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ON THE COHOMOLOGY OF NILPOTENT LIE ALGEBRAS

BY

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RÉSUMÉ. — A une algèbre de Lie graduée $\mathfrak{g}$ de dimension finie on associe un polynôme $P(\mathfrak{g})$. La longueur de $P(\mathfrak{g})$ donne une borne inférieure pour la dimension de la cohomologie totale de $\mathfrak{g}$. Il y a des applications au rang toral des variétés différentiables et à la cohomologie des algèbres stabilisateurs de Morava.

ABSTRACT. — We associate to a graded finite dimensional Lie algebra $\mathfrak{g}$ a polynomial $P(\mathfrak{g})$. The length of $P(\mathfrak{g})$ gives a lower bound for the dimension of the total cohomology of $\mathfrak{g}$. There are applications to the toral rank of differentiable manifolds and to the cohomology of the Morava stabilizer algebras.

1. Introduction

Our main objective in this note is to give lower bounds for the dimension of the total cohomology of finite dimensional nilpotent Lie algebras. According to DIXMIER [4] all Betti numbers with the exception of the zeroeth and the highest are at least two. Hence $\dim H^*(\mathfrak{g}) \geq 2\dim\mathfrak{g}$. For dimensions one to six except five there exist nilpotent Lie algebras with $\dim H^i(\mathfrak{g}) = 2$ for $0 < i < \dim\mathfrak{g}$. In dimension 5 however $\dim H^*(\mathfrak{g}) \geq 12$ as the classification of these Lie algebras in [5] shows. All other available evidence points to the fact that in higher dimensions Dixmier's estimate is by no means best possible.

We are mainly concerned with graded finite dimensional Lie algebras $\mathfrak{g} = \bigoplus_{j\geq 1}\mathfrak{a}_j$. These are necessarily nilpotent and admit nontrivial automorphisms. Calculating their Lefschetz numbers we show that $\dim H^*(\mathfrak{g}) \geq \operatorname{length} P$, where $P = \prod_j (1 - T^j)^{\dim \mathfrak{a}_j}$ and the length of a

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polynomial is defined as the sum of the absolute values of its coefficients. In many examples even the actual values of the individual Betti numbers can be read off from P.

The estimation of length P belongs to analytic number theory and combinatorics; however, we did not find any result in the literature which could be applied in our situation. In Section 3 we therefore develop a method based on the consideration of normed algebras which for example leads to the following result : For graded Lie algebras $\mathfrak{g} = \bigoplus_{j=1}^{n} \mathfrak{a}_j$ with n fixed, $\dim H^*(\mathfrak{g})$ grows exponentially with $\dim \mathfrak{g}$.

We have two applications in mind : The first is to the toral rank of a differentiable manifold [8]. This will be considered in (3.3). The second concerns the cohomology of the Morava stabilizer algebras and is dealt with in section 4.

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2. The polynomial associated to a graded Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K of characteristic zero and let f be a (Lie algebra-) endomorphism of $\mathfrak{g}$. We define the zeta function $\zeta_f(T) \in K(T)$ of f by the formula

$$\zeta_f(T) = \prod_i \det\big(1 - Tf^* | H^i(\mathfrak{g}; K)\big)^{(-1)^{i+1}}.$$

Although we will not need it we observe that $\zeta_f(T)$ has a functional equation if $\mathfrak{g}$ is nilpotent and $\lambda = \det f \neq 0$. Setting $N = \dim \mathfrak{g}$ and $\epsilon = \prod_i \det(f^* | H^i(\mathfrak{g}; K))^{(-1)^{N-i}}$ it takes the form

$$\zeta_f(T) = \epsilon \zeta_f\left(\frac{1}{\lambda T}\right)^{(-1)^N}.$$

For N even we have $\epsilon = \pm 1$ since the Euler characteristic of $\mathfrak{g}$ vanishes.

(2.1) LEMMA . — $\zeta_f(T) = \exp \sum_{\mu=1}^{\infty} \det(1 - f^\mu | \mathfrak{g}) \frac{T^\mu}{\mu}$ *in* $K[[T]]$.

Proof. — Clearly

$$\text{(2.1.1)} \qquad \det(1 - f^\mu | \mathfrak{g}) = \sum_i (-1)^i \operatorname{Tr}(f^{*^\mu} | \Lambda^i \mathfrak{g}^*)$$
$$= \sum_i (-1)^i \operatorname{Tr}\big(f^{*^\mu} | H^i(\mathfrak{g}; K)\big)$$

since f^* commutes with the differential on $\Lambda\mathfrak{g}^*$. Using the well-known formula

$$\det(1 - T\varphi)^{-1} = \exp\sum_{\mu=1}^{\infty} \operatorname{Tr}(\varphi^{\mu})\frac{T^{\mu}}{\mu}$$

which holds for any endomorphism φ of a finite dimensional K-vector space ([9], App. C), we obtain

$$\begin{aligned}\exp\sum_{\mu=1}^{\infty} \det(1 - f^{\mu}|\mathfrak{g})\frac{T^{\mu}}{\mu} &= \exp\sum_{i}(-1)^i\sum_{\mu=1}^{\infty}\operatorname{Tr}\big(f^{*^{\mu}}|H^i(\mathfrak{g};K)\big)\frac{T^{\mu}}{\mu}\\ &= \prod_i \det\big(1 - Tf^*|H^i(\mathfrak{g};K)\big)^{(-1)^{i+1}}.\end{aligned}$$

Let f have eigenvalues $\lambda_1,\dots,\lambda_N$ in the algebraic closure $\overline{K}$ of K. For an N-tuple $\kappa = (\kappa_1,\dots,\kappa_N)$ of non-negative integers we set $\lambda^{\kappa} = \lambda_1^{\kappa_1}\dots\lambda_N^{\kappa_N}$; we put $\Theta(f) = \{0,1\}^N/\sim$ where $\kappa \sim \kappa'$ iff $\lambda_{\kappa} = \lambda^{\kappa'}$. We have

$$\det(1 - f) = \prod_{i=1}^{N}(1 - \lambda_i) = \sum_{\kappa\in\Theta(f)} a_{\kappa}\lambda^{\kappa} \text{ with } a_{\kappa} \in \mathbf{Z}.$$

Replacing f by f^{μ} we find

$$\det(1 - f^{\mu}|\mathfrak{g}) = \sum_{\kappa\in\Theta(f)} a_{\kappa}\lambda^{\mu\kappa}.$$

Substituting this in (2.1), we conclude that the reduced form of the rational function $\zeta_f(T)$ in $\overline{K}(T)$ is

$$\zeta_f(T) = \prod_{\kappa\in\Theta(f)} (1 - \lambda^{\kappa}T)^{-a_{\kappa}}.$$

In particular :

(2.2) LEMMA . — $\dim H^*(\mathfrak{g};K) \geq \sum_{\kappa\in\Theta(f)} |a_{\kappa}|.$

This lemma is of course only useful if there are nonzero endomorphisms of $\mathfrak{g}$ with all eigenvalues different from 1. For the common nilpotent Lie algebras it is quite frequent that they admit automorphisms of this type. However, in [7] there was constructed a seven dimensional nilpotent Lie algebra with only nilpotent automorphisms.

We will consider finite dimensional Lie algebras $\mathfrak{g}$ which are multigraded in the following sense : $\mathfrak{g} = \bigoplus_{\nu\in\mathbf{Z}^n} \mathfrak{a}_{\nu}$ as vector spaces with $\mathfrak{a}_0 = 0$

and $[\mathfrak{a}_\nu, \mathfrak{a}_\mu] \subset \mathfrak{a}_{\nu+\mu}$ for $\nu, \mu \in \mathbb{Z}^n$; anticommutativity and Jacobi identity are supposed to hold in the usual and not in the graded sense.

Set $K(Y) = K(Y_1, \ldots, Y_n)$ where $Y_1, \ldots, Y_n$ are indeterminates and let $\mathfrak{g}_{/K(Y)}$ be the Lie algebra deduced from $\mathfrak{g}$ by extending the ground field from K to $K(Y)$. We obtain an automorphism f_Y of $\mathfrak{g}_{/K(Y)}$ by defining $f_Y(x) = Y^\nu x$ for $x \in \mathfrak{a}_\nu$. Applying (2.1.1) in this context we see that the following identity is valid in $K(Y)$:

$$(2.3) \qquad \sum_i (-1)^i \operatorname{Tr}\big(f_Y^* | H^i(\mathfrak{g}_{/K(Y)}; K(Y)\big) = \det(1 - f_Y | \mathfrak{g}_{/K(Y)}) = \prod_\nu (1 - Y^\nu)^{\dim \mathfrak{a}_\nu}.$$

The polynomial $P_\mathfrak{g}(Y) = \prod_\nu (1 - Y^\nu)^{\dim \mathfrak{a}_\nu}$ will be called the polynomial associated to the graded Lie algebra $\mathfrak{g}$.

Observe that the dimensions of cohomology groups don't change under field extension. From (2.2), or alternatively directly from (2.3) using the fact that the eigenvalues of f_Y on $\Lambda \mathfrak{g}_{/K(Y)}$ are $\mathbb{Z}$-linearly independent, we conclude :

(2.4) COROLLARY. — *Let $\mathfrak{g}$ be a multigraded finite dimensional Lie algebra as above. Write $P_\mathfrak{g}(Y) = \prod_\nu (1 - Y^\nu)^{\dim \mathfrak{a}_\nu} = \sum_\kappa a_\kappa Y^\kappa$ in $\mathbb{Z}[Y] = \mathbb{Z}[Y_1, \ldots, Y_n]$. Then $\dim H^*(\mathfrak{g}; K) \geq \sum_\kappa |a_\kappa|$.*

Clearly if an n-grading is refined to an m-grading for $n \leq m$ in (2.4) then the lower bound obtained from the latter will be at least as good as the one obtained from the former.

In many natural examples of multigraded Lie algebras $\mathfrak{g}$ the eigenvalues of f_Y belonging to different cohomology groups are different. If $\mathfrak{g}$ admits only a simple grading sometimes the following holds : If Y^{α_i} are eigenvalues of f_Y on H^i then $i < j$ implies $\alpha_i < \alpha_j$. In such cases by (2.3) the individual Betti numbers of $\mathfrak{g}$ can be obtained by grouping together consecutive terms of the same sign in the associated polynomial. These remarks apply in particular to the maximal nilpotent Lie algebras in semisimple Lie algebras, to all the Heisenberg Lie algebras and to all nilpotent Lie algebras of dimension at most five. However they are not true in complete generality as follows from (2.8) below. Perhaps for some natural class of graded Lie algebras a version of the Weil conjecture holds true asserting that the different factors $\det(1 - T f_Y^* | H^i(\mathfrak{g}_{/K(Y)}; K(Y))$ of the zeta function $\zeta_{f_Y}(T)$ are pairwise prime.

(2.5) *Remark.* — If $\mathfrak{g}_{/\mathbb{Z}}$ is a Lie algebra over $\mathbb{Z}$ let $\mathfrak{g}_{/\mathbb{F}_p}$ denote the $\mathbb{F}_p$-Lie algebra obtained by reducing the structure constants mod p and

$\mathfrak{g}_{/\mathbb{Q}}$ the associated $\mathbb{Q}$-Lie algebra. Then for all p we have

$$\dim H^i(\mathfrak{g}_{/\mathbb{F}_p};\mathbb{F}_p) \geq \dim H^i(\mathfrak{g}_{/\mathbb{Q}};\mathbb{Q})$$

with equality for almost all p, as follows by inspection of the Koszul complex. For suitable $\mathfrak{g}_{/\mathbb{Z}}$ this allows to extend the estimate of (2.4) to $\mathfrak{g}_{/\mathbb{F}_p}$.

Recall that a Lie algebra is called homogeneous if it is a quotient of a free Lie algebra on finitely many generators by a graded ideal. Any homogeneous Lie algebra is simply graded,

$$\mathfrak{g} = \bigoplus_{j\geq 1} \mathfrak{a}_j$$

and the vector space $\mathfrak{a}_j$ is isomorphic to the quotient $\mathfrak{g}^{(j-1)}/\mathfrak{g}^{(j)}$ where $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \ldots$ is the descending central series of $\mathfrak{g}$.

(2.6) COROLLARY. — *Let $\mathfrak{g}$ be a finite dimensional homogeneous Lie algebra and denote by n_j the dimension of $\mathfrak{g}^{(j-1)}/\mathfrak{g}^{(j)}$.*

Write $$\prod_j (1 - X^j)^{n_j} = \sum_k a_k X^k \text{ in } \mathbb{Z}[X].$$

Then $$\dim H^*(\mathfrak{g};K) \geq \sum_k |a_k|.$$

The next result may be used to improve the estimate of (2.2) in certain cases :

(2.7) PROPOSITION. — *Let $\mathfrak{g}$ be a nilpotent Lie algebra of odd dimension $\neq 1, 3, 7$ defined over $\mathbb{Q}$. Then* $\dim H^*(\mathfrak{g};\mathbb{Q}) \equiv 0 \pmod 4$.

Proof. — Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. By results of MALCEV [11] there exists a lattice Γ in G. It is classical that $b_i(\mathfrak{g}) = b_i(G/\Gamma)$. Since G/Γ is parallelizable, all characteristic numbers vanish. Hence the real and the mod 2 semicharacteristic coincide by [10]. Theorems of ADAMS and KERVAIRE (compare [2]) then imply that the semicharacteristic of G/Γ vanishes. The result follows.

(2.8) *Example.* — There is a Lie algebra $\mathfrak{g}$ of dimension 11 with the following properties :

(a) $\mathfrak{g}$ has a basis $x_1,\ldots,x_{11}$ with brackets $[x_i,x_j] = a_{ij}x_{i+j}$ for $i<j$ and $a_{ij} \in \mathbb{Q}, a_{ij} \neq 0$ for $i+j \leq 11$.

(b) $b_1(\mathfrak{g}) = b_2(\mathfrak{g}) = 2$.

Because of (a), $\mathfrak{g}$ is simply graded in a natural way and doesn't admit a nontrivial bigrading. Its associated polynomial is given by $P(T) = \prod_{i=1}^{11}(1 - T^i)$. It has length $54 \not\equiv 0 \pmod 4$. Hence $\dim H^*(\mathfrak{g}) \geq 56$ by (2.7).

3. Estimates for the length of associated polynomials

For simplicity we will restrict attention to polynomials in one variable. For $P(T) = \sum a_j T^j$ in $\mathbb{C}[T]$ we set $L(P) = \sum |a_j|$. If $\mathcal{A}$ is a normed algebra and $x \in \mathcal{A}$ with $\|x\| = 1$ we have $L(P) \geq \|P(x)\|$. We apply this to $\mathcal{A} = M_r(\mathbb{C})$, the algebra of $r \times r$ matrices with norm $\|x\| = \max_\mu(\sum_\nu |x_{\nu\mu}|)$ for $x = (x_{\nu\mu})$ in $\mathcal{A}$. For x choose the permutation matrix

$$x = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & . & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix} \text{ in } M_r(\mathbb{C}).$$

In order to calculate $P(x)$ we diagonalise x. Set $\rho = \exp(2\pi i/r)$ and $s = (s_{\nu\mu})_{0 \leq \nu,\mu \leq r-1}$ where $s_{\nu\mu} = \rho^{\nu\mu}$. Then $s^{-1} = 1/r(s_{\nu\mu}^{-1})$ and $x = s\,\mathrm{diag}(1, \rho, \dots, \rho^{r-1})s^{-1}$. Thus

$$P(x) = s\,\mathrm{diag}\big(P(1), P(\rho), \dots, P(\rho^{r-1})\big)s^{-1}.$$

Working out this matrix product we obtain the following estimate for the length of a polynomial.

(3.1) Lemma. — $$L(P) \geq \|P(x)\| = \frac{1}{r}\sum_{\mu=0}^{r-1}\Big|\sum_{\nu=0}^{r-1} P(\rho^\nu)\rho^{\mu\nu}\Big|.$$

We remark that for $r > \deg P$ we have $L(P) = \|P(x)\|$. Now in particular we consider polynomials of the form

$$P(T) = \prod_{j=1}^{k}(1 - T^j)^{n_j} \text{ with } n_j \geq 0.$$

(3.2) Proposition. — *$L(P) \geq 2^N$ where $N = \max\{n_j \mid j > k/2\}$.*

Proof. — Fix an integer j with $k/2 < j \leq k$. Then we have to show that $L(P) \geq 2^{n_j}$. We proceed in three steps :

Step 1. — Assume that $n_j \geq 2$. Define $\widetilde{P}(T) \in \mathbb{Z}[T]$ by $\widetilde{P}(T) = P(T)(1 - T^j)^{-1}$. We choose $r = 2j$ in the above discussion and show that

$$\|P(x)\| = 2\|\widetilde{P}(x)\|.$$

This follows immediately from (3.1) since

$$P(\rho^\nu) = 2\widetilde{P}(\rho^\nu) \text{ for } \rho = \exp(\pi i/j) \text{ and all } \nu.$$

(For even ν this requires $n_j \geq 2$).

Step 2. — We show that $\|P(x)\| \geq 2^{n_j-1}$: Define $Q(T) \in \mathbf{Z}[T]$ by $Q(T) = P(T)(1-T^j)^{-(n_j-1)}$. Then

$$\|P(x)\| = 2^{n_j-1}\|Q(x)\|$$

by *Step 1.* Therefore we have to show that $\|Q(x)\| \geq 1$. Now $Q(x)$ has the eigenvalue $Q(\rho)$, and $Q(\rho) \neq 0$ because of $j > k/2$. Hence $Q(x) \neq 0$. On the other hand, $Q(x)$ is an integral matrix and therefore $\|Q(x)\| \geq 1$.

Step 3. — We show that $\|P(x)\| \geq 2^{n_j}$: Let $m \in \mathbf{N}$ be arbitrary. Since $\|P(x)^m\| \leq \|P(x)\|^m$ we have $\|P(x)\| \geq \|P(x)^m\|^{1/m}$. Applying *Step 2* to the polynomial P^m we conclude that $\|P(x)\| \geq (2^{mn_j-1})^{1/m} = 2^{n_j-1/m}$. As this holds for all m, the assertion follows.

(3.3) *Remark.* — Let $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{a}_{2i}$ be an evenly graded finite dimensional Lie algebra with center $\mathfrak{z}$. As the authors learned from V. PUPPE, the study of the toral rank of a differentiable manifold (*e.g.*, [8]) led to the conjecture that $\dim H^*(\mathfrak{g}) \geq 2^{\dim \mathfrak{z}}$. From (3.2) we obtain the somewhat weaker conclusion

$$\dim H^*(\mathfrak{g}) \geq 2^{\dim \mathfrak{a}_{2n}}.$$

If $\mathfrak{g}$ has order of nilpotency 2 (*i.e.* if $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = 0$), this implies easily that $\dim H^*(\mathfrak{g}) \geq 2^{\dim \mathfrak{z}}$.

(3.4) *Remark.* — By a more careful argument, PROPOSITION (3.2) can be improved. For instance one can show

$$L(P) \geq 2^{n_k}\sqrt{k^m} \text{ with } m = \min\{n_1, \ldots, n_{k-1}\}.$$

The next theorem implies that for graded Lie algebras $\mathfrak{g} = \bigoplus_{j=1}^k \mathfrak{a}_j$ with k fixed, $\dim H^*(\mathfrak{g})$ grows exponentially with $\dim \mathfrak{g}$.

(3.5) THEOREM. — *Consider* $P(T) = \prod_{j=1}^k (1-T^j)^{n_j}$ *and set* $q = \sqrt[p-1]{p}$ *for any prime* $p > k$. *Then*

$$L(P) \geq q^{\Sigma n_j}.$$

In particular for $\mathfrak{g} = \bigoplus_{j=1}^{k} \mathfrak{a}_j$ *as above we have*

$$\dim H^*(\mathfrak{g}) \geq q^{\dim \mathfrak{g}}.$$

Proof. — We choose $r = p$ and hence $\rho = \exp(2\pi i/p)$ in the argument at the beginning of this section. Again x denotes the above permutation matrix.

Step 1. — For $j \not\equiv 0 \pmod p$ we have $\|P(x^j)\| = \|P(x)\|$: Since p is prime, $x^j = sxs^{-1}$ with a permutation matrix s. Obviously $\|s\| = \|s^{-1}\| = 1$ and thus from $P(x^j) = sP(x)s^{-1}$ we obtain $\|P(x^j)\| \leq \|P(x)\|$. Similarly $\|P(x^j)\| \geq \|P(x)\|$.

Step 2. — Set $\widetilde{Q}(T) = \prod_{i=1}^{p-1} P(T^i)$. According to *Step 1* $\|\widetilde{Q}(x)\| \leq \|P(x)\|^{p-1}$. Set $m = n_1 + \cdots + n_k$. As $x^p = 1$ we have $\widetilde{Q}(x) = Q(x)$ where $Q(T) = \prod_{i=1}^{p-1}(1-T^i)^m$. Hence $\|P(x)\| \geq \|Q(x)\|^{1/p-1}$. Thus it suffices to show that $\|Q(x)\| \geq p^m$.

Step 3. — We set $Q(T) = R(T)\prod_{i=1}^{p-1}(1-T^i)$. Arguing as in *Step 3* of PROPOSITION (3.2) it suffices to see that $\|Q(x)\| = p\|R(x)\|$ in case $m > 1$. By LEMMA (3.1) this will follow from $Q(\rho^\nu) = pR(\rho^\nu)$ for $1 \leq \nu \leq p-1$. But this amounts to $\prod_{j=1}^{p-1}(1-\rho^{\nu j}) = p$ which in turn is a consequence of the identity $X^p - 1 = \prod_{\zeta^p=1}(X-\zeta)$.

4. Some remarks on the cohomology of the Morava stabilizer algebras

We refer to [12], [13], [14] for background on the Morava stabilizer algebras and their relation with the E_2-term of the Novikov spectral sequence. According to [14] for $p > n+1$ the cohomology of the Morava stabilizer algebras is related to the $\mathbf{F}_p$-cohomology of the following Lie algebras $\mathcal{M}(n)$: For $n \in \mathbf{N}$ let $\mathcal{M}(n)$ be the Lie algebra with basis x_{ij} for $1 \leq i \leq n$, $j \in \mathbf{Z}/n$ and brackets

$$[x_{ij}, x_{k\ell}] = \begin{cases} \delta^{\ell}_{i+j} x_{i+k,j} - \delta^{j}_{k+\ell} x_{i+k,\ell} & \text{for } i+k \leq n, \\ 0 & \text{for } i+k > n. \end{cases}$$

Here $\delta_i^j = 1$ for $i \equiv j \pmod n$ and $\delta_i^j = 0$ otherwise. Clearly $\mathcal{M}(n)$ is defined over $\mathbf{Z}$. The center $\mathfrak{z}(n)$ of $\mathcal{M}(n)$ is generated by $x_{n,j}$, $j \in \mathbf{Z}/n$. We set $\widetilde{\mathcal{M}}(n) = \mathcal{M}(n)/\mathfrak{z}(n)$. The algebra $\mathcal{M}(n)$ should be visualized as follows : Let S^1 be the unit circle and $\mu_n \subset S^1$ the group of n-th roots of unity. The $x_{k\ell}$ are represented by intervals on S^1 intersected with μ_n, $x_{k\ell}$ has length k (*i.e.* contains k n-th roots of unity) and its starting point is $\exp(2\pi i\ell/n)$, going from there in the sense of the usual orientation. (For $k = n$, $x_{n\ell}$ is given as μ_n together with the distinguished point $\exp(2\pi i\ell/n)$). The bracket $[x_{ij}, x_{k\ell}]$ is zero if x_{ij} and $x_{k\ell}$ are not disjoint or if their union doesn't form an interval. In the remaining cases $[x_{ij}, x_{k\ell}]$ is given by $\pm x_{ij} \cup x_{k\ell}$ with the sign determined by orientation.

We set $e_k = (0\dots1\dots0)$ with 1 at the $\bar{k}$-th place where $1 \leq \bar{k} \leq n$ and $k \equiv \bar{k} \pmod n$. An n-grading on $\mathcal{M}(n)$ is defined by setting $\mathcal{M}(n) = \bigoplus_{\nu\in\mathbf{Z}^n} \mathfrak{a}_\nu$, where $x_{ij} \in \mathfrak{a}_{e_j+\dots+e_{j+i-1}}$. The automorphism f_Y of (2.3) on $\mathcal{M}(n)_{/K(Y)}$ is given by $f_Y(x_{ij}) = Y_j \dots Y_{j+i-1}x_{ij}$ where $Y_k = Y_{\bar{k}}$ with $\bar{k}$ as above. For the associated polynomial of $\mathcal{M}(n)$ with respect to this grading we obtain

$$
\begin{aligned}
P_{\mathcal{M}(n)}(Y) &= \det(1 - f_Y \mid \mathcal{M}(n)_{/K(Y)}) \\
&= \prod_{\substack{1\leq i\leq n \\ j \pmod n}} (1 - Y_j \dots Y_{j+i-1}) \\
&= \Big(\prod_{1\leq j<k\leq n} (1 - Y_j \dots Y_{k-1}) \Big(1 - \frac{Y_j \dots Y_{k-1}}{Y_1 \dots Y_n}\Big)\Big) \\
&\qquad \times (1 - Y_1 \dots Y_n)^n.
\end{aligned}
\tag{4.1}
$$

Analogously :

$$
P_{\widetilde{\mathcal{M}}(n)}(Y) = \prod_{1\leq j<k\leq n} (1 - Y_j \dots Y_{k-1}) \Big(1 - \frac{Y_j \dots Y_{k-1}}{Y_1 \dots Y_n}\Big).
$$

The length of $P_{\widetilde{\mathcal{M}}(n)}(Y)$ may be estimated as follows.

Clearly :

$$
\begin{aligned}
\operatorname{length} P_{\widetilde{M}(n)} &\geq \operatorname{length} \prod_{1\leq j<k\leq n} (1 - Y_j \dots Y_{k-1})^2 \\
&= \operatorname{length} \prod_{1\leq j<k\leq n} (T_k - T_j)^2
\end{aligned}
$$

as is seen by substituting $Y_j = T_j/T_{j+1}$. The Vandermonde determinant shows that $\prod_{1\leq j<k\leq n}(T_k - T_j)$ has length equal to $n!$, hence the discriminant $D_n = \prod_{1\leq j<k\leq n}(T_k - T_j)^2$ has length at most $(n!)^2$. We have length $(D_n) = (n!)^2$ for $n = 1, 2, 3, 4$. However in general the length of D_n is less then $(n!)^2$. According to (2.4) and (2.5) for all primes p we have

$$
\dim_{\mathbf{F}_p} H^*(\mathcal{M}(n); \mathbf{F}_p) \geq \operatorname{length} \prod_{i,j \pmod n} (1 - Y_j \dots Y_{j+i-1})
$$

and all evidence points to the fact that there is equality for large primes p. In particular this is true for $n = 1, 2$ (easy) and for $n = 3$ by Theorem (3.8) of [14].

We close this section by making some remarks on the relation between the cohomology of $\mathcal{M}(n)$ and the cohomology of maximal nilpotent subalgebras of semisimple Lie algebras. By $\mathfrak{n}(n)$ we denote the Lie algebra of strict upper triangular $n \times n$-matrices over a field K. Let $y_{ij} \in \mathfrak{n}(n)$ have 1 in position $(j, i+j)$ and zero elsewhere. We obtain a Lie algebra homomorphism $\mathcal{M}(n) \xrightarrow{\varphi} \mathfrak{n}(n)$ by defining $\varphi(x_{ij}) = y_{ij}$ for $1 \leq j \leq n-i$ and $\varphi(x_{ij}) = 0$ otherwise. Then $\mathfrak{a} = \operatorname{Ker}\varphi$ is an abelian ideal of $\mathcal{M}(n)$ of dimension $n(n+1)/2$.

(4.2) PROPOSITION. — *For any field K the spectral sequence*

$$H^p(\mathfrak{n}(n); \Lambda^q\mathfrak{a}) \quad \Longrightarrow \quad H^{p+q}(\mathcal{M}(n); K)$$

degenerates. Hence we have

$$H^i(\mathcal{M}(n); K) \cong \bigoplus_{p+q=i} H^p(\mathfrak{n}(n), \Lambda^q\mathfrak{a})$$

and $H^*(\mathfrak{n}(n); K)$ is a subalgebra of $H^*(\mathcal{M}(n); K)$.

Proof. — We may assume that $K = \mathbb{F}_p(T)$ or $K = \mathbb{Q}(T)$. We define an automorphism f by setting $f|\mathfrak{a} = T \cdot \mathrm{id}$ and $f(x_{ij}) = x_{ij}$ for $1 \leq j \leq n-1$ (f is obtained from f_Y by specialization $Y_1 = \cdots = Y_{n-1} = 1$ and $Y_n = T$). On $H^p(\mathfrak{n}(n); \Lambda^q\mathfrak{a})$ it is multiplication by T^q. Hence all differentials vanish.

Remark. — The ranks of $H^i(\mathfrak{n}(n); R)$ are known for $R = \mathbb{Q}$ by work of BOTT [1] and for $R = \mathbb{Z}_{(p)}$ with $p > n-1$ by [6]. The multiplicative structure of $H^*(\mathfrak{n}(n); \mathbb{Q})$ follows from a result of KOSTANT and CARTIER (*cf.* [3]), which applies to all maximal nilpotent Lie algebras in semisimple Lie algebras :

Let K be a real compact connected semisimple Lie group with maximal torus T, let $G = K_{\mathbb{C}}$ be the complexification of K and denote by $\mathfrak{k}$, $\mathfrak{t}$ and $\mathfrak{g}$ the respective Lie algebras. We denote by \$ the root system of K and by W the corresponding Weyl group. We choose a basis $\mathbf{B}$ of \$, write $\$^+$ for the positive roots, $\$^- = \$\backslash\$^+$ and set $\rho = 1/2\sum_{\alpha\in\$^+}\alpha$. The length $\ell(w)$ of an element $w \in W$ is defined as the minimal number q of reflections $s_1, \ldots, s_q$ in hyperplanes orthogonal to elements of $\mathbf{B}$ such that $w = s_1 \ldots s_q$. We have $\ell(w) = \#\$_w$ where $\$_w = \$^+_w \cap w^{-1}\$^-$. The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\$} \mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [u, x] = \alpha(u)x \text{ for all } u \in \mathfrak{t}\}$. Let $\mathfrak{n} = \bigoplus_{\alpha\in\$^+} \mathfrak{g}_\alpha$ be the maximal nilpotent subalgebra of $\mathfrak{g}$.

(4.3) Let σ be the representation of T on $H^*(\mathfrak{n}, \mathbb{C})$ induced by Ad : $T \to \mathrm{Aut}(\mathfrak{n})$. Then

a) $H^*(\mathfrak{n}, \mathbb{C}) = \bigoplus_{w \in W} V_w$ with $V_w = \{c \in H^*(\mathfrak{n}, \mathbb{C}) \mid \sigma(u)c = e^{(\rho - w\rho)(u)}c$ for all $u \in \mathfrak{t}\}$.

b) $V_w \subset H^{\ell(w)}(\mathfrak{n}, \mathbb{C})$, $\dim V_w = 1$.

c) There exist generators v_w in V_w such that the following holds : If $w, w' \in W$ are such that $\$_w \cap \$_{w'} = \emptyset$ and $\$_w \cup \$_{w'} = \$_{w''}$ for some $w'' \in W$ then

$$v_w \cup v_{w'} = \pm v_{w''}.$$

In any other case

$$v_w \cup v_{w'} = 0.$$

d) The sign ϵ in the relation $v_w \cup v_{w'} = \epsilon v_{w''}$ can be determined as follows. Choose an ordering of $\$^+$ and generators x_α of $\mathfrak{g}_\alpha^*$ for $\alpha \in \$^+$. Then v_w is represented by the cocycle $x_w = x_{\alpha_1} \wedge \ldots \wedge x_{\alpha_\ell} \in \Lambda\mathfrak{n}^*$ with $\$_w = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\alpha_1 < \cdots < \alpha_\ell$. Similarly for $v_{w'}$ and $v_{w''}$. Then $x_w \wedge x_{w'} = \epsilon x_{w''}$.

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