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WEIGHTED ESTIMATES FOR DIFFERENTIAL OPERATORS WITH OPERATOR-VALUED COEFFICIENTS

BY

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RÉSUMÉ. — On étudie des opérateurs différentiels de la forme $L=-(d/dt)M(t)d/dt+S(t)d/dt+T(t)$, où $M(t)$, $S(t)$ et $T(t)$ sont des opérateurs linéaires dans un espace de Hilbert H et $t\in I=(a,+\infty)\subset \mathbf{R}$. On démontre des inégalités du type $\|\exp(\varphi)u\|+\|d/dt(\exp(\varphi)u)\|\leqq c\|t(1+t\varphi')^{-1/2}\exp(\varphi)Lu\|$, où les normes sont dans $L^2(I;H)$ et $\varphi: I\to\mathbf{R}$ est croissante.

ABSTRACT. — We consider differential operators of the form $L=-(d/dt)M(t)d/dt+S(t)d/dt+T(t)$, where $M(t)$, $S(t)$ and $T(t)$ are linear operators in some Hilbert space H and $t\in I=(a,+\infty)\subset\mathbf{R}$. We prove weighted estimates of the type $\|\exp(\varphi)u\|+\|d/dt(\exp(\varphi)u)\|\leqslant c\|t(1+t\varphi')^{-1/2}\exp(\varphi)Lu\|$, where the norms are in $L^2(I;H)$ and $\varphi: I\to\mathbf{R}$ is increasing.

1. Introduction and abstract framework

Let L be a second order ordinary differential operator the coefficients of which are linear operators acting in a Hilbert space H. We are concerned here with the problem of getting information about the asymptotic behaviour of functions $u: I\equiv(a,\infty)\to H$ from that of Lu. More

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precisely we shall prove weighted estimates in the Hilbert space $L^2(I;\, H;\, dt)$ of the form

$$(1) \qquad \|e^{\varphi} u\| + \left\| \frac{d}{dt}(e^{\varphi} u) \right\| \leqslant c \, \| t (1 + t \varphi')^{-1/2} e^{\varphi} L u \|,$$

where $\varphi : I \to \mathbb{R}$ belongs to a class of increasing weight functions and u does not grow at infinity (in a sense to be specified).

Inequalities of the form (1) for simple first order differential operators have been known for a long time, viz. Hardy's inequality ([6], Theorem 330) and its generalizations. More recently AGMON [1] proved very precise weighted estimates in $L^2(\mathbb{R}^n)$ for second order elliptic partial differential operators for the case when $z=0$ does not belong to their essential spectrum. One of the motivations for our present work was to obtain similar inequalities for second order partial differential operators containing 0 in their essential spectrum. We shall indicate in the last part of this paper how our abstract formalism applies to such operators; only a simple situation will be considered, the applications to elliptic operators with coefficients that may be singular both locally and at infinity will be the topic of a separate publication. The body of the present paper consists of a derivation of inequalities of the type (1) in the more general framework of ordinary differential operators with operator-valued coefficients; we mention that this analysis can be pushed further and then leads to considerably more general and detailed estimates [2].

We fix a number $a \geqslant 1$ and denote by I the open interval $(a, +\infty)$. We denote by $|\,.\,|_H$ and $(.\,,.)_H$ the norm and the scalar product respectively in the complex Hilbert space H, and for $u, v \in \mathscr{H} \equiv L^2(I;\, H)$ we set

$$(u, v) = \int_I (u(t), v(t))_H \, dt$$

and $\|u\| = [(u, u)]^{1/2}$. We set $P = -i\,d/dt$ (the derivatives are always in the sense of distributions) and denote by $\mathscr{H}^m \equiv \mathscr{H}^m(I;\, H)$ the usual Sobolev space of order m $(m = 0, 1, 2, \ldots)$ of H-valued functions on I and by $\mathscr{H}^m_c$ the set of all functions $u \in \mathscr{H}^m$ with compact support in I. $\mathscr{H}^m$ is just the set of functions u in $L^2(I;\, H)$ such that $P^m u \in L^2(I;\, H)$, provided with the norm

$$\|u\|_{\mathscr{H}^m} = (\|u\|^2 + \|P^m u\|^2)^{1/2}$$

We say that a linear subspace $\mathscr{F}$ of $\mathscr{H}$ is *k-semilocal* ($k=0, 1, 2, \ldots$) if $\eta f \in \mathscr{F}$ for each $f \in \mathscr{F}$ and each $\eta \in C_0^k(I)$ (the set of all complex-valued functions of class C^k on I having compact support in I), and we then set

$$\mathscr{F}_c = \{ f \in \mathscr{F} \mid \text{supp } f \text{ is compact in } I \}$$

and

$$\mathscr{F}_{\text{loc}} = \{ f \in L^2_{\text{loc}}(I;\, H) \mid \eta f \in \mathscr{F} \text{ for each } \eta \in C_0^k(I) \}.$$

Clearly $\mathscr{F}_c$ and $\mathscr{F}_{\text{loc}}$ are stable under multiplication by functions from $C^k(I)$. We say that a mapping $T\colon \mathscr{F} \to \mathscr{H}$ is *k-ultralocal* if it is linear and $T(\eta f) = \eta T(f)$ for all $f \in \mathscr{F}$ and $\eta \in C_0^k(I)$. Then $T(\mathscr{F}_c) \subset \mathscr{H}_c$ and there is a unique mapping $\tilde{T}\colon \mathscr{F}_{\text{loc}} \to \mathscr{H}_{\text{loc}}$ which coincides with T on $\mathscr{F}$ and satisfies $\tilde{T}(\eta f) = \eta \tilde{T}(f)$ for each $f \in \mathscr{F}_{\text{loc}}$ and each $\eta \in C^k(I)$. We shall denote this canonical extension $\tilde{T}$ by the same letter T.

If $\mathscr{G} \subset \mathscr{H}_c$ is a linear subspace, we define its *k-semilocal closure* to be the smallest k-semilocal subspace of $\mathscr{H}$ that contains $\mathscr{G}$. Clearly it is equal to the linear subspace generated by elements of the form ηf, with $\eta \in C_0^k(I)$ and $f \in \mathscr{G}$.

Remark A. — The most important examples of k-semilocal subspaces we have in mind are the Sobolev spaces of sections of direct integrals of Hilbert spaces continuously embedded in H. More precisely, for each $t \in I$ let there be given a dense subspace $K(t)$ of H, provided with a new Hilbert structure such that the inclusion mapping $i\colon K(t) \to H$ is continuous. We say that an element u of $K \equiv \prod_{t \in I} K(t)$ is Borel if it is Borel when considered with values in H. Let us assume that for each such Borel element the real-valued function $|u(t)|_{K(t)}$ is Borel on I. Let $\mathscr{F}$ be the space of all equivalence classes with respect to Lebesgue measure of Borel elements $u \in K$ such that $t \mapsto |u(t)|^2_{K(t)}$ is integrable on I and $u \in \mathscr{H}^k(I;\, H)$. Then $\mathscr{F}$ is our standard example of a k-semilocal space. It occurs in our applications to elliptic operators the coefficients of the principal part of which are only locally Lipschitz. The fact that $\mathscr{F}$ has a quite complicated structure will be unimportant in these applications; it will be easy to show (without considering explicitly the spaces $K(t)$) that $\mathscr{F}$ is k-semilocal.

From now on we assume that the following objects are given:

(i) A 2-semilocal subspace $\mathscr{S}$ of $\mathscr{H}$ such that $\mathscr{S} \subset \mathscr{H}_c^2$.

(*ii*) A function $M\colon I \to \mathscr{B}(H)$ which is locally Lipschitz (in other terms such that its derivative M' is a weakly measurable locally bounded function) and such that $M(t)$ is a positive symmetric operator in H for each $t \in I$.

(iii) A 2-ultralocal mapping $Q:\mathscr{D}\to\mathscr{H}$ which is symmetric as an operator in $\mathscr{H}$.

(iv) A 1-ultralocal mapping $S:\mathscr{D}_1\to\mathscr{H}$, where $\mathscr{D}_1$ is the 1-semilocal closure of the linear space $\mathscr{D}+P\,\mathscr{D}$.

(v) A 2-ultralocal mapping $R:\mathscr{D}\to\mathscr{H}$.

As said before, we denote also by Q, $R:\mathscr{D}_{\mathrm{loc}}\to\mathscr{H}_{\mathrm{loc}}$ and $S:\mathscr{D}_{1,\mathrm{loc}}\to\mathscr{H}_{\mathrm{loc}}$ the canonical extensions of Q, R and S respectively. We observe that $\mathscr{D}_{\mathrm{loc}}\subset\mathscr{H}^2_{\mathrm{loc}}$ and introduce the following two operators from $\mathscr{D}_{\mathrm{loc}}$ into $\mathscr{H}_{\mathrm{loc}}$:

$$L_0=PMP+Q,\qquad L=L_0+SP+R. \tag{2}$$

If $\varphi\in C^2(I)$, we can define $L(\varphi)\equiv e^{\varphi}Le^{-\varphi}:\mathscr{D}_{\mathrm{loc}}\to\mathscr{H}_{\mathrm{loc}}$ and similarly $L_0(\varphi)$. We observe that

$$L(\varphi)=L_0+SP+R+2i\varphi' MP+i\varphi' S+(M\varphi')'-M\varphi'^2. \tag{3}$$

For each $\kappa\in\mathbb{R}$ we set $P_\kappa=P-i\kappa(2t)^{-1}$ and define the following symmetric sesquilinear form Q_κ on $\mathscr{D}$:

$$(v,Q_\kappa v)=(1-\kappa)(v,Q\,v)+2\,\mathrm{Re}(v',t\,Q\,v), \tag{4}$$

where $v'=dv/dt$. Formally $Q_\kappa=-\kappa Q-tQ'$, where $Q'=i[P,Q]$ is in some sense the derivative of Q with respect to t. By some straightforward integrations by parts one can then prove the following identity:

LEMMA 1. — *Let $\alpha\in\mathbb{R}$, $\varphi:I\to\mathbb{R}$ of class C^3 and $v\in\mathscr{D}$. Then*

$$\begin{aligned}2\,\mathrm{Im}(P_{1-\alpha}v,t\,L(\varphi)v)&=(v,[(2-\alpha)PMP+Q_\alpha]v)\\&\quad+(Pv,[4t\varphi' M-tM']Pv)+2\,\mathrm{Im}(Pv,t\,M'\varphi' v)\\&\quad+\left(v,\left[M\left(\alpha+t\frac{d}{dt}\right)\varphi'^2+t\,M'\varphi'^2-(t\,M\varphi'')'\right]v\right)\\&\quad+2\,\mathrm{Im}(P_{1-\alpha}v,t[SP+R+i\varphi' S]v).\end{aligned} \tag{5}$$

Remark B. — Equation (5), which we call the fundamental identity in [2], is a generalization of an identity that we first saw in a paper by Eidus [3]. Identities of a similar nature appeared in several other articles (see e. g. [4], [5], [7] and [8]).

To end this section, we introduce some additional notations. We provide the space $\mathscr{D}$ with the norm

$$\|u\|_2=[\|u\|^2+\|Pu\|^2+\|L_0 u\|^2]^{1/2}. \tag{6}$$

A norm $\|\cdot\|_1$ on $\mathscr{D}_1$ will be called *admissible* if, for some finite constant C, all $u\in\mathscr{D}$ and all $\eta\in C^1(I)$, one has:

$$\|\eta u\|_1\leqslant C[\|u\|_2+\|\eta Pu\|+\|(|\eta|^2+|\eta'|)u\|]. \tag{7}$$

For example, $\|\cdot\|_{\mathscr{H}^1}$ is an admissible norm, but in the applications stronger norms are usually needed.

For any subspace $\mathscr{G}\subset\mathscr{H}_{loc}$ and any $r>a$, we denote by $\mathscr{G}(r)$ the set of all $u\in\mathscr{G}$ which are zero on (a, r).

2. Weighted estimates

This section contains the main result of our paper, namely an estimate of type (1). We shall consider the difference of L and L_0 as some kind of a perturbation of L_0 and consequently assume that this perturbation is "small at infinity" in a certain sense. Explicitly, we shall assume from now on (in addition to the hypotheses (i)-(v) already made) that the coefficients M, Q, S and R of L satisfy the following conditions:

(vi) $\lim_{t\to\infty}|M(t)-1|_{\mathscr{B}(H)}=\lim_{t\to\infty}t|M'(t)|_{\mathscr{B}(H)}=0$.

(vii) There are real constants α, β, γ with $\alpha<2$, $\beta>0$, $\gamma>0$ such that, as sesquilinear forms on $\mathscr{D}$:

$$(2-\alpha)P^2+Q_\alpha\geqslant\beta+\gamma P^2. \tag{8}$$

(viii) An admissible norm $\|\cdot\|_1$ is given on $\mathscr{D}_1$ such that for each $\nu>0$ there is a number $r(\nu)\in(a, \infty)$ such that for all $u\in\mathscr{D}_1(r(\nu))$ and all $v\in\mathscr{D}(r(\nu))$:

$$\|tSu\|\leqslant\nu\|u\|_1, \tag{9}$$

$$\|t(SP+R)v\|\leqslant\nu\|v\|_2. \tag{10}$$

From now on we shall fix and denote by $r=r(\nu)$ a function $r:(0, 1)\to(a+1, \infty)$ which is decreasing, satisfies $r(\nu)\geqslant\nu^{-2}$, is such that for $t \geqslant r(\nu)$:

$$(11) \qquad |M(t)-1|_{\mathscr{B}(H)} \leqslant \nu, \qquad t|M'(t)|_{\mathscr{B}(H)} \leqslant \nu,$$

and such that (9) and (10) hold if $u \in \mathscr{D}_1(r(\nu))$ and $v \in \mathscr{D}(r(\nu))$. We also set, for each weight function φ of class C^2:

$$(12) \qquad \xi \equiv \xi_{\alpha,\beta}(\varphi) = \left(\alpha + t\frac{d}{dt}\right)\varphi'^2 + \beta = \alpha\varphi'^2 + 2t\varphi'\varphi'' + \beta,$$

$$(13) \qquad \psi \equiv \psi(\varphi) = (1+t\varphi')^{-1/2}.$$

We shall first prove an auxiliary estimate (Lemma 2) that follows from (vi) and (viii) and then use (vi)-(viii) to estimate the various terms in Lemma 1, which will lead to an inequality of type (1) for functions u of compact support (Proposition 3). This result will then be extended to general u in Theorem 4.

LEMMA 2. — *For each $q \in [1, \infty)$ there are constants $r_0 = r_0(q) \in I$ and $c_0 = c_0(q) \in (0, \infty)$ such that for all $v \in \mathscr{D}(r_0)$ and all real $\varphi \in C^3(I)$ satisfying*

$$(14) \qquad t(|\varphi''(t)| + |\varphi'''(t)|) \leqslant q(1+\varphi'(t)) \quad \text{and} \quad 0 \leqslant \varphi'(t) \leqslant qt,$$

the following inequality is true:

$$(15) \qquad \|\psi v\|_2 + \|Sv\| \leqslant c_0[\|\psi L(\varphi)v\| + \|Pv\| + \|(1+\varphi')v\|].$$

Proof. — Assume $q \in [1, \infty)$ to be fixed.

(i) By using (6) and the identity (3) to express L_0, one finds that for all $u \in \mathscr{D}$:

$$(16) \qquad \|u\|_2 \leqslant \|u\| + \|Pu\| + \|[L(\varphi) - i\varphi' S]u\| + 2\|\varphi' MPu\| + \|[\varphi' M' + (\varphi'' - \varphi'^2)M]u\| + \|(SP+R)u\|.$$

Now assume that $u \in \mathscr{D}(r(\nu))$ with $\nu \in (0, 1)$. Then, since $r(\nu) > a+1 \geqslant 2$, we have by (10):

$$\|(SP+R)u\| \leqslant \frac{1}{2}\|t(SP+R)u\| \leqslant \frac{1}{2}\|u\|_2.$$

Upon inserting this into (16) and then using (11) and (14), one sees that there is a constant c_1 such that, for all φ satisfying (14):

$$(17) \qquad \|u\|_2 \leqslant c_1[\|(L(\varphi) - i\varphi' S)u\| + \|(1+\varphi')Pu\| + \|(1+\varphi')^2 u\|].$$

(ii) One has the following commutation relation:

$$[L(\varphi)-i\varphi' S]\psi=\psi L(\varphi)-i(\varphi'\psi+\psi')S - 2i\psi' MP-\psi' M'-(\psi''+2\psi'\varphi')M.$$

By noticing that $0\leqslant\psi\leqslant 1$, $\varphi'\psi\leqslant q$, $|\psi'|\leqslant q\psi\leqslant q$, $|\psi''|\leqslant 2q^2$ and by using (17) with $u=\psi v$, one finds that there is a constant c_2 such that for all $\nu\in(0, 1)$, all $v\in\mathcal{S}(r(\nu))$ and all φ satisfying (14):

$$(18)\quad \|\psi v\|_2\leqslant c_2[\|\psi L(\varphi)v\| + \|Sv\| + \|Pv\| + \|(1+\varphi')v\| + \|(1+\varphi')P\psi v\| + \|(1+\varphi')^2\psi v\|].$$

Now, since $\psi^{-1}\leqslant(1+qt^2)^{1/2}\leqslant(2q)^{1/2}t$, we have by (9) and (7):

$$(19)\quad \|Sv\| = \|\psi^{-1}S\psi v\| \leqslant (2q)^{1/2}\|tS\psi v\| \leqslant(2q)^{1/2}\nu\|\psi v\|_1 \leqslant(2q)^{1/2}\nu C[\|\psi v\|_2+ \|P\psi v\| + \|\psi v\|].$$

Also $(1+\varphi')^2\psi\leqslant(1+q)(1+\varphi')$ and

$$(20)\quad \|P\psi v\| \leqslant \|(1+\varphi')P\psi v\| \leqslant \|(1+\varphi')\psi Pv\| + \|(1+\varphi')\psi' v\| \leqslant(1+q)\|Pv\| + q\|(1+\varphi')v\|.$$

Upon inserting these inequalities into (18), one finds that there is a constant $c_3\in(1, \infty)$ such that for all ν, v and φ as before:

$$(21)\quad \|\psi v\|_2+\frac{1}{2}\|Sv\|_1\leqslant c_3[\|\psi L(\varphi)v\| + \nu\|\psi v\|_2+ \|Pv\| + \|(1+\varphi')v\|].$$

(15) now follows from (21) upon choosing $\nu=\nu_0=(2c_3)^{-1}$, $c_0=2c_3$ and $r_0=r(\nu_0)$. ■

PROPOSITION 3. — *For each $q\geqslant 1$ there are constants $\nu_0=\nu_0(q)\in(0, 1)$ and $c=c(q)\in(1, \infty)$ such that for all $\nu\in(0, \nu_0)$, all $v\in\mathcal{S}(r(\nu))$ and all real $\varphi\in C^3(I)$ satisfying (14), the following inequality is true:*

$$(22)\quad (v, [\Xi-\nu c(1+\varphi')^2-t|\varphi'''|]v)+(Pv, [t\varphi'+\gamma-\nu c]Pv) +\nu\|Sv\|^2\leqslant \nu^{-1}c\|t\psi L(\varphi)v\|^2.$$

Proof. — The inequality (22) can be obtained from Lemma 1 by suitably estimating the l. h. s. and each of the five terms on the r. h. s. of (5). We shall use several times the inequality $2|(f, g)| \leqslant \nu\|f\|^2+\nu^{-1}\|g\|^2$ which is valid for all $\nu>0$ and $f, g\in\mathscr{H}$. We assume $\nu\in(0, 1)$ and $v\in\mathscr{D}(r(\nu))$ for the moment, and we fix a number $q\geqslant 1$. We then have for the l. h. s. of (5):

$$(23)\quad 2\,\mathrm{Im}(P_{1-\alpha}v, t\,L(\varphi)v)\leqslant\nu\|\psi^{-1}P_{1-\alpha}v\|^2+\frac{1}{\nu}\|t\,\psi\,L(\varphi)v\|^2.$$

For the first four terms on the r. h. s. of (5) we use (8), (11), and (14) (for the third term notice that $\varphi''\leqslant[r(\nu)]^{-1}q(1+\varphi')\leqslant\nu q^2(1+\varphi')$ on $[r(\nu), \infty))$ and obtain that

$$(24)\quad (v, [(2-\alpha)PMP+Q_\alpha]v)\geqslant\beta\|v\|^2+[\gamma-\nu(2-\alpha)]\|Pv\|^2,$$

$$(25)\quad (Pv, [\psi t\varphi' M-tM']Pv)\geqslant 4(1-\nu)(Pv, t\varphi' Pv)-\nu\|Pv\|^2,$$

$$(26)\quad \left(v, [M\left(\alpha+t\frac{d}{dt}\right)\varphi'^2+t\,M'\varphi'^2-(t\,M\varphi'')']v\right)\geqslant(v, [\xi-\beta]v)$$
$$-\nu(v, [|\alpha|\varphi'^2+2q\varphi'(1+\varphi')]v)-\nu(v, \varphi'^2 v)$$
$$-(v, t|\varphi'''|v)-\nu(q\nu+2+q)(v, (1+\varphi')v),$$

$$(27)\quad 2\,\mathrm{Im}(Pv, t\,M'\varphi' v)\geqslant-\nu\|Pv\|^2$$
$$-\frac{1}{\nu}\|t\,M'\varphi' v\|^2\geqslant-\nu\|Pv\|^2-\nu\|\varphi' v\|^2.$$

Finally the last term in (5) is majorized as follows:

$$(28)\quad 2\,\mathrm{Im}(P_{1-\alpha}v, t[SP+R+i\varphi' S]v)$$
$$\geqslant-\nu\|\psi^{-1}P_{1-\alpha}v\|^2-\frac{1}{\nu}\|t\,\psi(SP+R+i\varphi' S)v\|^2.$$

For the first norm on the r. h. s. of (28) — which also occurs in (23) — we use

$$\|\psi^{-1}P_{1-\alpha}v\|^2\leqslant 2\|(1+t\varphi')^{1/2}Pv\|^2+\frac{1}{2}(1-\alpha)^2(1+\sqrt{q})^2\|v\|^2.$$

To majorize the norm in the second term on the r. h. s. of (28), we set $\zeta=\varphi'+\psi'\psi^{-1}$, notice that $|\zeta\psi|\leqslant 2q$, $|\zeta\psi'|\leqslant 2q^2$,

$(\zeta^2+|\zeta'|)\psi\leqslant 5q^2(1+\varphi')$ and use (10), (9) and (7) to obtain that

$$\begin{aligned}\|t\psi(SP+R+i\varphi' S)v\|^2&=\|[t(SP+R)+it\zeta S]\psi v\|^2\\&\leqslant 2\nu^2\|\psi v\|_2^2+6\nu^2 C[\|\psi v\|_2^2+\|\zeta P\psi v\|^2+\|(\zeta^2+|\zeta'|)\psi v\|^2]\\&\leqslant 2(1+3C)\nu^2[\|\psi v\|_2^2+4q^2\|Pv\|^2+4q^4\|v\|^2+25q^4\|(1+\varphi')v\|^2].\end{aligned}$$

By using Lemma 2 to majorize $\|\psi v\|_2^2$ and after inserting all these inequalities into (5), one finds that (22) holds, without the term $\nu\|Sv\|^2$ on its l. h. s., provided that $\nu<\nu_0$, where ν_0 is chosen sufficiently small and c sufficiently large. By virtue of (15), (22) then also holds with the term $\nu\|Sv\|^2$ included (it suffices to choose the constant c suitably larger than that obtained before adding the term $\nu\|Sv\|^2$). ■

THEOREM 4. — *Assume that the conditions* (i)-(viii) *are satisfied. Let* $\varphi\in C^3(I)$ *be a real function such that* $\varphi'\geqslant 0$ *and*

$$\varphi'=O(t),\qquad \varphi''=O\left(\frac{1+\varphi'}{t}\right),\qquad \varphi'''=O\left(\frac{1+\varphi'}{t^2}\right),$$

$$\liminf_{t\to\infty}\frac{\xi_{\alpha,\beta}(\varphi)}{1+\varphi'^2}>0.$$

If $\alpha\leqslant 0$, *assume in addition that* $\varphi'=O(1)$. *Then there are constants* c, r *such that for all* $u\in\mathscr{D}_{\mathrm{loc}}(r)$ *satisfying* $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ *and* $t\exp(\varphi)Lu\in\mathscr{H}$:

$$(29)\qquad \|(1+\varphi')e^{\varphi}u\|+\|(1+t\varphi')^{1/2}Pe^{\varphi}u\|+\|Se^{\varphi}u\|\leqslant c\|t(1+t\varphi')^{-1/2}e^{\varphi}Lu\|.$$

Proof. — For $q\geqslant 1$ we denote by Φ_q the set of real functions $\varphi\in C^3(I)$ such that for $t\geqslant a+q$:

$$(30)\qquad 0\leqslant\varphi'\leqslant qt,\qquad |t\varphi''(t)|+|t^2\varphi'''(t)|\leqslant q(1+\varphi'(t)).$$

$$(31)\qquad \xi_{\alpha,\beta}(\varphi)(t)\geqslant\frac{1}{q}[1+\varphi'^2(t)].$$

Clearly, if φ is as in the statement of the theorem, then $\varphi\in\Phi_q$ for some $q\geqslant 1$.

(i) We observe that for each $q\geqslant 1$ there are constants c, r such that the inequality (29) holds for all $\varphi\in\Phi_q$ and all $u\in\mathscr{S}(r)$. indeed this follows from Proposition 3 by setting $v=\exp(\varphi)u$, by taking v small enough and by noticing that $L(\varphi)v=\exp(\varphi)Lu$ and $|t\varphi'''|\leqslant qr^{-1}(1+\varphi')^2$ if $t\geqslant r$. The extension of this result to functions u without compact support will be made below in several steps.

(ii) We first prove the following fact: let $q\geqslant 1$, let $\hat{\Phi}_q$ be a subset of Φ_q and suppose that for each individual $\varphi\in\hat{\Phi}_q$ one has proven the existence of constants $c=c(\varphi)$ and $r=r(\varphi)$ (i. e. which may depend on φ) such that the inequality (29) holds (with $c=c(\varphi)$) for all $u\in\mathscr{D}_{\mathrm{loc}}(r(\varphi))$ satisfying $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and $t\exp(\varphi)Lu\in\mathscr{H}$. Then there are other constants c, r (independent of φ, i. e. depending only on q) such that (29) holds for *all* $\varphi\in\hat{\Phi}_q$ and all $u\in\mathscr{D}_{\mathrm{loc}}(r)$ satisfying $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and $t\,exp(\varphi)Lu\in\mathscr{H}$.

To prove the above claim, we choose a non-increasing function $\eta\in C^\infty(\mathbb{R})$ satisfying $0\leqslant\eta\leqslant 1$, $\eta(t)=1$ for $t\leqslant 1$ and $\eta(t)=0$ for $t\geqslant 2$ and denote by η_ε the function $\eta_\varepsilon(t)=\eta(\varepsilon t)$, $\varepsilon>0$. Clearly

$$|t\eta_\varepsilon'(t)|+|t^2\eta_\varepsilon''(t)|\leqslant\mu\chi_{[\varepsilon^{-1},2\varepsilon^{-1}]}(t) \tag{32}$$

for some constant μ, where χ_Δ is the characteristic function of the set Δ.

By (i), there exist constants c and r (depending only on q) such that (29) holds for all $\varphi\in\Phi_q$ (and *a fortiori* for all $\varphi\in\hat{\Phi}_q$) and all functions $u\in\mathscr{D}$ that are equal to zero below r. We show that our claim holds for these constants c and r. So let $u\in\mathscr{D}_{\mathrm{loc}}(r)$ be such that $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and $t\exp(\varphi)Lu\in\mathscr{H}$, with $\varphi\in\hat{\Phi}_q$. Then

$$\|(1+\varphi')e^\varphi\eta_\varepsilon u\|+\|(1+t\varphi')^{1/2}Pe^\varphi\eta_\varepsilon u\|+\|\eta_\varepsilon Se^\varphi u\| \leqslant c\,\|t(1+t\varphi')^{-1/2}e^\varphi L\eta_\varepsilon u\|, \tag{33}$$

and it suffices to show that this inequality remains valid for $\varepsilon=0$ (i. e. with η_ε replaced by 1). For this we remark that

$$L(\eta_\varepsilon u)=\eta_\varepsilon Lu-2i\eta_\varepsilon'MPu-i\eta_\varepsilon'Su-(\eta_\varepsilon'M'+\eta_\varepsilon''M)u. \tag{34}$$

This implies, together with (32) and (11), that

$$|te^\varphi L(\eta_\varepsilon u)|_H\leqslant|te^\varphi Lu|_H+c_1\left(|e^\varphi Pu|_H+|Se^\varphi u|_H+\left|\frac{1}{t}e^\varphi u\right|_H\right). \tag{35}$$

where c_1 is a constant independent of ε and φ.

Since (29) is assumed to hold for $\varphi\in\hat{\Phi}_q$ (with some constants $c=c(\varphi)$ and $r=r(\varphi)$), one has

(36) $\qquad (1+\varphi')e^{\varphi}u\in\mathscr{H}, \qquad (1+t\varphi')^{1/2}Pe^{\varphi}u\in\mathscr{H}, \qquad Se^{\varphi}u\in\mathscr{H}.$

(If $r(\varphi)\leqslant r$, then $u\in\mathscr{D}_{\text{loc}}(r(\varphi))$; on the other hand, if $r(\varphi)>r$, write $u=\theta u+(1-\theta)u$, where $\theta\in C^{\infty}(I)$ is such that $\theta(t)=0$ for $t\leqslant r(\varphi)$ and $\theta(t)=1$ for $t\geqslant r(\varphi)+1$, and observe that $\theta u\in\mathscr{D}_{\text{loc}}(r(\varphi))$ and $(1-\theta)u\in\mathscr{D}$ and that φ and φ' are bounded on the support of $(1-\theta)u$). By using (36) and the identity $\exp(\varphi)Pu=P\exp(\varphi)u+i\varphi'\exp(\varphi)u$, one sees that the r. h. s. of (35) is a square-integrable function of t. Since $\eta_\varepsilon u\to u$, $P(\eta_\varepsilon u)\to Pu$ and $L(\eta_\varepsilon u)\to Lu$ almost everywhere on I, one can perform the limit $\varepsilon\to 0$ in (33) by writing the norms as integrals and using Fatou's Lemma on the l. h. s. and the Lebesgue dominated convergence theorem on the r. h. s. This completes the proof of our claim.

(iii) We now prove the theorem under the additional condition $\varphi=O(1)$. The argument is essentially the same as that of part (ii) above. We let η, η_ε and u be as above and observe that (33) and (35) are satisfied (with constants c and r depending only on q). Since $\exp(\varphi)$ is bounded, $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and $t\exp(\varphi)Lu\in\mathscr{H}$ by assumption and $t^{-1}u\in\mathscr{H}$ by Theorem 327 of [6], the r. h. s. of (35) is square-integrable and one can take the limit $\varepsilon\to 0$ in (33) as above. Thus, for each $q\geqslant 1$, there are constants c and r (depending only on q) such that (29) holds for all u as stated and all $\varphi\in\Phi_{q,0}\equiv\{\varphi\in\Phi_q\mid\varphi=O(1)\}$.

(iv) We now prove the theorem iteratively by considering successively the following three classes of weight functions:

$$\Phi_{q,1}\equiv\{\varphi\in\Phi_q\mid\varphi'=o(1)\},$$
$$\Phi_{q,2}\equiv\{\varphi\in\Phi_q\mid\varphi'=O(1)\}\qquad\text{and}\qquad(\text{if }\alpha>0)\,\Phi_{q,3}=\Phi_q.$$

We shall show by recursion on k (using the result of (iii) in the first step, i. e. for $k=1$) that the following statements (P_k) are true for $k=1, 2, 3$:

(P_k) for each $q\geqslant 1$ there are constants c and r such that (29) holds for all $\varphi\in\Phi_{q,k}$ and all $u\in\mathscr{D}_{\text{loc}}(r)$ satisfying $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and $t\exp(\varphi)Lu\in\mathscr{H}$.

For this, we fix $\varphi\in\Phi_{q,k}$ and approximate it from below by a sequence of functions $\{\varphi_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ such that $\varphi_\varepsilon\to\varphi$ as $\varepsilon\to 0$ and $\varphi_\varepsilon\in\Phi_{q',k-1}$ for some q' (depending on k, q and φ but not on ε). Then, by the recursion assumption, there are constants c and r, depending only on q', such that for each $\varepsilon\in(0,\varepsilon_0)$ and each $u\in\mathscr{D}_{\text{loc}}(r)$ satisfying $Pu\in\mathscr{H}$, $Su\in\mathscr{H}$ and

$t \exp(\varphi) L u \in \mathscr{H}$:

$$\|(1+\varphi_\varepsilon')e^{\varphi_\varepsilon}u\| + \|(1+t\varphi_\varepsilon')^{1/2} P e^{\varphi_\varepsilon}u\| + \|S e^{\varphi_\varepsilon}u\| \leqslant c\,\|t(1+t\varphi_\varepsilon')^{-1/2} e^{\varphi_\varepsilon} L u\| \tag{37}$$

(because $\exp(\varphi_\varepsilon) \leqslant \exp(\varphi)$, hence $t\exp(\varphi_\varepsilon) L u \in \mathscr{H}$ also). We let $\varepsilon \to 0$ in (37) and pass the limit under the integrals by using Fatou's Lemma on the l. h. s. and the Lebesgue dominated convergence theorem on the r. h. s. This shows that (37) holds also for $\varepsilon=0$, i. e. with φ_ε replaced by φ, and the validity of (P_k) now follows from the result of part (ii) above (take $\hat{\Phi}_q = \Phi_{q,k}$).

It remains to indicate for each class of weight functions $\Phi_{q,k}$ how to choose a sequence $\{\varphi_\varepsilon\}$ having the required properties. We begin with the case $k=1$ where we take

$$\varphi_\varepsilon(t) = \varphi(a+1) + \int_{a+1}^{t} \frac{\varphi'(s)}{1+\varepsilon s^2}\,ds. \tag{38}$$

Clearly $\varphi_\varepsilon^{(i)} \to \varphi^{(i)}$ as $\varepsilon \to 0$ uniformly on compact subsets of I for $i=0, 1, 2, 3$, $\varphi_\varepsilon \leqslant \varphi$ and $0 \leqslant \varphi_\varepsilon' \leqslant \varphi' \leqslant qt$ on $[a+q, \infty)$. It is also easy to show that

$$|t\varphi_\varepsilon''| \leqslant (q+2)(1+\varphi_\varepsilon')$$

and

$$|t^2\varphi_\varepsilon'''| \leqslant (5q+6)(1+\varphi_\varepsilon') \qquad \text{if} \quad t \geqslant a+q.$$

On the other hand, since $\varphi'(t) \to 0$ as $t \to \infty$ and $\xi_{\alpha,\beta}(\varphi) \geqslant q^{-1}(1+\varphi'^2)$ for $t \geqslant a+q$, we must have $\beta \geqslant q^{-1}$. Hence

$$\xi_{\alpha,\beta}(\varphi_\varepsilon) \equiv \alpha\varphi_\varepsilon'^2 + 2\varphi_\varepsilon'.\,t\varphi_\varepsilon'' + \beta \geqslant -c(q)\varphi' + \beta = o(1) + \beta.$$

These estimates imply the existence of a number $q' \in (q, \infty)$ such that $\varphi_\varepsilon' \in \Phi_{q'}$ for each $\varepsilon \in (0, 1)$, as claimed, and clearly $\varphi_\varepsilon = O(1)$ (not uniformly in ε, of course).

For $k=2$ we set

$$\varphi_\varepsilon(t) = \varphi(a+1) + \int_{a+1}^{t} \frac{\varphi'(s)}{1+\varepsilon s^4}\,ds \tag{39}$$

One has properties and estimates for φ_ε and its derivatives of the same type as those for $k=1$, with the exception of the lower bound for $\xi_{\alpha,\beta}(\varphi_\varepsilon)$ which requires a more careful analysis. For this we notice that, for $t \geqslant a+q$:

$$\xi_{\alpha,\beta}(\varphi_\varepsilon)=\frac{\alpha\varphi'^2+2t\varphi'\varphi''+\beta}{t^{2\varepsilon}}+\beta(1-t^{-2\varepsilon})-2\varepsilon\varphi_\varepsilon'^2$$

$$\geqslant \frac{1}{q}\,\frac{1+\varphi'^2}{t^{2\varepsilon}}+\beta(1-t^{-2\varepsilon})-2\varepsilon\varphi_\varepsilon'^2$$

$$=\frac{1}{q}(1+\varphi_\varepsilon'^2)+\left(\beta-\frac{1}{q}\right)(1-t^{-2\varepsilon})-2\varepsilon\varphi_\varepsilon'^2.$$

Since $\Phi_{q_1}\subset\Phi_{q_2}$ if $q_1<q_2$, we may assume without loss of generality that $1/q\leqslant\beta$. We then have

$$\xi_{\alpha,\beta}(\varphi_\varepsilon)\geqslant\frac{1+\varphi_\varepsilon'^2}{q}-2\varepsilon\varphi_\varepsilon'^2\geqslant\frac{1+\varphi_\varepsilon'^2}{2q}+\left(\frac{1}{2q}-2\varepsilon\right)\varphi_\varepsilon'^2.$$

By taking $\varepsilon\leqslant\varepsilon_0\equiv(4q)^{-1}$, we see that $\varphi_\varepsilon\in\Phi_{q'}$ for some fixed q', and since $\varphi_\varepsilon'(t)\to 0$ as $t\to\infty$ (because $\varphi'=O(1)$ if $k=2$), we have in fact $\varphi_\varepsilon\in\Phi_{q',1}$.

Finally let us consider the case $k=3$. Then $\alpha>0$, and we define φ_ε by

$$\varphi_\varepsilon(t)=\varphi(a+1)+\int_{a+1}^{t}\frac{\varphi'(s)}{1+\varepsilon\varphi'(s)}\,ds. \tag{40}$$

It is easy to show that $\varphi_\varepsilon\leqslant\varphi$, $0\leqslant\varphi_\varepsilon'\leqslant\varphi'\leqslant qt$, $\varphi_\varepsilon'=O(1)$, $|t\varphi_\varepsilon''|+|t^2\varphi_\varepsilon'''|\leqslant c(q)(1+\varphi_\varepsilon')$ on $[a+q,\infty)$. Let $\alpha_0=\alpha-q^{-1}$, $\beta_0=\beta-q^{-1}$. As above, we may assume without loss of generality that $\alpha_0\geqslant 0$ and $\beta_0\geqslant 0$. Then

$$\xi_{\alpha,\beta}(\varphi_\varepsilon)-\frac{1+\varphi_\varepsilon'^2}{q}=\xi_{\alpha_0,\beta_0}(\varphi_\varepsilon)$$

$$=\frac{\alpha_0\varphi'^2+2t\varphi'\varphi''}{(1+\varepsilon\varphi')^3}+\varepsilon\alpha_0\frac{\varphi'^3}{(1+\varepsilon\varphi')^3}+\beta_0$$

$$\geqslant\frac{\alpha_0\varphi'^2+2t\varphi'\varphi''}{(1+\varepsilon\varphi')^3}+\beta_0\geqslant-\frac{\beta_0}{(1+\varepsilon\varphi')^3}+\beta_0\geqslant 0.$$

Thus $\varphi_\varepsilon\in\Phi_{q',2}$ for some fixed q'. ■

Remark C. — (*a*) The condition $t\exp(\varphi)Lu\in\mathscr{H}$ in Theorem 4 is artificial if $t\varphi'$ is not bounded. It can be shown that this condition is not necessary if $t\varphi'(t)\to\infty$ as $t\to\infty$.

(*b*) The estimate (29) is essentially optimal, as can be shown by studying the asymptotic behaviour of solutions of $Lu=0$ in the case where the coefficients M, Q, S, R of L are real functions, hence where L is an ordinary differential operator ($H=\mathbb{C}$).

(*c*) The idea of approximating a weight function by a sequence of more slowly growing functions was already used in [1] and in [4] and [5] in somewhat different contexts.

3. An example

We shall consider here only a rather simple but non-trivial example, namely first order perturbations of the Laplace operator. Let $a\geqslant 1$,

$$\Omega=\{x\in\mathbb{R}^n \mid |x|>a\}, \qquad \mathscr{H}(\Omega)=L^2(\Omega; dx)$$

and $\mathscr{H}^m(\Omega)$ the usual Sobolev space of order m. We set $t=|x|$, $\omega=x/|x|$. If $D_j=-i\partial/\partial x_j$, let

$$P=\frac{1}{2}(\omega.D+D.\omega)\equiv\sum_{j=1}^{n}\omega_j D_j-i\frac{n-1}{2t}. \tag{41}$$

It can be shown that

$$-\Delta\equiv D^2\equiv\sum_{j=1}^{n}D_j^2=P^2+\frac{N}{t^2}, \tag{42}$$

where N is a second order operator that acts only on the angular variables ω (it is the spherical Laplace operator plus a constant). In particular N is independent of t. Consider now some Borel functions A_j, V, $W:\Omega\to\mathbb{C}$ ($j=1,\ldots,n$) and a constant $\lambda\in\mathbb{R}$ and the following formal differential operator:

$$\begin{aligned} L&=D^2+A.D+V+W-\lambda \\ &\equiv P^2+\left(\frac{N}{t^2}+W-\lambda\right)+(A.\omega)P+(t^{-1}A.\Lambda+V)=P^2+Q+SP+R, \end{aligned} \tag{43}$$

where $\Lambda=(\Lambda_1, \ldots, \Lambda_n)$ and

$$(44) \qquad \Lambda_j=\sum_{k=1}^{n} \omega_k (x_k D_j - x_j D_k) + i\frac{n-1}{2}\omega_j \qquad (j=1, \ldots, n).$$

The operators Λ_j are also independent of t and act only on the angular variables ω (and one has $N=\Lambda^2-(n-1)/2$). The last identity in (43) contains implicitly the definition of the operators Q, S and R, whereas $M\equiv 1$ in this case.

We say that the function W is *regular* if $Wu\in\mathscr{H}(\Omega)$ for each $u\in\mathscr{H}_c^2(\Omega)$ and if there is a constant c such that for all $u\in\mathscr{H}_c^2(\Omega)$:

$$(45) \qquad \|u\|_{\mathscr{H}^2(\Omega)}\leqslant c\,[\|u\|_{\mathscr{H}(\Omega)}+\|Pu\|_{\mathscr{H}(\Omega)}+\|(D^2+W)u\|_{\mathscr{H}(\Omega)}].$$

This class of functions is quite large: for example, it is enough to have $W=W_0+W_1$ such that $W_k\colon \Omega\to\mathbb{R}$, $W_k\mathscr{H}_c^2(\Omega)\subset\mathscr{H}(\Omega)$, W_0 is D^2-bounded on $\mathscr{H}_c^2(\Omega)$ with D^2-relative bound zero, W_1 is positive and $|\Delta W_1|\leqslant W_1^2+b$ for some $b<\infty$ (which admits functions that grow rapidly at infinity).

In the next theorem all derivatives are understood to be in the sense of distributions:

THEOREM 5. — *Let L be defined by* (43) *and assume that the following conditions are fulfilled:*

(*a*) *W is real, regular, and there are constants $\alpha<2$, $\beta>0$, $\gamma>0$ such that as sesquilinear forms on $C_0^\infty(\Omega)$:*

$$(46) \qquad \alpha W+x.\nabla W\leqslant(2-\alpha)D^2+\alpha\lambda-\beta-\gamma P^2,$$

(*b*) *for each $\varepsilon>0$ there is a number $\rho<\infty$ such that for all $u\in C_0^\infty(\Omega)$ with $u(x)=0$ for $|x|\leqslant\rho$:*

$$(47) \qquad \|\,|x|\,A_j u\|\leqslant\varepsilon\|u\|_{\mathscr{H}^1(\Omega)}, \qquad \|\,|x|\,Vu\|\leqslant\varepsilon\|u\|_{\mathscr{H}^2(\Omega)}.$$

Let φ be as in Theorem 4. Then there are constants c and r such that the following inequality is true (with $t(x)\equiv|x|$) for each $s\in[0, 2]$ and each $u\in\mathscr{H}_{\mathrm{loc}}^2(\Omega)$ having the properties $u(x)=0$ if $|x|<r$, $u\in\mathscr{H}^1(\Omega)$ and $|x|\exp(\varphi(|x|))Lu\in\mathscr{H}(\Omega)$:

$$(48) \qquad \|[1+\varphi'(t)]^{1-s}e^{\varphi(t)}u\|_{\mathscr{H}^s(\Omega)}\leqslant c\,\|t[1+t\varphi'(t)]^{-1/2}e^{\varphi(t)}Lu\|_{\mathscr{H}(\Omega)}.$$

If $t\varphi'(t)$ *is bounded or tends to infinity as* $t \to \infty$, *the condition* $|x| \exp(\varphi(|x|)) Lu \in \mathscr{H}(\Omega)$ *is not needed.*

We sketch the proof only for the case $s=0$ (our abstract Theorem 4 combined with an interpolation argument implies in fact the general case of (48)). The connection with the abstract formalism is made as follows. Let $\Sigma = \{\omega \in \mathbf{R}^n \mid |\omega| = 1\}$, provided with the usual measure $d\omega$, and $H = L^2(\Sigma; d\omega)$. Then we represent $\mathscr{H}(\Omega)$ as $\mathscr{H} = L^2(I; H)$ through the unitary operator $\mathscr{U}: \mathscr{H}(\Omega) \to \mathscr{H}$ defined by

$$(\mathscr{U} f)(t)(\omega) = t^{(n-1)/2} f(t\omega). \tag{49}$$

In this new representation of $\mathscr{H}(\Omega)$, P becomes $-id/dt$, and if η is a function from I to $\mathbb{C}$ then the operator of multiplication by the function $x \mapsto \eta(|x|)$ becomes the operator of multiplication by the function $t \mapsto \eta(t)$. We take $\mathscr{D} = \mathscr{U}\mathscr{H}_c^2(\Omega)$; the condition of 2-semilocality and the condition that $\mathscr{D} \subset \mathscr{H}_c^2$ (which is strictly larger than $\mathscr{U}\mathscr{H}_c^2(\Omega)$) are obviously satisfied. Since W is regular, we have $\|f\|_{\mathscr{H}^2(\Omega)} \leqslant c \|\mathscr{U} f\|_2$, from which one can easily deduce that $\|\,.\,\|_1$, defined by $\|u\|_1 = \|\mathscr{U}^{-1} u\|_{\mathscr{H}^1(\Omega)}$, is an admissible norm. Condition (b) of the theorem corresponds to condition (viii) in Section 2, whereas (a) corresponds to (vii) by virtue of the following identity (as sesquilinear forms on $\mathscr{H}_c^2(\Omega)$):

$$\mathscr{U}^{-1}[(2-\alpha) P^2 + Q_\alpha]\mathscr{U} = (2-\alpha) D^2 + \alpha\lambda - \alpha W - x.\nabla W. \tag{50}$$

Thus Theorem 5 is a straightforward consequence of Theorem 4.

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