BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 115 (1987), p. 71-93
<http://www.numdam.org/item?id=BSMF_1987__115__71_0>

Bull. Soc. math. France,
115, 1987, p. 71-95.

ARITHMETICALLY NORMAL SHEAVES

BY

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RÉSUMÉ. – Nous déterminons la borne supérieure pour la troisième classe de Chern d'un faisceau courbiligne à cohomologie seminaturelle; nous déterminons aussi toutes les triplets de classes de Chern pour lesquelles il y a deux différentes cohomologies seminaturelles. Nous faisons ceci en introduisant la notion de faisceau arithmétiquement normal.

ABSTRACT. – We find the upper bound for the third Chern class of a curvilinear sheaf with seminatural cohomology, and we determine all the triples of Chern classes for which there exist two different kinds of seminatural cohomology. This is done by introducing the notion of arithmetically normal sheaf.

1. Introduction and preliminaries

In this paper we study the cohomology of rank two reflexive sheaves on $\mathbf{P}^3$, with a particular attention for a simple kind of cohomology called "seminatural cohomology" [HH1], which is the most natural "minimal cohomology".

A sheaf F is said to be "reflexive" if the natural map $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism; there are several interesting reasons for studying such sheaves, explained for instance in [HA1], where their principal properties are exposed. In particular, reflexive sheaves can give informations about new curves, they "arise naturally from vector bundles of higher rank", and

(*) Texte reçu le 14 décembre 1985, révisé le 12 mai 1986

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The author prepared this work under the direction and with the constant help of A. Hirschowitz, during a stage in Nice supported by the Italian C N R

they arise naturally also "in the study of rank-two vector bundle on $\mathbb{P}^3$" (*see* the "reduction-step"-construction of [HA1] and [HA2]).

Since these sheaves are interesting mainly when their sections are smooth curves, Hartshorne and Hirschowitz introduced the notion of "curvilinear sheaf", that is a sheaf such that a suitable twist has smooth sections.

Our problems can be shortly formulated in this way: given a triple of integers (c_1, c_2, c_3), find, if it exists, a stable reflexive sheaf on $\mathbb{P}^3$ having seminatural cohomology and Chern classes (c_1, c_2, c_3). In particular, for fixed (c_1, c_2), determine $M[(c_1, c_2)]$ = maximum of the set of all c_3 such that there exists a sheaf with seminatural cohomology and Chern classes (c_1, c_2, c_3). These problems are closely related to those ones studied in [HH1].

We introduce the notion of arithmetically normal sheaf: a rank-two curvilinear reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$ is said to be arithmetically normal if $h^1(\mathbb{P}^3, \mathscr{F}(t)) = 0\ \forall t$. Suitable twists of an arithmetically normal sheaf have sections that are arithmetically normal curves. The main reason for introducing this notion is the following one: we can define a piecewise linear function $m[(c_1, c_2)]$ (whose asymptotical behaviour is $(4/3)c_2^{3/2}$) such that if $\mathscr{F}$ has seminatural cohomology, then it is arithmetically normal if and only if $m(c_1(\mathscr{F}), c_2(\mathscr{F})) \leqslant c_3(\mathscr{F})$.

In this way we can study the problem of determining the largest possible c_3 for sheaves with seminatural cohomology by using the results of Gruson and Peskine about the numerical characters of the arithmetically normal curves.

We define another piecewise linear function $M(c_1, c_2)$ (with $M(c_1, c_2) \sim (4/3)c_2^{3/2}$ and $M - m \sim c_2^{1/2}$) such that if $\mathscr{F}$ has seminatural cohomology then $c_3(\mathscr{F}) \leqslant M(c_1(\mathscr{F}), c_2(\mathscr{F}))$. Moreover, we determine all the existing arithmetically normal sheaves with seminatural cohomology; in particular we see that if $c_3 \in [m(c_1, c_2), M(c_1, c_2)]$, then there exists an arithmetically normal sheaf with seminatural cohomology and Chern classes (c_1, c_2, c_3). So, $c_3 = M(c_1, c_2)$ is the answer to the question of determining the largest possible third Chern class of a curvilinear sheaf with seminatural cohomology.

As another consequence, we get all the triples of Chern classes (c_1, c_2, c_3) for which there are two different kinds of seminatural cohomology. This gives us examples of reducible spaces of moduli of curvilinear sheaves with seminatural cohomology.

Our results suggest to state the following conjecture: given a triple (c_1, c_2, c_3) with $c_3 \leqslant M(c_1, c_2)$, $c_1 = 0, -1$, $(c_1, c_2, c_3) \neq (-1, 2, 0)$ or $(-1, 4, 0)$ (*see* [HH1]), $c_3 = c_1 c_2 \bmod 2$, there exists a stable curvilinear sheaf of rank-two on $\mathbb{P}^3$ with seminatural cohomology and Chern classes (c_1, c_2, c_3).

Remark that existence results [in a different range of (c_2, c_3)] for sheaves with seminatural cohomology have been proved by Hartshorne and Hirschowitz, and will appear in a forthcoming paper. In another paper we prove our conjecture for $c_2 \leqslant 10$, and we construct sheaves with seminatural cohomology and exactly one group $H^1(\mathbb{P}^3, \mathscr{F}(t))$ different from zero.

$\mathbb{P}^3$ always means $\mathbb{P}_k^3$, where k is an algebraically closed field of characteristic zero. We always use normalized sheaves (that is with $c_1 = 0, -1$). We often write $H^1(\mathscr{F}(n))$ and $h^1(\mathscr{F}(n))$ instead of $H^1(\mathbb{P}^3, \mathscr{F}(n))$ and $h^1(\mathbb{P}^3, \mathscr{F}(n))$.

Our general references about reflexive sheaves will be [HA1] and [HA2] (in particular for the well-known correspondance sheaves-curves).

DEFINITION 1.1. — Let $\mathscr{F}$ be a rank-two reflexive sheaf on $\mathbb{P}^3$. Then $\mathscr{F}$ is said to be *curvilinear* if it has the following property: If $\mathscr{F}(s)$ is globally generated, then the zero set of a general section of $\mathscr{F}(s)$ is a smooth curve.

DEFINITION 1.2. — Let $\mathscr{F}$ be a rank-two torsion free sheaf on $\mathbb{P}^3$ with $c_1 = 0, -1$. $\mathscr{F}$ has *seminatural cohomology* if for every $n \geqslant -2 - [c_1(\mathscr{F})/2]$ ([ ] means the integral part) at most one group $H^i(\mathscr{F}(n))$ is different from zero.

Remarks 1.3. — If $\mathscr{F}$ has seminatural cohomology, then the zero set of a section of $\mathscr{F}(p)$ is of maximal rank (*see* [BE]).

The condition $n \geqslant -2 - [c_1(\mathscr{F})/2]$ is necessary in order to get a good definition; indeed, if for every integer n at most one of the groups $H^i(\mathscr{F}(n))$ is different from zero, then necessarily $\mathscr{F}$ is locally free [HH1], and the problem is completely solved in [HH1].

It is known that if $\mathscr{F}$ has seminatural cohomology, then it is stable, except in four cases [BOL].

We need the properties of the spectrum of a reflexive sheaf. We collect here the results needed later, whose proofs are in [HA1] and [HA2].

PROPOSITION 1.4. — (A) *Let $\mathscr{F}$ be a rank-two reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ or -1 and $H^0(\mathscr{F}(-1))=0$. Then $\mathscr{F}$ has a spectrum, denoted* $\operatorname{Spec}(\mathscr{F})$, *that is an unique set of integers (k_i), $0\leqslant i\leqslant c_2(\mathscr{F})$, with the following properties:*

$$h^1(\mathbb{P}^3, \mathscr{F}(t))=h^0(\mathbb{P}^1, \oplus_i \mathcal{O}_{\mathbb{P}^1}(k_i+t+1) \qquad \textit{for} \quad t\leqslant -1$$

and

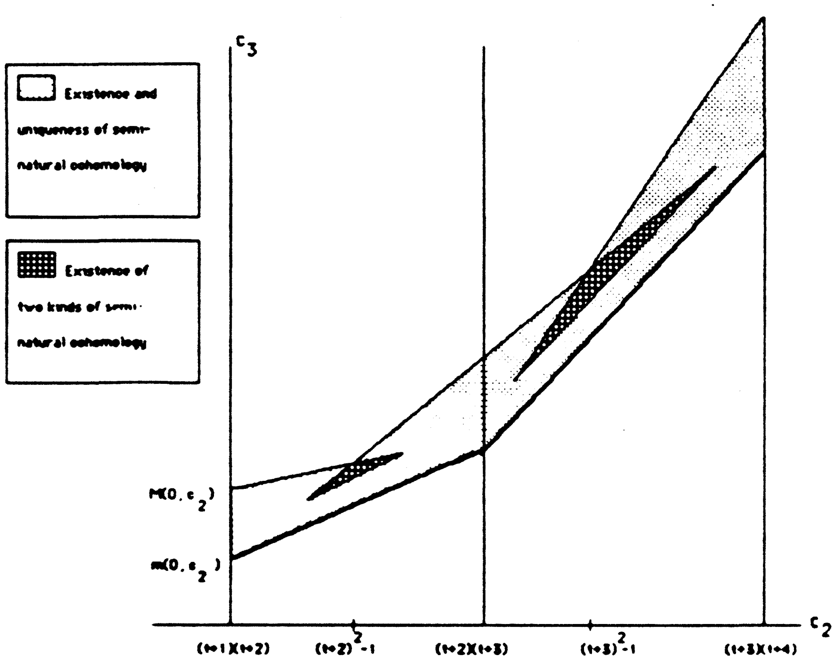
$$h^2(\mathbb{P}^3, \mathscr{F}(t))=h^1(\mathbb{P}^1, \oplus_i \mathcal{O}_{\mathbb{P}^1}(k_i+t+1) \qquad \textit{for} \quad t\geqslant -3-c_1.$$

(B) $c_3(\mathscr{F})=-2\sum_i k_i+c_1c_2$.

(C) *If there is a $k<-1$ in the spectrum, then -1, -2, $\ldots k$ also occur in the spectrum if $c_1=0$, and -2, -3, $\ldots$ k also occur if $c_1=-1$. If $\mathscr{F}$*

TABLEAU

Existence and uniqueness of arithmetically normal sheaves with seminatural cohomology and $c_1=0$.



is stable and $c_1=0$, *then either* 0 *occurs or* -1 *occurs at least twice; if* $c_1=-1$, *then* -1 *also occurs.*

(D) *Let* $\mathcal{F}$ *be stable, and let* $K=\max\{-k_i\}$. *If* $c_1=0$ *and there is a* k_0 *with* $-K<k_0\leqslant -1$ *which occurs just once in the spectrum, then each* k_i *with* $-K\leqslant k_i\leqslant k_0$ *occurs exactly once in the spectrum. If* $c_1=-1$ *and there is a* k_0 *with* $-K<k_0\leqslant -2$ *which occurs just once in the spectrum, then each* k_i *with* $-K\leqslant k_i\leqslant k_0$ *occurs exactly once in the spectrum.*

2. Arithmetically normal curves and arithmetically normal sheaves

We want to determine the largest possible c_3 for a sheaf with seminatural cohomology, thus bounding the range of Chern classes of these sheaves. So, we want to study the behaviour of this kind of cohomology if $c_3 \gg c_2$; in particular we are interested in the range where $\chi(\mathcal{F}(n))\geqslant 0$ for every $n\geqslant -2-c_1$. So, we use arithmetically normal curves, and our basic results are Gruson-Peskine's ones.

DEFINITION 2.1. — A rank-two reflexive sheaf $\mathcal{F}$ on $\mathbf{P}^3$ is said to be *arithmetically normal* if it is curvilinear and $H^1(\mathbf{P}^3, \mathcal{F}(n))=0$ for every n.

Remark 2.2. — For a suitable twist, the sections of an arithmetically normal sheaf are arithmetically normal curves.

DEFINITION 2.3 (*see* [GP], def. 2.4). — Let $Y\subset\mathbf{P}^r$ be a projectively Cohen-Macaulay two-codimensional subvariety, contained in an hypersurface of degree s and not contained in any hypersurface of degree s. A sufficientely general projection of Y on the hyperplane at the infinity gives an exact sequence

$$0\to\oplus_{i=0}^{s-1}\mathcal{O}_{\mathbf{P}^{r-1}}(-n_i)\to\oplus_{i=0}^{s-1}\mathcal{O}_{\mathbf{P}^{r-1}}(-i)\to\mathcal{O}_Y\to 0$$

with $n_0\geqslant n_1\geqslant\ldots\geqslant n_{s-1}\geqslant s$.

The sequence $(n_0, n_1, \ldots, n_{s-1})$ is called the *numerical character* of Y.

It is easy to see that if Y is integral this sequence is without gaps. Moreover, there is an important theorem:

THEOREM 2.4 [GP]. — *Let* $(n_i)_{0\leqslant i\leqslant s-1}$ *be a decreasing sequence of integers such that* $n_i\leqslant n_{i+1}+1$ $(i\leqslant s-2)$ *and* $s\leqslant n_{s-1}$. *Then there exists an arithmetically normal curve in* $\mathbf{P}^3$ *with numerical character* $(n_i)_{0\leqslant i\leqslant s-1}$.

Now, if $\mathcal{F}$ is a reflexive sheaf with seminatural cohomology and c_3 is large compared with c_2, $\chi(\mathcal{F}(p))\geqslant 0$ for every $p\geqslant -2$. So, if we take a section of $\mathcal{F}(n)$, $n\gg 0$, whose zero set Y is twocodimensional, we get an exact sequence

$$0\to \mathcal{O}_{\mathbf{P}^3}(-n)\to\mathcal{F}\to\mathcal{I}_Y(n)\to 0\qquad \text{if}\quad c_1=0,$$

or

$$0\to \mathcal{O}_{\mathbf{P}^3}(-n)\to\mathcal{F}\to\mathcal{I}_Y(n-1)\to 0\qquad \text{if}\quad c_1=-1,$$

where Y is a twocodimensional Cohen-Macaulay subscheme with $h^1(\mathcal{I}_Y(p))=0\ \forall p$. This follows from the fact that $h^1(\mathcal{F}(-1))=0$ implies that $\operatorname{Spec}\mathcal{F}$ contains only negative integers, and so $h^1(\mathcal{F}(p))=0\ \forall p\leqslant -1$. Since by hypothesis $h^1(\mathcal{F}(p))=0\ \forall p\geqslant -2$, we have $h^1(\mathcal{I}_Y(p))=0\ \forall p$.

The following proposition tell us the numerical character of a suitable section of a sheaf with seminatural cohomology.

PROPOSITION 2.5. — *Let $\mathcal{F}$ be a rank-two reflexive sheaf on $\mathbf{P}^3$ with seminatural cohomology and Chern classes* $(0, c_2, c_3)$ *such that* $\chi(\mathcal{F}(t))\geqslant 0$ $\forall t\geqslant -2$. *Let* $p=\max\{t\,|\,h^0(\mathcal{F}(t))=0\}$, *let f be a general section of* $\mathcal{F}(p+3)$, *and let Y be the zero set of F. Then*

$$s_{2p+4}=\chi(\mathcal{F}(p-2))-3(\chi(\mathcal{F}(p-1))-\chi(\mathcal{F}(p)))+1$$
$$s_{2p+5}=\chi(\mathcal{F}(p-1))-3\chi(\mathcal{F}(p))+1$$
$$s_{2p+6}=\chi(\mathcal{F}(p))$$
$$s_i=0\qquad \textit{for}\quad i\neq 2p+4,\ 2p+5,\ 2p+6,$$

where s_i is the number of elements equal to i in the numerical character of Y.

Remark 2.6. — It is also possible to compute the numerical characters of the sections of the further twists of $\mathcal{F}$, and thus to obtain the relation between the numerical characters of two curves which are sections of two different twists of the same reflexive sheaf.

Proof. — First of all, we prove that $p\geqslant -1$. In fact, $\mathcal{F}$ has seminatural cohomology, so [BOL],

(*a*) it is stable, and then $h^0(\mathcal{F})=0$ (and $p\geqslant 0$), (*b*) $h^0(\mathcal{F}(-1))=0$ if it is an exception to stability, and $p\geqslant -1$. By duality, $h^3(\mathcal{F}(t))=0$ if $t\geqslant -3$. Then $\mathcal{F}(p+3)$ is globally generated (thanks to Castelnuovo-Mumford's lemma).

Let $f\in H^0(\mathcal{F}(p+3))$ be a general section; its zero set Y is a Cohen-Macaulay curve, generically locally complete intersection, with $h^1(\mathcal{I}_Y(t))=0\ \forall t$. In fact, since $\chi(\mathcal{F}(t))\geqq 0\quad \forall t\geqslant -1$, we have $h^1(\mathcal{F}(t))=0\ \forall t$, and we have an exact sequence

$$0\to \mathcal{O}_{\mathbf{P}^3}(-p-3)\to \mathcal{F}\to \mathcal{I}_Y(p+3)\to 0$$

with

$$\deg Y=c_2+(p+3)^2$$

$$p_a(Y)=\frac{1}{2}[c_3-(4-2(p+3))(c_2+(p+3)^2)+2].$$

Moreover,

$$h^0(\mathcal{F}(p+1))\neq 0 \quad\Rightarrow\quad h^0(\mathcal{I}_Y(2p+4))\neq 0$$

$$h^0(\mathcal{F}(p))=0 \quad\Rightarrow\quad h^0(\mathcal{I}_Y(2p+3))=0.$$

So, $s=2p+4$, where s is, following [GP], the minimal degree of a surface containing Y.

Moreover,

$$H^0(\mathcal{F}(p+1))\neq 0 \quad\Rightarrow\quad H^2(\mathcal{I}_Y(2p+4))=0 \quad\Rightarrow\quad H^1(\mathcal{O}_Y(2p+4))=0.$$

But we have an exact sequence

$$(*)\quad 0\to H^1(\mathcal{O}_Y(2p+4))\to \oplus_{i=0}^{2p+3} H^2(\mathcal{O}_{\mathbf{P}^2}(2p+4-n_i))$$
$$\to \oplus_{i=0}^{2p+3} H^2(\mathcal{O}_{\mathbf{P}^2}(2p+4-i)).$$

Hence $H^2(\mathcal{O}_{\mathbf{P}^2}(2p+4-n_i))=0\ \forall i$, and then $2p+4-n_i\geqslant -2$, that is $n_i\leqslant 2p+6\ \forall i$. Thus $s_t=0$ for $t>2p+6$.

Since $h^0(\mathcal{F}(p))=0$ and $\chi(\mathcal{F}(p))\geqslant 0$, we have

$$\chi(\mathcal{F}(p))=h^2(\mathcal{F}(p))=h^2(\mathcal{I}_Y(2p+3))=h^1(\mathcal{O}_Y(2p+3)).$$

As before, the sequence $(*)$ gives

$$h^1(\mathcal{O}_Y(2p+3))=\sum h^2(\mathcal{O}_{\mathbf{P}^2}(2p+3-n_i))=s_{2p+6}.$$

In the same way we get

$$\chi(\mathcal{F}(p-1))=h^2(\mathcal{F}(p-1))=h^2(\mathcal{I}_Y(2p+2))-1$$
$$=h^1(\mathcal{O}_Y(2p+2))-1=3s_{2p+6}+s_{2p+5}-1,$$

and hence

$$s_{2p+5}=\chi(\mathcal{F}(p-1))-3\chi(\mathcal{F}(p))+1.$$

$$\chi(\mathcal{F}(p-2))=h^2(\mathcal{F}(p-2))=h^2(\mathcal{I}_Y(2p+1))-4$$
$$=h^1(\mathcal{O}_Y(2p+1))-4=6s_{2p+6}+3s_{2p+5}+s_{2p+4}-4,$$

and hence

$$s_{2p+4}=\chi(\mathcal{F}(p-2))-3\chi(\mathcal{F}(p-1))+3\chi(\mathcal{F}(p))+1.$$

An analogous result holds if $c_1=-1$.

PROPOSITION 2.7. — *Let $\mathcal{F}$ be a rank two reflexive sheaf on $\mathbf{P}^3$ with seminatural cohomology and Chern classes $(-1, c_2, c_3)$ such that $\chi(\mathcal{F}(t))\geqslant 0\ \forall t\geqslant -1$. Let $p=\max\{t \mid h^0(\mathcal{F}(t))=0\}$, let f be a general section of $\mathcal{F}(p+3)$, and let Y be the zero set of f. Then the numerical character of Y is given by*

$$s_{2p+5}=\chi(\mathcal{F}(p))$$
$$s_{2p+4}=\chi(\mathcal{F}(p-1))-3\chi(\mathcal{F}(p))+1$$
$$s_{2p+3}=\chi(\mathcal{F}(p-2))-3(\chi(\mathcal{F}(p-1))-\chi(\mathcal{F}(p)))+1$$
$$s_t=0 \qquad \textit{otherwise.}$$

Proof. — The proof is exactly as in 2.5; we only have to consider the sequence

$$0\to\mathcal{O}_{\mathbf{P}^3}(-p-3)\to\mathcal{F}\to\mathcal{I}_Y(p+2)\to 0.$$

LEMMA 2.8. — *Let $\mathcal{F}$ be a coherent sheaf with $c_1=0$, and let $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$. Then $\chi(\mathcal{F}(t))=\min\{\chi(\mathcal{F}(p))|p\geqslant -2\}$.*

Proof. — Let

$$f(x)=\frac{1}{2}c_3-(x+2)c_2+\frac{1}{2}(x+1)(x+2)(x+3), \qquad x\in\mathbf{R}.$$

Then $f'(x)=x^2+4x+(11/3)-c_2$, and $f'(x)=0$ if $x=-2\pm(c_2+1/3)^{1/2}$. So the relative maximum of $f(x)$ is obtained for some $x<-2$. In order to prove our lemma, it is sufficient to prove that $\chi(\mathscr{F}(t))\leqslant\chi(\mathscr{F}(t-1))$ and $\chi(\mathscr{F}(t))\leqslant\chi(\mathscr{F}(t+1))$. But it is now easy to check that

$$\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t))=c_2-(t+1)(t+2)\geqslant 0$$

and

$$\chi(\mathscr{F}(t+1))-\chi(\mathscr{F}(t))=(t+2)(t+3)-c_2>0.$$

LEMMA 2.9. — *Let $\mathscr{F}$ be a coherent sheaf with $c_1=-1$, and let $(t+1)^2\leqslant c_2<(t+2)^2$. Then $\chi(\mathscr{F}(t))=\min\{\chi(\mathscr{F}(p))\,|\,p\geqslant -1\}$.*

Proof. — Now $\chi(\mathscr{F}(p))-\chi(\mathscr{F}(p-1))=(p+1)^2-c_2$, and the proof is the same as for 2.8.

PROPOSITION 2.10. — *Let $\mathscr{F}$ be a rank-two curvilinear sheaf on $\mathbf{P}^3$ with seminatural cohomology and Chern classes $(0, c_2, c_3)$, $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$.*

Then the following conditions are equivalent:

(*a*) $\chi(\mathscr{F}(n))\geqslant 0$, $\forall n\geqslant -2$;

(*b*) $c_3\geqslant(2t+4)c_2-(2/3)(t+1)(t+2)(t+3)$;

(*c*) *$\mathscr{F}$ is arithmetically normal.*

Proof. — $(a)\Leftrightarrow(b)$. Thanks to lemma 2.8, $\chi(\mathscr{F}(n))\geqslant 0$, $\forall n\geqslant -2$, if and only if $\chi(\mathscr{F}(t))\geqslant 0$; but this is equivalent to (b) thanks to Riemann-Roch. $(a)\Leftrightarrow(c)$. If $\mathscr{F}$ is arithmetically normal, then it has no h^1; therefore the Euler characteristic of $\mathscr{F}(n)$, $n\geqslant -2$, is always equal to $h^0(\mathscr{F}(n))$ or $h^2(\mathscr{F}(n))$ (there is no h^3 thanks to Serre duality), that is it is non negative. Conservely, if $\mathscr{F}$ has seminatural cohomology and $\chi(\mathscr{F}(n))\geqslant 0$, $\forall n\geqslant -2$, then $h^1(\mathscr{F}(n))=0$, $\forall n\geqslant -2$; but this implies that the spectrum of $\mathscr{F}$ is strictly negative; therefore $h^1(\mathscr{F}(n))=0$, $\forall n$.

PROPOSITION 2.11. — *Let $\mathscr{F}$ be a rank-two curvilinear sheaf on $\mathbf{P}^3$ with seminatural cohomology and Chern classes $(-1, c_2, c_3)$, $(t+1)^2\leqslant c_2<(t+2)^2$.*

Then the following conditions are equivalent:

(*a*) $\chi(\mathscr{F}(n))\geqslant 0$, $\forall n\geqslant -1$;

(*b*) $c_3\geqslant(2t+3)c_2-(1/3)(t+1)(t+2)(2t+3)$;

(*c*) *$\mathscr{F}$ is arithmetically normal.*

Proof. — Almost as above.

Remark 2.12. — Remark that propositions 2.5 and 2.7 impose strong conditions on an arithmetically normal curve Y which is a section of a sheaf with seminatural cohomology. In particular we must have $e(Y)<s(Y)$, where $e(Y)=\max\{t \mid H^1(Y, \mathcal{O}_Y(t))\neq 0\}$ and

$s(Y)=\min\{t \mid H^0(\mathbb{P}^3, \mathcal{I}_Y(t))\neq 0\}$.

NOTATION 2.13. — If $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$, we put

$$m(0, c_2)=(2t+4)c_2-(2/3)(t+1)(t+2)(t+3);$$

if $(t+1)^2\leqslant c_2<(t+2)^2$, we put

$$m(-1, c_2)=(2t+3)c_2-(1/3)(t+1)(t+2)(2t+3).$$

Remark 2.14. — The asymptotical behaviour of m is $(4/3)c_2^{3/2}$.

3. Non existence of sheaves with seminatural cohomology and large c_3

Now we want to give bounds for the Chern classes of a reflexive wheaf, in order to determine the existence of the non-existence of a sheaf with those classes and seminatural cohomology. When we express c_3 in terms of c_2, the bounds that we find are not defined by polynomials, but by "piecewise linear" functions, whose asymptotical behaviour is easy to compute. The edges of these piecewise linear curves are usually in the points corresponding to the values $c_2=t^2+t$ (if $c_1=0$) or $c_2=t^2$ (if $c_1=-1$), where t is a nonnegative integer.

PROPOSITION 3.1. — *Let $\mathcal{F}$ be a rank-two curvilinear reflexive sheaf on $\mathbb{P}^3$ with Chern classes $(0, c_2, c_3)$, and let $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$ $(t\geqslant 1)$. If $\mathcal{F}$ has seminatural cohomology, then either* (a) $h^0(\mathcal{F}(t))=0$, *and* $c_3\leqslant(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)$; *or* (b) $h^0(\mathcal{F}(t))\neq 0$, *and either*

$$(2t+4)c_2-(2/3)t(t^2+6t+11)-2\leqslant c_3\leqslant(2t+3)c_2-(1/3)t(t+1)(2t+7),$$

or $c_2=t^2+5t+3$, $c_3=(2t+4)c_2-(1/3)t(t+1)(2t+7)+1$.

Proof. — (a) Let us suppose $c_3>(2t+5)c_2-(1/3)(t+2)(2t+9)$. First of all, we prove that $\chi(\mathcal{F}(n))>0$, $\forall n>-2$. Since

$$\chi(\mathcal{F}(t))=\min\{\chi(\mathcal{F}(n))\,|\,n\geqslant -2\},$$

it is enough to check this for $\chi(\mathcal{F}(t))$; that is to check that

$$c_3>(2t+4)c_2-(2/3)(t+1)(t+2)(t+3).$$

But

$$c_3>(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9),$$

and

$$(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)-(2t+4)c_2+ \\ +(2/3)(t+1)(t+2)(t+3)=c_2-(t+1)(t+2)\geqslant 0.$$

So the cohomology of $\mathcal{F}$ is

	-2	-1		$\dots$	$t-1$	t	$t+1$
h^0	0	0	0	$\dots$	0	0	*
h^1	0	0	0	$\dots$	0	0	0
h^2	*	*	*	$\dots$	*	*	0
h^3	0	0	0	$\dots$	0	0	0

since

$$\left.\begin{matrix}\chi(\mathcal{F}(t))>0\\ h^0(\mathcal{F}(t))=0\end{matrix}\right\}\Rightarrow h^2(\mathcal{F}(t))\neq 0,$$

and

$$\chi(\mathcal{F}(t+1))>\chi(\mathcal{F}(t))\Rightarrow h^2(\mathcal{F}(t+1))=0\Rightarrow h^0(\mathcal{F}(t+1))\neq 0.$$

Take a general section of $\mathcal{F}(t+3)$ (which is globally generated); its zero set Y is a smooth curve with $h^1(\mathcal{I}_Y)=0$ [since $h^1(\mathcal{F}(-t-3))=0$]; therefore Y is connected. With the notations of prop. 2.5, $t=p$ and $s_{2t+6}=\chi(\mathcal{F}(t))>0$. Moreover,

$$s_{2t+5}=\chi(\mathcal{F}(t-1))-3\chi(\mathcal{F}(t))+1 \\ =(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)+1-c_3\leqslant 0$$

by hypothesis. Since it must be non negative, we get $s_{2t+5}=0$. But this implies $s_{2t+4}=0$ too, since $s_{2t+6}>0$ and the numerical character must be without gaps (Y is integral). But

$$s_{2t+4}=\chi(\mathscr{F}(t-2))-3(\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t)))+1$$
$$=\frac{1}{2}c_3-(t+3)c_2+(1/3)(t+1)(t^2+8t+18)+1.$$

We claim that this is strictly positive. In fact,

$$c_3>(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)$$
$$\geqslant(2t+6)c_2-(2/3)(t+1)(t^2+8t+18)-2,$$

since

$$(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)-(2t+6)c_2$$
$$+(2/3)(t+1)(t^2+8t+18)+2=t^2+7t+8-c_2>0.$$

This is a contradiction.

(*b*) Since $0<h^0(\mathscr{F}(t))=\chi(\mathscr{F}(t))$, we have

$$\frac{1}{2}c_3\geqslant(t+2)c_2-(1/3)(t+1)(t+2)(t+3)+1,$$

that is $c_3\geqslant(2t+4)c_2-(2/3)t(t^2+6t+11)-2$.

So the cohomology of $\mathscr{F}$ is

	-2	-1		$\dots$	$t-1$	t	$t+1$
h^0	0	0	0	$\dots$	0	*	*
h^1	0	0	0	$\dots$	0	0	0
h^2	*	*	*	$\dots$	*	0	0
h^3	0	0	0	$\dots$	0	0	0

Let us suppose $c_3>(2t+3)c_2-(1/3)t(t+1)(2t+7)$.

Take a section of $\mathscr{F}(t+2)$ whose zero set is a smooth curve Y. With the notations of proposition 2.5, $p=t-1$.

Since $\chi(\mathscr{F}(t-1))\geqslant\chi(\mathscr{F}(t))>0$, we have $s_{2t+4}>0$.

Here $c_3>(2t+3)c_2-(1/3)t(t+1)(2t+7)$ implies $s_{2t+3}\leqslant 0$, and then actually $s_{2t+3}=0$ and $c_3=(2t+3)c_2-(1/3)t(t+1)(2t+7)+1$. So we must have $s_{2t+2}=0$, that is

$$0=\chi(\mathscr{F}(t-3))-3(\chi(\mathscr{F}(t-2))-\chi(\mathscr{F}(t-1)))+1=$$

$$= \frac{1}{2}c_3 - (t+2)c_2 + (1/3)t(t^2+6t+11)+1.$$

This leads to an equality

$$(2t+3)c_2 - (1/3)t(t+1)(2t+7)+1 = (2t+4)c_2 - (2/3)t(t^2+6t+11)-2$$

which gives

$$c_2 = t^2+5t+3$$
$$c_3 = (2t+3)c_2 - (1/3)t(t+1)(2t+7)+1.$$

This completes the proof.

We have a similar result if $c_1 = -1$.

PROPOSITION 3.2. — *Let $\mathscr{F}$ be a curvilinear reflexive sheaf with Chern classes $(-1, c_2, c_3)$ and let $(t+1)^2 \leqslant c_2 < (t+2)^2$ $(t \geqslant 1)$. If $\mathscr{F}$ has seminatural cohomology then either*

(a) $h^0(\mathscr{F}(t)) = 0$, and $c_3 \leqslant (2t+4)c_2 - (1/3)(t+1)(2t^2+10t+9)$;

or (b) $h^0(\mathscr{F}(t)) \neq 0$, and either

$$(2t+3)c_2 - (1/3)t(2t^2+9t+13) \leqslant c_3 \leqslant (2t+2)c_2 - (1/3)t(2t^2+6t+1),$$

or $c_2 = t^2+4t+1$, $c_3 = (2t+2)c_2 - (1/3)t(2t^2+6t+1)+1$.

Proof. — The proof of this proposition is almost identical to the proof of proposition 3.1; therefore it is omitted.

DEFINITION 3.3. — Let $t \geqslant 1$ be an integer and c_2 an integer such that $(t+2)^2 - 1 \leqslant c_2 < (t+3)^2 - 1$. We put

$$M(0, c_2) = (2t+5)c_2 - (1/3)(t+1)(t+2)(2t+9).$$

Let now c_2 be an integer such that $t^2+3t+(3/2) \leqslant c_2 < t^2+5t+(11/2)$. We put

$$M(-1, c_2) = (2t+4)c_2 - (1/3)(t+1)(2t^2+10t+9).$$

PROPOSITION 3.4. — *Let $t \geqslant 1$ be an integer and c_2 an integer such that $(t+1)(t+2) \leqslant c_2 < (t+2)(t+3)$. Then*

$$M(0, c_2) = \max\begin{cases} (2t+5)c_2 - (1/3)(t+1)(t+2)(2t+9) \\ (2t+3)c_2 - (1/3)t(t+1)(2t+7). \end{cases}$$

Proof. — It is enough to check that

$$(2t+3)c_2-(1/3)(t+1)(t+2)(2t+9)-(2t+5)c_2$$
$$+(1/3)(t+1)(t+2)(2t+9)=2(t^2+4t+3-c_2).$$

PROPOSITION 3.5. — *Let $t\geqslant 1$ be an integer and c_2 an integer such that $(t+1)^2\leqslant c_2<(t+2)^2$. Then*

$$M(-1, c_2)=\max\begin{cases}(2t+4)c_2-(1/3)(t+1)(2t^2+10t+9)\\ (2t+2)c_2-(1/3)t(2t^2+6t+1).\end{cases}$$

So we can summarize 3.1 and 3.2 with the following

COROLLARY 3.6. — *Let $\mathscr{F}$ be a normalized rank-two curvilinear reflexive sheaf on $\mathbf{P}^3$ with seminatural cohomology. Then*

$$c_3(\mathscr{F})\leqslant M(c_1(\mathscr{F}), c_2(\mathscr{F})).$$

Remark 3.7. — The asymptotical behaviour of this bound is $c_3\sim(4/3)c_2^{3/2}$.

4. Existence of reflexive sheaves with seminatural cohomology

In this chapter we construct arithmetically normal sheaves with seminatural cohomology, that is we work in the range where $\chi(\mathscr{F}(n))\geqslant 0$, $\forall n\geqslant -2$. We reverse the construction of propositions 3.1 and 3.2 using the fact that every numerical character without gaps is effective for some smooth curve Y.

The first result of this kind is the following one:

PROPOSITION 4.1. — *Let $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$, with $t\geqslant 1$, and let*

$$(2t+4)c_2-(2/3)t(t^2+6t+11)-2\leqslant c_3\leqslant(2t+3)c_2-(1/3)t(t+1)(2t+7),$$

or let $c_2=t^2+5t+3$, $c_3=(2t+3)c_2-(1/3)t(t+1)(2t+7)+1$,

c_3 even. Then there exists an arithmetically normal sheaf $\mathscr{F}$ with seminatural cohomology and Chern classes $(0, c_2, c_3)$, with $h^0(\mathscr{F}(t))\neq 0$.

Proof. — First observe that

$$(2t+3)c_2-(1/3)t(t+1)(2t+7)-(2t+4)c_2 \\ +(2/3)t(t^2+6t+11)+2=t^2+5t+2-c_2.$$

So, if $c_2 \leqslant (t+2)(t+3)-4$, there exist values of c_3 between

$$(2t+4)c_2+(2/3)t(t^2+6t+11)-2$$

and

$$(2t+3)c_2-(1/3)t(t+1)(2t+7).$$

Now,

$$(2t+4)c_2-(2/3)t(t^2+6t+11)-2>(2t+4)c_2-(2/3)(t+1)(t+2)(t+3);$$

so $c_3>(2t+4)c_2-(2/3)(t+1)(t+2)(t+3)$, and this is equivalent to say that $\chi(\mathscr{F}(t))>0$. Moreover,

$$\chi(\mathscr{F}(t-2))-3\chi(\mathscr{F}(t-1))=(2t+3)c_2-c_3-(1/3)t(t+1)(2t+7)\geqslant 0 \\ \Leftrightarrow \quad (2t+3)c_2-(1/3)t(t+1)(2t+7)\geqslant c_3;$$

and

$$\chi(\mathscr{F}(t-3))+1-3\chi(\mathscr{F}(t-2))+3\chi(\mathscr{F}(t-1)) \\ =\frac{1}{2}c_3-(t+2)c_2+(1/3)t(t^2+6t+11)+1\geqslant 0 \\ \Leftrightarrow \quad c_3\geqslant(2t+4)c_2-(2/3)t(t^2+6t+11)-2.$$

Let us consider now a smooth arithmetically normal curve Y with numerical character

$$\underbrace{2t+4, \ldots, 2t+4}_{s_{2t+4}\text{ times}}, \underbrace{2t+3, \ldots, 2t+3}_{s_{2t+3}\text{ times}}, \underbrace{2t+2, \ldots, 2t+2}_{s_{2t+2}\text{ times}}$$

where

$$s_{2t+4}=\chi(\mathscr{F}(t-1)) \\ s_{2t+3}=\chi(\mathscr{F}(t-2))-3\chi(\mathscr{F}(t-1))+1 \\ s_{2t+2}=\chi(\mathscr{F}(t-3))+1-3(\chi(\mathscr{F}(t-2))-\chi(\mathscr{F}(t-1))).$$

We have seen that

$$s_{2t+4} \geqslant 0 \qquad \text{(since } \chi(\mathcal{F}(t-1)) \geqslant \chi(\mathcal{F}(t)) \text{ as usual)}$$

$$s_{2t+3} \geqslant 1$$

$$s_{2t+2} \geqslant 0.$$

Moreover, $s_{2t+4}+s_{2t+3}+s_{2t+2}=2t+2$.

So this character is without gaps and $s=2t+2$ (with the notations of [GP]).

By means of easy but tedious computations we get the degree and the genus of the curve Y.

$$\operatorname{Deg}(Y)=s_{2t+4}(2t+4)+s_{2t+3}(2t+3)$$
$$+s_{2t+2}(2t+2)-\sum_{i=0}^{2t+1} i=c_2+(t+2)^2.$$

$$g(Y)=1+\frac{1}{2}[\sum_{i=0}^{s_{2t+4}-1}(2t+4-i)(2t+4+i-3)$$
$$+\sum_{i=s_{2t+4}}^{s_{2t+4}+s_{2t+3}-1}(2t+3-i)(2t+3+i-3)$$
$$+\sum_{i=s_{2t+4}+s_{2t+3}}^{2t+1}(2t+2-i)(2t+2+i-3)]=1+t^3+4t^2+4t+tc_2+\frac{1}{2}c_3.$$

We want to consider an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-t-2) \to \mathcal{F} \to \mathcal{I}_Y(t+2) \to 0.$$

This is possible if we can find a section f of $\omega_Y(-2t)$ which generates the sheaf $\omega_Y(-2t)$ except at finitely many points. But Y is smooth and connected; so it is enough to find a non trivial section. By Serre, $h^0(\omega_Y(-2t))=h^1(\mathcal{O}_Y(2t))$, and $h^1(\mathcal{O}_Y(2t))\neq 0$ since $s_{2t+3}\neq 0$ and there is an exact sequence

$$0 \to H^1(\mathcal{O}_Y(2t)) \to \oplus_{i=0}^{2t+1} H^2(\mathcal{O}_{\mathbf{P}^2}(2t-n_i)) \to 0.$$

It is easy to see that the Chern classes of F are

$$c_1(\mathcal{F})=0$$
$$c_2(\mathcal{F})=\deg Y-(t+2)^2=c_2$$
$$c_3(\mathcal{F})=2g(Y)-2t\deg Y-2=c_3.$$

At last, $\mathcal{F}$ has seminatural cohomology. In fact, Y is arithmetically normal, and so

$$h^1(\mathcal{F}(p))=h^1(\mathcal{I}_Y(p+t+2))=0, \qquad \forall p;$$
$$h^0(\mathcal{F}(p))=0 \qquad \text{if} \quad p\leqslant t-1$$

since

$$h^0(\mathcal{I}_Y(p+t+2))=0 \qquad \text{for} \quad p+t+2\leqslant 2t+1 \qquad (s=2t+2).$$

By Serre duality, this implies $h^3(\mathcal{F}(p))=0$ at least for $p\geqslant -2$.

Moreover, $h^2(\mathcal{F}(p))=0$ for $p\geqslant t$, since

$$0\to H^2(\mathcal{F}(p))\to H^2(\mathcal{I}_Y(p+t+2))$$

is exact, and $h^2(\mathcal{I}_Y(p+t+2))=h^1(\mathcal{O}_Y(p+t+2))=0$, $\forall p\geqslant t$. Therefore the cohomology of $\mathcal{F}$ is given by the diagram

	-2	-1	0	$\dots$	$t-2$	$t-1$	t	$t+1$
h^0	0	0	0	$\dots$	0	0	*	*
h^1	0	0	0	$\dots$	0	0	0	0
h^2	*	*	*	$\dots$	*	*	0	0
h^3	0	0	0	$\dots$	0	0	0	0

If f has only simple zeroes, $\mathcal{F}$ is curvilinear. But thanks to [GP], p. 41, Y can be chosen such that $\omega_Y(-e(Y))$ has a section without multiple points. In our case, $e(Y)=2t+1$ if $\chi(\mathcal{F}(t-1))>0$, and $e(Y)=2t$ otherwise.

If $c_2=t^2+5t+3$, $c_3=(2t+4)c_2-(1/3)t(t+1)(2t+7)+1$, take a smooth arithmetically normal curve Y with numerical character $s_{2t+4}=\chi(\mathcal{F}(t-1))$.

PROPOSITION 4.2. – *Let $(t+1)^2\leqslant c_2<(t+2)$, and let*

$$(2t+3)c_2-(1/3)t(2t^2+9t+13)\leqslant c_3\leqslant(2t+2)c_2-(1/3)t(2t^2+6t+1),$$

or let $c_2=t^2+4t+1$ *and* $c_3=(2t+2)c_2-(1/3)t(2t^2+6t+1)+1$, $c_3=c_2 \bmod 2$. *Then there exists an arithmetically normal sheaf with semi-natural cohomology, Chern classes $(-1, c_2, c_3)$ and $h^0(\mathcal{F}(t))\neq 0$.*

Proof (sketch). – Take an arithmetically normal smooth curve with numerical character

$$\underbrace{2t+3, \ldots, 2t+3}_{s_{2t+3}\ \text{times}},\ \underbrace{2t+2, \ldots, 2t+2}_{s_{2t+2}\ \text{times}},\ \underbrace{2t+1, \ldots, 2t+1}_{s_{2t+1}\ \text{times}}$$

where

$$s_{2t+3}=\chi(\mathcal{F}(t-1))$$
$$s_{2t+2}=\chi(\mathcal{F}(t-2))-3\chi(\mathcal{F}(t-1))+1$$
$$s_{2t+1}=\chi(\mathcal{F}(t-3))-3(\chi(\mathcal{F}(t-2))-\chi(\mathcal{F}(t-1)))+1$$

(here $s=2t+1$), and consider the sequence

$$0\to\mathcal{O}_{\mathbf{P}^3}(-t-2)\to\mathcal{F}\to\mathcal{I}_Y(t+1)\to 0.$$

Then $\mathcal{F}$ has Chern classes $(-1, c_2, c_3)$ and its cohomology is given by the diagram

	-1	0	1	$\dots$	$t-2$	$t-1$	t	$t+1$
h^0	0	0	0	$\dots$	0	0	*	*
h^1	0	0	0	$\dots$	0	0	0	0
h^2	*	*	*	$\dots$	*	*	0	0
h^3	0	0	0	$\dots$	0	0	0	0

In the same way we get another result:

PROPOSITION 4.3. — *Let* $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$, *and let*

$$(2t+4)c_2-(2/3)(t+1)(t+2)(t+3)\leqslant c_3 \leqslant(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9),$$

c_3 *even. Then there exists an arithmetically normal sheaf with seminatural cohomology, Chern classes* $(0, c_2, c_3)$ *and* $h^0(\mathcal{F}(t))=0$.

Proof. — First observe that

$$(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)-(2t+4)c_2 +(2/3)(t+1)(t+2)(t+3)=c_2-(t+1)(t+2)\geqslant 0.$$

Moreover,

$$c_3\geqslant(2t+4)c_2-(2/3)(t+1)(t+2)(t+3) \quad\Leftrightarrow\quad \chi(\mathcal{F}(t))\geqslant 0,$$

and

$$\chi(\mathcal{F}(t-1))-3\chi(\mathcal{F}(t))+1\geqslant 1$$

$$\Leftrightarrow \quad -c_3+(2t+5)c_2+(1/3)(t+1)(t+2)(2t+9)\geqslant 0$$
$$\Leftrightarrow \quad c_3\leqslant(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9).$$

At last,

$$\chi(\mathscr{F}(t-2))+1-3(\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t)))$$
$$=\frac{1}{2}c_3-(t+3)c_2+(1/3)(t+1)(t^2+8t+18)+1,$$

and then

$$\chi(\mathscr{F}(t-2))+1-3(\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t)))\geqslant 0$$
$$\Leftrightarrow \quad c_3\leqslant 2(t+3)c_2-(2/3)(t+1)(t^2+8t+18)-2.$$

But

$$(2t+4)c_2-(2/3)(t+1)(t+2)(t+3)$$
$$>2(t+3)c_2-(2/3)(t+1)(t^2+8t+18)-2.$$

In fact,

$$(2t+4)c_2-(2/3)(t+1)(t+2)(t+3)-(2t+6)c_2$$
$$+(2/3)(t+1)(t^2+8t+18)-2=2(t^2+5t+6-c_2)>0$$

since $c_2<t^2+5t+6$.

So we put

$$s_{2t+6}=\chi(\mathscr{F}(t))$$
$$s_{2t+5}=\chi(\mathscr{F}(t-1))-3(\chi(\mathscr{F}(t)))+1$$
$$s_{2t+4}=\chi(\mathscr{F}(t-2))+1-3(\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t)))$$

and we choose a smooth arithmetically normal curve with numerical character (we have verified that it is without gaps)

$$\underbrace{2t+6,\ \ldots,\ 2t+6}_{s_{2t+6}\text{ times}},\ \underbrace{2t+5,\ \ldots,\ 2t+5}_{s_{2t+5}\text{ times}},\ \underbrace{2t+4,\ \ldots,\ 2t+4}_{s_{2t+4}\text{ times}}$$

As before,

$$s=s_{2t+6}+s_{2t+5}+s_{2t+4}=2t+4,$$

$$\deg Y = c_2 + (t+3)^2$$

$$g(Y) = 1 + (t+1)^2 + 4(t+1)^2 + (t+1)c_2 + 4(t+1) + \frac{1}{2}c_3.$$

We consider the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-t-3) \to \mathcal{F} \to \mathcal{I}_Y(t+3) \to 0$$

(as before, this is possible) and we get a reflexive sheaf with Chern classes $(0, c_2, c_3)$, which is curvilinear if we suitably choose the section of the twist of ω_Y.

$\mathcal{F}$ has seminatural cohomology:

$$h^1(\mathcal{F}(p)) = h^3(\mathcal{F}(p)) = 0, \qquad \forall p \geq -2$$
$$H^0(\mathcal{F}(p)) = 0 \qquad \text{if} \quad p \leq t$$
$$H^2(\mathcal{F}(p)) = 0 \qquad \text{if} \quad p \geq t+1.$$

Therefore the cohomology of $\mathcal{F}$ is given by the diagram

	-2	-1	0	$\dots$	$t-2$	$t-1$	t	$t+1$
h^0	0	0	0	...	0	0	0	*
h^1	0	0	0	...	0	0	0	0
h^2	*	*	*	...	*	*	*	0
h^3	0	0	0	...	0	0	0	0

If $c_1 = -1$ we have the following

PROPOSITION 4.4. — *Let $(t+1)^2 \leq c_2 < (t+2)^2$ and let*

$$(2t+3)c_2 - (1/3)(t+1)(t+2)(2t+3) \leq c_3$$
$$\leq (2t+4)c_2 - (1/3)(t+1)(2t^2+10t+9), \qquad c_3 = c_2 \bmod 2.$$

Then there exists an arithmetically normal sheaf with seminatural cohomology, Chern classes $(-1, c_2, c_3)$ and $h^0(\mathcal{F}(t)) = 0$.

Proof (sketch). — Take a smooth arithmetically normal curve with numerical character

$$\underbrace{2t+5, \dots, 2t+5}_{s_{2t+5}\ \text{times}}, \underbrace{2t+4, \dots, 2t+4}_{s_{2t+4}\ \text{times}}, \underbrace{2t+3, \dots, 2t+3}_{s_{2t+3}\ \text{times}}$$

where

$$s_{2t+5}=\chi(\mathscr{F}(t))$$
$$s_{2t+4}=\chi(\mathscr{F}(t-1))-3\chi(\mathscr{F}(t))+1$$
$$s_{2t+3}=\chi(\mathscr{F}(t-2))-3(\chi(\mathscr{F}(t-1))-\chi(\mathscr{F}(t)))+1$$

and consider the sequence

$$0\to \mathcal{O}_{\mathbf{P}^3}(-t-3)\to \mathscr{F}\to \mathscr{I}_Y(t+2)\to 0.$$

$\mathscr{F}$ has Chern classes $(-1, c_2, c_3)$ and seminatural cohomology, given by

	-1	0	1	$\dots$	$t-1$	t		
h^0	0	0	0	$\dots$	0	0	$*$	$*$
h^1	0	0	0	$\dots$	0	0	0	0
h^2	$*$	$*$	$*$	$\dots$	$*$	$*$	0	0
h^3	0	0	0	$\dots$	0	0	0	0

We can summarize our results in the following

COROLLARY 4.5. — *Let* $c_1=0, -1$, *and* $m(c_1, c_2)\leqslant c_3\leqslant M(c_1, c_2)$, $c_3=c_1c_2 \bmod 2$. *Then there exists an arithmetically normal sheaf with Chern classes* (c_1, c_2, c_3) *and seminatural cohomology.*

Remark 4.6. — The interval $[m(0, c_2), M(0, c_2)]$ has a size, in the range $(t+1)(t+2)\leqslant c_2<(t+2)(t+3)$, decreasing from $2t+2$ to $t+1$, and then increasing to $2t+3$. The case $c_1=-1$ has a similar behaviour.

Remark 4.7. — We remark that there is here a difference between reflexive and locally free sheaves with (semi)-natural cohomology. In [HH1] Hartshorne and Hirschowitz proved that if a locally free sheaf $\mathscr{E}$ has seminatural cohomology, then all the numbers $h^i(\mathscr{E}(n))$ are uniquely determined by the Chern classes of $\mathscr{E}$.

On the contrary, we get the following results:

THEOREM 4.8. — *If* $\mathscr{F}$ *and* $\mathscr{G}$ *are curvilinear sheaves with seminatural cohomology, Chern classes* $(0, c_2, c_3)$ *and there exist* i, $t\geqslant -2$ *such that* $h^i(\mathscr{F}(t))\neq h^i(\mathscr{G}(t))$, *then both are arithmetically normal. The same is true if* $c_1=-1$.

Proof. — If there exists $n \geqslant -2-c_1$ such that $\chi(\mathcal{F}(n)) \leqslant 0$, then the dimensions of all the groups $H^i(\mathcal{F}(p))$ are uniquely determined, if F has seminatural cohomology.

In fact

$$h^3(\mathcal{F}(p))=0, \qquad \forall p \geqslant -2-c_1 \quad \text{(by duality);}$$
$$h^1(\mathcal{F}(p))=\chi(\mathcal{F}(p))$$

in the interval of integers $[a, b]$ where $\chi(\mathcal{F}(p)) \leqslant 0$; (by seminatural cohomology);

$$h^2(\mathcal{F}(p))=\chi(\mathcal{F}(p)) \quad \text{if} \quad p<a$$
$$h^2(\mathcal{F}(p))=0 \quad \text{if} \quad p \geqslant a$$
$$h^0(\mathcal{F}(p))=0 \quad \text{if} \quad p \leqslant b$$
$$h^0(\mathcal{F}(p))=\chi(\mathcal{F}(p)) \quad \text{if} \quad p>b.$$

($h^2(\mathcal{F}(p))$ is a decreasing function of p, if $p \geqslant -2$).

So $\chi(\mathcal{F}(n))=\chi(\mathcal{G}(n))>0$, $\forall n \geqslant -2$, and $\mathcal{F}$ and $\mathcal{G}$ are arithmetically normal. The same proof holds if $c_1=-1$.

THEOREM 4.9. — *Let $t \geqslant 1$, c_1, c_2, c_3 be integers, $c_3=c_1c_2 \bmod 2$.*

If $c_1=0$, $(t+1)(t+2) \leqslant c_2 < (t+2)(t+3)$, and

$$(2t+4)c_2-(2/3)t(t^2+6t+11)-2 \leqslant c_3 \leqslant \min\begin{cases}(2t+5)c_2-(1/3)(t+1)(t+2)(2t+9)\\(2t+3)c_2-(1/3)t(t+1)(2t+7)\end{cases}$$

or if $c_2=t^2+5t+3$, $c_3=(2t+3)c_2-(1/3)t(t+1)(2t+7)+1$, then there exist two different kinds of seminatural cohomology for reflexive sheaves with Chern classes $(0, c_2, c_3)$. As a consequence, the corresponding variety of moduli is reducible.

The same is true if $c_1=-1$, $(t+1)^2 \leqslant c_2 < (t+2)$ and

$$(2t+3)c_2-(1/3)t(2t^2+9t+13) \leqslant c_3 \leqslant \min\begin{cases}(2t+4)c_2-(1/3)(t+1)(2t^2+10t+9)\\(2t+2)c_2-(1/3)t(2t^2+6t+1)\end{cases}$$

or if $c_2=t^2+4t+1$, $c_3=(2t+2)c_2-(1/3)t(2t^2+6t+1)+1$.

Proof. — The proof follows from 4.1, 4.2, 4.3 and 4.4.

COROLLARY 4.10. — *These are the only cases for which this phenomenon of "double seminatural cohomology" can genuinely happen.*

Proof. — The proof follows from 4.8, 3.1 and 3.2.

Remark 4.11. — We have given a bound, in term of c_2 and c_3, which is the best possible one, at least for c_3 large. But we have supposed $\mathcal{F}$ curvilinear, since we must make sure that the zero set of a general section of $\mathcal{F}(s)$ is irreducible. We can find another bound, strictly larger, without any extra assumption. In fact, by using only the properties of the spectrum of a reflexive sheaf, we can prove the following

PROPOSITION 4.11.1. — *Let $t \geqslant 1$ be an integer, and let*

$$(t+1)(t+2)-1 \leqslant c_2 < (t+2)(t+3)-1.$$

Let $\mathcal{F}$ be a reflexive sheaf with Chern classes $(0, c_2, c_3)$ and seminatural cohomology. Then

$$c_3 < (2t+4)c_2 - (2/3)t(t+1)(t+5) - 2.$$

PROPOSITION 4.11.2. — *Let $c_2 \geqslant 4$, and let $(t+1)^2 - 1 \leqslant c_2 < (t+2)^2 - 1$. Let $\mathcal{F}$ be a reflexive sheaf with Chern classes $(-1, c_2, c_3)$ and seminatural cohomology. Then*

$$c_3 < (2t+3)c_2 - (1/3)t(2t^2+9t+1) - 2.$$

This will be done in a forthcoming paper.

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