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MORSE THEORY AND EXISTENCE OF PERIODIC SOLUTIONS OF CONVEX HAMILTONIAN SYSTEMS

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RÉSUMÉ. — On s'intéresse au problème d'existence des solutions périodiques non constantes des sytèmes hamiltoniens convexes. Les solutions correspondent aux points critiques de la fonctionnelle d'action duale ψ. On montre que si C est l'orbite (sous l'action naturelle de S^1) d'un point critique $u \neq 0$ de ψ et si U est un voisinage de C convenable, alors la paire $(\{\psi \leq \psi(u)\} \cap U, \{\psi \leq \psi(u)\} \cap U - C)$ a la structure d'un fibré relatif de base C. Utilisant cela et des formules d'itération pour l'indice de Morse on montre qu'il y a au moins deux trajectoires hamiltoniennes fermées sur une surface d'énergie convexe. Pour la démonstration on suppose qu'il n'existe qu'une trajectoire. On calcule alors certains nombres de Morse et on voit que les relations de Morse ne sont pas satisfaites.

ABSTRACT. — The paper is concerned with the problem of existence of nonconstant periodic solutions of convex Hamiltonian systems. The solutions correspond to critical points of the dual action functional ψ. It is shown that if C is the orbit (under the natural S^1-action) of a critical point $u \neq 0$ of ψ and if U is a suitable neighbourhood of C, then the pair $(\{\psi \leq \psi(u)\} \cap U, \{\psi \leq \psi(u)\} \cap U - C)$ has the structure of a fibre bundle pair with base space C. Using this and iteration formulas for the Morse index it is shown that there are at least two closed Hamiltonian trajectories on a convex energy surface. The proof is carried out by assuming that there is only one trajectory, computing certain Morse type numbers and showing that the Morse relations are not satisfied.

1. Introduction

In this paper we are concerned with the problem of existence of nonconstant periodic solutions of Hamiltonian systems of differential equations

$$\dot{x} = JH'(x).$$

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Here $H \in \mathcal{C}^2(\mathbb{R}^{2n}, \mathbb{R})$ and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the usual symplectic matrix. We shall denote by $(\ ,\ )$ the inner product and by $|\ |$ the norm in $\mathbb{R}^{2n}$. Suppose that H satisfies the following hypothesis :

$$\text{(H1)} \qquad \begin{cases} H \in \mathcal{C}^1(\mathbb{R}^{2n}, \mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^{2n} - \{0\}, \mathbb{R}) \quad \text{is strictly convex,} \\ H(x) > H(0) = 0 \quad \forall x \in \mathbb{R}^{2n}\,,\ x \neq 0, \\ H(x)|x|^{-1} \to \infty \quad \text{as } |x| \to \infty. \end{cases}$$

Let

$$G(y) = \sup\Big\{(x, y) - H(x) : x \in \mathbb{R}^{2n}\Big\}$$

be the Fenchel conjugate of H [1]. By (H1), G is strictly convex and of class $\mathcal{C}^1$. According to the Legendre reciprocity formula,

$$y = H'(x) \quad \text{if and only if} \quad x = G'(y).$$

Furthermore, if $x \neq 0$ and $H''(x)$ is invertible, G is $\mathcal{C}^2$ near $y = H'(x)$ and $G''(y) = H''(x)^{-1}$. In addition to (H1), suppose that there exist constants α, $\beta \in (1, \infty)$, $\alpha^{-1} + \beta^{-1} = 1$, and c_i such that

$$\text{(H2)} \qquad \begin{cases} \big|G'(y)\big| \leq c_1|y|^{\alpha-1} + c_2 \qquad \forall y \in \mathbb{R}^{2n}, \\ \big|H'(x)\big| \leq c_3|x|^{\beta-1} + c_4 \qquad \forall x \in \mathbb{R}^{2n}. \end{cases}$$

Set

$$L_0^\alpha = \Big\{u \in L^\alpha(0, T\,;\ \mathbb{R}^{2n}) : \int_0^T u\, dt = 0\Big\}$$

and let ψ be a functional given by

$$\psi(u) = \int_0^T \Big[\tfrac{1}{2}(Ju, Mu) + G(-Ju)\Big]\, dt,$$

where Mu is the primitive of u having mean value zero. Denote the duality pairing between L^β and L^α by $\langle\ ,\ \rangle$. It follows from the hypotheses (H1), (H2) that ψ is well defined, of class $\mathcal{C}^1$ on L_0^α, and

$$\big\langle \psi'(u), v\big\rangle = \int_0^T \Big(Mu - G'(-Ju), Jv\Big)\, dt.$$

It is well known [1, 6, 8] that if x is a solution of

$$\dot{x} = JH'(x), \quad x(0) = x(T), \tag{1}$$

then $u = \dot{x}$ is a critical point of ψ, and conversely, if u is a critical point of ψ, then $x = Mu + \xi$ is a solution of (1) for some $\xi \in \mathbb{R}^{2n}$. So under our hypotheses finding solutions of (1) is equivalent to finding critical points of ψ. Following [9] we shall call a solution $\bar{x}$ of (1) *admissible* if it is nonconstant and if $H''(\bar{x}(t))$ is invertible for all t.

The functional ψ is invariant under the S^1-action given by $S_\theta u(t) = u(\theta + t)$ (we identify T-periodic functions on $\mathbb{R}$ with their restrictions to $[0, T]$). Thus, if $\bar{u} \neq 0$ is a critical point of ψ, then so is $S_\theta \bar{u}$ for all θ. It follows that the orbit of $\bar{u}$,

$$C(\bar{u}) = \{S_\theta \bar{u} : 0 \leq \theta \leq T\}$$

consists of critical points. The set $C(\bar{u})$ will be called *a critical circle.* Note that if $\bar{u} = d\bar{x}/dt$, where $\bar{x}$ is an admissible solution of (1), then the function $\bar{u}$ is of class C^1 and so is the mapping $\theta \mapsto S_\theta \bar{u}$. Hence $C(\bar{u})$ is a C^1-submanifold of L_0^α.

In [8, 9] Ekeland and Hofer use Morse theory in order to find solutions of (1). Since ψ may not be of class C^2 on L_0^α and since L_0^α is not a Hilbert space (unless $\alpha = 2$), Morse theory cannot be applied directly. To get around this difficulty, they reduce the problem to a finite-dimensional one which, however, no longer has the S^1-symmetry (still, it does have $\mathbb{Z}_p$-symmetry for an appropriate p).

In this paper we propose a different approach to (1). Let

$$\psi_b = \{u \in L_0^\alpha : \psi(u) \leq b\}.$$

In Section 2 we show that if $\bar{u}$ is a critical point of ψ corresponding to an admissible solution $\bar{x}$ of (1), if $\psi(\bar{u}) = b$ and $C(\bar{u})$ is the orbit of $\bar{u}$, then for a suitable neighbourhood U of $C(\bar{u})$ the pair $(\psi_b \cap U, \psi_b \cap U - C(\bar{u}))$ has the structure of a fibre bundle pair with base space $C(\bar{u})$. In Section 3 we demonstrate that the corresponding fibre pair has the homotopy type of $(\chi_b, \chi_b - \{0\})$, where χ is a function of class C^2 defined in a neighbourhood of the origin in a finite dimensional space. The remaining sections are devoted to the proof of the fact that there exist at least two closed Hamiltonian trajectories on a convex hypersurface in $\mathbb{R}^{2n}$, $n \geq 3$. Arguing by contradiction, we assume that there is only one such trajectory, and then, using the results of Sections 2–3 and iteration formulas for the index [8, 17], we compute certain critical groups and corresponding Morse type numbers M_q [5, 22, 23]. The conclusion follows by observing that the M_q do not satisfy the Morse relations. Existence of two closed Hamiltonian trajectories has also been proved by EKELAND and LASSOUED [12], *cf.* also [11], by means of different methods.

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2. A fibre bundle structure

Throughout this section we assume that (H1), (H2) are satisfied, $\bar{x}$ is an admissible solution of (1), $\bar{u} = d\bar{x}/dt$ and $C = C(\bar{u})$ is a corresponding critical circle of ψ in L_0^α. Recall that C is a 1-dimensional $\mathcal{C}^1$-submanifold of L_0^α (diffeomorphic to S^1).

According to [16, Proposition III.5.8], the restriction of the tangent bundle of L_0^α to C, $T(L_0^\alpha)|_C$, splits and

$$T(L_0^\alpha)\big|_C = T(C) \oplus N(C).$$

The normal bundle $N(C)$ may be chosen in such a way that the fibre at $u \in C$ consists of all $v \in L_0^\alpha$ which satisfy $\langle Ju, v\rangle = 0$ (note that $\dot{u} \in T_u(C)$ and $\langle Ju, \dot{u}\rangle \neq 0$ because for each fixed t,

$$\begin{aligned}\big(Ju(t), \dot{u}(t)\big) = \big(Ju(t), \ddot{x}(t)\big) &= \Big(Ju(t), JH''(x(t))u(t)\Big)\\ &= \Big(u(t), H''(x(t))u(t)\Big)\end{aligned}$$

and $H''(x(t))$ is positive definite). On $T(L_0^\alpha)$ we may define an exponential mapping by $\exp_u(v) = u + v$. Using the argument of [16, Sec. IV.5] it is easy to show that the mapping $(u, v) \mapsto u + v$, where $u \in C$, $v \in N_u(C)$, is a homeomorphism in a neighbourhood of the zero section of C in $N(C)$. Summarizing, we obtain the following

LEMMA 2.1. — *There exists a neighbourhood U of C in L_0^α such that each $w \in U$ can be uniquely represented as $w = u + v$, where $u \in C$ and $\langle Ju, v\rangle = 0$.*

Suppose that $\bar{u}$ has minimal period T/k, $k \geq 1$ an integer. Let U_0 be a neighbourhood of $\bar{u}$ in the set

$$\bar{u} + N_{\bar{u}}(C) \equiv \Big\{\bar{u} + v : v \in L_0^\alpha\,,\ \langle J\bar{u}, v\rangle = 0\Big\}$$

and let $U = S^1 U_0$ (*i.e.*, U is obtained from U_0 by taking orbits under the S^1-action). If U_0 is small enough, U satisfies the conclusion of LEMMA 2.1 and all $u \in U$ have minimal period greated than or equal to T/k.

PROPOSITION 2.2. — *Set $\psi(\bar{u}) = b$ and let U_0, U be as above. Then $(\psi_b \cap U,\ \psi_b \cap U - C)$ has the structure of a fibre bundle pair with base space C and fibre pair $(\psi_b \cap U_0,\ \psi_b \cap U_0 - \{\bar{u}\})$.*

Proof. — The definition of fibre bundle pair may be found in [24, Sec. 5.7]. Let $I \subset \mathbb{R}$ be an open interval of length less than T/k and let

$$V = \Big\{u \in C : u = S_\theta \bar{u} \quad \text{for some} \quad \theta \in I\Big\}.$$

Such sets V cover C. Consider the mappings $p : \psi_b \cap U \to C$ given by $p(u+v) = u$, where $u \in C$ and $\langle Ju, v\rangle = 0$ (*cf.* LEMMA 2.1), and

$$\alpha : V \times \big(\psi_b \cap U_0\,,\ \psi_b \cap U_0 - \{\bar{u}\}\big) \to \big(p^{-1}(V)\,,\ p^{-1}(V) - C\big)$$

given by $\alpha(u, \bar{u} + v) = u + S_\theta v$, where θ is the unique number in I such that $u = S_\theta \bar{u}$. One readily verifies that α is a homeomorphism and $p\alpha : V \times (\psi_b \cap U_0) \to V$ is the projection on the first factor. $\square$

3. Structure of the fibre pair

In this section we assume again that (H1) and (H2) are satisfied, $\bar{x}$ is an admissible solution of (1) and $\bar{u} = d\bar{x}/dt$. Recall that $\bar{u}$ is of class C^1.

It has been shown in [9, Lemma II.1] that the symmetric bilinear form $Q(\bar{u}) : L_0^2 \times L_0^2 \to \mathbb{R}$ given by

$$(2) \qquad Q(\bar{u})(v_1, v_2) = \int_0^T \Big[(Jv_1, Mv_2) + \big(G''(-J\bar{u})Jv_1, Jv_2\big)\Big]\,dt$$

is well defined. Formally, $Q(\bar{u})(v_1, v_2) = \langle \psi''(\bar{u})v_1, v_2\rangle$, but $\psi''(\bar{u})$ may not exist (in particular, it can never exist if $1 < \alpha < 2$). Let K, $A : L_0^2 \to L_0^2$ be given by

$$\begin{aligned} Kv &= -JMv, \\ Av &= -JG''(-J\bar{u})Jv + \frac{1}{T}\int_0^T JG''(-J\bar{u})Jv\,dt. \end{aligned}$$

Then $G(\bar{u})(v, v) = \langle Kv, v\rangle + \langle Av, v\rangle$. Recall that *the index* of the quadratic form $Q(\bar{u})$ is the maximal dimension of a subspace on which $Q(\bar{u})$ is negative definite and *the nullity* is the dimension of the kernel of $K + A$.

LEMMA 3.1. — *There exists a base $(e_i)_{i=1}^{\infty}$ of L_0^2, $e_i \in C([0,T], \mathbb{R}^{2n})$, and a corresponding sequence of real numbers (λ_i) such that $Ke_i = \lambda_i A e_i$,*

$\langle Ae_i, e_j\rangle = \delta_{ij}$, $\lambda_i \to 0$ and $\lambda_i \neq 0$. Furthermore, if $v_1 = \sum \alpha_i e_i$ and $v_2 = \sum \beta_i e_i$, then

$$Q(\bar u)(v_1, v_2) = \sum (1+\lambda_i)\alpha_i\beta_i \tag{3}$$

(in particular, setting $v_1 = v_2$ one sees that the index and the nullity of $Q(\bar u)$ are finite).

Proof. — The argument we sketch here is essentially contained in [8, pp. 36–37]. Since the operator A is selfadjoint and positive definite, it has a square root which is invertible in L_0^2. Moreover, K is compact and $Kv \neq 0$ if $v \neq 0$. It follows that there exist sequences (e_i) and (λ_i) such that $Ke_i = \lambda_i A e_i$, $\langle Ae_i, e_j\rangle = \delta_{ij}$, $\lambda_i \to 0$, $\lambda_i \neq 0$ and (e_i) is a base of L_0^2. A simple computation gives (3). The equality $Ke_i = \lambda_i A e_i$ is equivalent to

$$G''(-J\bar u)Je_i = \lambda_i^{-1} Me_i + \xi,$$

where $\xi \in \mathbb{R}^{2n}$. Since Me_i and $G''(-J\bar u)^{-1}$ are continuous, so is e_i. $\square$

LEMMA 3.2. [9, Lemma II.5]. — *There exist $\delta > 0$, $k > 0$ and $\overline{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ such that $kI \leq \overline{H}''(x) \leq k^{-1}I$ $\forall x \in \mathbb{R}^{2n}$, and if $\mathrm{Min}_t|x - \bar x(t)| \leq \delta$, then $\overline{H}(x) = H(x)$.*

Let $\overline{G}$ denote the Fenchel conjugate of $\overline{H}$ and let

$$\overline{\psi}(u) = \int_0^T \Big[\tfrac12 (Ju, Mu) + \overline{G}(-Ju)\Big]\,dt. \tag{4}$$

Using LEMMA 3.2 and the fact that $\bar u(t) = JH'(\bar x(t))$ one readily obtains the following result which is implicitly contained in [9, § II].

LEMMA 3.3.

(i) *$kI \leq \overline{G}''(y) \leq k^{-1}I$ $\quad\forall y \in \mathbb{R}^{2n}$.*

(ii) *There exists a constant $\delta^* > 0$ such that if $\mathrm{Min}_t|u - \bar u(t)| \leq \delta^*$, then $\overline{G}(-Ju) = G(-Ju)$.*

(iii) *$\overline{\psi}$ is of class C^2 in L_0^p for each $p \in (2,\infty)$ and $\overline{\psi}$ is twice Gâteaux differentiable in L_0^2.*

Next we perform a finite dimensional reduction of $\overline{\psi}$ near $\bar u$. Let (e_i) be the base of L_0^2 given in LEMMA 3.1. Set $W = \mathrm{span}\{e_1, \ldots, e_j\}$ and $Z_0 = \mathrm{span}\{e_{j+1}, e_{j+2}, \ldots\}$. Then $L_0^2 = W \oplus Z_0$. Since $-JM$ is compact and

$$\begin{aligned}\langle \overline{\psi}''(u)v, v\rangle &= \int_0^T \Big[(Jv, Mv) + \big(\overline{G}''(-Ju)Jv, Jv\big)\Big]\,dt \\ &\geq \int_0^T (Jv, Mv)\,dt + k\|v\|_2^2 \qquad \forall u, v \in L_0^2\end{aligned}$$

($\| \ \|_p$ denotes the norm in L^p), one may find a number j and a constant $c > 0$ such that

(5) $$\langle \overline{\psi}''(u)z, z\rangle \geq c\|z\|_2^2 \qquad \forall u \in L_0^2,\ z \in Z_0.$$

Let $u = \bar{u} + w + z$, where $w \in W$, $z \in Z_0$. By (5), the functional $z \mapsto \overline{\psi}(\bar{u} + w + z)$ is strictly convex $\forall w \in W$. Fix a number $p \in (2, \infty)$, $p \geq \alpha$. The restriction of $\overline{\psi}$ to L_0^p (still denoted by $\overline{\psi}$) is of class C^2 and is strictly convex in the z-variable. Let $Z_1 = Z_0 \cap L_0^p$. Then $L_0^p = W \oplus Z_1$. Since $\overline{G}''(y) \leq k^{-1}I \quad \forall y \in \mathbb{R}^{2n}$, $|\overline{G}'(y)| \leq k^{-1}|y| + |\overline{G}'(0)|$ and $\overline{\psi}'$ maps L_0^p into itself (so $\forall u \in L_0^p$, $\overline{\psi}'(u) \in L_0^p \subset L_0^q$, where $p^{-1} + q^{-1} = 1$).

Below we use an argument close to the one which may be found in [15, pp. 597–598] and [3, pp. 120–121]. Let $P : L_0^p = W \oplus Z_1 \to Z_1$ be the projection along W, let P^* be the adjoint of P and consider the mapping $P^*\overline{\psi}'(\bar{u} + .) : W \oplus Z_1 \to Z_1^*$, where $Z_1^* = P^*(L_0^p) \subset P^*(L_0^q)$. Note that $P^*\overline{\psi}'(\bar{u}) = 0$. Since $\overline{\psi} \in C^2$, $P^*\overline{\psi}' \in C^1$. For $z \in Z_1$,

$$\langle P^*\overline{\psi}''(\bar{u})z, z\rangle = \langle \overline{\psi}''(\bar{u})z, z\rangle \geq c\|z\|_2^2$$

according to (5). Hence the derivative $D_z(P^*\overline{\psi}'(\bar{u})) = P^*\overline{\psi}''(\bar{u})|_{Z_1}$ is injective. Furthermore, by (i) of LEMMA 3.3, the mapping $\overline{A}$ given by

$$\overline{A}v = -J\overline{G}''(-J\bar{u})Jv + \frac{1}{T}\int_0^T J\overline{G}''(-J\bar{u})Jv\,dt$$

is an isomorphism of L_0^p onto itself (note that $A|_{L_0^p} = \overline{A}$ and recall A is invertible on L_0^2). Since $-JM$ is compact, $\overline{\psi}''(\bar{u}) = -JM + \overline{A} : L_0^p \to L_0^p$ is a Fredholm operator of index zero. The projections P and P^* have ranges of the same codimension $j < \infty$. Accordingly, also the mapping $P^*\overline{\psi}''(\bar{u})|_{Z_1} : Z_1 \to Z_1^*$ is Fredholm of index zero, and therefore an isomorphism (recall it is injective). It follows now from the implicit function theorem that there exist open balls, B_W in W and B_{Z_1} in Z_1, centered at $0 \in W$ and $0 \in Z_1$ respectively, and a C^1-mapping $w \mapsto z(w)$ from B_W to B_{Z_1} such that for $w \in B_W$ and $z \in B_{Z_1}$, $P^*\overline{\psi}'(\bar{u}+w+z) = 0$ if and only if $z = z(w)$. In other words, for each $w \in B_W$ there is a unique $z = z(w) \in B_{Z_1}$ such that

(6) $$\Big\langle \overline{\psi}'\big(\bar{u} + w + z(w)\big), y\Big\rangle = 0 \qquad \forall y \in Z_1.$$

Note that $z(0) = 0$ because $\overline{\psi}'(\bar{u}) = 0$, and

(7) $$\overline{\psi}\big(\bar{u} + w + z(w)\big) < \overline{\psi}(\bar{u} + w + z) \qquad \forall z \in B_{Z_1},\ z \neq z(w)$$

by strict convexity of $\overline{\psi}$. Let $\bar{\varphi}(w) = \overline{\psi}(\bar{u} + w + z(w))$. Then, using (6) and the fact that $z'(w)v \in Z_1$,

$$\begin{aligned} \langle \overline{\varphi}'(w), v\rangle &= \langle \overline{\psi}'\big(\bar{u} + w + z(w)\big), v + z'(w)v\rangle \\ &= \langle \overline{\psi}'\big(\bar{u} + w + z(w)\big), v\rangle \qquad \forall v \in W. \end{aligned} \tag{8}$$

So w is a critical point of $\overline{\varphi}$ if and only if $\bar{u} + w + z(w)$ is a critical point of $\overline{\psi}$. It is easily seen from (8) that $\varphi \in C^2$ and

$$\langle \bar{\varphi}''(0)v, v\rangle = \Big\langle \overline{\psi}''(\bar{u})\big(v + z'(0)v\big), v\Big\rangle \qquad \forall v \in W.$$

Since $z'(0)v \in Z_1$, it follows from (3) that $\langle \overline{\psi}''(\bar{u})z'(0)v, v\rangle = Q(\bar{u})(z'(0)v, v) = 0$. Hence

$$\langle \overline{\varphi}''(0)v, v\rangle = \langle \overline{\psi}''(\bar{u})v, v\rangle = Q(\bar{u})(v, v)\,,$$

and $\overline{\varphi}''(0)$ has the same index and nullity as $Q(\bar{u})$. Summarizing, we have the following

PROPOSITION 3.4. — *There exist open balls, B_W in W and B_{Z_1} in Z_1, centered at the origin, and a unique C^1-mapping $w \mapsto z(w)$ from B_W to B_{Z_1} such that (6) and (7) are satisfied. Furthermore, w is a critical point of the function $\overline{\varphi} : B_W \to \mathbb{R}$ given by $\overline{\varphi}(w) = \overline{\psi}(\bar{u} + w + z(w))$ if and only if $\bar{u} + w + z(w)$ is a critical point of $\overline{\psi}$, and $\overline{\varphi}''(0)$ has the same index and nullity as $Q(\bar{u})$.*

LEMMA 3.5. — *For each $w \in B_W$, $z(w) \in C^1([0,T], \mathbb{R}^{2n})$. Furthermore, the mapping $w \mapsto z(w)$ is continuous from B_W to $C([0,T], \mathbb{R}^{2n})$.*

Proof. — It follows from (6) and (8) that

$$\overline{\psi}'\big(\bar{u} + w + z(w)\big) = \overline{\varphi}'(w) \in W^*, \tag{9}$$

where $W^* = (I - P^*)(L_0^p)$. Set $u = \bar{u} + w + z(w)$. By (4) and (9),

$$-JMu + J\overline{G}'(-Ju) = \xi + \overline{\varphi}'(w)\,, \tag{10}$$

where $\xi \in \mathbb{R}^{2n}$. Integrating (10) we obtain

$$\xi = \frac{1}{T}\int_0^T J\overline{G}'(-Ju)\,dt.$$

If $w \to w_0$, then $u \to u_0 = \bar{u} + w_0 + z(w_0)$ in L^p, and thus ξ is continuous as a function of w. According to the Legendre reciprocity formula, (10) is equivalent to

$$u = J\overline{H}'\big(Mu - J\xi - J\overline{\varphi}'(w)\big). \tag{11}$$

It is easy to see from LEMMA 3.1 that $W^* = \operatorname{span}\{Ae_1, \dots, Ae_j\} = \operatorname{span}\{JMe_1, \dots, JMe_j\}$. So all elements of W^* are continuously differentiable functions of t. Since Mu is continuous, it follows from (11) that u and $z(w) = u - \bar{u} - w$ are continuous. Hence $Mu \in C^1$, and by (11) again, $z(w) \in C^1$. Also,

$$\|M(u-u_0)\|_\infty \le C_1 \big\|M(u-u_0)\big\|_{H^{1,p}} \le C_2\|u-u_0\|_p$$

for appropriate constants C_1 and C_2. So if $w \to w_0$, then $Mu \to Mu_0$ in L^∞, and according to (11), $z(w) \to z(w_0)$ in L^∞. $\square$

Now we reformulate PROPOSITION 3.4 in terms of L_0^α and ψ. Let Z be the closure of Z_1 in the L^α-topology. Then $L_0^\alpha = W \oplus Z$.

PROPOSITION 3.6. — *There exist open balls, B_W in W and B_Z in Z, centered at the origin, and a unique C^1-mapping $w \mapsto z(w)$ from B_W to B_Z such that if $w \in B_W$, then*

$$\big\langle \psi'(\bar{u}+w+z), y\big\rangle = 0 \qquad \forall y \in Z \quad \text{if and only if} \quad z = z(w), \tag{12}$$

$$\psi\big(\bar{u}+w+z(w)\big) < \psi(\bar{u}+w+z) \qquad \forall z \in B_Z,\ z \neq z(w). \tag{13}$$

Furthermore, w is a critical point of the function $\varphi : B_W \to \mathbb{R}$ given by $\varphi(w) = \psi(\bar{u}+w+z(w))$ if and only if $\bar{u}+w+z(w)$ is a critical point of ψ, and $\varphi''(0)$ has the same index and nullity as $Q(\bar{u})$.

Proof. — Our argument is similar to [9, Proof of Lemma II.7]. Since $\int_0^T G(-Ju)\,dt$ is continuous and convex and $-JM$ is compact, ψ is weak lower semicontinuous. So for each $w \in B_W$, the infimum of $\psi(\bar{u}+w+z)$ over $\overline{B}_Z$ is attained at some $z = z_0$. For such z_0 we have

$$\big\langle \psi'(u) + \lambda z_0|z_0|^{\alpha-2}, y\big\rangle = 0 \qquad \forall y \in Z, \tag{14}$$

where $u = \bar{u}+w+z_0$ and $\lambda \ge 0$ is the Lagrange multiplier. We shall show that if B_W and B_Z are sufficiently small, then $z_0 = z(w)$.

Let (B_Z^m) be a sequence of open balls in Z, centered at the origin, with radii converging to zero as $m \to \infty$. Since $z(w) \to z(0) = 0$ in L^p as $w \to 0$ and since $p \ge \alpha$, we may choose a sequence (B_W^m) of open balls in

W, centered at the origin, with radii tending to zero, in such a way that $z(w) \in B_Z^m \quad \forall w \in B_W^m$. Let $u_m = \bar{u} + w_m + z_m$, where $w_m \in B_W^m$ and z_m is a point at which $\psi(\bar{u} + w_m + z)$ attains its infimum in $\overline{B}_Z^m$. Then (14) with $z_0 = z_m$, $u = u_m$ and $\lambda = \lambda_m$ is satisfied. We claim that

$$\|z_m\|_\infty \to 0 \qquad \text{as} \quad m \to \infty, \tag{15}$$

uniformly with respect to the choice of $w_m \in B_W^m$.

Set $y = z_m/\|z_m\|_\alpha$ in (14). Then

$$\frac{\langle \psi'(u_m), z_m\rangle}{\|z_m\|_\alpha} + \lambda_m \|z_m\|_\alpha^{\alpha-1} = 0.$$

Since $\psi'(u_m) \to \psi'(\bar{u}) = 0$, $\lambda_m\|z_m\|_\alpha^{\alpha-1} \to 0$. Hence $\lambda_m z_m |z_m|^{\alpha-2} \to 0$ in L^β (because $\|z\|_\alpha^{\alpha-1} = \|z|z|^{\alpha-2}\|_\beta$). (14) with $z = z_m$, $u = u_m$ and $\lambda = \lambda_m$ is equivalent to

$$-JMu_m + JG'(-Ju_m) + \lambda_m z_m |z_m|^{\alpha-2} = a_m + \xi_m, \tag{16}$$

where $a_m \in W^*$ and $\xi_m \in \mathbb{R}^{2n}$. Since the left-hand side of (16) approaches $\bar{\xi} = T^{-1}\int_0^T JG'(-J\bar{u})\,dt$ as $m \to \infty$, $a_m + \xi_m \to \bar{\xi}$. Let

$$G_m(t,z) = G\Big(-J\bar{u}(t) - Jw_m(t) - Jz\Big) + \lambda_m \alpha^{-1}|z|^\alpha.$$

Then $G_m(t,.)$ is strictly convex, and denoting the derivative of G_m with respect to the second variable by G'_m,

$$\begin{aligned}\lambda_m\alpha^{-1}|z|^\alpha \le G_m(t,z) &\le \big(G'_m(t,z),z\big) + G_m(t,0)\\ &\le \big|G'_m(t,z)\big|\,|z| + G_m(t,0).\end{aligned} \tag{17}$$

According to (16), $G'_m(t, z_m(t)) = JMu_m(t) + a_m(t) + \xi_m \equiv v_m(t)$, or equivalently, $z_m(t) = H'_m(t, v_m(t))$, where $H_m(t,.)$ is the Fenchel conjugate of $G_m(t,.)$. It is easy to verify that H'_m is continuous in both variables. It follows that z_m is a continuous function (because v_m is) and (16) is satisfied pointwise for all t. Suppose (15) is false. Then we may find $w_m \in B_W^m$ and $t_m \to \bar{t}$ such that, possibly after passing to a subsequence, $|z_m(t_m)|$ is bounded away from zero. Since

$$\|Mu_m - M\bar{u}\|_\infty \le C_1\|Mu_m - M\bar{u}\|_{H^{1,\alpha}} \le C_2\|u_m - \bar{u}\|_\alpha,$$

$Mu_m \to M\bar{u}$ in L^∞. So $|G'_m(t, z_m(t))| = |v_m(t)| \le C$, where C is a constant independent of t. Hence by (17), $\lambda_m|z_m(t_m)|^{\alpha-1}$ is bounded.

Using (16), it follows that also $y_m(t_m) \equiv G'(-Ju_m(t_m))$ is bounded, and by the Legendre reciprocity formula, so is $u_m(t_m) = JH'(y_m(t_m))$. Consequently, taking a subsequence if necessary, $z_m(t_m) = u_m(t_m) - \bar{u}(t_m) - w_m(t_m) \to \bar{z} \neq 0$. Since $\lambda_m |z_m(t_m)|^{\alpha-1}$ is bounded and $\bar{z} \neq 0$, we may assume that $\lambda_m \to \bar{\lambda}$. Recall that (16) is satisfied pointwise and set $t = t_m$ in (16). Passing to the limit we obtain

$$JG'\big(-J\bar{u}(\bar{t}) - J\bar{z}\big) + \bar{\lambda}\bar{z}|\bar{z}|^{\alpha-2} = JM\bar{u}(\bar{t}) + \bar{\xi} = JG'\big(-J\bar{u}(\bar{t})\big).$$

Taking the inner product (in $\mathbb{R}^{2n}$) with $\bar{z}$ gives

$$\Big(G'(-J\bar{u}(\bar{t}) - J\bar{z}) - G'\big(-J\bar{u}(\bar{t})\big), -J\bar{z}\Big) + \bar{\lambda}|\bar{z}|^{\alpha} = 0.$$

Since G is strictly convex (and therefore G' is strictly monotone) and $\bar{\lambda} \geq 0$, $\bar{z} = 0$. This is the desired contradiction. According to (15), we may find B_W and B_Z such that if $w \in B_W$ and $u = \bar{u} + w + z_0$ satisfies (14), then $\|u - \bar{u}\|_\infty < \delta^*$, where δ^* is the constant in (ii) of LEMMA 3.3. So $\psi(u) = \bar{\psi}(u)$. Note that hitherto we have not used the minimization property of z_0 but only the fact that (14) is satisfied. By LEMMA 3.5, $z(w) \to z(0) = 0$ in L^∞ as $w \to 0$. We may therefore choose B_W in such a way that $\|(\bar{u} + w + z(w)) - \bar{u}\|_\infty < \delta^*$. Hence by (7),

$$\psi(\bar{u} + w + z_0) = \bar{\psi}(\bar{u} + w + z_0) > \bar{\psi}\big(\bar{u} + w + z(w)\big) = \psi\big(\bar{u} + w + z(w)\big)$$

if $z_0 \neq z(w)$. It follows that $z_0 = z(w)$ and (13) is satisfied. In order to verify (12), note first that $z(w) \in B_Z$, and therefore (14) holds with $z_0 = z(w)$ and $\lambda = 0$. If $\langle \psi'(\bar{u} + w + z_0), y\rangle = 0 \quad \forall y \in Z$, then (14) with $\lambda = 0$ is satisfied. So $\|u - \bar{u}\|_\infty < \delta^*$, where again $u = \bar{u} + w + z_0$. Consequently,

$$\begin{aligned} 0 &= \frac{d}{dt}\psi(\bar{u} + w + z_0 + ty)\big|_{t=0} = \frac{d}{dt}\bar{\psi}(\bar{u} + w + z_0 + ty)\big|_{t=0} \\ &= \big\langle \bar{\psi}'(\bar{u} + w + z_0), y\big\rangle \end{aligned}$$

for all $y \in L^\infty \cap Z$, and by continuity, for all $y \in Z_1$. Hence $z_0 = z(w)$ according to (6).

To complete the proof, note that $\varphi(w) = \psi(\bar{u} + w + z(w)) = \bar{\psi}(\bar{u} + w + z(w)) = \bar{\varphi}(w)$ and use the second part of PROPOSITION 3.4. □

Let $C = C(\bar{u})$ be the critical circle corresponding to $\bar{u}$ and recall from § 2 that $d\bar{u}/dt \in T_{\bar{u}}(C)$. Since $\bar{u} = d\bar{x}/dt = JH'(\bar{x})$, $\bar{x} = G'(-J\bar{u})$, and therefore $M(d\bar{u}/dt) = \bar{u} = d\bar{x}/dt = -G''(-J\bar{u})Jd\bar{u}/dt$. So by (2),

$Q(\bar{u})(d\bar{u}/dt,.) = 0$, and $d\bar{u}/dt \in \mathrm{Ker}(K + A) \subset W$. We may assume that $d\bar{u}/dt = ke_1$, where k is a constant. Let

$$W_0 = \mathrm{span}\{e_2, \dots, e_j\}$$

and $B_{W_0} = B_W \cap W_0$. Define $\chi = \varphi|_{B_{W_0}}$.

PROPOSITION 3.7. — *0 is a critical point of χ and* $\mathrm{Index}(\chi''(0)) = \mathrm{Index}(Q(\bar{u}))$, $\mathrm{Nullity}(\chi''(0)) = \mathrm{Nullity}(Q(\bar{u})) - 1$. *Furthermore, 0 is an isolated critical point of χ if and only if C is an isolated critical circle.*

Proof. — We need only show that if C is an isolated critical circle, then 0 is an isolated critical point of χ. Other conclusions follow from PROPOSITION 3.6.

Recall that $W^* = \mathrm{span}\{Ae_1, \dots, Ae_j\}$ and suppose $\chi'(w) = 0$. Then $\varphi'(w) \in \mathrm{span}\{Ae_1\} = \mathrm{span}\{A(d\bar{u}/dt)\}$. Since $A(d\bar{u}/dt) = JM(d\bar{u}/dt) = J\bar{u}$, $\psi'(u) = \varphi'(w) = sJ\bar{u}$, where $u = \bar{u} + w + z(w)$ and s is a real number. Using the S^1-invariance of ψ and the fact that $u \in C^1$ (*cf.* LEMMA 3.5) and $\langle d\bar{u}/dt, J\bar{u}\rangle = \langle d\bar{u}/dt, A(d\bar{u}/dt)\rangle = k^2$ we obtain

$$\begin{aligned} 0 = \frac{d}{d\theta}\psi(S_\theta u)\Big|_{\theta=0} &= \big\langle \psi'(u), \dot{u}\big\rangle = s\langle J\bar{u}, \dot{u}\rangle = s\big\langle d\bar{u}/dt, Ju\big\rangle \\ &= sk^2 + s\big\langle d\bar{u}/dt, Jw + Jz(w)\big\rangle. \end{aligned}$$

Hence

$$|s|k^2 \le C|s| \, \big\|w + z(w)\big\|_\alpha ,$$

where C is a constant independent of w. So if $\|w\|_\alpha$ is sufficiently small, $s = 0$ and $\psi'(u) = 0$. It follows that $u = \bar{u}$ and $w = 0$. $\square$

Next we shall show that $D_z\psi$ satisfies the following compactness condition.

LEMMA 3.8. — *Each sequence (u_m) such that $u_m = \bar{u} + w_m + z_m$, $w_m \in B_{W_0}$, $z_m \in B_Z$ and $D_z\psi(u_m) \to 0$, possesses a convergent subsequence.*

Proof. — We may assume that $w_m \to w$ and $z_m \to z$ weakly. Then $u_m \to u$ weakly. Since $D_z\psi(u_m) \to 0$, $\psi'(u_m) = \alpha_m + \epsilon_m$, where $\alpha_m \in W^*$ and $\epsilon_m \to 0$ in L_0^β, or equivalently,

$$-JMu_m + JG'(-Ju_m) = \xi_m + \alpha_m + \epsilon_m \tag{18}$$

for some $\xi_m \in R^{2n}$. Since the left-hand side of this equality is bounded in L^β (by (H2)), $\alpha_m \to \alpha$ and $\xi_m \to \xi$, possibly after passing to a subsequence. By the Legendre reciprocity formula, (18) is equivalent to

$$u_m = JH'\big(Mu_m - J(\xi_m + \alpha_m + \epsilon_m)\big).$$

Taking limits we see that $u_m \to JH'(Mu - J\xi - J\alpha)$ (*cf.* [8, Proof of Proposition III.4]. $\square$

Let

$$V = \Big\{ w + z \in B_{W_0} \oplus Z : \psi(\bar{u} + w + z) < \psi\big(\bar{u} + w + z(w)\big) + \epsilon_0 \Big\},$$

where $\epsilon_0 > 0$ is given, and let V_0 be the connected component of V containing the set $\{w + z : z = z(w)\}$.

LEMMA 3.9. — *If ϵ_0 is sufficiently small, $V_0 \subset B_{W_0} \oplus B_Z$.*

Proof. — We may assume without loss of generality that the conclusions of PROPOSITION 3.6 hold in slightly larger balls, $B'_{W_0} \supset \overline{B}_{W_0}$ and $B'_Z \supset \overline{B}_Z$. Suppose that the assertion of the lemma is false. Then we may find $w_m \in B_W$ and $z_m \in \partial B_Z \equiv \overline{B}_Z - B_Z$ such that $\psi(\bar{u} + w_m + z_m) < \psi(\bar{u} + w_m + z(w_m)) + (2m)^{-1}$. Using the fact that ψ is Lipschitz continuous on bounded sets we may assume after passing to a subsequence that $w_m \to w$ and $\psi(\bar{u} + w + z_m) < \psi(\bar{u} + w + z(w)) + m^{-1}$. By Ekeland's variational principle [7, Corollary 11], there is a $z'_m \in \overline{B}'_Z$ such that $\|z_m - z'_m\|_\alpha \le m^{-1/2}$ and

$$\psi(\bar{u} + w + z) - \psi(\bar{u} + w + z'_m) \ge -\frac{1}{\sqrt{m}} \|z - z'_m\|_\alpha \qquad \forall z \in \overline{B}'_Z.$$

Since $\overline{B}_Z \subset B'_Z$, $z'_m \in B'_Z$ for almost all m. So setting $z = z'_m + ty$, $t > 0$, in the inequality above, dividing by t and letting $t \to 0$ we obtain $\|D_z\psi(\bar{u} + w + z'_m)\|_\beta \le m^{-1/2}$. By LEMMA 3.8, $z'_m \to \bar{z} \in \partial B_Z$ and $D_z\psi(\bar{u} + w + \bar{z}) = 0$, a contradiction to (12). $\square$

Choose now ϵ_0 so that $V_0 \subset B_{W_0} \oplus B_Z$.

PROPOSITION 3.10. — *Let $j : B_{W_0} \to L^\alpha_0$ be the embedding given by $j(w) = \bar{u} + w + z(w)$ and let $U_0 = \bar{u} + V_0$. Then the pair $(j(B_{W_0})$, $j(B_{W_0} - \{0\}))$ is a deformation retract of $(U_0, U_0 - (\bar{u} + B_Z))$. Moreover, the deformation r may be chosen so that for each $u \in U_0$, $r(0, u) = u$ and $\psi(r(t, u))$ is a nonincreasing function of t.*

Proof. — On U_0, $D_z\psi(\bar{u} + w + z) = 0$ if and only if $z = z(w)$ according to (12). Using the method of [20, Lemma 1.6], it is easy to construct a mapping $F : V_0 - \{w + z : z = z(w)\} \to Z$ which is locally Lipschitz continuous and satisfies

$$\begin{aligned} \big\|F(w + z)\big\|_\alpha &\le 2 \big\|D_z\psi(\bar{u} + w + z)\big\|_\beta, \\ \big\langle \psi'(\bar{u} + w + z), F(w + z) \big\rangle &\ge \big\|D_z\psi(\bar{u} + w + z)\big\|^2_\beta, \end{aligned} \tag{19}$$

for all $w+z$ in the domain of F (note that for each fixed w, $F(w+.)$ is a pseudogradient vector field for the functional $z \mapsto \psi(\bar u+w+z)$). Consider the flow η given by

$$\begin{cases} \dot\eta(t,w+z) = -\big(\psi(\bar u+w+\eta) - \psi(\bar u+w+z(w))\big)F(w+\eta) \\ \qquad\qquad\qquad \equiv F_0(w+\eta), \\ \eta(0,w+z) = z, \end{cases}$$

where $w+z \in V_0$, $z \neq z(w)$. The vector field F_0 is bounded on its domain and locally Lipschitz continuous (because $z(w)$ is differentiable). By (19), (12) and (13),

$$\begin{aligned} \frac{d}{dt}&\psi\big(\bar u+w+\eta(t,w+z)\big) \\ &= \big\langle \psi'(\bar u+w+\eta), F_0(w+\eta)\big\rangle \\ &\le -\big(\psi(\bar u+w+\eta) - \psi(\bar u+w+z(w))\big) \\ &\qquad\qquad \times \big\|D_z\psi(\bar u+w+\eta)\big\|_\beta^2 < 0 \end{aligned} \tag{20}$$

whenever $\eta(t,w+z) \neq z(w)$. Hence $w+\eta$ cannot leave V_0 for $t>0$. Moreover, by (19),

$$\begin{aligned} \frac{d}{dt}&\big[\psi(\bar u+w+\eta) - \psi(\bar u+w+z(w))\big] \\ &= \langle \psi'(\bar u+w+\eta), F_0(w+\eta)\rangle \\ &\ge -C\big(\psi(\bar u+w+\eta) - \psi(\bar u+w+z(w))\big), \end{aligned}$$

where $C>0$ is a constant independent of w, z and t. Thus,

$$\psi(\bar u+w+\eta)-\psi\big(\bar u+w+z(w)\big) \ge \big[\psi(\bar u+w+z)-\psi(\bar u+w+z(w))\big]e^{-Ct} > 0,$$

and $\eta(t,w+z) \neq z(w)$ whenever $t \ge 0$. It follows that η is defined for all $t \ge 0$ (*cf.* [20, (1.13)]).

We shall prove that if $w \to w_0$, $z \to z_0$ and $t \to \infty$, then

$$\eta(t,w+z) \to z(w_0). \tag{21}$$

Let

$$V_\epsilon = \Big\{w+z \in V_0 : \|w-w_0\|_\alpha < \epsilon, \psi(\bar u+w+z) < \psi\big(\bar u+w+z(w)\big)+\epsilon\Big\}.$$

If N is a neighbourhood of $w_0 + z(w_0)$, then, by the argument of LEMMA 3.9, $V_\epsilon \subset N$ for all sufficiently small ϵ. So it remains to show that one can find δ and T such that $w + \eta(t + w + z) \in V_\epsilon$ whenever $\|w - w_0\|_\alpha < \delta$, $\|z - z_0\|_\alpha < \delta$ and $t > T$. Given $\delta \leq \epsilon$, it follows from LEMMA 3.8 that if $\|w - w_0\|_\alpha < \delta$ and $w+\eta \notin V_\epsilon$, then $\|D_z\psi(\bar{u}+w+\eta)\|_\beta \geq \delta_0$ for some $\delta_0 > 0$. So by (20),

$$\begin{aligned}\frac{d}{dt}&\psi(\bar{u} + w + \eta)\\ &\leq -\big(\psi(\bar{u} + w + \eta) - \psi(\bar{u} + w + z(w))\big)\big\|D_z\psi(\bar{u} + w + \eta)\big\|_\beta^2\\ &\leq -\epsilon\,\delta_0^2.\end{aligned}$$

Furtheremore, by (20) again, $w + \eta$ can enter but not leave the set V_ϵ. Consequently, if $w + \eta(t, w + z) \notin V_\epsilon$,

$$\psi\big(\bar{u} + w + \eta(t, w + z)\big) - \psi(\bar{u} + w + z) \leq -\epsilon\,\delta_0^2 t.$$

Since $C_1 \leq \psi(\bar{u} + w + z) \leq C_2$, where the constants C_1 and C_2 are independent of the choice of $w + z \in V_0$, $t \leq (C_2 - C_1)\epsilon^{-1}\delta_0^{-2} \equiv T$. It follows that $w + \eta \in V_\epsilon$ for all $t > T$. This completes the proof of (21).

Now it remains to define the deformation retraction r by setting

$$r(t, \bar{u} + w + z) = \begin{cases} \bar{u} + w + \eta\big(t(1-t)^{-1}, w + z\big) \\ \qquad\qquad \text{if } z \neq z(w) \text{ and } 0 \leq t < 1, \\ \bar{u} + w + z(w) \\ \qquad\qquad \text{if } t = 1 \text{ or } z = z(w) \text{ and } 0 \leq t \leq 1. \end{cases} \quad \square$$

By shrinking the balls B_{W_0} and B_Z if necessary, we may assume that U_0 is so small that the conclusions of PROPOSITION 2.2 are valid. Recall that

$$W_0 \oplus Z = N_{\bar{u}}(C) = \big\{v \in L_0^\alpha : \langle J\bar{u}, v\rangle = 0\big\}.$$

Now we state the main result of this section.

THEOREM 3.11. — *Let $\psi(\bar{u}) = b$ and $C = C(\bar{u})$. Then there exists a neighbourhood U_0 of $\bar{u}$ in $\bar{u} + (W_0 \oplus Z)$ and a corresponding neighbourhood $U = S^1 U_0$ of C in L_0^α such that $(\psi_b \cap U, \psi_b \cap U - C)$ is a fibre bundle pair with base space C and fibre pair $(\psi_b \cap U_0, \psi_b \cap U_0 - \{\bar{u}\})$. The fibre pair has the homotopy type of $(\chi_b, \chi_b - \{0\})$, where the function $\chi \in C^2(B_{W_0}, \mathbb{R})$ is given by $\chi(w) = \psi(\bar{u} + w + z(w))$ and has the properties that $\chi'(0) = 0$,* $\operatorname{Index}\chi''(0) = \operatorname{Index} Q(\bar{u})$, $\operatorname{Nullity}\chi''(0) = \operatorname{Nullity} Q(\bar{u}) - 1$, *and 0 is an isolated critical point of χ if and only if C is an isolated critical circle.*

Proof. — The first part of the theorem coincides with PROPOSITION 2.2. The statement concerning the homotopy type follows from PROPOSITION 3.10 upon observing that $\psi(\bar{u}+z) > \psi(\bar{u}) = b$ if $z \in B_Z$, $z \neq 0$. Finally, the properties of χ are given in PROPOSITION 3.7. $\square$

4. Existence of two closed Hamiltonian trajectories

Let S be the boundary of a compact convex subset A of $\mathbb{R}^{2n}$. Suppose that the interior of A (denoted $\mathrm{Int}(A)$) is nonempty, $0 \in \mathrm{Int}(A)$, and S is of class C^2 and has strictly positive Gaussian curvature. For $x \in S$, denote the outward unit normal vector by $N(x)$. We want to find the number of closed trajectories of the flow

$$\dot{x} = JN(x) \qquad \text{on } S. \tag{22}$$

This problem may be put in Hamiltonian form

$$\dot{x} = JH'(x), \quad x(0) = x(T_0), \quad H(x) = 1, \tag{23}$$

where H is strictly convex, of class C^2 in a neighbourhood of S and $H'(x) \neq 0$ on S [8, § 2]. Here x and T_0 are the unknown. We may assume that

$$H(\lambda x) = \lambda^{\beta} H(x) \qquad \forall x \in \mathbb{R}^{2n},\ \lambda > 0, \tag{24}$$

where $\beta \in (1,2)$. Then $H \in C^2(\mathbb{R}^{2n} - \{0\}, \mathbb{R})$. It is easy to see by homothesy [8, Lemma II.4] that $x(t)$ is a solution of (23) if and only if $x_h(t) = h^{1/\beta} x(h^{1-2/\beta} t)$ is a solution of

$$\dot{x}_h = JH'(x_h), \quad x_h(0) = x_h(T_0 h^{2/\beta - 1}), \quad H(x_h) = h.$$

Consequently, the fixed energy problem (23) is equivalent to the fixed period problem (1). Furthermore, if x_1 is a solution of (1) with minimal period T, then for each positive integer k, $x_k(t) = k^{1/(\beta-2)} x_1(kt)$ is a solution of (1) with minimal period T/k. Observe that the trajectory of the flow (22) corresponding to x_k is obtained by covering the one corresponding to x_1 k times.

Denote the Fenchel conjugate of H by G. If $\alpha^{-1} + \beta^{-1} = 1$, then, according to (24),

$$G(\lambda y) = \lambda^{\alpha} G(y) \qquad \forall y \in \mathbb{R}^{2n},\ \lambda > 0. \tag{25}$$

Since the hypersurface S has positive Gaussian curvature, $H''(x)$ is invertible $\forall x \neq 0$ and $G \in C^2(\mathbb{R}^{2n} - \{0\}, \mathbb{R})$. Moreover, since $\alpha > 2$, it follows from (25) that G has continuous second derivative at the origin. So $G \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. Note that the hypotheses (H1), (H2) of Section 2 are satisfied and each nonzero solution of (1) is admissible.

Using the above results and the ones quoted in Section 1 we obtain the following

PROPOSITION 4.1.

(i) *Let* $\psi : L_0^\alpha \to \mathbb{R}$ *be defined by*

$$\psi(u) = \int_0^T \left[\tfrac{1}{2}(Ju, Mu) + G(-Ju)\right] dt.$$

Then ψ *is of class* C^2, u *is a critical point of* ψ *if and only if* $x = Mu + \xi$ *is a solution of* (1) *for some* $\xi \in \mathbb{R}^{2n}$, *and* x *is admissible whenever* $u \neq 0$.

(ii) *If* $u_1(t)$ *is a critical point of* ψ, *then so is* $u_k(t) = k^{1/(2-\alpha)} u_1(kt)$ *for each integer* $k \geq 2$. *Furthermore, if* u_1 *has minimal period* T *and* x_k *is a trajectory of the flow* (22) *corresponding to* u_k, *then* x_k *is obtained by covering the trajectory* x_1 k *times* (*so all* u_k *correspond to trajectories which are geometrically the same*).

It is known that there always exists one closed trajectory of the flow (22) (see [21, § 2] and the references therein), there exist at least n such trajectories if for some $r > 0$, $r \leq |x| < r\sqrt{2}$ $\forall x \in S$ ([10], *cf.* also [2]), and generically, they are infinitely many ([8], *cf.* also [25]). We shall show that there always exist at least two closed trajectories. The same result has been obtained by EKELAND and LASSOUED [11, 12] by means of different methods.

THEOREM 4.2. — *Let* S *be the boundary of a compact convex subset* $A \subset \mathbb{R}^{2n}$, $n \geq 3$, *such that* $0 \in \operatorname{Int}(A)$. *Suppose that* S *is of class* C^2 *and has strictly positive Gaussian curvature. Then there exist at least two geometrically distinct closed trajectories of the flow* (22).

The proof will be given in Section 6.

5. Index of iterated solutions and Morse relations

First we summarize some results which may be found in [8, 17]. Let u_1 be a critical point of ψ, with minimal period T, and let u_k be as in PROPOSITION 4.1. Denote the index and the nullity of the quadratic form $Q(u_k) = \psi''(u_k)$ by i_k and n_k respectively. Then

$$i_k = \sum_{\omega^k = 1} j(\omega), \qquad n_k = \sum_{\omega^k = 1} m(\omega), \tag{26}$$

where j and m are functions from $S^1 = \{\omega \in \mathbb{C} : |\omega| = 1\}$ to the set of nonnegative integers. Let x_1 be the solution of (1) corresponding to u_1 and let $R(t)$ be the solution of

$$\begin{cases} \dot{R}(t) = JH''(x_1(t))R(t), \\ R(0) = I, \end{cases}$$

(I is the unit matrix). Recall that the eigenvalues of $R(T)$ are called *the Floquet multipliers of* x_1. It is known that 1 is an eigenvalue of multiplicity at least 2. The function m is defined by $m(\omega) = \dim_{\mathbb{C}} \operatorname{Ker}(\omega I - R(T))$. Denote $j(\omega^{\pm}) = \lim_{\epsilon \to 0^{\pm}} j(e^{i\epsilon}\omega)$. We shall need the following properties of j [12, 17] :

PROPOSITION 5.1.

(i) *If $\omega \in S^1$ is not a Floquet multiplier, then j is continuous at ω ;*

(ii) $j(1) = i_1$;

(iii) $j(\bar{\omega}) = j(\omega) \qquad \forall \omega \in S^1$;

(iv) *if $\omega \in S^1 - \{1\}$ is a Floquet multiplier of type (p, q) in the sense of Krein and of multiplicity m (so that $p+q = m$), then $j(\omega^+)-j(\omega^-) = q-p$ and $j(\omega) \leq j(\omega^+) \leq j(\omega) + q$, $j(\omega) \leq j(\omega^-) \leq j(\omega) + p$;*

(v) *$j(1^{\pm}) \geq i_1 + n$, and $j(1^{\pm}) = i_1 + n + 1$ provided $n_1 = 1$ and α in (25) is sufficiently close to 2.*

A proof of this proposition may be found in [12].

COROLLARY 5.2. — *Suppose that α is close to 2. Then*

(i) $j(-1) \geq 2$;

(ii) *if $j(-1) = 2$, we have $i_{k+1} - i_k \geq 2$ for all k ;*
if in addition $n \geq 3$, $i_{k+1} - i_k > 2$ for some k.

Proof.

(i) By the S^1-invariance of ψ, $n_1 \geq 1$. Suppose $n_1 > 1$. Then 1 is an eigenvalue of $R(T)$ of multiplicity at least 4, so there are at most $n - 2$ Floquet multipliers (counted with their multiplicities) on the open upper half-circle of S^1. It follows from (iv) and (v) of PROPOSITION 5.1 that $j(1^+) \geq i_1 + n$ and $j(e^{i\theta})$ can drop by at most $n - 2$ as θ goes from 0^+ to π. Hence $j(-1) \geq i_1 + n - (n - 2) \geq 2$. If $n_1 = 1$, the same argument shows that $j(-1) \geq i_1 + n + 1 - (n - 1) \geq 2$.

(ii) If $j(-1) = 2$, one sees from (iv) of PROPOSITION 5.1 that at each Floquet multiplier on the open upper half-circle, $j(\omega^+) - j(\omega^-) = -m$. So $p = m$, $q = 0$ and $j(\omega) = j(\omega^+)$. It follows that $j(e^{i\theta})$ is nonincreasing as θ increases from 0^+ to π. In particular, $j(\omega) \geq 2 \quad \forall \omega \in S^1 - \{1\}$ and $j(e^{i2\pi m/(k+1)}) \geq j(e^{i2\pi m/k})$ for all integers $m \in [1, k/2]$. Thus, according to (26), $i_{k+1} - i_k \geq 2$. Finally, if $n \geq 3$, $j(1^{\pm}) \geq 3$. So $i_{k+1} - i_k \geq 3$ for some k. $\square$

The functional ψ of PROPOSITION 4.1 is of class $\mathcal{C}^2$, satisfies the Palais-Smale condition [8, Proposition III.4] and is bounded below (because $\psi(u) \to \infty$ as $\|u\|_\alpha \to \infty$, *cf.* [8, Proof of Proposition III.4] or (28)). One can show (see *Remark* in § 6) that if $u \neq 0$ is a critical point of ψ, then $\psi(u) < 0$. *Suppose from now on that all critical circles of* ψ *are isolated.* It follows that the nonzero critical levels may be ordered into an increasing sequence $(b_k)_{k=1}^{\infty}$ with $b_k \to 0$. Denote by K_b the set of critical points of ψ at level b. Observe that 0 is a non-isolated critical level, $K_0 = \{0\}$, and each K_{b_k} is the union of a finite number of disjoint critical circles.

LEMMA 5.3. — *Let* $b < c$ *and suppose that there are no critical levels in the interval* (b, c). *Then* ψ_b *is a deformation retract of* $\psi_c - K_c$.

Proof. — Since our argument is similar to the one given in [4, pp. 385–387], we only point out the differences. If $b = 0$ or b is not a critical level, the proof in [4] applies. So let $b = b_k$. Then K_b is the union of a finite number of critical circles $C_1, \ldots, C_p$. For each $i = 1, \ldots, p$ choose $\bar{u}_i \in C_i$ and an open neighbourhood U_{i0} of $\bar{u}_i$ in

$$\bar{u}_i + N_{\bar{u}_i}(C_i) = \left\{\bar{u}_i + v : v \in L_0^{\alpha}, \langle J\bar{u}_i, v\rangle = 0\right\}$$

(*cf.* § 2). Let $U_i = S^1 U_{i0}$, $U = U_1 \cup \ldots \cup U_p$ and $\psi_i = \psi|_{U_{i0}}$. If all U_{i0} are sufficiently small, then the sets U_i are pairwise disjoint, $U - K_b$ contains no critical points, and according to PROPOSITIONS 3.6 and 3.7, $\psi_i'(u) = 0$ if and only if $u = \bar{u}_i$. By [20, Lemma 1.6], there exists a pseudogradient vector field F_i for ψ_i on $U_{i0} - \{\bar{u}_i\}$. Setting $F_0(S_\theta u) = S_\theta F_i(u)$ $\forall u \in U_{i0} - \{\bar{u}_i\}$ we obtain an equivariant field F_0 on $\widetilde{U} = U - K_b$. Let $N \subset U$ be a closed neighbourhood of K_b and $\widetilde{V} = \psi_c - (\psi_b \cup N \cup K_c)$. On $\widetilde{V}$ there exists a pseudogradient vector field F for ψ. Set

$$\widetilde{F}(u) = \rho(u)F(u) + \rho_0(u)F_0(u) \qquad \forall u \in \widetilde{U} \cup \widetilde{V},$$

where ρ, ρ_0 are Lipschitz continuous functions which vanish outside $\widetilde{V}$ and $\widetilde{U}$ respectively and satisfy $\rho(u), \rho_0(u) \geq 0$, $\rho(u) + \rho_0(u) = 1$ $\forall u \in \widetilde{U} \cup \widetilde{V}$. Define a flow η by

$$\begin{cases} \dot{\eta}(t, u) = -\dfrac{(\psi(u) - b)\widetilde{F}(\eta)}{\langle \psi'(\eta), \widetilde{F}(\eta)\rangle}, \\ \eta(0, u) = u, \quad u \in \widetilde{U} \cup \widetilde{V}. \end{cases}$$

Note that if $u = u_i + v \in N - K_b$, where $u_i \in C_i$ and $v \in N_{u_i}(C_i)$, then $\widetilde{F}(u) = F_0(u) \in N_{u_i}(C_i)$. Furthermore, $\langle \psi'(u), F(u)\rangle \geq \|\psi'(u)\|_\alpha^2$ $\forall u \in \widetilde{V}$ and $\langle \psi'(u), F_0(u)\rangle = \langle \psi_i'(u), F_i(u)\rangle \geq \|\psi_i'(u)\|_\alpha^2$ $\forall u \in U_{i0} - \{\bar{u}_i\}$. Since ψ and ψ_i satisfy the Palais-Smale condition, it follows that if $A \subset \widetilde{U} \cup \widetilde{V}$ and $\overline{A} \cap K_b = \overline{A} \cap K_c = \emptyset$, there is a constant $d > 0$ (depending on A) such that $\langle \psi'(u), \widetilde{F}(u)\rangle \geq d$ $\forall u \in A$. Using these facts one shows as in [4] that $\eta(t,u)$ is defined for $t \in [0,1)$, $\lim_{t\to 1} \eta(t,u)$ exists, $\lim_{t\to 1} \psi(\eta(t,u)) = b$ and the mapping

$$r(t,u) = \begin{cases} \eta(t,u) & \text{if } 0 \leq t < 1 \text{ and } u \in \widetilde{U} \cup \widetilde{V}, \\ \lim_{t\to 1} \eta(t,u) & \text{if } t = 1 \text{ and } u \in \widetilde{U} \cup \widetilde{V}, \\ u & \text{if } 0 \leq t \leq 1 \text{ and } u \in \psi_b, \end{cases}$$

is a deformation retraction of $\psi_c - K_c$ onto ψ_b. $\square$

Denote by H_* the (unreduced) singular homology with coefficients in $\mathbb{Z}_2$. Since ψ has no positive critical values, it follows from LEMMA 5.3 that $H_q(\psi_b) \approx H_q(\psi_0)$ for all $b > 0$ and all q ($\approx$ means isomorphic). Let $\beta_q = \operatorname{rank} H_q(\psi_0)$. We shall show that

$$\beta_q = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Choose u with $\|u\|_\alpha = 1$ and consider the function $\psi(su)$, $s \geq 0$. Since G is homogeneous and $G(y) > 0$ $\forall y \in \mathbb{R}^{2n} - \{0\}$, there exists a constant $C_1 > 0$ such that $\forall y \in \mathbb{R}^{2n}$, $G(y) \geq C_1|y|^\alpha$. Consequently

$$\begin{aligned} \psi(su) &= \int_0^T \Big[\tfrac{1}{2}s^2(Ju, Mu) + s^\alpha G(-Ju)\Big]\, dt \geq C_1 s^\alpha - C_2 s^2, \\ \frac{d}{ds}\psi(su) &= \int_0^T \Big[s(Ju, Mu) + \alpha s^{\alpha-1} G(-Ju)\Big]\, dt \geq C_1 \alpha s^{\alpha-1} - 2C_2 s. \end{aligned} \tag{28}$$

Note that the constants C_1 and C_2 are independent of the choice of u. Thus, if b is sufficiently large, one may find a ball B, centered at the origin and having the property that $\overline{B} \subset \psi_b$ and the radial retraction of ψ_b onto $\overline{B}$ is contained in ψ_b. This implies that $H_q(\psi_0) \approx H_q(\psi_b) \approx H_q(\overline{B})$. Hence (27).

Choose $\epsilon > 0$ so that $-\epsilon$ is not a critical level. Choose also $\delta > 0$ in such a way that b_k is the only critical level in $[b_k - \delta, b_k + \delta]$ for all $b_k < -\epsilon$. Define

$$M_q^\epsilon = \sum_{b_k < -\epsilon} \operatorname{rank} H_q(\psi_{b_k+\delta}, \psi_{b_k-\delta}) + \operatorname{rank} H_q(\psi_0, \psi_{-\epsilon}). \tag{29}$$

By an algebraic argument due to PITCHER [19, § 11], see also [5, 22, 23], it follows that whenever all M_q^ϵ, $q < q_0$, are finite, we have the Morse relations

$$M_q^\epsilon - M_{q-1}^\epsilon + \cdots + (-1)^q M_0^\epsilon \geq \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0 \qquad \forall q \leq q_0,$$

or taking into account (27),

$$M_q^\epsilon - M_{q-1}^\epsilon + \cdots + (-1)^q M_0^\epsilon \geq (-1)^q \qquad \forall q \leq q_0. \tag{30}$$

Suppose that C is a critical circle at level b. Define the q-th critical group of C by

$$c_q(\psi, C) = H_q(\psi_b \cap U, \psi_b \cap U - C), \tag{31}$$

where U is a neighbourhood of C whose closure does not contain other critical points than those in C. By excision, c_q does not depend on the choice of U. Using LEMMA 5.3, excision and the fact that the homology of the union of path components is isomorphic to the direct sum of homologies of each path component, it follows that

$$\begin{aligned} H_q(\psi_{b_k+\delta}, \psi_{b_k-\delta}) &\approx H_q(\psi_{b_k}, \psi_{b_k-\delta}) \\ &\approx H_q(\psi_{b_k}, \psi_{b_k} - K_{b_k}) \\ &\approx \bigoplus_{i \in I_k} c_q(\psi, C_i), \end{aligned} \tag{32}$$

where I_k is the set of all indices i such that $\psi(C_i) = b_k$ (*cf.* [5, 22, 23]). So according to (29), if I is the set of all indices i with $\psi(C_i) < -\epsilon$, then

$$M_q^\epsilon = \sum_{i \in I} \operatorname{rank} c_q(\psi, C_i) + \operatorname{rank} H_q(\psi_0, \psi_{-\epsilon}). \tag{33}$$

Let C be a critical cricle and $\bar{u} \in C$, $\psi(\bar{u}) = b$. We want to compute $c_q(\psi, C)$. By THEOREM 3.11, a neighbourhood U of C may be chosen so that $(\psi_b \cap U,\ \psi_b \cap U - C)$ is a fibre bundle pair whose base space C is homeomorphic to S^1 and whose fibre pair has the homotopy type of $(\chi_b, \chi_b - \{0\})$. Recall that $\chi \in \mathcal{C}^2(B_{W_0}, \mathbb{R})$, $\dim B_{W_0} < \infty$, $\chi'(0) = 0$ and $\operatorname{Index} \chi''(0) = \operatorname{Index} \psi''(\bar{u})$, $\operatorname{Nullity} \chi''(0) = \operatorname{Nullity} \psi''(\bar{u}) - 1$. Since C is isolated, we may assume that 0 is the only critical point of χ. Let $W_0 = W_0^- \oplus W_0^0 \oplus W_0^+$ be the decomposition of W_0 into subspaces spanned by those e_i for which $1 + \lambda_i$ is negative, zero and positive respectively (*cf.* LEMMA 3.1). Then $\chi''(0)$ is negative definite on W_0^-, positive definite on W_0^+ and zero on W_0^0. Note that $\dim W_0^- = \operatorname{Index}\psi''(\bar{u})$. According to the generalized Morse lemma [9, Theorem II.4; 14, Theorem 3; 18, Theorem 1], there is an open neighbourhood N of the origin in B_{W_0}, an origin-preserving homeomorphism $\gamma : N \to \gamma(N) \subset B_{W_0}$ and a function $\xi \in C^2(N \cap W_0^0, \mathbb{R})$ such that

$$\chi(\gamma(w)) = \tfrac{1}{2}\langle \chi''(0)w^-, w^-\rangle + \tfrac{1}{2}\langle \chi''(0)w^+, w^+\rangle + \xi(w^0), \tag{34}$$

where $w = w^- + w^0 + w^+ \in W_0^- \oplus W_0^0 \oplus W_0^+$ and $\xi'(0) = 0$, $\xi''(0) = 0$.

PROPOSITION 5.4. — *Suppose that* $\operatorname{Index}(\psi''(\bar{u})) = i$. *Then*

(i) $c_q(\psi, C) \approx 0 \quad \forall q < i$;

(ii) *if* 0 *is a local mimimum of* ξ,

$$c_q(\psi, C) \approx \begin{cases} \mathbb{Z}_2 & \text{for } q = i,\ i+1\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

(iii) *if* 0 *is not a local minimum of* ξ, $c_q(\psi, C) \approx 0 \quad \forall q \le i$.

Proof.

(i) By the shifting theorem [5], *cf.* also [13],

$$c_q(\chi, 0) \approx \begin{cases} 0 & \text{if } q < i, \\ c_{q-i}(\xi, 0) & \text{if } q \ge i. \end{cases} \tag{35}$$

Consequently, $H_q(\chi_b, \chi_b - \{0\}) = c_q(\chi, 0) \approx 0 \quad \forall q < i$ (recall that 0 is the only critical point of χ). Since $(\psi_b \cap U, \psi_b \cap U - C)$ is a fibre bundle pair whose fibre pair has the homotopy type of $(\chi_b, \chi_b - \{0\})$, $c_q(\psi, C) = H_q(\psi_b \cap U, \psi_b \cap U - C) \approx 0 \quad \forall q < i$ [24, Lemma 5.7.16].

(ii) Since

$$c_q(\xi, 0) \approx \begin{cases} \mathbb{Z}_2 & \text{if } q = 0, \\ 0 & \text{otherwise}, \end{cases}$$

it follows from (35) that

$$c_q(\chi, 0) \approx \begin{cases} \mathbb{Z}_2 & \text{if } q = i, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, the fibre pair has the homology of the pair (D^i, S^{i-1}), where D^i is the i-dimensional closed ball with boundary S^{i-1}. So $(\psi_b \cap U, \psi_b \cap U - C)$ is a spherical bundle and therefore orientable over $\mathbb{Z}_2$ [24, Corollary 5.7.18]. Using Thom's isomorphism [24, Theorem 5.7.10] and the homology of S^1 we obtain the conclusion.

(iii) Since $c_0(\xi, 0) \approx 0$, $c_q(\chi, 0) \approx 0 \quad \forall q \leq i$ according to (35). The conclusion follows by repeating the argument of (i). □

Note that if $\text{Nullity}(\psi''(\bar{u})) = 1$, then $W_0^0 = \{0\}$ and (ii) of PROPOSITION 5.4 applies.

PROPOSITION 5.5. — *For each q_0 there exists an $\epsilon > 0$ such that $H_q(\psi_0, \psi_{-\epsilon}) \approx 0 \quad \forall q \leq q_0$.*

Proof. — First we note that there is an $\epsilon > 0$ such that for each critical point u with $\psi(u) > -\epsilon$, $\text{Index}(\psi''(u)) > q_0 + 1$. Indeed, if this is not the case, we may find a sequence (u_m) of critical points such that $\psi(u_m) \to 0$ and $\text{Index}(\psi''(u_m)) \leq q_0 + 1$. By the Palais-Smale condition, $u_m \to \bar{u}$ (possibly after passing to a subsequence). Since $\psi(\bar{u}) = 0$, and $\psi'(\bar{u}) = 0$, $\bar{u} = 0$. Hence $\text{Index}(\psi''(0)) \leq q_0 + 1$. On the other hand, $\text{Index}(\psi''(0)) = \infty$ because

$$\langle \psi''(0)v, v \rangle = \int_0^T (Jv, Mv)\, dt.$$

Next we show that

$$H_q(\psi_0, \psi_{-\epsilon}) \approx H_q(\psi_0, \overset{\circ}{\psi_0}) \qquad \forall q \leq q_0, \tag{36}$$

where $\overset{\circ}{\psi_0} = \{u \in L_0^\alpha : \psi(u) < 0\}$. Since $\psi_{-\epsilon} \subset \overset{\circ}{\psi_0} \subset \psi_0$, we have the exact sequence of the triple $(\psi_0, \overset{\circ}{\psi_0}, \psi_{-\epsilon})$ [24, Section 4.5] :

$$H_q(\overset{\circ}{\psi_0}, \psi_{-\epsilon}) \to H_q(\psi_0, \psi_{-\epsilon}) \to H_q(\psi_0, \overset{\circ}{\psi_0}) \to H_{q-1}(\overset{\circ}{\psi_0}, \psi_{-\epsilon}).$$

So (36) will follow if we prove that $H_q(\overset{\circ}{\psi_0}, \psi_{-\epsilon}) \approx 0 \quad \forall q \leq q_0$. Suppose $H_q(\overset{\circ}{\psi_0}, \psi_{-\epsilon}) \not\approx 0$. For a relative q-cycle c we shall denote its support (*i.e.*, the union of the images of all singular simplexes of c) by $|c|$ and its homology class by $[c]$. If c_1 and c_2 are homologous, we shall write $c_1 \sim c_2$. Let c be such that $[c] \in H_q(\overset{\circ}{\psi_0}, \psi_{-\epsilon})$ and $c \not\sim 0$. Then $\max_{u \in |c|} \psi(u) = d$, where $-\epsilon < d < 0$. According to LEMMA 5.3, we may assume that $d = b_j$, where b_j is a critical level. Choose $\delta > 0$ so that the interval $[b_j - \delta, b_j)$ contains no critical level. By the first part of the proof, (i) of PROPOSITION 5.4 and (32), $H_q(\psi_{b_j}, \psi_{b_j - \delta}) \approx 0 \quad \forall q \leq q_0 + 1$. So in the exact sequence

$$H_{q+1}(\psi_{b_j}, \psi_{b_j-\delta}) \to H_q(\psi_{b_j-\delta}, \psi_{-\epsilon}) \to H_q(\psi_{b_j}, \psi_{-\epsilon}) \to H_q(\psi_{b_j}, \psi_{b_j-\delta})$$

the first and the last term are zero. Hence the middle terms are isomorphic. We may therefore find $c' \in [c]$ such that $\max_{u \in |c'|} \psi(u) < b_j$. Proceeding in this way, we eventually find $c'' \in [c]$ such that $\max_{u\in|c''|}\psi(u) \le -\epsilon$. Therefore $c \sim 0$, a contradiction.

We complete the proof by using (36) and showing that

$$H_q(\psi_0, \psi_0^{\circ}) \approx 0 \qquad \forall q \le q_0.$$

This is clear for $q = 0$. Let $q \ge 1$ and let c be a relative q-cycle with boundary ∂c. Since $|\partial c|$ is compact and $|\partial c| \subset \psi_0^{\circ}$, we may find a finite dimensional subspace X_0 of L_0^{α} and a projection $P : L_0^{\alpha} \to X_0$ such that $(1-t)u + tPu \in \psi_0^{\circ}$ $\forall u \in |\partial c|$, $0 \le t \le 1$. It follows that ∂c is chain homotopic (and therefore homologous) to a $(q-1)$-cycle in ψ_0° whose support lies in X_0. Thus we may assume $|\partial c| \subset X_0$. Since $\operatorname{Index}(\psi''(0)) = \infty$, we may choose X_0 so that $\operatorname{Index}[(\psi|_{X_0})''(0)] > q$. Note that $\operatorname{Nullity}(\psi''(0)) = 0$. Accordingly, 0 is a nondegenerate critical point of $\psi|_{X_0}$. Let c' be the relative q-cycle obtained from ∂c by taking linear segments joining all points of $|\partial c|$ to the origin. For $u \in |\partial c|$ and all $s \in (0,1]$

$$\psi(su) = s^2 \int_0^T \Big[\tfrac{1}{2}(Ju, Mu) + s^{\alpha-2} G(-Ju)\Big]\, dt \le s^2 \psi(u) < 0.$$

So $[c'] \in H_q(\psi_0, \psi_0^{\circ})$. Consider the exact sequence

$$H_q(\psi_0) \to H_q(\psi_0, \psi_0^{\circ}) \xrightarrow{\partial_*} H_{q-1}(\psi_0^{\circ}).$$

Since $\operatorname{rank} H_q(\psi_0) = \beta_q = 0$, $H_q(\psi_0) \approx 0$. Thus ∂_* is a monomorphism. It follows that $c' \sim c$. The mapping $u \mapsto (1-s)u + s\alpha u$, $0 \le s \le 1$, $\alpha > 0$ small and fixed, induces a chain homotopy between c' and a relative q-cycle c'' whose support is contained in a small neighbourhood of $0 \in X_0$. So $c \sim c' \sim c''$. Since $\operatorname{Index}[(\psi|_{x_0})''(0)] > q$, $\;c_q(\psi|_{X_0}, 0) \approx 0$. Consequently, $c \sim c'' \sim 0$. $\square$

6. Proof of Theorem 4.2

Let C_1 be a critical circle obtained by minimizing ψ. Then all $u \in C_1$ have minimal period T [6, 9]. By PROPOSITION 4.1, there exists a sequence $C_1, C_2, \ldots$ of critical circles which correspond to the same trajectory of (22). Also, $\psi(C_k) < \psi(C_{k+1})$ $\forall k$ and $\psi(C_k) \to 0$ as $k \to \infty$. Suppose that ψ has no other critical circles (which means that there is only one closed trajectory of the flow (22)). We shall consider two cases, $j(-1) \ge 3$ and $j(-1) = 2$ (*cf.* (i) of COROLLARY 5.2).

Let $j(-1) \geq 3$. It is easy to see that $c_0(\psi, C_1) \approx c_1(\psi, C_1) \approx \mathbb{Z}_2$ and $c_q(\psi, C_1) \approx 0 \quad \forall q \geq 2$. By (26), $i_2 = j(1) + j(-1) \geq 3$. So according to COROLLARY 5.2 and (i) of PROPOSITION 5.4, $c_q(\psi, C_k) \approx 0$ for $q = 0, 1, 2$ and $k \geq 2$. Hence if $\epsilon > 0$ is small enough, $M_0^\epsilon = M_1^\epsilon = 1$ and $M_2^\epsilon = 0$ (*cf.* (33) and PROPOSITION 5.5). This contradicts (30) with $q = 2$.

Suppose now $j(-1) = 2$. We shall show that for all $k \geq 1$,

$$i_k = 2k - 2\,, \tag{37}$$

$$c_q(\psi, C_k) \approx \begin{cases} \mathbb{Z}_2 & \text{if } q = 2k-2,\ 2k-1; \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

This will complete the proof of the theorem because $i_k > 2k - 2$ for some k according to (ii) of COROLLARY 5.2. It is clear that (37) and (38) are satisfied if $k = 1$. Assume they are satisfied $\forall k \leq k_0$. Using COROLLARY 5.2, (33) and PROPOSITIONS 5.4(i) and 5.5, it follows that $i_{k_0+1} \geq 2k_0$ and $M_q^\epsilon = 1 \quad \forall q \leq 2k_0 - 1$. So according to (30) with $q = 2k_0$, $M_{2k_0}^\epsilon \geq 1$. Since $c_{2k_0}(\psi, C_k) \approx 0 \quad \forall k \geq k_0 + 2, \quad c_{2k_0}(\psi, C_{k_0+1}) \not\approx 0$. Consequently, $i_{k_0+1} = 2k_0$. Also, by (ii) and (iii) of PROPOSITION 5.4, (38) with $k = k_0 + 1$ is satisfied. ∎

Remark.

(i) If $u \neq 0$ is a critical point of ψ, then $\psi(u) < 0$. Indeed, $G(y) > 0$ $\forall y \in \mathbb{R}^{2n} - \{0\}$, and by homogeneity, $(G'(y), y) = \alpha G(y) \quad \forall y \in \mathbb{R}^{2n}$. Hence

$$\begin{aligned} 0 = \langle \psi'(u), u \rangle &= \int_0^T \Big[(Ju, Mu) + (G'(-Ju), -Ju)\Big]\, dt \\ &= \int_0^T \Big[(Ju, Mu) + \alpha G(-Ju)\Big]\, dt. \end{aligned}$$

It follows that

$$\psi(u) = \int_0^T (1 - \tfrac{1}{2}\alpha) G(-Ju)\, dt < 0.$$

(ii) Using the above fact and (ii) of PROPOSITION 5.4, it is easy to show by our arguments that if all critical circles are nondegenerate (*i.e.*, have nullity 1), then for each even number $k \geq 0$ there exists a critical circle of index k. This is the main result in [8, § III].

Note. — In a very recent manuscript "Hamiltonian flows with finitely many trajectories", I. EKELAND and H. HOFER have shown that if the critical circle C_1 is degenerate (*i.e.*, nullity $n_1 \geq 2$), then $j(-1) \geq 3$.

Using this and (v) of PROPOSITION 5.1, it follows easily that $i_{k+1} - i_k > 2$ for some k also when $n = 2$ (*cf.* COROLLARY 5.2). So the conclusion of THEOREM 4.2 remains valid for $n = 2$.

Ekeland's and Hofer's paper contains still another proof of THEOREM 4.2.

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