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BIVARIANT COHOMOLOGY AND S^1 -SPACES

BY

ANDRÉA SOLOTAR (*)

RÉSUMÉ. — Le but de cet article est d'étendre au cadre bivariant le théorème de Jones, Goodwillie et Burghlelea-Fiedorowicz (cf. [J], [G], [B-F]), qui prouve l'isomorphisme entre la cohomologie cyclique du complexe singulier d'un S^1 -espace X et la cohomologie S^1 -équivariante de X . Nous faisons également la comparaison entre la longue suite exacte de Connes (théorie cyclique) et la longue suite exacte de Gysin (théorie S^1 -équivariante).

Nous prouvons aussi que dans quelques cas, la cohomologie cyclique bivariante peut être calculée comme la cohomologie cyclique (monovariante) d'un certain complexe mixte.

ABSTRACT. — The purpose of the following work is to provide a generalization to the bivariant setting of the theorem of Jones, Goodwillie and Burghlelea-Fiedorowicz (cf. [J], [G], [B-F]), which proves the existence of an isomorphism between the cyclic cohomology of the singular complex module of an S^1 -space X and the S^1 -equivariant cohomology of X . We also compare Connes' long exact sequence in the cyclic theory with Gysin's long exact sequence in the S^1 -equivariant theory.

We see that in some cases the bivariant cyclic cohomology can be computed as the (monovariant) cyclic cohomology of a mixed complexe.

0. Introduction

The bivariant version of cyclic cohomology was introduced by JONES and KASSEL in [J-K]. In the other hand, there is a topological definition for S^1 -spaces X and Y of the bivariant S^1 -equivariant cohomology, denoted $H_{S^1}^*(X, Y)$ which can be found in [C].

In the following work we prove the bivariant version of the theorem of JONES [J], GOODWILLIE [G] and BURGHELEA-FIEDOROWICZ [B-F], which

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says that if X is an S^1 -space, then its equivariant cohomology is isomorphic to the cyclic cohomology of the singular complex module of X :

$$H_{S^1}^*(X) = HC^*(\mathbb{S}(X)).$$

One of the main results is the following :

THEOREM. — *Let X and Y have the homotopy type of CW-complexes equipped with a pointed S^1 -action, such that Y has the homotopy type of a finite complex. Then there exists a natural isomorphism :*

$$H_{S^1}^n(X, Y) \cong \overline{HC}^n(\mathbb{S}X, \mathbb{S}Y)$$

where $\overline{HC}^n(\mathbb{S}X, \mathbb{S}Y)$ is the reduced bivariant cyclic cohomology of $\mathbb{S}X$ and $\mathbb{S}Y$.

This isomorphism sends Connes' long exact sequence in bivariant cyclic cohomology to a Gysin long exact sequence of X and Y in the topological context.

We also prove that in certain cases the bivariant cyclic cohomology of two cyclic modules M . et N . can be computed as the cyclic cohomology of the mixed complex $M \otimes DN$., where DN . is the dual of N ., considered as a chain complex.

The paper is organized as follows :

In sections 1–3 we recall the definitions and some properties of bivariant cohomology, bivariant cyclic cohomology and the stable homotopy category Stab , respectively. This category, studied principally in [D-P], is used to provide an intermediate result during the proof.

In section 4 we extend the theorem of [J], [G] and [B-F] to the Stab category (PROPOSITION 4.1).

Section 5 gives the following preliminary result which is used in the proof of the main theorem :

PROPOSITION. — *Under the hypotheses of the theorem, there is a natural isomorphism between $HC^n(\mathbb{S}X, \mathbb{S}Y)$ and $HC^n(\mathbb{S}(X \wedge DY))$, where DY is the Spanier-Whitehead dual of Y in the Stab category and \mathbb{S} . denotes the reduced singular complex module.*

This result is proved in section 6.

Finally, in section 7, we show that the diagram which relates the reduced Gysin long exact sequence of $ES^1 \times_{S^1}(X \wedge DY)$ and Connes' long exact sequence of the reduced bivariant cyclic cohomology of $\mathbb{S}X$ and $\mathbb{S}Y$ is commutative.

All the spaces that we are going to consider have the homotopy type of a CW-complex, are connected and base pointed.

1. Bivariant cohomology

Given two CW-complexes X and Y , their bivariant cohomology with integral coefficients is defined, using maps of spectra as $[\Sigma^\infty X, \Sigma^\infty Y \wedge \mathbf{H}]$, where \mathbf{H} is the Eilenberg-Mac Lane spectrum, $\mathbf{H}_i = K(\mathbb{Z}, i)$, $\Sigma^\infty X$ is the spectrum defined by $(\Sigma^\infty X)_i = S^i \wedge X$ and $[,]$ denotes homotopy classes of morphisms that fix the base point [C, p. 3].

As $(\Sigma^\infty Y \wedge \mathbf{H})_n = Y \wedge \mathbf{H}_n$ [S, Cor. 13.39], we may define :

$$H^i(X, Y) = \varinjlim_j [\Sigma^j X, Y \wedge K(\mathbb{Z}, j + i)] \quad (i \in \mathbb{Z}).$$

There are other definitions of the same object which are equivalent, such as : $H^i(X, Y)$ is the group of chain homotopy classes of chain maps of degree i from the reduced singular chain complex of X to the reduced singular chain complex of Y [C-S, p. 398]), and one has a split short exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}^{n+1}(\bar{H}_*(X), \bar{H}_*(Y)) &\longrightarrow H^n(X, Y) \\ &\longrightarrow \text{Hom}_{-n}(\bar{H}_*(X), \bar{H}_*(Y)) \rightarrow 0, \end{aligned}$$

where \bar{H} denotes reduced homology : $\bar{H}(X) = H(X)/H(*)$.

As a consequence, $H^n(X, Y)$ is $H_{-n}(\text{Hom}(\mathcal{S}.X, \mathcal{S}.Y))$.

2. Bivariant cyclic cohomology

For the definition and properties of cyclic k -modules (where k is a commutative ring with unit), we refer to [Co] , [L1] and [L2].

We recall that a mixed complex (M, b, B) is a nonnegatively graded k -module $(M_n)_{n \in \mathbb{N}}$ endowed with a degree -1 morphism b and a degree $+1$ morphism B , such that $b^2 = B^2 = [B, b] = 0$.

The cyclic homology of a cyclic k -module is defined in [Co] and [L2], and the cyclic homology of a mixed complex is defined in [B] and [K1].

Given cyclic k -modules $M.$ and $N.$, KASSEL [K2] has defined the bivariant Hochschild cohomology of $M.$ and $N.$ as follows :

Definition. — $HH^n(M, N) = H_{-n}(\text{Hom}(M., N.), d)$ ($n \in \mathbb{Z}$), where $d(f) = b_N \cdot f - (-1)^{\text{deg}(f)} f \cdot b_M$ and $\text{Hom}(M., N.)$ is the complex such that $(\text{Hom}(M., N.))_j = \Pi_p \text{Hom}(M_p, N_{p-j})$.

Given augmented cyclic k -modules $M.$ and $N.$, their reduced bivariant Hochschild cohomology $\overline{HH}^n(M., N.)$ is defined as $HH^n(\mathbb{M}., \mathbb{N}.)$, where $M. = k \oplus \mathbb{M}., N. = k \oplus \mathbb{N}.$ It verifies that :

$$HH^n(M., N.) = \overline{HH}^n(M., N.) \oplus HH^n(k) \oplus \overline{HH}^n(M.) \oplus HH^n(k., \mathbb{N}.)$$

So we observe that $H^n(X, Y) \cong \overline{HH}^n(\mathbb{S}.X, \mathbb{S}.Y)$.

JONES and KASSEL [J-K] have also defined the bivariant cyclic cohomology of $M.$ and $N.$ in the following way : to the mixed complex (M, b, B) is associated the complex

$$\beta(M) = k[u] \otimes M,$$

(where $\text{deg}(u) = 2$), with differential

$$d(u^n \otimes m) = \begin{cases} u^n \otimes bm + u^{n-1} \otimes Bm & \text{if } n > 0, \\ u^n \otimes bm & \text{if } n = 0. \end{cases}$$

The natural projection $S : \beta(M) \rightarrow \beta(M)[2]$ is given by :

$$S(u^n \otimes m) = \begin{cases} u^{n-1} \otimes m & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

which is a morphism of complexes.

The module $\beta(M)$ is then called an S -module.

We consider $\text{Hom}_S(\beta(M), \beta(N))$, the submodule of $\text{Hom}(\beta(M), \beta(N))$ consisting of elements which commute with S .

Definition. — $HC^n(M, N) = H_{-n}(\text{Hom}_S(\beta(M), \beta(N)))$ ($n \in \mathbb{Z}$).

Remarks :

- 1) HC^n is a contravariant functor in M and a covariant functor in N .
- 2) If $N = k$, $HC^i(M, k) = HC^i(M)$.
- 3) If $M = k$, $HC^i(k, N) = HC^-_i(N)$.

(For a definition of HC^*_* , see [J].)

Following the ideas of [L-Q, § 4], the following definition of the reduced bivariant cyclic cohomology is given in [K3, 8.2].

Definition.—If $M.$ and $N.$ are augmented cyclic k -modules, the reduced bivariant cyclic cohomology of $M.$ and $N.$ is $\overline{HC}^n(M., N.) = HC^n(\mathbb{M}., \mathbb{N}.)$, where $M. = k. \oplus \mathbb{M}.$ and $N. = k. \oplus \mathbb{N}.$

Remark :

$$HC^n(M., N.) = \overline{HC}^n(M., N.) \oplus \overline{HC}^n(M.) \oplus \overline{HC}^{-n}(N.) \oplus HC^n(k.).$$

Examples. — If X and Y are S^1 -spaces, then their reduced singular complex k -modules, denoted $\mathcal{S}.X$ and $\mathcal{S}.Y$ are generated by their reduced singular complexes $\mathcal{S}.X$ and $\mathcal{S}.Y$. These k -modules are not only simplicial k -modules but also cyclic k -modules, with the cyclic action defined by (see [G]) :

$$\begin{aligned} C_n \times \mathcal{S}_n(X) &\longrightarrow \mathcal{S}_n(X) \\ (t_n, s) &\longmapsto t_n \cdot \sigma, \end{aligned}$$

where $t_n \cdot \sigma(u_0, \dots, u_n) = e^{2\pi i u_0} \cdot \sigma(u_1, \dots, u_n, u_0)$.

In this case $\mathcal{S}.X = k \oplus \mathcal{S}.X$ and the same for Y , so the S^1 -spaces X and Y give rise to the bivariant cyclic cohomology groups $HC^n(\mathcal{S}.X, \mathcal{S}.Y)$ ($n \in \mathbb{Z}$), which by definition are the reduced bivariant cyclic cohomology groups of $\mathbb{S}.X$ and $\mathbb{S}.Y$, $\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y)$.

3. The stable homotopy category

From now on all the spaces considered are base pointed, compactly generated CW -complexes.

3.1. — We recall from [D-P] that the stable homotopy category Stab is the category whose objects are pairs (X, n) , where X is a space, $n \in \mathbb{Z}$, and whose maps are :

$$\text{Stab}((X, n); (Y, m)) = \varinjlim_k [\Sigma^{n+k} X, \Sigma^{m+k} Y].$$

The product $(X, n) \otimes (Y, m) = (X \wedge Y, n + m)$ makes Stab a monoidal category.

We shall make use of the following objects :

Definitions :

- (1) Given (X, n) , if (X', n') is an object in Stab such that

$$\text{Stab}((X, n) \otimes (Z, k), (S^0, 0))$$

is isomorphic to $\text{Stab}((Z, k); (X', n') \otimes (S^0, 0))$, for every object (Z, k) in Stab , then (X', n') is called (up to canonic isomorphism) the *weak dual* of (X, n) and denoted $D(X, n)$.

(2) If $\text{Stab}((X, n) \otimes ((Z, k), (W, \ell)))$ is isomorphic to

$$\text{Stab}((Z, k), (X', n') \otimes (W, \ell))$$

for every pair of objects (Z, k) and (W, ℓ) in Stab , then (X', n') is called (up to canonic isomorphism) the *strong dual* of (X, n) and also denoted $D(X, n)$.

The uniqueness of a (weak) dual object is assured by [S, cor. 14.25] and the existence of a strong dual for a finite CW -complex, by [S, th.14.34].

Then $D : \text{Stab}^* \rightarrow \text{Stab}$ is a contravariant functor, where Stab^* is the full subcategory whose objects are those (X, n) such that X is a finite CW -complex.

According to the definition of the homology of spectra [S], it is clear that :

$$H_n((X, h)) = H_{n-h}(X).$$

Similarly, for cohomology we have that $H^n((X, h)) = H^{n-h}(X)$.

4. S^1 -spaces

Let Z be an S^1 -space, such that the action of S^1 over Z is pointed, that is, it preserves the base point (ex. : $Z = \text{Map}(S^1, Z')$), and consider, for $n \in \mathbb{Z}$, the S^1 -space $\Sigma^n Z = S^n \wedge Z$ with the trivial action of S^1 on S^n .

In this situation we want to define $ES^1 \times_{S^1} (Z, n)$, for $(Z, n) \in \text{Stab}$. We observe that if $n = 0$, then

$$ES^1 \times_{S^1} (Z, 0) = ES^1 \times_{S^1} Z = (ES^1 \times_{S^1} Z, 0).$$

If $n > 0$, we identify ES^1 with $(ES^1, 0)$, then

$$ES^1 \times_{S^1} (Z, n) = ES^1 \times_{S^1} (\Sigma^n Z, 0) = (ES^1 \times_{S^1} \Sigma^n Z, 0).$$

As the space $ES^1 \times_{S^1} \Sigma^n Z$ is homotopy equivalent to $\Sigma^n (ES^1 \times_{S^1} Z)$, we define $ES^1 \times_{S^1} (Z, n) = (ES^1 \times_{S^1} Z, n)$ and we want to show now that, as in the case of S^1 -spaces, we have :

PROPOSITION 4.1. — $H_*((ES^1 \times_{S^1} Z, n)) = HC_*(\mathbb{S}.(Z, n))$.

Proof. — We first observe that $\mathbb{S}.(Z, n)$ though not a cyclic module is a mixed complex.

We have already seen that :

$$H_*((ES^1 \times_{S^1} Z, n)) = H_{*-n}(ES^1 \times_{S^1} Z)$$

and by [J], [G] and [B-F], this last term equals $HC_{*-n}(\mathbb{S}.Z)$.

In order to calculate $HC_*(\mathbb{S}.(Z, n))$, we use the β -complex of [L-Q], whose total complex is such that $(\text{Tot } \beta)i = (\text{Tot } \beta')_{i-n}$ (where β' is the double complex of $\mathbb{S}.Z$). We obtain that $HC_*(\mathbb{S}.(Z, n))$ is isomorphic to $HC_{*-n}(\mathbb{S}.Z)$.

COROLLARY 4.2. — $\bar{H}C^*(\mathbb{S}.(Z, n))$ is isomorphic to $\bar{H}_{S^1}^*((Z, n))$, where $\bar{H}_{S^1}^*$ is the BS^1 reduced cohomology (i.e. $\bar{H}_{S^1}^n(\text{pt}) = \bar{H}^n(BS^1) = 0$ for all n).

5. Duality, bivariant Hochschild cohomology and bivariant cyclic cohomology

Let us consider now the category $\text{Ho}(\partial\text{-mod}_k)$, whose objects and morphisms are respectively chain k -complexes and homotopy classes of chain maps.

The tensor product of complexes makes $\text{Ho}(\partial\text{-mod}_k)$ a monoidal category, with neutral object $I = (I_q)_{q \in \mathbb{Z}}$ (where $I_q = k$ if $q = 0$ and 0 if not).

Every chain complex A has a weak dual DA defined by $(DA)_q = \text{Hom}_k(A_{-q}, k)$ and, by [D-P], a chain complex A is strongly dualizable in $\text{Ho}(\partial\text{-mod}_k)$ if and only if it has the homotopy type of a finitely generated and projective chain complex.

DOLD and PUPPE give an extension S' to the category Stab of the functor which associates to a pointed space (X, x_0) , its singular complex module $S.X$, such that

$$H_n(S' \cdot (X, k)) = \begin{cases} H_{n-k}(\mathbb{S}.X) & \text{if } n - k \geq 0, \\ 0 & \text{if } n - k < 0. \end{cases}$$

Remark. — Singular reduced homology with coefficients in k in the Stab category coincides with the definition given in this category by means of spectra, and similarly for cohomology.

In this context we have the following result :

PROPOSITION 5.1. — *If M . and N . are simplicial modules, there is an isomorphism :*

$$HH^*(M., DN) \cong HH^*(M \otimes N.).$$

Proof.

$$\begin{aligned} HH^*(M., DN.) &= H_{-*}(\text{Hom}(M., DN.)) \\ &= H_{-*}(\text{Hom}(M. \otimes N., k)) \\ &= HH^*(M. \otimes N.). \end{aligned}$$

The second equality is obtained by definition of the weak dual in the category.

Similarly, we can prove the following results :

COROLLARY 5.2.

(i) *If $N.$ is such that $DDN.$ is homotopically equivalent to $N.$ (for example, is $N.$ is finitely generated and projective), then :*

$$HH^*(M., N.) \text{ is isomorphic to } HH^*(M. \otimes DN.).$$

(ii) *If $M.$ is such that $DDM.$ is homotopically equivalent to $M.$ (for example, is $M.$ is finitely generated and projective), then :*

$$HH^*(M., N.) \text{ is isomorphic to } HH_{-*}(DM. \otimes N.).$$

We have already seen that $\bar{H}_*^{S^1}(X \wedge DY) = HC_*(\mathcal{S}.(X \wedge DY))$ and the same for cohomology.

Now we want to show that $HC^n(\mathcal{S}.(X \wedge DY))$ is isomorphic to $\bar{H}C^n(\mathcal{S}.X, \mathcal{S}.Y)$. We shall consider a more general framework.

Let C be the subcategory of $\text{Ho}(\partial\text{-mod}_k)$ whose objects are the chain complexes which have a degree -2 action S and whose morphisms are those ones of $\text{Ho}(\partial\text{-mod}_k)$ that commute with S .

The subcategory C consists, then, of the complexes which are $k[u]$ -comodules ($\text{dg}(u) = 2$), where $k[u]$ is a coalgebra with coproduct

$$\Delta(u^n) = \sum_{i+j=n} u^i \otimes u^j$$

and counit $\kappa(u^i) = 1$ if $i = 0$ and 0 if not.

The cotensor product $\square_{k[u]}$ makes C a monoidal category.

PROPOSITION 5.4. — *Let (M, b, B) be a mixed complex and let $DM.$ denote the weak dual of $M.$ in $\text{Ho}(\partial\text{-mod}_k)$, and ${}_B M = k[u] \underline{\otimes} M.$ the associated total complex, with differential*

$$\partial(u^i \underline{\otimes} m) = u^i \underline{\otimes} bm + u^{i-1} \underline{\otimes} Bm.$$

Then $DM.$ is also a mixed complex and ${}_B DM$ is the weak dual of ${}_B M$ in C . Moreover, if $M.$ has the homotopy type of a projective finitely generated chain complex (and so $DM.$ is the strong dual of $M.$ in $\text{Ho}(\partial\text{-mod}_k)$), then ${}_B DM$ is the strong dual of ${}_B M$ in C .

Proof. — We have the mixed complex (M, b, B) . Then $DM.$ is the complex defined by $(DM)_j = \text{Hom}_k(M_{-j}, k)$ ($DM.$ is zero in positive degrees).

We define $b : DM_j \rightarrow DM_{j-1}$ and $B : DM_j \rightarrow DM_{j+1}$ from b and B by composition. Then $(DM., b, B)$ is also a mixed complex.

The fact that the evaluation $\varepsilon : DM. \otimes M. \rightarrow k$ is a morphism of mixed complexes implies that ${}_B DM$ is the dual of ${}_B M$ in C .

If $DM.$ is the strong dual of $M.$ in $\text{Ho}(\partial\text{-mod}_k)$, we consider the morphisms of $k[u]$ -comodules :

$$\varepsilon' : {}_B M \square_{k[u]} {}_B DM \rightarrow k[u] \quad (\text{evaluation}) \text{ and}$$

$$\nu' : k[u] \rightarrow {}_B M \square_{k[u]} {}_B DM \quad (\text{coevaluation}).$$

The last one is defined by $\nu'(u^j) = \sum_i (u^j \otimes a_i) \otimes f_i$ (where $\{a_1, \dots, a_n\}$ is a basis of $\bigcup_{q \in \mathbb{Z}} M_q$ and $\{f_1, \dots, f_n\}$ is the dual basis in $\bigcup_{q \in \mathbb{Z}} (DM)_q$). They are such that :

(i) $(\text{id} \otimes \varepsilon') \circ (\nu' \otimes \text{id}) = \text{id}$;

(ii) $(\varepsilon' \otimes \text{id}) \circ (\text{id} \otimes \nu') = \text{id}$.

So the proof is finished.

Remark. — $\text{Hom}_C({}_B M, {}_B N) = \text{Hom}_S({}_B M, {}_B N)$.

We have then the following result :

THEOREM 5.5. — *If $M.$ and $N.$ are mixed complexes and $N.$ is such that $DDN.$ is homotopically equivalent to $N.$, there is an isomorphism :*

$$HC^n(M., N.) \cong HC^n(M. \otimes DN.).$$

Proof. — By the previous Proposition, $DN.$ is also a mixed complex and ${}_B DN$, being the strong dual of ${}_B N$ verifies :

$$\begin{aligned} HC^n(M., N.) &= H_{-n}(\text{Hom}_S(k[u] \otimes M., k[u] \otimes N.)) \\ &= H_{-n}(\text{Hom}_S((k[u] \otimes M.) \square_{k[u]} (k[u] \otimes DN.), k[u])) \\ &= H_{-n}(\text{Hom}_S(k[u] \otimes (M. \otimes DN.), k[u])), \end{aligned}$$

and this, by definition, is $HC^n(M. \otimes DN.)$.

The last isomorphism is due to Eilenberg-Moore [E-M].

COROLLARY 5.6. — *If $M.$ has the homotopy type of a projective finitely generated chain complex, there is an isomorphism :*

$$HC^n(M., N.) \approx HC_{-n}^-(DM. \otimes N.)$$

COROLLARY 5.7. — *If X and Y are pointed paces provided of a pointed action of S^1 , and Y is homotopically equivalent to DDY , then $\overline{HC}^m(\mathcal{S}.X, \mathcal{S}.Y)$ is isomorphic to $\overline{H}_{S^1}^n(X \wedge DY)$.*

Proof. — In this conditions, $\mathcal{S}.(DY)$ is the dual of $\mathcal{S}.(Y)$ in $\text{Ho}(\partial\text{-mod}_k)$ and so :

$$\begin{aligned} \overline{HC}^n(\mathcal{S}.X, \mathcal{S}.Y) &\cong HC^n(\mathcal{S}.X, \mathcal{S}.Y) \cong HC^n(\mathcal{S}.X \otimes \mathcal{S}.DY) \\ &\cong HC^n(\mathcal{S}.(X \wedge DY)) \cong \overline{H}_{S^1}^n(X \wedge DY). \end{aligned}$$

6. Bivariant S^1 -equivariant cohomology and bivariant cyclic cohomology

In [C], CRABB defines the bivariant S^1 -equivariant cohomology of two CW-complexes equipped with an S^1 -action, X and Y , as :

$$H_{S^1}^i(X, Y) = H_B^i(ES^1 \times_{S^1} X, ES^1 \times_{S^1} Y)$$

and this last object is defined as

$$\varinjlim_k \left[(B \times S^k) \wedge_B (ES^1 \times_{S^1} X), (ES^1 \times_{S^1} Y) \wedge_B (B \times \mathbb{H})_{k+i} \right]_B$$

(the homotopy classes of morphisms that commute with the projections) where $B = BS^1$ and given two fibrations $Z \rightarrow BS^1$ and $Z' \rightarrow BS^1$ with sections s and s' , $Z \wedge_B Z'$ is the push-out of the diagram :

$$\begin{array}{ccc} BS^1 & \xrightarrow{s} & Z \\ s' \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \wedge_B Z'. \end{array}$$

From now on, if X is an S^1 space, we denote $\mathbb{X} = ES^1 \times_{S^1} X$.

If $D_{BS^1}(Y)$ is the B -dual [D-P, 6] of \mathbb{Y} , then the last expression in the definition of $H_{S^1}^i(X, Y)$ is :

$$\varinjlim_k \left[S^k \wedge (\mathbb{X} \wedge_B D_B(\mathbb{Y})/B), \mathbb{H}_{k+i} \right] = \overline{H}^i((\mathbb{X} \wedge_B D_B(\mathbb{Y})/B).$$

But, if DY is the dual of $(Y, 0)$ in Stab , following ([B-G, 4]), then $ES^1 \times_{S^1} DY$ is canonically isomorphic to the B -dual of \mathbb{Y} , that will be denoted $\mathbb{D}\mathbb{Y}$, so :

$$H_{S^1}^i(X, Y) = \overline{H}^i((\mathbb{X} \wedge_B \mathbb{D}\mathbb{Y})/B).$$

This is a reduced cohomology, in the sense that if $X = Y = \text{pt}$, then $H_{S^1}^i(X, Y) = 0$, for every i .

PROPOSITION 6.1. — $\bar{H}^n(\mathbb{X} \wedge_B \mathbb{D}Y) = \bar{H}_{S^1}^n(X \wedge DY)$ and we have exact sequences (for $k \in \mathbb{Z}$)

$$0 \rightarrow H^{2k}((\mathbb{X} \wedge_B \mathbb{D}Y)/B) \rightarrow H^{2k}(\mathbb{X} \wedge_B \mathbb{D}Y) \xrightarrow{i'} H^{2k}(B) \\ \rightarrow H^{2k+1}((\mathbb{X} \wedge_B \mathbb{D}Y)/B) \rightarrow H^{2k+1}(\mathbb{X} \wedge_B \mathbb{D}Y) \rightarrow 0,$$

where i' splits by means of the section $s : B \rightarrow \mathbb{X} \wedge_B \mathbb{D}Y$.

Proof. — We observe that $\mathbb{X} \wedge_B \mathbb{D}Y \rightarrow B$ is a fibration of fibre $X \wedge DY$, so we have the long Gysin exact sequence for this fibration.

There is another fibration $ES^1 \times_{S^1}(X \wedge DY) \rightarrow B$ also of fibre $X \wedge DY$, and a morphism $F : ES^1 \times_{S^1}(X \wedge DY) \rightarrow \mathbb{X} \wedge_B \mathbb{D}Y$ given by $F((e, x, y')) = ((e, x), (e, y'))$ which induces the identity on the fibre (where (e, x) is the class of (e, x)). Then $H^n(\mathbb{X} \wedge_B \mathbb{D}Y) = H_{S^1}^n(X \wedge DY)$ for $n \in \mathbb{N}$ and we also have the corresponding isomorphisms for the reduced cohomology theory.

Next, if we regard (using the section $s : B \rightarrow \mathbb{X} \wedge_B \mathbb{D}Y$), B as a subspace of $\mathbb{X} \wedge_B \mathbb{D}Y$, whose quotient space is $(\mathbb{X} \wedge_B \mathbb{D}Y)/B$ we get the desired exact sequences.

From now on we shall take $k = \mathbb{Z}$.

THEOREM 6.2. — Under the conditions of Theorem 5.5, there is an isomorphism :

$$\bar{H}C^n(\mathbb{S}.X, \mathbb{S}.Y) \cong H_{S^1}^n(X, Y).$$

Proof. — By COROLLARY 6.7, $\bar{H}C^n(\mathbb{S}.X, \mathbb{S}.Y) = HC^n(\mathbb{S}.X, \mathbb{S}.Y)$ is isomorphic to $H_{S^1}^n(X \wedge DY)$, and by the first part of PROPOSITION 6.1, the latter is $\bar{H}^n(\mathbb{X} \wedge_B \mathbb{D}Y)$.

Then, using the exact sequences of this proposition, we obtain that :

$$H^n(\mathbb{X} \wedge_B \mathbb{D}Y) = \begin{cases} H^n((\mathbb{X} \wedge_B \mathbb{D}Y)/B) & \text{if } n = 2k + 1, \\ H^n((\mathbb{X} \wedge_B \mathbb{D}Y)/B) \oplus \mathbb{Z} & \text{if } n = 2k. \end{cases}$$

Therefore, $\bar{H}^n(\mathbb{X} \wedge_B \mathbb{D}Y)$ is isomorphic to $H^n((\mathbb{X} \wedge_B \mathbb{D}Y)/B)$, which is, by definition, $H_{S^1}^n(X, Y)$.

Example. — If the action of S^1 on Y is trivial, we observe that $Y = BS^1 \times Y$ and $H^n(\mathbb{X}, Y)$ is $H_B^n(\mathbb{X}, Y)$ because :

$$H^n(\mathbb{X}, Y) = \varinjlim_k [\Sigma^k \mathbb{X}, Y \wedge \mathbb{H}_{k+n}],$$

while

$$\begin{aligned} H_B^n(\mathbb{X}, \mathbb{Y}) &= \varinjlim_k [\Sigma_B^k \mathbb{X}, (B \times Y) \wedge_B (B \times \mathbb{H})_{k+n}]_B \\ &= \varinjlim_k [\Sigma_B^k \mathbb{X}, B \times (Y \wedge \mathbb{H})_{k+n}]_B \\ &= \varinjlim_k [\Sigma^k \mathbb{X}, Y \wedge \mathbb{H}_{k+n}] \quad (\text{see [C-S]}). \end{aligned}$$

If we consider now the structure of $\mathbb{S}.Y$ as a cyclic module, we find that it is trivial, and it is known then that $HC_*(\mathbb{S}.Y) = k[u] \otimes H_*(\mathbb{S}.Y)$, with trivial S coaction on $H_*(\mathbb{S}.Y)$.

In this case, $HC^n(\mathbb{S}.X, \mathbb{S}.Y) = \text{Hom}(HC_*(\mathbb{S}.X), H_*(\mathbb{S}.Y))_{-n}$ and this is $\text{Hom}(H_*(X), H_*(\mathbb{S}.Y))_{-n}$ (cf. [J-K, § 7]).

Taking the reduced bivariant cyclic cohomology, we get the following identification :

$$\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y) = \text{Hom}(\overline{H}_*(X), \overline{H}_*(\mathbb{S}.Y))_{-n}.$$

By the isomorphism of the last theorem, $\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y)$ is $H^n(\mathbb{X}, Y)$, which is also $\text{Hom}(\overline{H}_*(\mathbb{X}), \overline{H}_*(Y))_{-n}$ by using the short exact sequence of (1) if we suppose, as in [J-K, § 4], that $HC_*(\mathbb{S}.X) = H_*(\mathbb{X})$ is k -projective.

7. Connes' long exact sequence

KASSEL has shown [K2, I.2.3] that there is a long exact sequence (l.e.s.) for bivariant cyclic cohomology :

If $M.$ and $N.$ are cyclic k -modules and $M.$ is k -projective, there is a long exact sequence :

$$\begin{aligned} \dots \rightarrow HC^{n-2}(M., N.) \xrightarrow{S} HC^n(M., N.) \\ \xrightarrow{I} HH^n(M., N.) \xrightarrow{B} HC^{n-1}(M., N.) \rightarrow \dots \end{aligned}$$

and he has also described the morphisms S, B and I .

There is also a long (Gysin) exact sequence :

$$\begin{aligned} \dots \rightarrow H_{S^1}^{n-2}(X \wedge DY) \xrightarrow{S'} H_{S^1}^n(X \wedge DY) \\ \xrightarrow{I'} H^n(X \wedge DY) \xrightarrow{B'} H_{S^1}^{n-1}(X \wedge DY) \rightarrow \dots \end{aligned}$$

and we want to show that, taking $M. = \mathbb{S}.X$ and $N. = \mathbb{S}.Y$, if we relate the reduced versions of both sequences by the isomorphism of the above paragraphs, then the diagram is commutative :

PROPOSITION 7.1. — *If X and Y are S^1 -spaces satisfying the conditions of Theorem 5.5, then the following diagram is commutative :*

$$\begin{array}{ccccccc}
 \dots \rightarrow & \bar{H}C^{n-2}(\mathbb{S}.X, \mathbb{S}.Y) & \xrightarrow{S} & \bar{H}C^n(\mathbb{S}.X, \mathbb{S}.Y) & & & \\
 & \downarrow \phi_{n-2} & & \downarrow \phi_n & & & \\
 \dots \longrightarrow & H_{S^1}^{n-2}(X, Y) & \longrightarrow & H_{S^1}^n(X, Y) & & & \\
 & & & & & & \\
 & \xrightarrow{I} \bar{H}\bar{H}^n(\mathbb{S}.X, \mathbb{S}.Y) & \xrightarrow{B} & \bar{H}C^{n-1}(\mathbb{S}.X, \mathbb{S}.Y) & \rightarrow & \dots & \\
 & \downarrow \phi'_n & & \downarrow \phi_{n-1} & & & \\
 & \longrightarrow & H^n(X, Y) & \longrightarrow & H_{S^1}^{n-1}(X, Y) & \longrightarrow & \dots
 \end{array}$$

Proof. — We can introduce an additional row in the middle and consider the following diagram :

$$\begin{array}{ccccccc}
 \dots \longrightarrow & \bar{H}\bar{H}^n(\mathbb{S}.X, \mathbb{S}.Y) & \longrightarrow & \bar{H}C^{n-1}(\mathbb{S}.X, \mathbb{S}.Y) & & & \\
 & \downarrow & & \downarrow & & & \\
 \dots \longrightarrow & \bar{H}^n(\mathbb{S}.(X \wedge DY)) & \longrightarrow & \bar{H}C^{n-1}(\mathbb{S}.(X \wedge DY)) & & & \\
 & \downarrow & & \downarrow & & & \\
 \dots \longrightarrow & H^n(X, Y) & \longrightarrow & H_{S^1}^{n-1}(X, Y) & & & \\
 & & & & & & \\
 & \longrightarrow \bar{H}C^{n+1}(\mathbb{S}.X, \mathbb{S}.Y) & \longrightarrow & \bar{H}\bar{H}^{n+1}(\mathbb{S}.X, \mathbb{S}.Y) & \longrightarrow & \dots & \\
 & \downarrow & & \downarrow & & & \\
 \longrightarrow & \bar{H}C^{n+1}(\mathbb{S}.(X \wedge DY)) & \longrightarrow & \bar{H}^{n+1}(\mathbb{S}.(X \wedge DY)) & \longrightarrow & \dots & \\
 & \downarrow & & \downarrow & & & \\
 \longrightarrow & H_{S^1}^{n+1}(X, Y) & \longrightarrow & H^{n+1}(X, Y) & \longrightarrow & \dots
 \end{array}$$

As the lower part commutes [J, thm 3.3]), we have to show that the upper part also commutes.

The first exact sequence is a consequence of [K2, I, prop. 2.1], while the second one follows from the short exact sequence :

$$0 \rightarrow \text{Ker}(\underline{\text{AdS}})_{-n+2} \rightarrow \text{Ker}(\underline{\text{AdS}})_{-n} \rightarrow \text{Hom}_{-n}(\mathbb{S}.(X \wedge DY), k) \rightarrow 0,$$

where

$$\begin{aligned} \underline{\text{AdS}} : \text{Hom}_{-n+2}(k[u] \otimes \mathbb{S}.(X \wedge DY), k[u]) \\ \longrightarrow \text{Hom}_{-n}(k[u] \otimes \mathbb{S}.(X \wedge DY), k[u]) \end{aligned}$$

is defined by $(\underline{\text{AdS}})(f) = S \circ f - f \circ S$.

So, the proof reduces to verify the commutativity of the following square, as the maps ϕ_i, ϕ'_i are defined between the complexes before taking homology

$$\begin{array}{ccc} \text{Ker}(\text{AdS})_{-n+2} & \longrightarrow & \text{Ker}(\text{AdS})_{-n} \\ \phi'_{n-2} \downarrow & & \downarrow \phi'_n \\ \text{Ker}(\underline{\text{AdS}})_{-n+2} & \longrightarrow & \text{Ker}(\underline{\text{AdS}})_{-n}. \end{array}$$

We observe that if f is an element of $\text{Hom}_{-n+2}(k[u] \otimes \mathbb{S}.X, k[u] \otimes \mathbb{S}.Y)$ then $\phi'_{n-2}(f) = (\text{id} \otimes \varepsilon) \circ (f \square_{k[u]} \text{id}_{\mathbb{S}.(DY)})$.

So, if $f : (k[u] \otimes \mathbb{S}.X)_j \rightarrow (k[u] \otimes \mathbb{S}.Y)_{j-n+2}$ and

$$f \in \text{Ker}(\text{AdS}), \quad \phi'_{n-2}(f) \in \text{Ker}(\text{AdS}),$$

then we have that :

$$\begin{array}{ccc} f & \longrightarrow & (Sf) \\ \downarrow & & \downarrow \\ (\text{id} \otimes \varepsilon) \circ (f \square_{k[u]} \text{id}) & \longrightarrow & S\phi'_{n-2}(f) = \phi'_n(Sf) \end{array}$$

is commutative.

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