BULLETIN DE LA S. M. F.

JINGCHENG TONG

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Bulletin de la S. M. F., tome 117, nº 1 (1989), p. 59-67

<http://www.numdam.org/item?id=BSMF_1989__117_1_59_0>

Bull. Soc. math. France,
117, 1989, p. 59–67.

SYMMETRIC AND ASYMMETRIC DIOPHANTINE APPROXIMATION OF CONTINUED FRACTIONS

BY

JINGCHENG TONG (*)

RÉSUMÉ. — Soit ξ un nombre irrationnel avec l'expansion de la fraction continue simple $\xi = [a_0\,;\ a_1, \dots, a_i, \dots]$ et soit p_i/q_i son $i^{\text{ème}}$ convergent. Dans cet article, on donne explicitement deux suites de nombres réels (α_n), (β_n) et on démontre les résultats suivants :

(i) Entre trois convergents consécutifs p_i/q_i de ξ $(i = n-2, n-1, n)$, un au moins satisfait $|\xi - p_i/q_i| < 1/(\alpha_n q_i q_{i+1})$ et un au moins ne satisfait pas cette inégalité ;

(ii) Soit $\tau > 0$. Entre quatre convergents consécutifs p_i/q_i de ξ $(i = n-2, n-1, n, n+1)$, un au moins satisfait $-1/(\beta_n q_i q_{i+1}) < |\xi - p_i/q_i| < \tau/(\beta_n q_i q_{i+1})$ et un au moins ne satisfait pas ces inégalités.

ABSTRACT. — Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0\,;\ a_1, \dots, a_i, \dots]$, and p_i/q_i be its i-th convergent. In this paper, we give explicitly two sequences of real numbers (α_n), (β_n), and prove the following results :

(i) Among any three consecutive convergents p_i/q_i of ξ $(i = n-2,\ n-1,\ n)$, at least one satisfies $|\xi - p_i/q_i| < 1/(\alpha_n q_i q_{i+1})$, and at least one does not satisfy this inequality ;

(ii) For any $\tau > 0$, among any four consecutive convergents p_i/q_i of ξ $(i = n-2,\ n-1,\ n,\ n+1)$, at least one satisfies $-1/(\beta_n q_i q_{i+1}) < |\xi - p_i/q_i| < \tau/(\beta_n q_i q_{i+1})$, and at least one does not satisfy this inequality.

1. Introduction

Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0\ a_1, \dots, a_i, \dots]$, and p_i/q_i be its ith-convergent. The following inequality has been extensively investigated for a century :

$$(1) \qquad \left|\xi - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2}.$$

(*) Texte reçu le 30 août 1987, révisé le 2 février 1988.

Jingcheng TONG, Institute of Applied Mathematics, Academia Sinica, BEIJING, China, and Department of Mathematical Sciences, University of North Florida, Jacksonville, FL 32216, U.S.A.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/1989/59/$ 5.00

We state the main results about inequality (1).

(i) By the works of HURWITZ [5], BOREL [2], FUJIWARA [4], MÜLLER [7] (or [1]) and the present author [12], the following theorem on symmetric approximation is obtained :

Among any three consecutive convergents p_i/q_i of ξ ($i = n-1$, n, $n+1$), at least one satisfies $|\xi - p_i/q_i| < 1/(\sqrt{a_n^2 + 4\,q_i^2})$, and at least one satisfies $|\xi - p_i + q_i| > 1/(\sqrt{a_n^2 + 4\,q_i^2})$.

(ii) By the works of SEGRE [10], ROBINSON [9], LEVEQUE [6], SZÜSZ [11] and the present author [15], the following theorem on asymmetric approximation is obtained :

Let τ be a positive number and c_n be defined as follows :

$$c_n = \begin{cases} a_{n+1} & \text{if } n \text{ is odd,} \\ a_{n+2} & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Then among any four consecutive convergents p_i/q_i of ξ ($i = n-1$, n, $n+1$, $n+2$), at least one satisfies the following inequality :

$$-\frac{1}{\sqrt{c_n^2 + 4\tau q_i^2}} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\sqrt{c_n^2 + 4\tau q_i^2}}, \tag{3}$$

and at least one does not satisfy inequalities (3).

The study of the following inequality can be traced back to DIRICHLET [3] :

$$\left|\xi - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}}. \tag{4}$$

But surprisingly, although inequality (1) is obtained from (4), there are no corresponding results on (4) in literature. The purpose of this paper is to do some investigations on (4). We will use the method established by the present author in [12–15].

2. Preliminaries

Let $\xi = [a_0, a_1, \ldots, a_n, \ldots]$ with convergent p_n/q_n. It is known [8] that

$$\xi - \frac{p_n}{q_n} = \frac{(-1)^n}{M_n q_n^2}, \tag{5}$$

where $M_n = [0\,;\ a_{n+2}, a_{n+3}, \ldots] + [a_{n+1}\,;\ a_n, \ldots, a_1]$; thus

$$\xi - \frac{p_n}{q_n} = \frac{(-1)^n}{C_n q_n q_{n+1}}, \tag{6}$$

where $C_n = (M_n q_n)/q_{n+1}$. Let $D_n = [a_{n+1}\,;\ a_n, \ldots, a_1][a_{n+2}\,;\ a_{n+3}, \ldots]$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1}$, and $q_{n+1}/q_n = [a_{n+1}\,;\ a_n, \ldots, a_1]$. It is easily seen that

$$C_n = 1 + \frac{1}{D_n}. \tag{7}$$

We need some simple lemmas.

LEMMA 1. — *Let $f(x) = (a + x^{-1})/(rx^{-1} - a)$ with $a > 0$, $r > 0$. Then $f(x)$ is increasing for $0 < x < r/a$.*

Proof.

$$f'(x) = \frac{ax^{-2}(1 + r)}{(rx^{-1} - a)^2} > 0.$$

LEMMA 2. — *Let $g(x) = (a + bx)/(x - a)$ with $a > 0$, $b > 0$. Then $g(x)$ is decreasing for $x > a$.*

Proof.

$$g'(x) = -\frac{a(b + 1)}{(x - a)^2} < 0.$$

LEMMA 3. — *Let $h(x) = \sqrt{x + 4} - \sqrt{x}$. Then $h(x)$ is decreasing for $x > 0$.*

Proof.

$$h'(x) = \frac{1}{2\sqrt{x + 4}} - \frac{1}{2\sqrt{x}} < 0.$$

3. Main theorems

Let $P = [a_{n+1}\,;\ a_{n+2}, \ldots]$, $Q = [a_n\,;\ a_{n-1}, \ldots, a_1]$. It is easily seen that

$$D_{n-2} = \frac{a_n + P^{-1}}{Q - a_n}, \tag{8}$$

$$D_{n-1} = PQ, \tag{9}$$

$$D_n = \frac{a_{n+1} + Q^{-1}}{P - a_{n+1}}. \tag{10}$$

Now we prove the main theorems.

THEOREM 1. — *Let r be a positive real number and $n \geq 3$ be a positive integer. The following statements are true.*

(I) *If $D_{n-1} < r$, then*

$$\max(D_{n-2}, D_n) > \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

(II) *If $D_{n-1} = r$, then*

$$\max(D_{n-2}, D_n) > \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

Proof. — (I) Since $D_{n-1} < r$, from (9) we have

$$Q < r/P, \tag{11}$$

$$Q^{-1} > P/r. \tag{12}$$

From (8) and (11) we have

$$D_{n-2} > \frac{a_n + P^{-1}}{rP^{-1} - a_n}. \tag{13}$$

By LEMMA 1, D_{n-2} is an increasing function for $P < r/a_n$.

From (10) and (12) we have

$$D_n > \frac{a_{n+1} + P/r}{P - a_{n+1}}. \tag{14}$$

By LEMMA 2, D_n is a decreasing function for $P > a_{n+1}$.

Since $D_n > 0$, from (10) we have $P > a_{n+1}$. Since $D_{n-2} > 0$, from (8) we have $Q > a_n$, hence $P = D_{n-1}/Q < r/a_n$. We discuss two possible cases on P.

(i) $r/a_n > P \geq \sqrt{a_{n+1} r/a_n}$. From (13) we have

$$D_{n-2} > \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

(ii) $a_{n+1} < P < \sqrt{a_{n+1} r/a_n}$. From (14) we have

$$D_n > \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

Thus THEOREM 1 (I) is established.

(II) To prove THEOREM 1 (II), we first prove that $P \neq \sqrt{a_{n+1}r/a_n}$. Suppose $P = \sqrt{a_{n+1}r/a_n}$. Since $D_{n-1} = r$, we have $Q = D_{n-1}/P = \sqrt{a_n r/a_{n+1}}$. It is easily seen that $Q = [a_n\,;\ a_{n-1}, \ldots, a_1]$ is a rational number, while $P = [a_{n+2}\,;\ a_{n+3}, \ldots]$ is an irrational number. Let $Q = k/h$, where k, h are two co-prime positive integers. Then we have $\sqrt{a_n r/a_{n+1}} = k/h$, with $r = k^2 a_{n+1}/(h^2 a_n)$. Therefore $P = \sqrt{a_{n+1}r/a_n} = (ka_{n+1})/(ha_n)$, and P is a rational number! This contradiction shows $P \neq \sqrt{a_{n+1}r/a_n}$.

Similar to the proof of THEOREM 1 (I), we have

$$D_{n-2} = \frac{a_n + P^{-1}}{rP^{-1} - a_n},$$
$$D_n = \frac{a_{n+1} + P/r}{P - a_{n+1}}.$$

Since $P \neq \sqrt{a_{n+1}r/a_n}$, case (i) in THEOREM 1 (I) becomes $r/a_n > P > \sqrt{a_{n+1}r/a_n}$, the required result is correct.

By similar arguments, we can reverse the directions of the inequalities in THEOREM 1 to obtain a conjugate theorem.

THEOREM 2. — *Let r be a positive real number and $n \geq 3$ be a positive integer. The following statements are true.*

(I) *If $D_{n-1} > r$, then*

$$\min(D_{n-2}, D_n) < \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

(II) *If $D_{n-1} = r$, then*

$$\min(D_{n-2}, D_n) < \frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}}.$$

4. Symmetric approximation

Now we apply THEOREM 1 and 2 to investigate symmetric approximation to the irrational number ξ.

THEOREM 3. — *Let $\alpha_n = 1 + \frac{1}{4}(\sqrt{a_n a_{n+1} + 4} - \sqrt{a_n a_{n+1}})^2$. Then among any three consecutive convergents p_i/q_i of ξ ($i = n-2,\ n-1,\ n$), at least one satisfies*

$$\left|\xi - \frac{p_i}{a_i}\right| < \frac{1}{\alpha_n q_i q_{i+1}}, \tag{15}$$

and at least one satisfies

$$\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{\alpha_n q_i q_{i+1}}. \tag{16}$$

Proof. — Letting $r = \frac{1}{4}(\sqrt{a_n a_{n+1} + 4} + \sqrt{a_n a_{n+1}})^2$, it is easily seen that $\alpha_n = 1 + 1/r$ and

$$\frac{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}}{\sqrt{r} - \sqrt{a_n a_{n+1}}} = r.$$

If $D_{n-1} < r$, then $C_{n-1} = 1 + 1/D_{n-1} > 1 + 1/r = \alpha_n$. From (6) we know that (15) holds for $i = n-1$. By THEOREM 1 (I), $\max(D_{n-2}, D_n) > r$, hence $\min(C_{n-2}, C_n) < 1 + 1/r = \alpha_n$, (16) is true for $i = n - 2$ or n.

If $D_{n-1} > r$, then $C_{n-1} = 1 + 1/D_{n-1} < 1 + 1/r = \alpha_n$. From (6) we know that (16) holds for $i = n-1$. By THEOREM 2 (II), $\min(D_{n-2}, D_n) < r$, hence $\max(C_{n-2}, C_n) > 1 + 1/r = \alpha_n$, (15) is true for $i = n - 2$ or n.

If $D_{n-1} = r$, then by THEOREM 1 (II) and THEOREM 2 (II), we have $\max(D_{n-2}, D_n) > r$ and $\min(D_{n-2}, D_n) < r$. Thus $\min(C_{n-1}, C_n) < 1 + 1/r = \alpha_n$, $\max(C_{n-2}, C_n) > 1 + 1/r = \alpha_n$. By (6) we know that (15) is true for i such that $C_i = \max(C_{n-2}, C_n)$, (16) is true for i such that $C_i = \min(C_{n-2}, C_n)$.

Because we always have $a_n a_{n+1} \geq 1$, by LEMMA 3 the following corollary is correct.

COROLLARY 1. — *Among any three consecutive convergents p_i/q_i of ξ ($i = n - 2$, $n - 1$, n), at least one satisfies the following inequality :*

$$\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{\frac{1}{2}(5 - \sqrt{5}) q_i q_{i+1}}. \tag{17}$$

If ξ is not equivalent to $\frac{1}{2}(\sqrt{5} - 1) = [1\,;\, 1, \ldots]$, then there are infinitely many pairs a_n, a_{n+1} such that $a_n a_{n+1} \geq 2$. Therefore the following corollary is true.

COROLLARY 2. — *Let ξ be an irrational number not equivalent to $\frac{1}{2}(\sqrt{5} - 1)$. Then there are infinitely many convergents satisfying the following inequality.*

$$\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{(3 - \sqrt{3}) q_i q_{i+1}}. \tag{18}$$

5. Asymmetric approximation

Now we apply THEOREM 1 and 2 to investigate asymmetric approximation to the irrational number ξ.

Let τ be a positive real number. If

$$r = \frac{\left(\sqrt{a_n a_{n+1} + 4\tau} + \sqrt{a_n a_{n+1}}\right)^2}{4\tau^2},$$

it is easily checked that

$$1 + \frac{\sqrt{r} - \sqrt{a_n a_{n+1}}}{\sqrt{a_n a_{n+1}} + 1/\sqrt{r}} = \frac{1}{\tau}\left(1 + \frac{1}{r}\right).$$

THEOREM 4. — *Let τ be a positive real number and $\nu_n = 1 + \frac{1}{4}(\sqrt{a_n a_{n+1} + 4\tau} - \sqrt{a_n a_{n+1}})^2$. If n is an even positive integer, then among three consecutive convergents p_i/q_i of ξ ($i = n-2,\ n-1,\ n$), at least one satisfies the following inequalities :*

$$-\frac{1}{\nu_n q_i q_{i+1}} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\nu_n q_i q_{i+1}}, \tag{19}$$

and at least one satisfies one of the following inequalities :

$$\xi - \frac{p_i}{q_i} < -\frac{1}{\nu_n q_i q_{i+1}}, \tag{20}$$

$$\xi - \frac{p_i}{q_i} > \frac{\tau}{\nu_n q_i q_{i+1}}. \tag{21}$$

Proof. — If $D_{n-1} < r$, then by (7) and THEOREM 1 (I) we have $C_{n-1} > 1 + 1/r = \nu_n$ and $\min(C_{n-2}, C_n) < \tau^{-1}(1 + 1/r) = \tau^{-1}\nu_n$. Hence by (6) the left hand side of (19) is true for $i = n-1$, and (21) is true for $i = n-2$ or n.

If $D_{n-1} > r$, then by (7) and THEOREM 2 (I) we have $C_{n-1} < 1 + 1/r = \nu_n$ and $\max(C_{n-2}, C_n) > \tau^{-1}(1 + 1/r) = \tau^{-1}\nu_n$. Hence by (6) we know that (20) is true for $i = n-1$, and the right hand side of (19) is true for $i = n-2$ or n.

If $D_{n-1} = r$, then by THEOREM 1 (II) and THEOREM 2 (II), we have $\min(C_{n-2}, C_n) < \tau^{-1}\nu_n$ and $\max(C_{n-2}, C_n) > \tau^{-1}\nu_n$. Hence (21) is true for i such that $C_i = \min(C_{n-2}, C_n)$, the right hand side of (19) is true for i such that $C_i = \max(C_{n-2}, C_n)$.

Let

$$d_n = \begin{cases} a_n & \text{if } n \text{ is even}, \\ a_{n+1} & \text{if } n \text{ is odd}; \end{cases}$$

such that

$$d_{n+1} = \begin{cases} a_{n+1} & \text{if } n \text{ is even}, \\ a_{n+2} & \text{if } n \text{ is odd}, \end{cases}$$

and $\beta_n = 1 + \frac{1}{4}(\sqrt{d_n d_{n+1} + 4\tau} - \sqrt{d_n d_{n+1}})^2$. Using THEOREM 4 if n is even, or replacing $n-1$ by n in THEOREM 4 if n is odd, we have the following theorem.

THEOREM 5. — *Let τ be a positive real number. Among four consecutive convergents p_i/q_i of ξ $(i = n-2, n-1, n, n-1)$, at least one satisfies the following inequalities :*

$$-\frac{1}{\beta_n q_i q_{i+1}} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\beta_n q_i q_{i+1}}, \tag{22}$$

and at least one satisfies one of the following inequalities :

$$\xi - \frac{p_i}{q_i} < -\frac{1}{\beta_n q_i q_{i+1}}, \tag{23}$$

$$\xi - \frac{p_i}{q_i} > \frac{\tau}{\beta_n q_i q_{i+1}}. \tag{24}$$

Since we always have $d_i d_{i+1} \geq 1$, the following corollary is immediate.

COROLLARY 3. — *Let τ be a positive real number. Among any four consecutive convergents p_i/q_i of ξ $(i = n-2, n-1, n, n+1)$, at least one does not satisfy the following inequalities :*

$$-\frac{1}{\frac{1}{2}(3 + 2\tau - \sqrt{1+4\tau})q_i q_{i+1}} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\frac{1}{2}(3 + 2\tau - \sqrt{1+4\tau})q_i q_{i+1}}.$$

Acknowledgement. — The author thanks the referee sincerely for his very valuable suggestions to improve this paper.

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