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ABOUT STEFAN'S DEFINITION OF A FOLIATION WITH SINGULARITIES : A REDUCTION OF THE AXIOMS

BY

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RÉSUMÉ. — L'article présent concerne la définition d'un feuilletage de Stefan. Le résultat principal de cet article est le fait qu'un axiome de la définition d'un feuilletage de Stefan [4, chap. 1] est une conséquence des autres.

ABSTRACT. — The aim of this paper is to give an accurate proof of the fact formulated in [3, p. 45] that one of the axioms of Stefan's foliations [4, chap. 1] follows from the remaining ones.

The following definitions of a *foliation with singularities* comes from the work by P. STEFAN [4].

Suppose V is a connected Hausdorff C^∞ and paracompact (equivalently and with a countable basis) manifold of dimension n. By a *foliation of V with singularities* we mean a partition $\mathcal{F}$ of V into sets such that :

(1) for each element $L \in \mathcal{F}$, there exists a structure of differentiable manifold σ on L such that

(i) (L, σ) is a connected immersed submanifold of V,

(ii) (L, σ) is a *leaf* of V with respect to all locally connected topological spaces, i.e. if X is an arbitrary locally connected topological space and $f : X \to V$ is a continuous function such that $f[X] \subset L$, then $f : X \to (L, \sigma)$ is continuous;

(2) for each $x \in V$, there exists a local chart φ on V around x with the following properties :

(a) φ is a surjection $D_\varphi \to U_\varphi \times W_\varphi$ where U_φ, W_φ are open neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively, and k is the dimension of the leaf through x (denoted by L_x);

(b) $\varphi(x) = (0, 0)$;

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(c) if $L \in \mathcal{F}$, then $\varphi[L \cap D_\varphi] = U_\varphi \times \ell_{\varphi,L}$ where

$$\ell_{\varphi,L} = \{w \in W_\varphi : \varphi^{-1}(0,w) \in L\}\cdot$$

A chart φ which fulfils the above condition is called *distinguished* around x.

THEOREM. — *Let $\mathcal{F}$ be a partition of V into connected immersed submanifolds of V, fulfilling* (2). *Then $\mathcal{F}$ is a foliation with singularities.*

REMARK. — This theorem is formulated in [3, p. 45] without an accurate proof. The author say that it easily follows in the same way as in the case without singularities, indicating [1]. It turns out that this theorem needs a subtler proof. The reasoning as in [1] gives the proof provided some added assumption

$$(1) \qquad \frac{\partial}{\partial\varphi^i}\Big|_y \in T_y(L_x) \text{ for } i \leq k \text{ and all } y \in L_x \cap D_\varphi,\ k = \dim L_x,$$

is satisfied, which is exactly the body of Stefan's lemma [4, Lemma 3.1]. That this added condition follows from the remaining ones is the aim of our paper.

Proof of the Theorem : according to STEFAN ([4, Lemma 3.1]), it is sufficient to show that each distinguished chart $\varphi = (\varphi^1, \ldots, \varphi^n)$ around x has the property (1).

Assume to the contrary that, for a distinguished chart φ around x, this property does not hold at a point $y_0 \in L_x \cap D_\varphi$. Then, of course, there exists a vector $v \in T_{y_0}(L_x)$ such that

$$(2) \qquad \varphi_{*y_0}(v) \notin T_{(\hat{y}_0,c_0)}\big(U_\varphi \times \{c_0\}\big)$$

where $(\hat{y}_0, c_0) = \varphi(y_0)$, $\hat{y}_0 \in U_\varphi$, $c_0 \in W_\varphi$.

Take any smooth curve

$$c : (-\varepsilon, \varepsilon) \longrightarrow L_x, \quad \varepsilon > 0,$$

such that $c(0) = y_0$, $\dot{c}(0) = v$ and $\operatorname{Im} c \subset D_\varphi$. Consider the curve $\varphi \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$. Let $pr_2 : U_\varphi \times W_\varphi \to W_\varphi$ denotes the projection onto the second factor. By (2) :

$$(pr_2)_*\big((\varphi \circ c)^{\cdot}(0)\big) \neq 0.$$

Diminishing $\varepsilon > 0$, if necessary, we may assume that

$$pr_2 \circ \varphi \circ c : (-\varepsilon, \varepsilon) \longrightarrow W_\varphi$$

is an embedding. Denote the set $\operatorname{Im}(pr_2 \circ \varphi \circ c)$ by I. Of course,

$$U_\varphi \times I \subset U_\varphi \times \ell_{\varphi, L_x}$$

(because $I \subset pr_2 \circ \varphi[D_\varphi \cap L_x] = \ell_{\varphi, L_x}$) and $U_\varphi \times I$ is a $k+1$-dimensional hypersurface of $\mathbb{R}^n$, thus a locally compact space. Put — for brevity —

$$M := L_x \cap D_x$$

understanding it as an open submanifold of L_x, and consider the injective immersion

$$\bar{\varphi} : M \longrightarrow \mathbb{R}^n, \quad x \longmapsto \varphi(x).$$

By the above $\bar{\varphi}[M] \supset U_\varphi \times I$.

For each point $x \in M$, we choose a neighbourhood $U(x) \subset M$ of x such that

$$\bar{\varphi}|_{U(x)} : U(x) \longrightarrow \mathbb{R}^n$$

is an embedding. By the assumption of the second axiom of countability of V, each connected immersed submanifold of V fulfils this axiom (see Appendix). Then M, as an open submanifold of the manifold L_x, has a countable basis. Choose a countable open covering $\{U_i\,;\ i \in \mathbb{N}\}$ of M such that each $\bar{U}_i$ is compact and contained in some $U(x_i)$. We prove that

$$\bar{\varphi}[\bar{U}_i] \cap (U_\varphi \times I)$$

— as a subset of the space $U_\varphi \times I$ — has no interior. We have a little more, namely that the set $\bar{\varphi}[U(x_i)] \cap (U_\varphi \times I)$ has no interior. If it were not, then by taking an nonempty and open subset $U \subset U_\varphi \times I$ such that $X \subset \bar{\varphi}[U(x_i)]$, we would obtain the mapping

$$\left(\bar{\varphi}|_{U(x_i)}\right)^{-1}|_X : X \to U(x_i)$$

from a $(k+1)$-dimensional manifold to a k-dimensional one, being an immersion, which is not possible. Thus $U_\varphi \times I$ is an union of a countable sequence of nowhere dense sets

$$\left\{\bar{\varphi}[\bar{U}_i] \cap (U_\varphi \times I)\,;\ i \in \mathbb{N}\right\},$$

which leads to a contradiction with Baire's theorem for locally compact spaces. The theorem is proved. $\square$

Appendix : The following theorem is well known; here we give a simple proof of it.

THEOREM. — *Each connected immersed submanifold L of a C^∞ Hausdorff paracompact manifold V has a countable basis.*

Proof. — Let $f : L \to V$ be an immersion. The assumptions imply the existence of a Riemann tensor G on V. Its pullback f^*G is a Riemann tensor on L. A connected manifold which possesses a Riemann tensor is separable [2], therefore it has a countable basis. □

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