geq 0}$ of T-invariant, cyclic subspaces of Γ_g such that $V_t^g\subset V_{t+1}^g$, $t\geq 0$, and $\Gamma_g=\bigcup_{t=0}^{\infty}V_t^g$. Put

$$C_0^h=\{y\in\Omega_x;\ y[0]=h\}\qquad (h\in G)$$
$$C_t^h=\{b^0\times b^1\times\ldots\times b^{t-1}\times y;\ y\in\Omega_{x_t} \text{ and } y[0]=h\}$$
$$(h\in G,\ t>0)$$

and

$$v_t^g=\sum_{h\in G}\hat{g}(h)\,1_{C_t^h}\qquad (t\geq 0)$$

where $1_{C_t^h}$ is the characteristic function of C_t^h. It holds

$$(2)\qquad v_t^g=\sum_{h\in G}[\hat{g}(h).(\textstyle\sum_{i\in K_h}T^{-i.n_t}(v_{t+1}^g))]\qquad (t\geq 0,\ n_{-1}=1)$$

where

$$K_h = \{ i;\ 0 \leqslant i \leqslant \lambda_t - 1 \text{ and } b^t[i] = h \} \qquad (h \in G).$$

Let V_t^g be the cyclic subspace of Γ_g defined by v_t^g. The equality (2) is of a form

$$v_t^g = P_t^g(T^{-1}) v_{t+1}^g \qquad (t \geqslant 0)$$

where P_t^g is a polynomial.

Since the spectral measures of v_{t+1}^g are continuous and the function $z \to P_t^g(\bar{z})$ has a finite many zeroes then $v_{t+1}^g \in V_t^g$, $t \geqslant 0$. Hence $V_0^g = V_1^g = \ldots$ and the function v_0^g determines the maximal spectral type of T on $\overline{\bigcup_{t=0}^{\infty} V_t^g}$. Note that $v_0^g = h_g$. To finish the proof it suffices to show that $\overline{\bigcup_{t=0}^{\infty} V_t^g} = \Gamma_g$. Now we define

$$C_t^h(l) = T^l C_t^h \qquad (h \in G,\ t \geqslant 0,\ l = 0, 1, \ldots, n_{t-1} - 1).$$

For every fixed t, the sets $C_t^h(l)$, $h \in G$, $l = 0, 1, \ldots, n_{t-1} - 1$, form a partition ζ_t of Ω_x. It follows from [12] that the sets $C_t^h(l)$ are pairwise disjoint and $\zeta_t \nearrow \varepsilon$. Thus the functions v_t^g, $g \in G$, $t \geqslant 0$, span $L^2(\Omega_x, \mu_x)$. Since $v_t^g \in \Gamma_g$, $t \geqslant 0$, and $L^2(\Omega_x, \mu_x) = \oplus_{g \in G} \Gamma_g$ then we obtain $\Gamma_g = \overline{\bigcup_{t=0}^{\infty} V_t^g}$ for every $g \in G$.

Remark 1. — M. QUEFFELEC ([14], *lemma* 4.1) *has shown Theorem* 1 *in the case when* $b^0 = b^1 = b^2 = \ldots$ *using a similar method.*

Let μ_g^t be the measure on R/Z determined by the function $g_g : \Omega_{x_t} \to C$, $t \geqslant 0$. We will write μ_g instead of μ_g^0.

Remark 2. — *The measure* $\sum_{g \neq 0} \mu_g$ *determines the maximal spectral type of* T *on* $\Gamma = \oplus_{g \neq 0} \Gamma_g$ *and it is defined by the function*

$$h(y) = \begin{cases} |G| - 1, & y[0] = 0 \\ -1, & y[0] \neq 0 \end{cases} \qquad (y \in \Omega_x)$$

Now repeating the same arguments as in [7] we come to the following theorem.

THEOREM 2. — *Each two measures* $\mu_{g'}$, μ_g, $g \neq g' \in G$, $g \neq 0 \neq g'$, *are either orthogonal or equivalent. If* $\mu_g \sim \mu_{g'}$ *then*

$$\| \mu_g^t - \mu_{g'}^t \| \xrightarrow[t \to \infty]{} 0 \tag{3}$$

where $\| . \|$ *is the variation norm.*

By the same reasoning as in [8] we can establish that (3) implies the following

$$\sup_{l \in Z} |\hat{\mu}_g^t(l) - \hat{\mu}_{g'}^t(l)| \xrightarrow[t \to \infty]{} 0. \tag{4}$$

Using the spectral theorem we can write (4) in the other form. We have

$$\hat{\mu}_g^t(l) = (T^l h_g, h_g)_{\Omega_{x_t}} = \int_{\Omega_{x_t}} (h_g \circ T^l)(y) . \overline{h_g(y)} \mu_{x_t}(dy)$$

$$= \int_{\Omega_{x_t}} \hat{g}(y[l] - y[0]) \mu_{x_t}(dy) = \sum_{r \in G} \hat{g}(r) p_l^t(r)$$

where

$$p_l^t(r) = \mu_{x_t}\{ y \in \Omega_{x_t};\ y[l] - y[0] = r \} \tag{5}$$

$(r \in G,\ l \in Z)$.

Therefore (4) is equivalent to the following condition

$$\sup_{l \in Z} |\sum_{r \in G} p_l^t(r) . (\hat{g}(r) - \hat{g}'(r))| \xrightarrow[t \to \infty]{} 0. \tag{6}$$

The condition

$$\sup_{1 \leq l < y_t} |\hat{u}_g^t(l) - \hat{\mu}_{g'}^t(l)| \xrightarrow[t \to \infty]{} 0 \tag{7}$$

can be obtained from [13], p. 399. In fact, the measure μ_g, $g \in G$, $g \neq 0$, is a spectral leasure of $\bar{\lambda}$-multiplicative sequence $\mathscr{L}(k) = \hat{g}(x[k])$, $k \geq 0$, where $\bar{\lambda} . \hat{}\, \lambda_t \}_{t \geq 0}$. Thus we have:

$$(l \geq 0) \quad \begin{array}{ll} (a) \ \hat{\mu}_g(l) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathscr{L}(k+l) . \overline{\mathscr{L}(k)}, & \hat{\mu}_g(-l) = \overline{\hat{\mu}_g(l)} \\ (b) \ \mathscr{L}(j + k' n_t) = \mathscr{L}(j) . \mathscr{L}(k . n_t) & (t \geq 0,\ k \geq 0,\ 0 \leq j < n_t). \end{array}$$

The first statement follows from a strictly ergodicity of the (Ω_x, T, μ_x) and the second from the equalities

$$x[j+k.n_t]=c_{t+1}[j+k.n_t]=c_{t+1}[j]+c_{t+1}[k.n_t]=x[j]+x[k.n_t].$$

Using (a) and the equality

$$x[i+k.\lambda_0]-x[i]=x_1[s+k]-x_1[s]$$
$$(i=s.\lambda_0+t,\ k,\ s\geqslant 0,\ t=0,\ 1,\ \ldots,\ \lambda_0-1)$$

it is easy to check that

$$\hat{\mu}_g^1(k)=\hat{\mu}_g(k.\lambda_0) \qquad (k\in Z)$$

and in general

$$\hat{\mu}_g^t(k)=\hat{\mu}_g(k.n_{t-1}) \qquad (t\geqslant 0,\ k\in Z). \tag{8}$$

If $\mu_g\sim\mu_{g'}$ then

$$\sum_{t=0}^{\infty}|\hat{\mu}_g(b_t.n_t)-\hat{\mu}_{g'}(b_t.n_t)|^2<\infty$$

for every sequence $\{b_t\}$, $1\leqslant b_t<\lambda_{t+1}$. The last condition and (8) implies (7)

3. *G*-symbolic Morse shifts with simple spectra

The condition (6) permits to answer some questions concerning of G-symbolic Morse dynamical systems. One of them is the question of spectral multiplicity of such systems. On the one hand all Morse shifts over Z_2 have simple spectra. On the other hand the GOODSON's example [4] shows that this statement is false in the class of all Morse shifts. We will throw some light on this question.

We assume that

$$x=b^0\times b^1\times\ldots$$

is a continuous Morse sequence over G. Let $S_{|G|}$ be a set of all $|G|$-dimensional probability vectors $\bar{p}=\langle p(r)\rangle_{r\in G}$. Given $g,\ g'\in G,\ g\neq g'$, $g\neq 0\neq g'$, we denote by $S(g,\ g')$ a subset of $S_{|G|}$ consisting of all vectors $\bar{p}$ satisfying

$$\sum_{r\in G}p(r)(\dot{g}(r)-\dot{g}'(r))=0. \tag{9}$$

The sequence x determines a set $\{\vec{p}_l^t\}$, $t\geqslant 0$, $l\in Z$, of elements of $S_{|S|}$, where $\vec{p}_l^t=\langle p_l^t(r)\rangle_{r\in G}$, and the numbers $p_l^t(r)$ are defined by (5).

Now we can interpret (6) *in a geometrical sense.* First we observe that (9) defines a $|G|-2$-dimensional plane in $R^{|G|-1}$ [we omitt $p(0)$] so the set $S g, g')$ is the intersection of that plane and $S_{|G|}$. Then (6) means that the $R^{|G|-1}$-distances of the vectors $\{\vec{p}_l^t\}$ to $S(g, g')$ converge to zero as $t\to\infty$ uniformly with respect to $l=+1, +2, \ldots$

The above considerations lead to a sufficient condition for Morse shift (Ω_x, T) to have simple spectrum. Namely, we set

$$S=\bigcup_{g, g'\in G} S(g, g').$$

Remark 3. — *A Morse dynamical system* (Ω_x, T, μ_x) *has simple spectrum if a sequence of probability vectors* $\{\vec{p}_l^t\}_{t=0}^{\infty}$ *has a limit point contained in* $S_{|G|}\setminus S$ *for at least one* $l\in Z$. *In particular this statement is true for* $l=1$.

Proof. — To show this we observe that the assumption of the remark quarantee that (6) is not satisfied for all $g, g'\in G$, so $\mu_g\perp\mu_{g'}$, $g\neq g'$, $g\neq 0\neq g'$. Considering that T has a simple apectrum on each Γ_g, $g\in G$, we obtain that T has a simple spectrum on $L^2(\Omega_x, \mu_x)$.

Now we will construct a large class of Morse shifts over G having simple spectra. First we will prove that for every $\bar{p}.\langle p(r)\rangle_{r\in G}\in S_{|G|}$ with $p(r)>0$, $r\in G$, there exists a continuous Morse sequence x such that $p=\lim_t\vec{p}_1^t$, where $\vec{p}_1^t$, $t\geqslant 0$, are defined by (5) for $l=1$. For this purpose we will need formulas on $p_1^t(r)$, $t\geqslant 0$, $r\in G$. Put

$$s_l^t(r)=\operatorname{card}\{0\leqslant i\leqslant\lambda_t-l-1;\ b^t[i+l]-b^t[i]=r\}. \tag{10}$$

$r\in G$, $l=1, 2, \ldots, \lambda_t-1$, $t\geqslant 0$.

Since $\mu_{x_t}(B)$, B is a block, may be obtained as the the relative average frequency B in X_t then we have

$$p_l^t(r)=\lim_n\frac{1}{n}\operatorname{card}\{0\leqslant i\leqslant n-l-1;\ x_t[i+l]-x_t[i]=r\}, \tag{11}$$

for $l=1, 2, \ldots$, $t\geqslant 0$, $r\in G$.

Using (11) and some combinatorial arguments we obtain

$$p_1^t(r)=\frac{s_1^t(r)}{\lambda_t}+\frac{1}{\lambda_t}p_1^{t+1}(r+r_{t+1}). \tag{12}$$

where $r_t = b^t[\lambda_t - 1]$, $t \geqslant 0$, $r \in G$.

The equality (12) show that if $\lambda_t \to \infty$ then the sequences of vectors $\{\vec{p}_1^t\}$ and $\{\langle (1/(\lambda_t - 1)) . s_1^t(r) \rangle_{r \in G}\}_{t \geqslant 0}$ have the same set of limit points. Thus Remark 3 remains true if we replace the vectors $\{\vec{p}_1^t\}$ by $\{\langle (1/(\lambda_t - 1)) . s_1^t(r) \rangle_{r \in G}\}_{t \geqslant 0}$.

Take $\bar{p} \in S_{|G|}$ with positive members $p(r)$, $r \in G$, and choose a sequence of positive integers $\{\lambda_t\}_{t \geqslant 0}$ such that $\lambda_t \to \infty$ and for every prime factor p of $|G|$, $p \mid \lambda_t$, $t \geqslant 0$. Next we find a sequence of probability vectors $\{\vec{q}^t\}_{t \geqslant 0}$ such that

$$\vec{q}^t \underset{t}{\to} \bar{p},$$

$$\vec{q}^t = \frac{w^t(r)}{\lambda_t - 1} {}_{r \in G},$$

$$\frac{w^t(r)}{\lambda_t - 1} \geqslant \rho > 0 \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad r \in G.$$

It is easy to construct blocks b^t over G, $t \geqslant 0$, satisfying

$$b^t[0] = 0, \qquad |b^t| = \lambda_t, \qquad s_1^t(r) = w^t(r),$$

$$fr(r, b^t) \geqslant \rho' > 0$$

for $t \geqslant 0$ and $r \in G$, where $s_1^t(r)$, $r \in G$, $t \geqslant 0$, are defined by (10).

The above conditions quarantee that

$$x = b^0 \times b^1 \times \dots$$

is a strictly transitive Morse sequence and the condition $p \mid \lambda_t$ for every prime factor p of $|G|$, $t \geqslant 0$, implies that x is continuous (*see* [12]).

Let $\mathcal{M}_0$ be a class of all Morse sequences over G constructed as above for all $p = \langle p(r) \rangle_{r \in G} \in S_{|G|} \cap \pi \setminus S$ with $p(r) > 0$, $r \in G$. Then for every $x \in \mathcal{M}_0$ the corresponding Morse shift (Ω_x, T, μ_x) has simple spectrum. The class $\mathcal{M}_0$ may be treated as a "large" class by this reason that $l_{|G|}(G_{|G|}) > 0$ while $l_{|G|}(S) = 0$, where $l_{|G|}$ is the $|G| - 1$-dimensional Lebesque measure.

4. Spectral multiplicity of special Morse shifts

In this part we will assume

$$x = b \times b \times \ldots, \qquad |b| = \lambda, \quad b[0] = 0 \tag{13}$$

$|G| = n \geq 3$ *and* b *contains all symbols of* G. Note that if x has the form (13) then $x_t = x$ for $t \geq 0$ and therefore *the measures* μ_g, $\mu_{g'}$, $g \neq g' \in G$, $g \neq 0 \neq g'$ *are either orthogonal or equal* [*see* (4)].

LEMMA 1. — *For each* $n \geq 3$ *and for* $g \neq g' \in G$, $g \neq 0 \neq g'$, *the measures* μ_g, $\mu_{g'}$ *are equal iff*

$$C(g, g') \qquad \sum_{r \in G} s_l(r) \cdot (\hat{g}(r) - \hat{g}'(r)) = 0 \qquad \text{for} \quad l = 1, \ldots, \lambda - 1$$

where $s_l(r) = \operatorname{card}\{i;\ 0 \leq i \leq \lambda - l - 1$ *and* $b[i+l] - b[i] = r\}$ $(r \in G)$.

Proof. — Lemma is a simple consequence of the lemmas 1 and 3 from [13]. Let

$$s_l^n(r) = \operatorname{card}\{i;\ 0 \leq i \leq \lambda^n - 1 - l \text{ and } \overbrace{b \times \ldots \times b}^{n}[i+l] - \overbrace{b \times \ldots \times b}^{n}[i] = r\} \qquad (r \in G,\ n \geq 1).$$

If ω_n denotes the Haar measure of the group $D_n \subset R/Z$ generated by $1/\lambda^n$ then

$$\frac{d\mu_g}{d\mu_g * \omega_n)} = f_n^g \qquad (n \geq 1),$$

where $*$ denotes the convolution of the measures and

$$f_n^g(x) = 1 + \sum_{1 \geq l \leq \lambda^n - 1} \Big[\sum_{r \in G} s_l^n(r) \cdot \hat{g}(r)\Big] \exp(-2\pi i l x)$$
$$+ \sum_{1 - \lambda^n \leq l \leq -1} \Big[\overline{\sum_{r \in G} s_{-l}^n(r) \cdot \hat{g}(r)}\Big] \exp(-2\pi i l x).$$

Now, if $\mu_g = \mu_{g'}$ then $f_1^g = f_1^{g'}$ so $C(g, g')$ is satisfied.

In the other hand let the condition $C(g, g')$ be satisfied. It is easy to obtain

$$s_l^{n+1} = s_q^n * s_p + \breve{s}_{\lambda^n - q}^n * s_{p+1} \qquad (n \geq 1)$$
where $1 \leq l \leq \lambda^{n+1} - 1$, $l = p \cdot \lambda^n + q$, $0 \leq q \leq \lambda^n - 1$.

Using the above and the fact that

$$\sum_{r\in G} s_k(r)\,.\,\hat{g}(r)=\hat{s}_k(g)$$

where $\hat{s}_k$ is the Fourier transform of s_k, we have

$$\begin{aligned}\hat{s}_l^{n+1}(g)-\hat{s}_l^{n+1}(g')&=(s_q^n * s_p)(g)+(\check{s}_{\lambda^n-q}^n * s_{p+1})(g)\\&\quad-(s_q^n * s_p)(g')-(\check{s}_{\lambda^n-q}^n * s_{p+1})(g')\\&=\hat{s}_q^n(g)\,.\,\hat{s}_p(g)-\hat{s}_q^n(g')\,\hat{s}_p(g')\\&\quad+\hat{s}_{\lambda^n-q}^n(-g)\,\hat{s}_{p+1}(g)-\hat{s}_{\lambda^n-q}^n(-g')\,\hat{s}_{p+1}(g')=0.\end{aligned}$$

Then $f_n^g=f_n^{g'}$ for $n\geqslant 1$ and applying lemma 3 from [13] we obtain $\mu_g=\mu_{g'}$.

Remark 4. — *The spectrum of a Morse shift* (Ω_x, T, μ_x) *is simple, whenever* $x=b\times b\times\ldots$, b *is a block over* Z_n, $n\geqslant 3$ *and* $b[\lambda-1]$ *is a generator of* Z_n.

It sufficies to notice that $\sum_{r\in G} s_{\lambda-1}(r)\,.\,(g(r)-g'(r))\neq 0$ for every $g\neq g'\in Z_n$, $g\neq 0\neq g'$ because $s_{\lambda-1}(b[\lambda-1])=1$ and $s_{\lambda-1}(r)=0$ for $r\neq b[\lambda-1]$.

Now we also assume that n is a prime number. Then $G=Z_n$. In this case we are able to compute the spectral multiplicity of T on $\Gamma=\oplus_{g\neq 0}\Gamma_g$. We will prove that the multiplicity is either one or two and it is two iff b is a symmetrical block.

LEMMA 2. — *Let* $\bar{s}=(s(0),\ldots,s(n-1))$ *be a vector with* $s(r)$ *rational,* $r\in Z_n$, *let* j, $k\in Z_n^*$ *and let* H *be a multiplicative subgroup of* Z_n^* *generated by* $j\,.\,k^{-1}$. *Then* $\bar{s}$ *satisfies the condition* $C(j,k)$ *iff* $s(m)=s(r)$ *whenever* $m\,.\,r^{-1}\in H$.

Proof. — Because Z_n is a field then the condition $C(j,k)$ can be written as

$$\sum_{r=1}^{n-1} \mathscr{L}^r\,.\,[s(k^{-1}\,.\,r)-s(j^{-1}\,.\,r)]=0$$

where $\mathscr{L}=\exp(2\pi i/n)$.

Dividing the above equality by $\mathscr{L}$ we obtain a polynomial over Q of the degree $\leqslant n-2$ having $\mathscr{L}$ as a zero. Thus

$$s(k^{-1}\,.\,r)=s(j^{-1}\,.\,r)\qquad (r\in Z_n^*)$$

because the minimal polynomial of $\mathscr{L}$ over Q has the degree $n-1$. It is easy to see that the last condition is equivalent to the thesis of the lemma.

It follows from Lemma 2 that $\mu_j=\mu_k$, $j, k\in Z_n^*$, iff $j.k^{-1}\in H$, where H is a subgroup of Z_n^*. Hence the spectrum of T on Γ is homogeneous and its spectral multiplicity is equal to the index of H in Z_n^*.

THEOREM 3. — *The spectral multiplicity of a Morse shift* (Ω_x, T, μ_x) *defined by* (13) *with n prime is less or equal than two. It is equal two iff b is a symmetrical block i. e.* $b[k]=b[\lambda-1-k]$ for $k=0, 1, \ldots, \lambda-1$.

Proof. — According to the result in [12] if n is prime then the maximal spectral type of T on Γ is continuous for all sequences defined by (13).

Let H be a subgroup of Z_n^* such that $|H|\geqslant 3$ and

$$(14)\qquad s_l(j)=s_l(k)\quad \text{whenever } j.k^{-1}\in H,$$

for all $l=1, 2, \ldots, \lambda-1$. Applying (14) for $l=\lambda-1, \lambda-2, \ldots$ we obtain $b=00\ldots0$. Hence either $H=\{1, -1\}$ or $H=\{1\}$.

Let b be a symmetrical block. Then for each $r\in Z_n$ and $l=1, \ldots, \lambda-1$ we have $s_l(r)=s_l(-r)$ so the functions s_l are constans on the cosets of Z_n^* modulo $H=\{1, -1\}$. The last denotes that the spectral multiplicity is two.

Let b be a nonsymmetrical block. If $b[\lambda-1]\neq 0$ then Remark 3 implies the simplicity of the spectrum. Assume that $b[\lambda-1]=0$. *Taking* $l_0=\lambda-1-\min\{i;\ b[i]\neq b[\lambda-i-1]\}$ we have $s_{l_0}(b[l_0])\neq s_{l_0}(-b[l_0])$. Thus the function s_{l_0} is not constant on the cosets of Z_n^* modulo $H=\{1, -1\}$.

The next lemma shows that the multiplicity greater than one is possible without the assumption b is a symmetrical block if n is nonprime.

LEMMA 3. — *Let* $x=b\times b\times\ldots$ *be a continuous Morse sequence over* Z_4. *The spectral multiplicity of* T *on* Γ *is two iff* $s_l(1)=s_l(3)$ *for* $l=1, \ldots, \lambda-1$. *Then the spectrum is non-homogeneous. In the remaining cases the spectrum of* Γ *is simple.*

Proof. — Using Lemma 1 for $G=Z_4$ by simple calculations we can check that $\mu_1=\mu_2$ iff $s_l(1)=s_l(2)=s_l(3)$ for $l=1, \ldots, \lambda-1$. Similarly $\mu_2=\mu_3$ iff $s_l(1)=s_l(2)=s_l(3)$ so by preceding considerations $\mu_1\perp\mu_2$ and $\mu_2\perp\mu_3$. Finally the same calculations show that $\mu_1=\mu_3$ iff $s_l(1)=s_l(3)$ for $l=1, \ldots, \lambda-1$. For example if $b=0132$ *then* $x=b\times b\times\ldots$ is continuous Morse sequence (*see* [12]), the block b satisfies the conditions in

Lemma 3 so the spectrum multiplicity of T on Γ is two while b is not a symmetrical block.

LEMMA 4. — *For each $n \geqslant 3$ exists a continuous Morse sequence $x = b \times b \times \ldots$ that the spectrum of T on Γ is not simple.*

Proof. — Let $G = Z_n$, $n \geqslant 3$ and $b = 0110 \times 0110 \times \ldots \times 0110$ ($n-1$ times). Since $x[k . n_t] = b[k]$ for $t \geqslant 0, \ldots, \lambda - 1$ hence the condition of continuoity of the spectral measure from [2] is satisfied for our measures μ_g, $g \in Z_n$, $g \neq 0$. Moreover, since $s_{\lambda-2}(-1) = s_{\lambda-2}(1) = 1$ and $s_{\lambda-2}(r) = 0$ for $r \notin \{-1, 1\}$ so the condition

$$\sum_{r=0}^{n-1} s_{\lambda-2}(r)(\mathscr{L}^{j \cdot r} - \mathscr{L}^{k \cdot r}) = 0$$

implies $j = -k$.

In the other hand the block b is symmetrical i. e. $s_l(r) = s_l(-r)$ for $r \in Z_n$ and $l = 1, \ldots, \lambda - 1$ then the condition $C(j, -j)$ is satisfied for each $j \in Z_n$, $j \neq 0$.

We end this paper by the following.

THEOREM 4. — *Let $n \geqslant 4$ and let T be a Morse shift given by a continuous Morse sequence $x = b \times b \times \ldots$, where b is a block over Z_n. Then the spectral multiplicity T on Γ is less than $n - 1$.*

Proof. — Let us suppose that the spectral multiplicity is equal to $n-1$. Then the numbers $s_l(r)$, $r \in Z_n$, $l = 1, \ldots, \lambda - 1$ satisfy the conditions $C(1, 2)$, $C(1, 3)$, $\ldots$, $C(1, n-1)$ that is:

$$(15) \qquad \begin{aligned} &\sum_{r=1}^{n-1} s_l(r)(\mathscr{L}^{r} - \mathscr{L}^{2r}) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\sum_{r=1}^{n-1} s_l(r)(\mathscr{L}^{r} - \mathscr{L}^{(n-1) \cdot r}) = 0 \end{aligned}$$

The conditions (15) are $n-2$ equations with $n-1$ unknows $s_l(1), \ldots, s_l(n-1)$. Since $\det[\mathscr{L}^{r} - \mathscr{L}^{r \cdot k}]_{1 \leqslant r \leqslant n-2,\ 2 \leqslant k \leqslant n-1} \neq 0$ then the dimension of the solution space is 1. Observe that $y(1) = y(2) = y(n-1) = 1$ is a solution of (15). Hence we conclude that any solution of (15) has a form $(d, d, \ldots, d)$. Thus $s_l(1) = s_l(2) = \ldots = s_l(n-1)$ for $l = 1, \ldots, \lambda - 1$ what is impossible.

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