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MINIMAL SURFACES OF FINITE TYPE

PAR

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RÉSUMÉ. — Une surface minimale complète M est de type fini si M est de type conforme fini et si M peut être paramétrisé par des formes méromorphes sur une surface de Riemann compacte (ou l'intégrale de telles formes). Si M est de courbure totale finie, M est de type fini. L'hélicoïde est de type fini (et de courbure totale infinie). Nous donnons une condition sur la croissance de la courbure totale, dans le sens de Nevanlinna, qui entraîne que M est de type fini. On donne un exemple d'une surface minimale complète, simplement connexe, transverse à chaque plan horizontal de $\mathbb{R}^3$, et conformément le disque unité. Nous démontrons que si un tel M est plongé, conformément $\mathbb{C}$, de courbure totale de croissance finie, alors M est un hélicoïde ou un plan.

ABSTRACT. — A complete minimal surface M is said to be of finite type if M is of finite conformal type and M can be parametrized by meromorphic data on a compact Riemann surface (or integrals of such data). Finite total curvature M are of finite type. The helicoid is of finite type (and infinite total curvature). We give a condition on the growth of the total curvature, in the sense of Nevanlinna, which implies M is of finite type. We give an example of a simply connected complete minimal surface M, transverse to every horizontal plane of $\mathbb{R}^3$, and conformally the unit disk. We prove that if such an M is embedded, conformally $\mathbb{C}$, and of finite growth, then M is a helicoid or plane.

1. Introduction

In this paper we describe conditions on the growth of the total curvature of a surface that permit the surface to be parametrized by meromorphic data on a compact Riemann surface. Henceforth we assume M is a complete minimal surface in $\mathbb{R}^3$ or $\mathbb{R}^3/G$, G a groupe of isometries of $\mathbb{R}^3$.

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We say M is of *finite type* if M is of finite conformal type and each end of M (which is conformally a parametrized disk D^*) can be parametrized by meromorphic forms that extend meromorphically to the puncture. More precisely, M of finite type means the topology of M is finite, each end is conformally a punctured disk D^* and after a (possible) rotation of the end in $\mathbb{R}^3$, $\mathrm{d}g/g$ and μ extend meromorphically to the puncture. In our notation, g is the Gauss map of M (stereographic projection of the unit normal field of M) and μ is the holomorphic one form on M which is the complex differential of the third coordinate function on M. Thus the Weierstrass parametrization of M is

$$X(z) = \operatorname{Re} \int_{z_0}^{z} (g^{-1} - g, i(g^{-1} + g), 2)\mu.$$

For $E \subset M$, we denote by

$$A(E) = \int_E |K|,$$

the total curvature of E. This is the area of the spherical image of $g(E)$ counted with multiplicity.

It is well known that finite total curvature of $M \subset \mathbb{R}^3$ implies M is of finite type [7]. If M is a properly embedded minimal surface in a flat non-simply connected 3-manifold, and if M has finite topology then M has finite type [4]. The helicoid in $\mathbb{R}^3$ is of finite type: here $M = \mathbb{C}$, $g(z) = \mathrm{e}^z$ so $\mathrm{d}g/g = \mathrm{d}z$ and $\mu = i\,\mathrm{d}z$. Simply connected examples of finite type can be constructed from any polynomial $P(z)$ on $\mathbb{C}$, by letting

$$g(z) = \exp(P(z)), \quad \mu = \mathrm{d}z.$$

One can add one handle to a helicoid in $\mathbb{R}^3$ to obtain a surface of finite type [2], in fact the surface is constructed by starting with a meromorphic pair $(\mathrm{d}g/g, \mu)$ on a torus.

We will prove that a properly embedded minimal surface in $\mathbb{R}^3$ that is of finite growth type, simply connected, and transverse to the planes $x_3 =$ constant, is a helicoid or plane (THEOREM 1.4). Also, a properly embedded minimal surface in $\mathbb{R}^3$ of finite topology and at least two ends is of finite type when it has finite conformal growth type (COROLLARY 1.3).

Conformal Growth Type.

Let E be an end of M, conformally parametrized by

$$D^* = \{0 < |z| \leq 1\}.$$

For $0 < t \leq 1$, let $A(t)$ denote the total curvature of $\{t \leq |z| \leq 1\}$. Let T be the (Ahlfors-Shimizu) characteristic function of E:

$$T(r) = \int_r^1 \frac{A(t)}{t}\,\mathrm{d}t, \qquad 0 < r \leq 1.$$

We say E has *finite (conformal) growth type* equal to $|\lambda|$ if λ is finite and

$$\overline{\lim_{r\to 0}} \frac{\ln T(r)}{\ln(r)} = \lambda.$$

For example, if E has finite total curvature or if $A(t)$ grows like $|\ln(t)|$, then E has finite growth type equal to zero. If $A(t)$ grows like t^{-m}, for some positive integer m (as $t \to 0$), then E has finite growth type m.

We say M has *finite growth type* if M has finite conformal type and if each end of M has finite growth type.

Notice that if $\mathrm{d}g/g$ extends meromorphically to the puncture of D^*, then g has at most a finite number of poles in D^* (assuming g is not constant).

THEOREM 1.1. — *Assume g has a finite number of zeros and poles in D^* and is of finite growth type. Then $\mathrm{d}g/g$ extends meromorphically to the puncture.*

Proof. — By passing to a neighborhood of 0 in D^*, we can assume g has no zeros or poles in D^*. Then g can be written as $cz^k \exp(h(z))$, for some constant c, integer k, and holomorphic function $h(z)$ in D^*. This representation is elementary but since I know no reference for this, I include a proof. Let α be the period of $\mathrm{d}g/g$ in D^*,

$$\alpha = \frac{1}{2\pi i}\int_{|z|=1} \frac{\mathrm{d}g}{g}\cdot$$

Then $\dfrac{g'(z)}{g(z)} - \dfrac{\alpha}{z}$ has no period in D^* so

$$\tilde{g}(z) = \exp\Big(\int^z \Big(\frac{g'}{g} - \frac{\alpha}{z}\Big)\,\mathrm{d}z\Big)$$

can be defined to be single valued in D^*. We have:

$$\Big(\frac{\tilde{g}}{g}\Big)' = -\frac{\alpha}{z}\Big(\frac{\tilde{g}}{g}\Big).$$

Writing the Laurent expansion for $\tilde{g}/g$, the desired representation for g follows. We can assume $c = 1$ so $g(z) = z^k \exp(h(z))$.

Now we proceed as in Ahlfors proof of Nevanlinna's First Main theorem [6]. We have

$$\Delta \ln\big(1+|g(z)|^2\big) = \frac{4\,|g'(z)|^2}{(1+|g(z)|^2)^2},$$

where Δ is the Laplacian in the z-coordinate. For Ω a domain in D^*, this yields:

$$\int_\Omega \frac{4\,|g'(z)|^2}{(1+|g(z)|^2)^2} = A(\Omega) = \int_{\partial\Omega} \frac{\partial}{\partial n}\big(\ln(1+|g(z)|^2)\big)\,\mathrm{d}s.$$

Thus for $0<r\le 1$, we have:

$$A(r) = \int_{|z|=1} \frac{\partial}{\partial n}\ln\big(1+|g(z)|^2\big)\,\mathrm{d}s - r\frac{\partial}{\partial r}\int_0^{2\pi} \ln\big(1+|g(r\,\mathrm{e}^{i\theta})|^2\big)\,\mathrm{d}\theta,$$

$$\frac{A(r)}{r} = \frac{c_1}{r} - \psi'(r),$$

where

$$c_1 = \int_{|z|=1} \frac{\partial}{\partial n}\ln\big(1+|g(z)|^2\big)\,\mathrm{d}s, \quad \psi(r) = \int_0^{2\pi} \ln\big(1+|g(r\,\mathrm{e}^{i\theta})|^2\big)\,\mathrm{d}\theta.$$

Then:

$$T(r) = \int_r^1 \frac{A(t)}{t}\,\mathrm{d}t = c_1\ln(r) - \big(\psi(1)-\psi(r)\big).$$

Since $T(r)$ has finite growth type as $r\to 0$, we have $T(r) = \mathcal{O}(r^{-m})$, as $r\to 0$, for some integer m. Solving for $\psi(r)$ we conclude:

$$\psi(r) = T(r) - c_1\ln(r) + c_2 = \mathcal{O}(r^{-m}) \quad \text{as } r\to 0.$$

Now, for any real number a,

$$\ln^+|a| \le \ln\sqrt{1+|a|^2} \le \ln^+|a| + 2,$$

where $\ln^+|a|$ is $\ln|a|$ if $|a|\ge 1$ and 0 otherwise. Hence:

$$\int_0^{2\pi} \ln^+\big|g(r\,\mathrm{e}^{i\theta})\big|\,\mathrm{d}\theta \le \frac{1}{r^m},$$

for r sufficiently small.

Recall that $g = z^k\exp(h(z))$ in D^*, so

$$\ln^+\big|g(r\,\mathrm{e}^{i\theta})\big| = \begin{cases} u(z)+k\ln(r) & \text{if } u(z)+k\ln(r)\ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $u(z) = \mathrm{Re}(h(z))$, $z = r\,\mathrm{e}^{i\theta}$.

Let u^+ denote the positive part of u, and write $u^+ = u_1 + u_2$ where

$$u_1(z) = \begin{cases} u(z) & \text{if } u(z) + k\ln(r) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_2(z) = \begin{cases} u(z) & \text{if } 0 \leq u(z) \leq |k\ln(r)|, \\ 0 & \text{otherwise.} \end{cases}$$

We have:

$$\begin{aligned} \int_0^{2\pi} u^+(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta &= \int_0^{2\pi} \big(u_1(z) + k\ln(r)\big)\,\mathrm{d}\theta - \int_0^{2\pi} k\ln(r)\,\mathrm{d}\theta + \int_0^{2\pi} u_2\,\mathrm{d}\theta \\ &\leq \int_0^{2\pi} \ln^+\big|g(r\,\mathrm{e}^{i\theta})\big|\,\mathrm{d}\theta - 2\pi k\ln(r) + \int_0^{2\pi} u_2\,\mathrm{d}\theta \\ &= \mathcal{O}(r^{-m}) \quad \text{as} \quad r \to 0, \end{aligned}$$

i.e.

$$(*) \qquad \int_0^{2\pi} u^+(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta \leq \mathcal{O}(r^{-m}) \quad \text{as} \quad r \to 0.$$

Now repeat the above argument with g replaced by $1/g$. For any $\Omega \subset D^*$, the area of the spherical image of Ω by g (counted with multiplicity) equals the area of the spherical image of Ω by $1/g$; *i.e.* $A_g(\Omega) = A_{1/g}(\Omega)$. Thus the characteristic function of $1/g$ has the same growth type as that of g and the same proof as above proves:

$$\int_0^{2\pi} u^-(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta = \mathcal{O}(r^{-m}) \quad \text{as} \quad r \to 0,$$

where u^- is the negative part of u. Now it is well known that this implies h has at most a pole at 0. For completeness, here is the argument. Write:

$$h(z) = \sum_{-\infty}^{+\infty} c_n z^n \quad \text{in } D^*.$$

Then:

$$u(z) = \tfrac{1}{2} \sum_{-\infty}^{+\infty} c_n z^n + \overline{c_n z^n} = \tfrac{1}{2} \sum_{-\infty}^{+\infty} \big(c_n r^n + \bar{c}_{-n} r^{-n}\big)\,\mathrm{e}^{in\theta}.$$

Hence:

$$\frac{1}{2\pi} \int_0^{2\pi} u(r\,\mathrm{e}^{i\theta})\,\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = \frac{1}{2}\big(c_n r^n + \bar{c}_{-n} r^{-n}\big),$$

$$|c_n r^n + \bar{c}_{-n} r^{-n}| \leq \frac{1}{\pi} \int_0^{2\pi} \big|u(r\,\mathrm{e}^{i\theta})\big|\,\mathrm{d}\theta = \mathcal{O}(r^{-m}) \quad \text{as} \quad r \to 0.$$

Thus $c_{-n} = 0$ for $m > n$, and THEOREM 1.1 is proved. □

Now the question arises of extending μ meromorphically to the punctures; *i.e.* when is M of finite type? In general it is not enough to assume the (conformal) growth type of M is finite; one must relate the metric to the conformal parametrization. If E is an end of M conformally parametrized by D^*, the Gaussian curvature of E is given by:

$$K = \frac{-4}{(|g| + |g|^{-1})^4} \cdot \frac{|\mathrm{d}g/g|^2}{|\mu|^2}.$$

Assuming $\mathrm{d}g/g$ extends meromorphically to the puncture, and $|K| \geq r^j$, for some integer j and r sufficiently small, $r \neq 0$ it follows that $|\mu| \leq r^m$ for r small, $r \neq 0$, and some m. Hence μ extends meromorphically as well.

Next suppose E is an annular end of M which intersects every plane $x_3 = \text{constant} \geq 0$, in a compact, nonempty immersed curve. Since x_3 is harmonic on M it follows easily that a subend of E can be conformally parametrized by D^* so that $x_3 = -\ln r$; hence $\mu = -\mathrm{d}z/z$. Now combining this with THEOREM 1.1 we have proved:

COROLLARY 1.2. — *Suppose $M \subset \mathbb{R}^3$ has finite growth type and each end E of M has the property that some linear function on M (linear combination of the coordinate functions) is proper on a subend of E and tends to infinity on E. Then M has finite type.*

There are many examples of surfaces with annular ends as in the above corollary. Let $M = \mathbb{C}^* = \mathbb{C} - (0)$, and let $F(z)$ be a holomorphic function on M. Then

$$g(z) = z \exp\bigl(F(z) - \overline{F(\bar{z}^{-1})}\bigr), \quad \mu = (\mathrm{d}z/z)$$

defines a complete minimal immersion of M in $\mathbb{R}^3$, transverse to every horizontal plane $x_3 = \text{constant}$; here $x_3 = \ln(r)$, so M meets every plane $x_3 = \text{constant}$ [10]. When F is meromorphic on $\overline{\mathbb{C}}$, M is of finite type. It is believed the only embedded such example is the catenoid — $F(z) = z$ —; this is the *Nitsche conjecture*(*), since any complete minimal annulus transverse to every horizontal plane $x_3 = \text{constant}$ is of this form for some holomorphic function F on $\mathbb{C}^*$.

COROLLARY 1.3. — *Let M be a properly embedded minimal surface in $\mathbb{R}^3$ of finite topology and at least two ends. If M has finite (conformal) growth type, then M has finite type.*

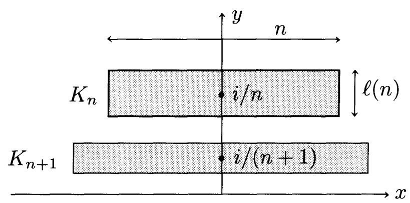
(*) Added in proof: the Nitsche conjecture has now been proven by Pascal COLLIN.

Proof. — If M is a properly embedded minimal surface of finite topology in $\mathbb{R}^3$ with at least two ends then M has finite conformal type and for each end of M there is a linear function which is proper on a subend [5]. Thus COROLLARY 1.3 follows from THEOREM 1.1. $\square$

Simply Connected Surfaces.

We do not know if the helicoid and plane are the only embedded complete minimal surfaces in $\mathbb{R}^3$. It is natural to begin the study of such surfaces by considering those M transverse to every plane $x_3 =$ constant. For minimal annuli with this property we know the conformal type is finite, *i.e.* such annuli are $\mathbb{C}^*$ (since x_3 is a proper harmonic function on M going from $-\infty$ to $+\infty$). However when M is simply connected this fails; we now give an example of such a surface conformally the disk. The idea using Runge's theorem (as we do in this example) originates in [3].

Let M be the upper half plane $z = x + iy$, with $y > 0$. We will construct a holomorphic g on M, with no zeros or poles, so that the metric $\mathrm{d}s = \frac{1}{2}(|g| + |g|^{-1})|\mathrm{d}z|$ is a complete metric on M. Then g and $\mu = \mathrm{d}z$ are the Weierstrass data of a complete minimal surface in $\mathbb{R}^3$, transverse to every plane $x_3 =$ constant (since on M, $x_3(z) = x$), going from $-\infty$ to $+\infty$, and conformally the disk.



It remains to construct g. Let K_n be a rectangle in M centered at i/n, of height $\ell(n)$ and length n; the sides of K_n are parallel to the x, y axes. Choose the $\ell(n)$ so the rectangles are pairwise disjoint. As $n \to \infty$, the K_n converge to the entire x axis. By Runge's theorem, there is a holomorphic function $h(z)$ on M such that $|h(z) - c_n| < 1$ for $z \in K_n$, where c_n are real numbers we will specify shortly.

Define $g(z) = \exp(h(z))$. Then g is a holomorphic function on M with no zeros and for $z \in K_n$, $|g(z)| \geq \mathrm{e}^{c_n - 1} = \mathrm{e}^{c_n}/\mathrm{e}$.

Now if γ is any divergent path in M, either γ has infinite Euclidean length, or γ crosses all but a finite number of the K_n (going from the top to the bottom). In the former case, $\int_\gamma |\mathrm{d}z| = \infty$, so

$$\int_\gamma \mathrm{d}s = \int_\gamma \tfrac{1}{2}\left(|g| + |g|^{-1}\right)|\mathrm{d}z| \geq \tfrac{1}{2}\int_\gamma |\mathrm{d}z|,$$

and γ has infinite length for the metric $\mathrm{d}s$. In the second case,

$$\int_\gamma \mathrm{d}s \geq \sum_{n=k}^{\infty} \ell(n) \frac{\mathrm{e}^{c_n}}{\mathrm{e}},$$

since when γ traverses K_n,

$$\int_{\gamma \cap K_n} \mathrm{d}s \geq \ell(n) \frac{\mathrm{e}^{c_n}}{\mathrm{e}}\cdot$$

So if c_n are chosen to make the above series diverge, the metric $\mathrm{d}s$ is complete.

I do not know if such a surface M can be properly immersed in $\mathbb{R}^3$. Perhaps a simply connected properly immersed minimal surface in $\mathbb{R}^3$, meeting every horizontal plane transversely must be conformally $\mathbb{C}$. It is reasonable that the helicoid and plane are the only properly embedded such surfaces. It is known that the helicoid and plane are the only simply connected properly embedded minimal surfaces that admit a nontrivial symmetry [4].

We can obtain some information about such surfaces when the growth type is finite.

THEOREM 1.4. — *Let M be a simply connected properly embedded minimal surface in $\mathbb{R}^3$ of finite growth type. If M is transverse to the planes $x_3 =$ constant, then M is a helicoid or plane.*

Proof. — Since M is simply connected and of finite growth type, M is conformally $\mathbb{C}$. Let u be the harmonic function on M given by the third coordinate function and let v be the conjugate harmonic function of u. The level curves of u are connected and u has no critical points so u is linear. To see this notice that v is strictly monotone on the level curves of u and the level curves are connected so $f = u + iv$ takes on each value at most once. Since f is an entire function, f is linear and $z = u + iv$ is a global parameter on M. Then $\mu = \mathrm{d}z$ on M and g is an entire nonvanishing holomorphic function. The growth of g in the sense of Nevanlinna is finite so by the Hadamard-Nevanlinna representation theorem, we have $g(z) = c\exp(Q(z))$, for some polynomial $Q(z)$ and $c \in \mathbb{C}$ [1]. These surfaces have been analyzed in the thesis of Pascal ROMON; the only embedded such surface is $Q(z) = z$ or Q constant [9]. $\square$

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