BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 120, nº 1 (1992), p. 113-128
<http://www.numdam.org/item?id=BSMF_1992__120_1_113_0>

Bull. Soc. math. France,
120, 1992, p. 113–128.

CAUCHY-FANTAPPIÈ-LERAY FORMULAS WITH LOCAL SECTIONS AND THE INVERSE FANTAPPIÈ TRANSFORM

BY

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RÉSUMÉ. — Nous déduisons une formule du type Cauchy-Fantappiè-Leray n'utilisant que des sections localement définies. A l'aide de cela, nous construisons une formule d'inversion pour la transformation de Fantappiè dans le cas $\mathbb{C}$-convexe général. Ceci rétablit la moitié non démontrée d'une conjecture de Aïzenberg, Trutnev et Znamenskij affirmant qu'un domaine est $\mathbb{C}$-convexe si et seulement si la transformation de Fantappiè y est un isomorphisme.

ABSTRACT. — We derive a Cauchy-Fantappiè-Leray formula that requires only locally defined sections. We use it to construct an inversion formula for the Fantappiè transform in the general $\mathbb{C}$-convex case. This establishes the unproved half of a conjecture of Aizenberg, Trutnev and Znamenskij that states that a domain is $\mathbb{C}$-convex if and only if the Fantappiè transform is an isomorphism.

0. Introduction

If D is a domain in $\mathbb{C}^n$ (or $\mathbb{P}^n$) then the Fantappiè transform F maps $H'(D)$ into $H(D^*)$, where D^* is the set of all hyperplanes not intersecting D (for exact definitions see § 3). If D is convex, then F is an isomorphism. This was proved by MARTINEAU, see [2]. In the seventies AIZENBERG and TRUTNEV conjectured :

THEOREM 0. — *The Fantappiè transform F is an isomorphism if and only if D has simply connected intersections with all complex lines.*

(*) Texte reçu le 13 février 1991.
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Partially supported by the Swedish Natural Science Research Council.

Keywords : Cauchy-Fantappiè-Leray formula, Fantappiè transform.

Classification AMS : 32E30, 32E10.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037–9484/1992/113/$ 5.00

We call such a domain D $\mathbb{C}$-*convex*. This conjecture was announced by ZNAMENSKIJ in [8] to be true, and it is subject to a more elaborated treatment in [9], where the necessity of $\mathbb{C}$-convexity is established. However, the proposed arguments for the sufficiency are long and involved and seem to have gaps.

The aim of this paper is to give a rigorous and comprehensive proof of the sufficiency, i.e. that F is an isomorphism if D is $\mathbb{C}$-convex. Along the way we obtain a quite explicit inversion formula for F. This is constructed by means of a Cauchy-Fantappiè-Leray representation formula for holomorphic functions, in which the Cauchy-Leray sections are defined only locally. We derive the representation formula in § 1 and use it in § 2 to give a simple Hahn-Banach proof of a (known) Runge type theorem which we need later on. In § 3 we construct the inversion formula for the Fantappiè transform F when D is $\mathbb{C}$-convex, and prove that F is an isomorphism. However, the proof requires some topological facts about $\mathbb{C}$-convex domains, some of which we have not found in the literature, and we collect them with proofs in an appendix. For instance, we prove (although it is not needed in our proof of THEOREM 1) that a $\mathbb{C}$-convex domain is contractible.

We conclude this paragraph by suggesting the idea behind the Cauchy-Fantappiè representation formula. Suppose K is a compact set in $\mathbb{C}^n$ and $f \in H(\Omega)$ where $\Omega \supset\supset K$. Then there is an open set ω with smooth boundary such that $K \subset \omega \subset\subset \Omega$. If $n = 1$ we can represent f on K by the Cauchy formula :

$$f(z) = \frac{1}{2\pi i}\int_{\partial\omega} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}, \quad z \in K. \tag{1}$$

A simple generalization of (1) to $\mathbb{C}^n$ is the Bohner-Martinelli formula

$$f(z) = c_n \int_{\partial\omega} \frac{\sum(\bar\zeta_j - \bar z_j)\,\mathrm{d}\zeta \wedge \big(\sum \mathrm{d}\bar\zeta_j \wedge \mathrm{d}\zeta_j\big)^{n-1}}{|\zeta - z|^{2n}} f(\zeta), \quad z \in K, \tag{2}$$

which however has the disadvantage that the kernel is not holomorphic in z, which is crucial in certain applications.

In order to obtain a representation formula with holomorphic kernel one, roughly speaking, has to find a complex hypersurface, not intersecting K, through each point $\zeta \in \partial\omega$, and moreover do this in a C^1-manner on $\partial\omega$. More explicitly, if we have a mapping $s(\zeta, z) : \partial\omega \times \Omega \to \mathbb{C}^n$ which is holomorphic in z and such that the hypersurfaces $\{z\,;\ s(\zeta, z)\cdot(\zeta - z) = 0\}$ do not intersect K, then the Cauchy-Fantappiè-Leray formula (see § 1)

$$f(z) = c_n \int_{\partial\omega} \frac{s(\zeta, z) \wedge \big(\bar\partial s(\zeta, z)\big)^{n-1}}{\big(s(\zeta, z)\cdot(\zeta - z)\big)^n} f(\zeta), \quad z \in K, \tag{3}$$

holds, where $s(\zeta, z)$ is identified with the form $\sum s_j(\zeta, z)\,\mathrm{d}\zeta_j$. We will present a method for constructing a global representation formula of this kind from local (on $\partial\omega$) choices of $s(\zeta, z)$. The resulting formulas will inherit some properties of $s(\zeta, z)$, e.g. being holomorphic or algebraic in z. In case of domains with piecewise smooth boundary, our formulas are connected to Norguet's formula, see the last remark in § 1.

I am grateful to Ragnar SIGURDSSON for valuable comments on the manuscripts.

1. Cauchy-Fantappiè-Leray formulas with locally defined sections

We present the formulas in $\mathbb{P}^n$-formalism since this is most natural when applied to the Fantappiè transform in § 3. However, there is a simple way to translate to the $\mathbb{C}^n$-form (see below).

Let $\zeta = (\zeta_0, \ldots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ be homogeneous coordinates for the point $[\zeta]$ in $\mathbb{P}^n$ and let π denote the natural projection :

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

Sometimes we abusively write $\zeta \in \mathbb{P}^n$ rather than $[\zeta] \in \mathbb{P}^n$. Via the natural pairing $\langle\ ,\ \rangle$ of $\mathbb{C}^{n+1}$ and its dual $(\mathbb{C}^{n+1})^*$, the elements in $(\mathbb{P}^n)^* = \big\{[\xi]\,;\ \xi \in (\mathbb{C}^{n+1})^* \setminus \{0\}\big\}$ are identified with the hyperplanes in $\mathbb{P}^n$, i.e. $[\xi] \sim \big\{[\zeta] \in \{[\zeta] \in \mathbb{P}^n\,;\ \langle \zeta, \xi\rangle = 0\big\}$ and vice versa.

A fixed choice of hyperplane $\eta^* \in (\mathbb{P}^n)^*$ in $\mathbb{P}^n$ (called the *hyperplane at infinity*) defines a unique affine structure on $\mathbb{P}^n \setminus \eta^*$, making it affine-isomorphic to $\mathbb{C}^n$.

If Ω is an open set in $\mathbb{P}^n$ there is a 1-1 correspondence between holomorphic functions in Ω, $H(\Omega)$, and zero-homogeneous holomorphic functions in $\pi^{-1}\Omega \subset \mathbb{C}^{n+1} \setminus \{0\}$. Let

$$X = \big\{([\zeta], [\xi]) \in \mathbb{P}^n \times (\mathbb{P}^n)^*\,;\ \langle \zeta, \xi\rangle = 0\big\}$$

and $c_n = (2\pi i)^{-n}$.

PROPOSITION 1. — *If $F(\zeta)$ and $\Phi(\xi)$ are $-n$-homogeneous and holomorphic in (some open subsets of) $\mathbb{C}^{n+1} \setminus \{0\}$ and $(\mathbb{C}^{n+1})^* \setminus \{0\}$, respectively, then*

$$\alpha(\xi, \zeta) = c_n \Phi(\xi) \sum_0^n \xi_j\,\mathrm{d}\zeta_j \wedge \Big(\sum_0^n \mathrm{d}\xi_j \wedge \mathrm{d}\zeta_j\Big)^{n-1} F(\zeta) \tag{1}$$

is a well-defined closed form on (some open subset of) X.

Proof. — If for instance $\zeta_0 \neq 0$ and $\xi_0 \neq 0$, then

$$(2) \qquad \alpha(\xi,\zeta) = c_n \xi_0^n \Phi(\xi) \sum_1^n \frac{\xi_j}{\xi_0} \,\mathrm{d}\frac{\zeta_j}{\zeta_0} \wedge \Big(\sum_1^n \mathrm{d}\frac{\xi_j}{\xi_0} \wedge \mathrm{d}\frac{\zeta_j}{\zeta_0}\Big)^{n-1} \zeta_0^n F(\zeta)$$

which shows that α indeed is well-defined on $\mathbb{P}^n \times (\mathbb{P}^n)^*$ and in particular on the submanifold X. Since $F(\zeta)$ is holomorphic it follows of bidegree reasons that $\mathrm{d}_\zeta \alpha = 0$, but since $\sum \xi_j \,\mathrm{d}\zeta_j = -\sum \zeta_j \,\mathrm{d}\xi_j$ on X and also $\Phi(\xi)$ is holomorphic, we similarly have $\mathrm{d}_\xi \alpha = 0$ on X. $\square$

Definition. — A CL-*section* (Cauchy-Leray section) $s(\zeta)$ is a C^1-mapping from some subset of $\mathbb{P}^n$ into $(\mathbb{P}^n)^*$ such that $\langle \zeta, s(\zeta)\rangle = 0$, i.e. such that $s(\zeta)$ is a hyperplane through ζ.

If $\Phi(\xi)$ is holomorphic on the image of $s(\zeta)$ it follows from PROPOSITION 1 that $\alpha(s(\zeta),\zeta)$ is a closed $(n, n-1)$-form in ζ.

PROPOSITION 2 (The Cauchy-Fantappiè-Leray formula). — *Suppose that $\omega \subset\subset \mathbb{P}^n \setminus \eta^*$ has smooth boundary and that $f \in H(\bar\omega)$. If $z \in \omega$ and $s(\zeta)$ is a* CL-*section over $\partial\omega$ such that $s(\zeta)$ does not contain z, then*

$$(3) \qquad f(z) = c_n \int_{\partial\omega} \frac{\langle z,\eta^*\rangle^n}{\langle z,\xi\rangle^n}\, \xi \wedge (\mathrm{d}\xi)^{n-1}|_{\xi = s(\zeta)}\, f(\zeta)\, \frac{1}{\langle \zeta,\eta^*\rangle^n}$$

where ξ is identified with the form $\sum_0^n \xi_j \,\mathrm{d}\zeta_j$.

This well-known formula is a special case of THEOREM 4 below.

Remark. — If $\eta = \eta^* = (1,0,0,\ldots)$ and we make the identifications $[\zeta] = [(1,\zeta')] \sim \zeta' \in \mathbb{C}^n, [z] = [(1,z')] \sim z' \in \mathbb{C}^n$ and $[s] = [(s_0, s')] \sim s'$, noting that $s(\zeta)$ being a CL-section means that $s_0(\zeta) = -s'(\zeta)\cdot\zeta'$, then (3) becomes (3) in § 0. In particular, if $s(\zeta)$ is the complex tangent plane to the level surface of the distance function $d(\zeta, z)$ in $\mathbb{C}^n$, then (3) becomes the Bochner-Martinelli formula (2) in § 0. $\square$

If we for each z in, say, $K \subset \omega$ have a section $\zeta \mapsto (\zeta, z)$ that does not intersect z, then of course (3) holds for all $z \in K$. Then $s(\zeta, z)$ also may have some additional desirable property ; e.g. being constant, a polynomial or at least holomorphic in z. We are going to construct formulas when such a required $s(\zeta, z)$ only is found locally on $\partial\omega$. To this end, we will use the following formalism :

Let $z \in \mathbb{P}^n/\eta^*$ be fixed and let

$$\alpha(\xi,\zeta) = c_n \frac{\langle z,\eta^*\rangle^n}{\langle z,\xi\rangle^n}\, \xi \wedge (\mathrm{d}\xi)^{n-1} \frac{1}{\langle \zeta,\eta^*\rangle^n}.$$

For CL-sections $s_0(\zeta)$, $s_1(\zeta), \dots$ not intersecting z, we define

$$H^{k+1}\big(s_0, s_1, \dots, s_k\big) = \int_{t\in\Delta_k} \alpha\Big(\sum_0^k \frac{t_j s_j}{\langle z, s_j\rangle}, \zeta\Big), \tag{4}$$

where Δ_k is the standard simplex $\{t \in \mathbb{R}_+^{k+1}; \sum_0^k t_j = 1\}$ with the usual orientation, and the integrand is the pullback of α under the mapping $(t,\zeta) \mapsto (\xi,\zeta) = (\sum t_j s_j(\zeta)/\langle z, s_j(\zeta)\rangle, \zeta)$. Hence $H^{k+1}(s_0,\dots,s_k)$ is a form in ζ and we have :

PROPOSITION 3. — *$H^{k+1}(s_0,\dots,s_k)$ has bidegree $(n, n-k)$ and vanishes if $k \geq n$. It is alternating in s_j and*

$$\begin{aligned} \mathrm{d}H^{k+1}\big(s_0, s_1, \dots, s_k\big) &= H^k\big(s_1,\dots,s_k\big) - H^k\big(s_0, s_2, \dots, s_k\big) \\ &\quad + \cdots + (-1)^k H^k\big(s_0, s_1, \dots, s_{k-1}\big). \end{aligned} \tag{5}$$

Explicitly,

$$H^k\big(s_1,\dots,s_k\big) = c_n \langle z, \eta^*\rangle^n s_1 \wedge \cdots \wedge s_k \wedge \mathcal{S}, \tag{6}$$

with

$$\mathcal{S} = \sum_{|\alpha|=n-k} \frac{(\bar\partial s_1)^{\alpha_1} \wedge \cdots \wedge (\bar\partial s_k)^{\alpha_k}}{\langle z, s_1\rangle^{\alpha_1+1} \cdots \langle z, s_k\rangle^{\alpha_k+1}} \frac{1}{\langle \zeta, \eta^*\rangle^n}.$$

Proof. — By PROPOSITION 1, $\alpha(\xi,\zeta)$ is a closed form and hence $(\mathrm{d}_\zeta + \mathrm{d}_t)\alpha(\sum_0^k t_j s_j/\langle z, s_j\rangle, \zeta) = 0$. Thus

$$\begin{aligned} \mathrm{d}_\zeta H^{k+1}\big(s_0,\dots,s_k\big) &= -\int_{t\in\Delta_k} \mathrm{d}_t \alpha\Big(\sum_0^k \frac{t_j s_j}{\langle z, s_j\rangle}, \zeta\Big) \\ &= -\int_{t\in\partial\Delta_k} \alpha\Big(\sum_0^k \frac{t_j s_j}{\langle z, s_j\rangle}, \zeta\Big) \\ &= H^k\big(s_1,\dots,s_k\big) - H^k\big(s_0, s_2, \dots, s_k\big) + \cdots \end{aligned}$$

if $\partial\Delta_k$ is oriented in the usual way. This proves (5) and the other statements follow from formula (6) which we now are going to derive.

Since both sides of (6) are zero-homogeneous in each s_j, we may assume that $\langle z, s_j\rangle = 1$. We also assume that

$$\langle z, \eta^*\rangle = \langle \zeta, \eta^*\rangle = 1.$$

Then, letting $\tau = \sum_0^n \tau_j \, \mathrm{d}\zeta_j$ (and assuming $\langle z, \xi \rangle > 0$) we can write

$$n!\, \alpha(\xi, \zeta) = c_n \int_{\sigma=0}^{\infty} \mathrm{e}^{-\langle z, \tau \rangle} (\mathrm{d}\tau)^n |_{\tau = \sigma \xi}$$

so that

$$H^k(s_1, \ldots, s_k) = \frac{c_n}{n!} \int_{t \in \Delta_{k-1}} \int_{\sigma=0}^{\infty} \mathrm{e}^{-\sigma \Sigma t_j} (\mathrm{d}\tau)^n |_{\tau = \sigma \Sigma t_j s_j}.$$

Making the change of variables $x_j = \sigma t_j$ we get

$$\begin{aligned} H^k &= \frac{c_n}{n!} \int_{x \in \mathbb{R}_+^k} \mathrm{e}^{-\Sigma x_j} \Big(\sum \mathrm{d}x_j \wedge s_j + \sum x_j \, \mathrm{d}s_j \Big)^n \\ &= \frac{c_n}{n!} \binom{n}{k} \int_{\mathbb{R}_+^k} \mathrm{e}^{-\Sigma x_j} k!\, s_1 \wedge \cdots \wedge s_k \wedge \sum_{|\alpha| = n-k} \frac{(n-k)!}{\alpha!} x^\alpha (\mathrm{d}s)^\alpha \, \mathrm{d}x \\ &= c_n s_1 \wedge \cdots \wedge s_k \sum_{|\alpha| = n-k} (\mathrm{d}s_1)^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}s_k)^{\alpha_k}. \end{aligned}$$

Finally, each $\mathrm{d}s_j$ can be replaced by $\bar{\partial} s_j$ for bidegree reasons. $\square$

We are now ready for our main result of this paragraph.

THEOREM 4. — *Suppose that $f \in H(\bar{\omega})$, $z \in \omega \subset\subset \mathbb{P}^n \setminus \eta^*$ and $\partial\omega$ is smooth. If $\{\omega_\alpha\}$ is an open cover of $\partial\omega$, $s_\alpha(\zeta)$ are* CL*-sections over ω_α which do not intersect z and $\{\psi_\alpha\}$ is a partition of unity subordinated $\{\omega_\alpha\}$, then*

$$\begin{aligned} (7) \quad f(z) = \int_{\partial\omega} \sum_\alpha \psi_\alpha H^1(s_\alpha) f + \int_{\partial\omega} \sum_{\alpha_2, \alpha_1} \psi_{\alpha_2} \, \mathrm{d}\psi_{\alpha_1} \wedge H^2(s_{\alpha_1}, s_{\alpha_2}) f \\ + \cdots + \int_{\partial\omega} \sum_{\alpha_1, \ldots, \alpha_n} \psi_{\alpha_n} \, \mathrm{d}\psi_{\alpha_{n-1}} \wedge \cdots \wedge \mathrm{d}\psi_{\alpha_1} \wedge H^n(s_{\alpha_1}, \ldots, s_{\alpha_n}) f. \end{aligned}$$

If $n = 1$, eq. (7) reduces to Cauchy's formula. If $s_\alpha(\zeta, z)$ is defined for $z \in K$ and holomorphic in z, then (7) is a representation formula with holomorphic kernel. Note that if there are only $m < n$ different s_α, then only H^k for $k \leq m$ occur. In particular, if $m = 1$ we get back (3).

Proof. — Let $s(\zeta)$ be the complex tangent space to the level surface of same appropriate distance function (cf. the remark above). Then by the Bochner-Martinelli formula (2) in § 0,

$$\begin{aligned} f(z) &= \int_{\partial\omega} H^1(s) f = \int_{\partial\omega} \sum_{\alpha_1} \psi_{\alpha_1} H^1(s) f \\ &= \int_{\partial\omega} \sum \psi_{\alpha_1} H^1(s_{\alpha_1}) f - \int_{\partial\omega} \sum \psi_{\alpha_1} \big(H^1(s_{\alpha_1}) - H^1(s) \big) f. \end{aligned}$$

By (5), $H^1(s_{\alpha_1}) - H^1(s) = \mathrm{d}H^2(s_{\alpha_1}, s)$, so the holomorphicity of f and Stockes' theorem yield :

$$f(z) = \int_{\partial\omega} \sum \psi_{\alpha_1} H^1(s_{\alpha_1}) f + \int_{\partial\omega} \sum_{\alpha_1,\alpha_2} \psi_{\alpha_2} \mathrm{d}\psi_{\alpha_1} \wedge H^2\big(s_{\alpha_1}, s\big) f.$$

Now, $H^2(s_{\alpha_1}, s) = H^2(s_{\alpha_1}, s_{\alpha_2}) + H^2(s_{\alpha_2}, s) - \mathrm{d}H^3(s_{\alpha_1}, s_{\alpha_2}, s)$ by (5). Moreover we notice that

$$\sum_{\alpha_1,\alpha_2} \psi_{\alpha_2} \mathrm{d}\psi_{\alpha_1} \wedge H^2\big(s_{\alpha_2}, s\big) = \sum_{\alpha_1} \mathrm{d}\psi_{\alpha_1} \wedge \sum_{\alpha_2} \psi_{\alpha_2} H^2(s_{\alpha_2}, s) = 0,$$

so we get

$$f(z) = \int_{\partial\omega} \sum_{\alpha_1} \psi_{\alpha_1} H^1(s_{\alpha_1}) f + \int_{\partial\omega} \sum_{\alpha_1,\alpha_2} \psi_{\alpha_2} \mathrm{d}\psi_{\alpha_1} \wedge H^2\big(s_{\alpha_1}, s_{\alpha_2}\big) f$$
$$- \int_{\partial\omega} \sum_{\alpha_1,\alpha_2,\alpha_3} \psi_{\alpha_3} \mathrm{d}\psi_{\alpha_2} \wedge \mathrm{d}\psi_{\alpha_1} \wedge H^3\big(s_{\alpha_1}, s_{\alpha_2}, s\big) f.$$

With a repeated application of (5) we finally arrive at (7). □

Remark. — If D is a domain with piecewise smooth boundary $\partial D = \sum S_j$, where S_j are the smooth pieces, and s_j are CL-sections over S_j that do not contain $z \in D$, then in our notation the Norguet formula (for details see [3] and [4]) is

$$f(z) = \sum_I \int_{S_I} H^{|I|}\big(s_{I_1}, \dots, s_{I_{|I|}}\big) f, \tag{8}$$

where $S_I = \bigcap_{j \in I} S_j$, appropriately oriented, for multiindices I. Hence (8) can be viewed as a limit case of (7) when the ψ_j : s tend to characteristic functions.

2. A Runge type theorem

THEOREM 1. — *A domain of holomorphy $D \subset \mathbb{C}^n$ is a Runge domain if and only if for any compact $K \subset D$ and $z \in D$ sufficiently near ∂D (depending on K) there is a curve of algebraic hypersurfaces from z to the hyperplane at infinity that does not intersect K.*

We recall that D is a Runge domain if the polynomials are dense in $H(D)$. The condition in the theorem means that there is a continuous mapping $t \mapsto F_t$, with $a \leq t \leq b$, where F_t are polynomials, such that $F_a(z) = 0$, F_b is a non-zero constant and $K \cap \{F_t = 0\} = \emptyset$ for $a \leq t \leq b$.

We first observe that the "only if" part is quite trivial. Namely, if $z \in D \setminus \widehat{K}_D$, then there is a $f \in H(D)$ and hence a polynomial f such that $f(z) = 1$ but $|f| < 1$ on K. Thus $F_t(z) = 1 - f(z)/t$, for $1 \leq t \leq \infty$ yields the curve of algebraic hypersurfaces.

The "if" part is for $n = 1$ just Runge's theorem and for $n > 1$ it is a consequence of the Oka-Stolzenberg theorem, see [6], and the (not quite trivial) fact that D is a Runge domain if the polynomial hull $\widehat{K}$ is contained in D whenever K is. However, we will give a direct Hahn-Banach proof here. It is clearly enough to prove :

PROPOSITION 2. — *Suppose K is compact in the domain $\omega \in \mathbb{C}^n$, $\partial\omega$ is smooth and that through each point on $\partial\omega$ there is a curve of algebraic hypersurfaces ending at the hyperplane at infinity and not intersecting K. Then any $f \in H(\bar{\omega})$ can be uniformly approximated by polynomials on K.*

Proof. — We may assume that $0 \in K$. For fixed $\zeta_0 \in \partial\omega$, let $\{z\,;\ p(z) = 0\}$ be a hypersurface through ζ_0 as in the hypothesis. We can write $p(z) = a(z) \cdot (\zeta_0 - z)$ where $a(z)$ is a polynomial. For ζ near ζ_0 it is clear by continuity that $\{z, a(z) \cdot (\zeta - z) = 0\}$ is a hypersurface through ζ which can be joined to the hyperplane at infinity by a curve that does not intersect K. Locally on $\partial\omega$ we now put

$$s(\zeta, z) = \frac{a(z)}{a(0)}\,\zeta, \tag{1}$$

so that $s(\zeta, z) \cdot (\zeta - z) = 1 + \sum_{0<|\beta|\leq M} a_\beta z^\beta$.

Now let $\{\omega_\alpha\}$ be an open cover of $\partial\omega$ such that we have a $s_\alpha(\zeta, z)$ of the form (1) in ω_α, and let ψ_α be a partition of unity. We can then form a representation formula for $f \in H(\bar{\omega})$ in accordance to THEOREM 4 in § 1.

Since $s_\alpha(\zeta, z)$ is holomorphic in ζ, (cf. (6) in § 1) only the last sum of terms in (7) in § 1 actually occurs and we thus have (compare with the proof of PROPOSITION 2 in § 1).

$$f(z) = \int_{\partial\omega} \sum_{\alpha_1,\dots,\alpha_n} \psi_{\alpha_n}\, \mathrm{d}\psi_{a_{n-1}} \wedge \cdots \wedge \mathrm{d}\psi_{\alpha_1} \frac{s_{\alpha_1} \wedge \cdots \wedge s_{\alpha_n}}{s_{\alpha_1} \cdot (\zeta - z) \cdots s_{\alpha_n} \cdot (\zeta - z)}\, f(\zeta). \tag{2}$$

This formula may be compared to the Weil formula for analytic polyhedra.

Now we are ready for the Hahn-Banach argument. Thus let μ be a measure on K that annihilates (the restrictions to K of) all polynomials.

We then have to show that also $\mu(f) = 0$. However, by (2) and Fubini's theorem it is enough to show that, for each fixed $\zeta \in \partial\omega$,

$$\int_K \frac{s_{\alpha_1} \wedge \cdots \wedge s_{\alpha_n} d\mu(z)}{s_{\alpha_1} \cdot (\zeta - z) \cdots s_{\alpha_n} \cdot (\zeta - z)} = 0.$$

The integral has the form

$$\int_K \frac{p(z)\,\mathrm{d}\mu(z)}{\left(1 + \sum_{|\beta| \le M} a_\beta^1 z^\beta\right) \cdots \left(1 + \sum_{|\beta| \le M} a_\beta^n z^\beta\right)}$$

where $p(z)$ is a polynomial. By the assumption on the polynomials, this integral is holomorphic in the coefficients α_β^j in a connected open set that contains the point $a_\beta^i = a_\beta^j(\zeta)$ as well as $a_\beta^j = 0$. But for a_β^j near 0 we can expand in a power series in z, so by the assumption on μ, the integral vanishes for these a_β^j and hence by uniqueness also for $a_\beta^j = a_\beta^j(\zeta)$. $\square$

3. The Fantappiè transform

We begin with some definitions and simple facts. If E is a set in $\mathbb{P}^n$ we let E^* be the set in $(\mathbb{P}^n)^*$ consisting of all hyperplanes in $\mathbb{P}^n$ which do not intersect E. A set E is called linearly convex if $\mathbb{P}^n \setminus E$ is a union of hyperplanes, i.e. if $E^{**} = E$. If E is compact (open) then E^* is open (compact) and if $E \subset E'$ then $(E')^* \subset E^*$.

A complex line in $\mathbb{P}^n$ is the generic intersection of $(n-1)$ hyperplanes and thus isomorphic to $\mathbb{P}^1$.

Definition. — An open set $D \subset \mathbb{P}^n$ is called $\mathbb{C}$-*convex* if its intersection with each complex line is simply connected and $\neq \mathbb{P}^1$, i.e. homeomorphic to the unit disc.

Any $\mathbb{C}$-convex domain is linearly convex, see [8], and the converse is true if D has C^2-boundary, see [1], but not in general, see e.g. example 1 below.

Suppose we have chosen a hyperplane at infinity η^* in $\mathbb{P}^n$ and a point $\eta \in \mathbb{P}^n \setminus \eta^*$ (η and η^* can be thought of as the origins in $\mathbb{P}^n$ and $(\mathbb{P}^n)^*$, respectively). For a linearly convex domain D in $\mathbb{P}^n$ wet let $H_0(D)$ be the space of holomorphic functions in D that vanish on the hyperplane η^*. Similarly, $H_0(D^*)$ is the space of functions, that are holomorphic in some neighborhood of D^* and vanish on η. Thus $H(D) = H_0(D)$ and $H(D^*) = H_0(D^*)$ if $\eta \in D \subset \mathbb{P}^n \setminus \eta^*$. All spaces are taken with the usual topologies, see [2].

If μ is an analytic functional, $\mu \in H_0'(D)$, then the Fantappiè transform of μ, with respect to η and η^*, is

$$F\mu(\zeta) = \mu\Big(\frac{\langle \eta, \zeta\rangle\langle \cdot, \eta^*\rangle}{\langle \cdot, \zeta\rangle}\Big),$$

which is an element in $H_0(D^*)$, since μ has some compact carrier in D, and $F : H_0'(D) \to H_0(D^*)$ is continuous. Its adjoint, $F^* : H_0'(D^*) \to H_0(D)$ is

$$F^*\mu(z) = \mu\Big(\frac{\langle \eta, \cdot\rangle\langle z, \eta^*\rangle}{\langle z, \cdot\rangle}\Big).$$

MARTINEAU [2] showed that F is an isomorphism if and only if F^* is. (A set D for which this holds is called *strongly linearly convex* by MARTINEAU.) Now we can reformulate THEOREM 0 in § 0 :

THEOREM 1. — *If D is a linearly convex domain in $\mathbb{P}^n$, $n > 1$ then F (and F^*) are isomorphisms if and only if D is $\mathbb{C}$-convex.*

The "only if" part is proved in [9] and we are now going to prove the "if" part. Our first objective is to reduce to the case when $\eta \in D \subset\subset \mathbb{P}^n \setminus \eta^*$. To this end, choose $\tilde\eta$ and $\tilde\eta^*$ so that $\tilde\eta \in D \subset\subset \mathbb{P}^n \setminus \tilde\eta^*$, and let $\widetilde{F}$ be the corresponding Fantappiè transform. Let τ and σ be the isomorphisms $\tau : H_0'(D) \to H'(D)$, $\sigma : H_0(D^*) \to H(D^*)$ defined by

$$\tau\mu(f) = \mu\Big(\frac{\langle \cdot, \tilde\eta^*\rangle}{\langle \cdot, \eta^*\rangle} f\Big) \quad \text{and} \quad \sigma\varphi(\zeta) = \frac{\langle \tilde\eta, \zeta\rangle}{\langle \eta, \zeta\rangle}\varphi(\zeta).$$

Then the diagram

$$\begin{tikzcd} H_0'(D) \arrow[r, "F"] \arrow[d, "\tau"'] & H_0(D^*) \arrow[d, "\sigma"] \\ H'(D) \arrow[r, "\widetilde{F}"] & H(D^*) \end{tikzcd}$$

is commutative, so F is an isomorphism if and only if $\widetilde{F}$ is. Hence we may assume in the sequel that $\eta \in D \subset\subset \mathbb{P}^n \setminus \eta^*$. Sometimes it is convenient to assume that $\eta^* = \eta = (1, 0, \ldots, 0)$ so that in "$\mathbb{C}^n$-formalism"

$$F\mu(\zeta) = \mu\Big(\frac{1}{1 + \zeta \cdot \xi}\Big).$$

LEMMA 2. — *If D is $\mathbb{C}$-convex then D^* is connected.*

Proof. — See COROLLARY 4 in Appendix. □

PROPOSITION 3. — *If D is an open linearly convex domain in $\mathbb{C}^n \simeq \mathbb{P}^n \setminus \eta^*$ such that D^* is connected, then D is a Runge domain and $F : H'(D) \to H(D^*)$ is injective.*

Proof. — The Runge property immediately follows from THEOREM 1 in § 2. Then suppose that $F\mu(\zeta) = \mu\big(1/(1+\zeta\cdot\xi)\big) = 0$ for all $\zeta \in D^*$. In particular this holds for ζ in a neighborhood of the origin. After differentiation one finds that $\mu(p) = 0$ for all polynomials $p(\xi)$ and hence $\mu = 0$. ∎

It remains to prove that F is surjective. Take a fixed $\varphi \in H(D^*)$, say $\varphi \in H(\Omega)$ where $D^* \subset\subset \omega \subset\subset \Omega$ and ω has smooth boundary. Locally in ω_α on $\partial\omega$ we can find CL-sections $s_\alpha(\tau)$ such that $s_\alpha(\tau)$ avoids D^*, and hence by THEOREM 4 in § 1 we have :

$$(1)\quad \varphi(\zeta) = \int_{\partial\omega} \sum_\alpha \psi_\alpha H^1(s_\alpha)\varphi + \int_{\partial\omega} \sum_{\alpha_1,\alpha_2} \psi_{\alpha_2}\,\mathrm{d}\psi_{\alpha_1} \wedge H^2(s_{\alpha_1}, s_{\alpha_2})\varphi + \cdots$$

Note that the right hand side of (1) consists of a sum of terms, cf. PROPOSITION 3 in § 1, of the form

$$(2)\qquad \int_{\tau\in\partial\omega} h(\tau)\langle\eta,\zeta\rangle^n \prod_{j=1}^n \frac{\big\langle s_{\alpha_j}(\tau),\eta^*\big\rangle}{\big\langle s_{\alpha_j}(\tau),\zeta\big\rangle}, \quad \zeta \in D^*,$$

where $h(\tau)$ is smooth and has its support where all the occuring s_{α_j} are defined. It is thus sufficient to find an inverse Fantappiè transform of any function in ζ of the form (2). Since the image of $\tau \mapsto s_{\alpha_j}(\tau)$ is a compact set in D, this is accomplished by :

LEMMA 4. — *If D is $\mathbb{C}$-convex, $\eta \in D \subset\subset \mathbb{P}^n \setminus \eta^*$, then for points $a_1, \ldots, a_k \in D$ there is a $\mu_{a_1,\ldots,a_k} \in H'(D)$ such that :*

$$(3)\qquad F\mu_{a_1,\ldots,a_k}(\zeta) = \langle\eta,\zeta\rangle^k \prod_1^k \frac{\langle a_j,\eta^*\rangle}{\langle a_j,\zeta\rangle}.$$

Moreover, for any compact $K \subset D$ there is a compact $K' \subset D$ and a constant $C_K > 0$ such that :

$$(4)\qquad \big|\mu_{a_1,\ldots,a_k}(f)\big| \le C_K \sup_{K'} |f|, \quad a_j \in K.$$

It is clear from the lemma that $f \mapsto \int_{\partial\omega} h(\tau)\mu_{s_{\alpha_1}(\tau),\ldots,s_{\alpha_n}(\tau)} f$ defines an element in $H'(D)$ and that its Fantappiè transform is (2).

Proof of the lemma. — For $a, b \in D$ and, say, entire f we consider the functional :

$$(5) \qquad \mu_{a,b}(f) = \int_0^1 \eta \cdot \partial_\xi \Big(\frac{f(\xi)}{\langle \xi, \eta^* \rangle}\Big)\Big|_{\xi = ta/\langle a,\eta^*\rangle + (1-t)b/\langle b,\eta^*\rangle} \,\mathrm{d}t.$$

One readily verifies that $\mu_{a,b} f$ is holomorphic in a and b and that :

$$(6) \qquad F\mu_{a,b}(\zeta) = \langle \eta, \zeta \rangle^2 \frac{\langle a, \eta^* \rangle \langle b, \eta^* \rangle}{\langle a, \zeta \rangle \langle b, \zeta \rangle}.$$

We can change path of integration in the integral such that (5) is defined for all $f \in H(D)$. In fact, for any fixed $K \subset D$ there is another compact $K' \subset D$ and $C_K > 0$ such that (4) holds (with $k = 2$). Now we put :

$$\mu_{a_1,\dots,a_k}(f) = \mu_{a_k,a_{k-1}}\Big(\mu_{a_{k-1},\cdot}\big(\cdots(\mu_{a_1,\cdot}(f))\cdots\big)\Big).$$

Then (3) follows from (6) and (4) follows by induction. □

Thus the "if" part of THEOREM 1 is completely proved.

Example 1. — Let $D = \Omega \times \Delta \subset \mathbb{C}^2 \subset \mathbb{P}^2$ where Δ is the unit disc and Ω is a non-convex domain in $\mathbb{C}$. Then D and D^* are both connected so that both of F^* and F are injective, but D is not $\mathbb{C}$-convex so none of them are surjective. In fact, if L is a line such that $L \cap D$ is not connected and a and b belong to different components of $L \cap D$, then the function

$$\varphi(\zeta) = \frac{1}{(1 + a \cdot \zeta)(1 + b \cdot \zeta)}$$

does not belong to the image of F unless L meets the origin. In fact, if L meets the origin (this is equivalent to that the planes $a \cdot z = 1$ and $b \cdot z = 1$ are parallel) then $\mu_{a,b}$ can be realized as Dirac measures in the points a and b. However, in the generic case when $0 \notin L$, then any carrier of $\mu_{a,b}$ must connect a and b in L. This follows by computing $\mu_{a,b}$ according to (5) and comparing with the "only if" proof of Theorem 1 in [9]. □

Remark. — The proof of the surjectivity of F can be turned in a slightly different way, which reveals the connection to usual inversion formulas for F when e.g. ∂D^* is smooth.

So let again $\varphi \in H(\Omega)$ where $D^* \subset\subset \omega \subset\subset \Omega$, and let $s(\tau)$ be a CL-section over $\partial\omega$ that avoids $\eta^* \in D^*$. Let :

$$\beta(f, \xi, \tau) = (\eta \cdot \partial_\xi)^{n-1}\Big(\frac{f(\xi)}{\langle \xi, \eta^* \rangle}\Big)\xi \wedge \frac{(\mathrm{d}\xi)^{n-1}\varphi(\tau)}{\langle \eta, \tau \rangle^n}.$$

Then

$$f \longmapsto \int_{\partial\omega} \beta\big(f, s(\tau), \tau\big) = \mu(f)$$

is defined for all f which are holomorphic in some big domain in $\mathbb{P}^n \setminus \eta^*$ that contains the image of $s(\tau)$. Also note that $F\mu(\zeta)$ is defined and equals $\varphi(\zeta)$ for all ζ near η^*. We want to show that μ has a continuous extension to $H(D)$. It then follows by uniqueness of analytic continuation that $F\mu \equiv \varphi$. In particular, if $s(\tau)$ takes values in D, we are already done. Otherwise we choose s_α in ω_α as before and then repeating the argument in the proof of THEOREM 4 in § 1, we can successively eliminate $s(\zeta)$ and get :

$$\begin{aligned}\mu(f) = &\int_{\partial\omega} \sum_\alpha \psi_\alpha \beta(f, s_\alpha, \tau)\\ &+ \int_{\partial\omega} \sum_{\alpha_1,\alpha_2} \psi_{\alpha_2}\,\mathrm{d}\psi_{\alpha_1} \wedge \int_{\Delta_1} \beta\big(f, t_1 s_{\alpha_1} + t_2 s_{\alpha_2}, \tau\big)\\ &+ \cdots + \int_{\partial\omega} \sum_{\alpha_1,\ldots,\alpha_n} \psi_{\alpha_n}\,\mathrm{d}\psi_{\alpha_{n-1}} \wedge \cdots\\ &\qquad\qquad \cdots \wedge \mathrm{d}\psi_{\alpha_1} \wedge \int_{\Delta_{n-1}} \beta\Big(f, \sum_1^n t_j s_{\alpha_j}, \tau\Big).\end{aligned}$$

We show in Appendix that the simplices Δ_k can be deformed in such a way that the integrals remain unchanged but the supports will be contained in D, which thus gives an extension of μ to $H(D)$. □

Notice in particular that :

$$\mu_{a_1,\ldots,a_n}(f) = \int_{t\in\Delta_{n-1}} (\eta\cdot\partial_\xi)^{n-1}\Big(\frac{f(\xi)}{\langle\xi,\eta^*\rangle}\Big)\Big|_{\xi=\Sigma t_j a_j/\langle a_j,\eta^*\rangle} \,\mathrm{d}t_2\wedge\cdots\wedge\mathrm{d}t_n.$$

This follows by uniqueness since both sides are equal when applied to the Fantappiè kernel.

Appendix Topology of $\mathbb{C}$-convex domains

PROPOSITION 1. — *A $\mathbb{C}$-convex domain D in $\mathbb{P}^n$ is simply connected, i.e. any closed curve in D is homotopic to a point, i.e.* $\pi_1(D) = 1$.

Proof. — Let σ be a closed curve in D. Since D is open, we may assume that σ is piecewise linear (in some affinization) and we let $a_0, \ldots, a_m$

denote the corners of σ. Let a be a fixed point in D. There are curves $\gamma_j(t)$, $0 \le t \le 1$, from a_j to a lying entirely in the intersection of D and the complex line spanned by a_j and a. To prove that σ is homotopic to a, it is enough to show that the segment between $a_j = \gamma_j(0)$ and $a_{j+1} = \gamma_{j+1}(0)$ can be continuously (in t) deformed to a curve from $\gamma_j(t)$ to $\gamma_{j+1}(t)$, lying in D and the complex line spanned by thse two points. Since D is open, this is clearly possible locally in t, say, in the intervals $[0,t_1]$, $[t_1,t_2],\dots,[t_m,1]$; and since the intersection of D and the complex line spanned by $\gamma_j(t_k)$ and $\gamma_{j+1}(t_k)$ is simply connected, the two different choices of curves, joining these two points, are homotopic. $\square$

COROLLARY 2. — *The affine projection of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ onto a complex line is simply connected.*

Proof. — It is enough to show that any closed curve $\gamma(t)$ in the projection can be lifted to a closed curve in D. Since D is open, this is clearly possible locally in t, say, on $[0,t_1]$, $[t_1,t_2],\dots,[t_m,1]$. The two different points in D corresponding to t_k can then be connected by a curve in the complex line through these two points, and this line is orthogonal to the projection. $\square$

A mapping $T : \mathbb{P}^m \to \mathbb{P}^n$ is projective if it is induced by an injective mapping $\widetilde{T} : \mathbb{C}^{m+1} \to \mathbb{C}^{n+1}$. Then we also have a mapping $T^* : (\mathbb{P}^n)^* \setminus X \to (\mathbb{P}^m)^*$ where $X = \{[z] \in (\mathbb{P}^n)^* \,;\, \widetilde{T}^* z = 0\}$.

If X is contained in the hyperplane at infinity for some affinization of $(\mathbb{P}^n)^*$, then T^* is a affine mapping from $(\mathbb{P}^n)^* \setminus \eta$ to $(\mathbb{P}^m)^* \setminus T\eta^{-1}$.

PROPOSITION 3. — *If $T : \mathbb{P}^m \to \mathbb{P}^n$ is projective and $\operatorname{Im} T \cap E^* \neq \emptyset$, then T^* is defined on E, i.e. $E \subset \mathbb{P}^n \setminus \{T^* = 0)$, and $(T^*E)^* = T^{-1}E^*$ in $\mathbb{P}^m$.*

Proof. — $z \in T^{-1}E^* \Leftrightarrow Tz \in E^* \Leftrightarrow \langle Tz, \xi\rangle \neq 0$ for all $\xi \in E \Leftrightarrow \langle z, T^*\xi\rangle \neq 0$ for all $\xi \in E$ (in particular $T^*\xi \neq 0$) $\Leftrightarrow z \in (T^*E)^*$. $\square$

COROLLARY 4. — *Suppose D is a $\mathbb{C}$-convex domain in $\mathbb{P}^n$. If L is any complex line, then $(D^* \cap L)^*$ is simply connected*; *in particular $D^* \cap L$ is connected.*

Proof. — Let L be the image of the projective map $T : (\mathbb{P}^1)^* \to (\mathbb{P}^n)^*$. Then $D^* \cap L \simeq T^{-1}D^* = (T^*D)^*$ and T^*D is simply connected by COROLLARY 2. $\square$

THEOREM 5 (Zelinskij). — *Any $\mathbb{C}$-convex domain D is acyclic, i.e. $H_k(D,\mathbb{Z}) = 0$ for $k > 0$ and $H_0(D,\mathbb{Z}) = \mathbb{Z}$.*

Proof. — See [7]. $\square$

THEOREM 6. — *Any $\mathbb{C}$-convex domain D is contractible.*

Proof. — By THEOREM 5, PROPOSITION 1 and Hurewicz theorem (7.5, Theorem 5 in [5]) it follows that $\pi_k(D) = 1$ for all k, i.e. all homotopy groups are trivial. Since D is a CW-complex then Whitehead's theorem (7.6, Corollary 24 in [5]) implies that the mapping $f : D \to$ point is a homotopy equivalence, i.e. D is contractible. $\square$

COROLLARY 7. — *Suppose $D \subset \mathbb{P}^n$ is a $\mathbb{C}$-convex domain and $p_0, \dots, p_k \in D$. Then there is a differentiable k-simplex $\sigma : \Delta_k \to D$ with corners p_j such that each r-face, $r \leq k$, lies the (at most) r-dimensional subspace spanned by its $r+1$ corners.*

Proof. — We construct σ successively. Suppose we have a σ with the proposed properties from the $(r-1)$ skeleton of Δ_k, i.e. from the union of the $(r-1)$-faces. Now let F be a fixed r-face and Π the (at most) r-dimensional complex subspace spanned by its corners. Since $\Pi \cap D$ is $\mathbb{C}$-convex, and thus contractible by THEOREM 6, we can extend σ from ∂F to a continuous $\sigma : F \to \Pi \cap D$. We may also assume that σ is smooth since otherwise we approximate it by a smooth map into Π which coincides with σ on ∂F. $\square$

Let $p_0, \dots, p_k$ be points in $\mathbb{C}^n$ and let $\tau : \Delta_k \to \mathbb{C}^n$ be the k-simplex with corners p_j. For holomorphic $(k,0)$-forms f, defined in some neighborhood of τ, we define the functional $\Lambda_p f = \Lambda_{p_0,\dots,p_k} f = \int_\tau f$.

PROPOSITION 8. — *Suppose $D \subset \mathbb{P}^n$ is a $\mathbb{C}$-convex domain and $p_0, \dots p_k \in D$. Then the functional Λ_p has a continuous extension to all holomorphic $(k,0)$-forms in D.*

In particulars, Λ_p is defined intrinsically on $\mathbb{P}^n$ for appropriate holomorphic $(k,0)$-forms.

Proof. — Suppose $D \subset \mathbb{C}^n$ and let $\tau : \Delta_k \to D$ be the geometrical simplex and $\sigma : \Delta_k \to D$ the one from COROLLARY 7. We now claim that there are k-chains c_j, contained in $(k-1)$-dimensional complex planes, such that $\sigma - \tau - \sum c_j$ is a cycle and hence a boundary in $\mathbb{C}^n$.

We show the claim by induction over k, so assume that it holds for $(k-1)$. Let Δ^j be a $(k-1)$-face of Δ_k. Then $\tau_{|\Delta^j}$ and $\sigma_{|\Delta^j}$ are in the same $(k-1)$-plane and also fulfills the other requirements, so by assumption there is a chain c_j such that $\sigma_{|\Delta^j} - \tau_{|\Delta^j} - c_j$ is a cycle. Hence $\sigma - \tau - \sum c_j$ is a cycle.

Now let f be a e.g. entire $(k,0)$-form. Since f is a closed form in the k-plane spanned by $p_0, \dots, p_k$, it follows by the claim that :

$$\int_\sigma f - \int_\tau f = \sum \int_{c_j} f.$$

But since $f = 0$ in each $(k-1)$-plane, we find that $\int_\sigma f = \int_\tau f$, and then we define the proposed extension to holomorphic $(k,0)$-forms in D by $\Lambda_p = \int_\sigma f$. $\square$

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