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CRYSTALLINE DIEUDONNÉ THEORY OVER EXCELLENT SCHEMES

BY A.J. DE JONG AND W. MESSING (*)

In respectful memory of Bernard Dwork (1923–1998)

ABSTRACT. — We write $\mathbb{D}$ for the crystalline Dieudonné module functor on p-divisible groups over a base S of characteristic p. The main results are: the full faithfulness of $\mathbb{D}$ over excellent local complete intersection schemes, and the full faithfulness of $\mathbb{D}$ up to isogeny when S is local excellent. We make use of the desingularization of D. Pospescu and the extension theorem of A.J. de Jong.

RÉSUMÉ. — THÉORIE DE DIEUDONNÉ CRISTALLINE SUR LES SCHÉMAS EXCELLENTS. Notons $\mathbb{D}$ le foncteur « module de Dieudonné cristallin » des groupes p-divisibles sur une base S de caractéristique p. Les résultats principaux de cet article sont : la pleine fidélité de $\mathbb{D}$ sur une base locale excellente d'intersection complète, et sa pleine fidélité à isogénie près lorsque S est local excellent. Nous utilisons la désingularisation de D. Popescu et le théorème d'extension de A.J. de Jong.

In this paper we add to the faithfulness results that have been obtained for the crystalline Dieudonné functor, see [3] and [4]. We pay particular attention to complete intersections and obtain fairly good results. The strongest results can be phrased loosely as follows: (a) we remove the hypothesis of finite p-basis from [4], (b) for p-divisible groups the functor is fully faithful up to isogeny over excellent local rings, and (c) for p-divisible groups $\mathbb{D}$ is fully faithful over excellent schemes which are locally complete intersections.

There are two new tools used in this paper. First, we make full use of the strong Néron desigularization of Popescu [8]. Second, we use the extension theorem for homomorphisms of p-divisible groups that was obtained in [5].

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1. Finite group schemes

Let S be a scheme in characteristic p. Let us write FG_S for the category of finite locally free group schemes over S, commutative and annihilated by some power of p. On the other hand, we denote C_S the category of Dieudonné crystals of finite presentation with action of F and V over S (see [3, 2.4]). We consider the functor [2, 3.1.5]

$$\mathbb{D}\colon \mathrm{FG}_S \longrightarrow \mathrm{C}_S. \tag{1.1}$$

We want to know for which base schemes S this functor is faithful. This is not true in general; a counter example of Koblitz and Ogus can be found in [3, 4.4.1].

Note that if $A \to B$ is an injective $\mathbb{F}_p$-algebra homomorphism and (1.1) is faithful over $S = \operatorname{Spec} B$ then (1.1) is faithful over $\operatorname{Spec} A$. The same holds for injective families of ring homomorphisms $A \to B_i$. In this way we see that (1.1) is faithful over any reduced scheme S, since $\mathbb{D}$ is (fully) faithful over a field, see [3, 4.1.1].

The other case we can handle is that of locally complete intersections. We recall that a Noetherian local ring A is said to be a *complete intersection*, if its completion $A^\wedge$ is of the form R/I, where R is a regular local ring and I is generated by a regular sequence in R.

PROPOSITION 1.2. — *Suppose the scheme S is locally Noetherian and all its local rings are complete intersections. The functor $\mathbb{D}$ (1.1) is faithful.*

Proof. — Suppose that $\varphi\colon G_1 \to G_2$ is a morphism of FG_S and that $\mathbb{D}(\varphi) = 0$. We want to prove that $\varphi = 0$. The question is local on S, hence we may assume there exists an $n \in \mathbb{N}$ such that both G_1 and G_2 are killed by p^n. Consider the scheme

$$S' = G_1 \times_S G_2^D$$

over S. Here G_2^D is the Cartier dual of G_2, see [7] for example. Note that the scheme S' also satisfies the hypothesis of the proposition, since $S' \to S$ is a locally complete intersection morphism, see [10, III 4.15]. Over S' we have the morphism of group schemes $\mathbb{Z}/p^n\mathbb{Z}_{S'} \to G_{1,S'}$ determined by $\mathrm{pr}_1\colon S' \to G_1$. Similarly, we have $G_{2,S'} \to \mu_{p^n,S'}$ given by $\mathrm{pr}_2\colon S' \to G_2^D$. Composing these with $\varphi_{S'}$ we get an element

$$\zeta_\varphi \in \operatorname{Hom}_{S'}(\mathbb{Z}/p^n\mathbb{Z}_{S'}, \mu_{p^n,S'}) = \mu_{p^n}(S').$$

By functoriality, we have $\mathbb{D}(\zeta_\varphi) = 0$. Finally, we remark that $\varphi = 0$ if and only if $\zeta_\varphi = 1$. Indeed, $\varphi \neq 0$ if and only if there is an S-scheme T,

$x \in G_1(T)$, $\mu \in G_2^D(T)$ such that $\mu(\varphi(x)) \neq 1$ in $\mathcal{O}_T^*$. The universal pair (x,μ) is $(\mathrm{pr}_1, \mathrm{pr}_2)$. For this pair $\mathrm{pr}_2(\varphi(\mathrm{pr}_1)) = \zeta_\varphi$. Hence if $\zeta_\varphi = 1$, then $\varphi = 0$ and conversely.

In this way we are reduced to the case $G_1 = (\mathbb{Z}/p^n\mathbb{Z})_S$ and $G_2 = \mu_{p^n,S}$. Thus we are given $\zeta \in \mu_{p^n}(S)$ with $\ell_n(\zeta) = 0$, see [3, 2.2.4] for explanation. We want to show that $\zeta = 1$.

To do this we may replace S by the spectrum of one of its local rings: $S = \operatorname{Spec} A$. Since $A \to A^\wedge$ is injective, we reduce to the case of a complete local ring. Hence by the Cohen structure theorem we have $A = k[[x_1, \dots, x_N]]/I$. By our assumption on S we know that I can be generated by a regular sequence $f_1, \dots, f_c$ in $k[[\underline{x}]]$. The ring $k[[\underline{x}]]/(f_i)$ is the projective limit of Artinian complete intersection rings of the form

$$k[[x_1, \dots, x_N]]/(f_1, \dots, f_c, g_{c+1}, \dots, g_N).$$

Thus, by the remarks preceding the proposition, it suffices to do the case of an Artinian complete intersection

$$A = k[x_1, \dots, x_N]/(f_1, \dots, f_N).$$

Clearly, we may suppose k is algebraically closed.

Let $W = W(k)$ be the ring of Witt vectors of k. Choose lifts $\tilde{f}_i \in W[x_1, \dots, x_N]$ of the polynomials f_i. Let

$$D\colon = W[\underline{x}]\left[\frac{\tilde{f}_i^m}{m!}\right] \subset W[\underline{x}] \otimes \mathbb{Q}.$$

The ring D is the divided power envelope of $(\tilde{f}_1, \dots, \tilde{f}_N)$ in $W[\underline{x}]$, divided powers compatible with the canonical ones on (p). The argument now runs as in [3, 4.1.5]. Let $D_n = D/p^nD$. Take a lift $\hat{\zeta} \in W[\underline{x}]$ of $\zeta \in k[\underline{x}]/(f_i)$. The element $\ell_n(\zeta)$ is given by

$$\ell_n(\zeta) = \log(\hat{\zeta}^{p^n}) \in D_n.$$

The PD-completion of D_n is denoted $D_n^\wedge$. The nullity of $\ell_n(\zeta)$ implies

$$1 = \exp\log(\hat{\zeta}^{p^n}) = \hat{\zeta}^{p^n} \in D_n^\wedge.$$

However, the map $D_n \to D_n^\wedge$ is injective (see *loc. cit.*), hence $\hat{\zeta}^{p^n} = 1$ in D_n also. Note, as in [3, 4.1.4] that the map $\Phi : A \longrightarrow D/pD$ is injective. Thus $\hat{\zeta}^{p^n} = 1$ implies $\zeta^{p^{n-1}} = 1$ in A. In this case $\ell_n(\zeta) = p\ell_{n-1}(\zeta)$. As the ring D is without p-torsion, we see that $\ell_n(\zeta) = 0$ implies $\ell_{n-1}(\zeta) = 0$. We win by induction on n. $\square$

REMARK 1.3.

(a) Using the proposition we may conclude that $\mathbb{D}$ is faithful for more general base schemes S. For example if S allows a locally complete intersection morphism $f: S \to T$ to a reduced scheme T. Indeed, we may assume S and T are affine and that f is given by the ring homomorphism $R \to A$. By a limit argument we reduce to the case that R is Noetherian. In this case the family of homomorphisms $A \to A \otimes_R k$, indexed by homomorphisms $R \to k$ into fields, is injective. To each of the $A \otimes_R k$ we may apply our proposition. This result will not be used in the sequel.

(b) If R is a reduced ring with only finitely many minimal primes (*e.g.* a Noetherian ring or an integral domain) and A is a syntomic R-algebra, then the map $A \mapsto \prod_{\mathfrak{p} \in \operatorname{Spec} R} A/\mathfrak{p}A$ is obviously injective. We ask: Is this still true for an arbitrary reduced R? If so, then one can avoid the passage to the limit argument just given.

REMARK 1.4. — Over S as in the proposition the functor (1.1) is not fully faithful in general, see [3, 4.4.1]. In addition, over such S, the functor $\mathbb{D}$ is not fully faithful on the category of truncated Barsotti-Tate groups. An explicit example of this, answering [3, question 4.4.3] in the negative, is given as follows. With regard to homomorphisms of $\mathbb{Z}/p\mathbb{Z}$ into μ_p, the question is whether the map

$$\ell_1 : \mu_p(S) \longrightarrow \Gamma(S/\Sigma_1, \mathcal{O}_{S/\Sigma_1})^{F=0}$$

is surjective ($\Sigma_1 = \operatorname{Spec} \mathbb{F}_p$). If $S = \operatorname{Spec} \mathbb{F}_p[t]/(t^p)$ the right hand side contains the infinity of elements $(t^p)^{[n]}$, $n \geq 1$, compare [3, 4.4.1]. But $\mu_p(S)$ is finite in this case.

2. Removing the hypothesis of finite p-basis from [4]

Let k be a field of characteristic p. For a scheme S of finite type over $\operatorname{Spec} k$, we consider the question whether $\mathbb{D}$ is fully faithful up to isogeny on p-divisible groups over S. In [4] it was shown that if k has a finite p-basis, then the answer is yes. We will show here that the general case follows from the special case, more precisely from [4 , 5.1.1].

As in [4] it suffices to do the case that S is reduced. Let us choose a p-basis $\{x_i\}_{i \in I}$ of k. For any subset $J \subset I$ we put

$$k_J : = \bigcup_{n \in \mathbb{N}} k[y_{j,n}, j \in J]/({y_{j,n}}^{p^n} - x_j).$$

This is a purely inseparable field extension of k with p-basis $\{x_i\}_{i\in I\setminus J}$. Hence, if $\#(I\setminus J)<\infty$, then k_J has a finite p-basis. Furthermore, the fields k_{J_1} and k_{J_2} are linearly disjoint if $J_1\cap J_2=\emptyset$.

We claim there is a finite subset $V\subset I$ such that $S\otimes k_J$ is a reduced scheme whenever $J\cap V=\emptyset$. To see this, let $\{\eta_\ell\}$ be the (finite) set of generic points of S. For each ℓ, take a finite subset $V_\ell\subset I$ such that $k(\eta_\ell)\cap k^{\mathrm{perf}}\subset k_{V_\ell}$. The field $k(\eta_\ell)$ is a finitely generated field extension of k, hence this is possible. Put $V=\bigcup\limits_\ell V_\ell$; this is still finite. For any $J\subset I$ with $J\cap V=\emptyset$ the tensor products $k(\eta_\ell)\otimes_k k_J$ are fields. For $U\subset S$ open, the map

$$\Gamma(U\otimes k_J,\mathcal{O})\longrightarrow\prod_{\eta_\ell\in U}k(\eta_\ell)\otimes_k k_J$$

is injective since it is the flat base change of $\Gamma(U,\mathcal{O})\to\prod k(\eta_\ell)$. This proves the claim.

Let G_1, G_2 be p-divisible groups over S and let $\varphi:\mathbb{D}(G_2)\to\mathbb{D}(G_1)$ be given. As in [4, 5.1] we are going to prove the stronger statement: there exists a unique $\psi:G_1\to G_2$ such that $\mathbb{D}(\psi)-\varphi$ is torsion. Put

$$J:=I\setminus V.$$

We denote by an index '$_J$' the base change of an object over S to $S\otimes k_J$. By [4, 5.1.1] there exists a unique $\psi_1:G_{1,J}\to G_{2,J}$ such that $\mathbb{D}(\psi_1)-\varphi_J$ is torsion. For any $n\in\mathbb{N}$ the map $\psi_1[p^n]:G_1[p^n]_J\to G_2[p^n]_J$ exists over $k_{J'}$ for some *finite* $J'\subset I$. Next, choose

$$J_2=I\setminus(V\cup J').$$

As before we get a unique $\psi_2:G_{1,J_2}\to G_{2,J_2}$ such that $\mathbb{D}(\psi_2)-\varphi_{J_2}$ is torsion. By uniqueness we see that $\psi_{2,J}=\psi_1$. Therefore the morphisms $\psi_1[p^n]$ and $\psi_2[p^n]$ are defined over the linearly disjoint fields $k_{J'}$ and k_{J_2}, and agree over the compositum $k_J\cdot k_{J_2}\subset k^{\mathrm{perf}}$. Clearly this gives $G_1[p^n]\to G_2[p^n]$ over S.

Putting these together for varying n gives a homomorphism

$$\psi:G_1\longrightarrow G_2.$$

We know that $\mathbb{D}(\psi)-\varphi$ becomes torsion over $S\otimes k_J$. By Lemma 4.1 for example we see that $\mathbb{D}(\psi)-\varphi$ is torsion. This completes the proof of the following theorem.

TEOREM 2.1. — *The crystalline Dieudonné module functor is fully faithful up to isogeny for p-divisible groups over any scheme of finite type over any field of characteristic p.*

3. Locally complete intersections over fields

We fix a field k of characteristic p. Let $S \to \operatorname{Spec} k$ be a locally complete intersection morphism. Consider p-divisible groups G_1, G_2 over S and a morphism of Dieudonné crystals $\varphi : \mathbb{D}(G_2) \to \mathbb{D}(G_1)$. Assume in addition that S is quasi-compact, then by the theorem above we can find $\psi : G_1 \to G_2$ over S such that $\mathbb{D}(\psi) = p^n\varphi$ for some $n \in \mathbb{N}$. This means that $\psi[p^n] : G_1[p^n] \to G_2[p^n]$ satisfies

$$\mathbb{D}(\psi[p^n]) = 0.$$

By Proposition 1.2 we get $\psi[p^n] = 0$. This implies that ψ is divisible by p^n, *i.e.*, $\psi = p^n\psi'$. We derive the relation

$$p^n\mathbb{D}(\psi') = p^n\varphi.$$

This in turn implies $\mathbb{D}(\psi') = \varphi$, since the universal divided power envelopes of affine opens of S are p-torsion free by our assumption on S (compare Lemma 4.7). For a not necessarily quasi-compact S, we can glue the local ψ' because of the faithfulness given in Proposition 1.2. We have proved the following theorem.

THEOREM 3.1. — *If $S \to \operatorname{Spec} k$ is a locally complete intersection morphism then $\mathbb{D}$ is fully faithful on p-divisible groups over S.*

While this theorem will be made obsolete by the stronger Theorem 4.6 below, the above proof is (much) more elementary.

4. Barsotti-Tate groups over excellent schemes

Let A be an $\mathbb{F}_p$-algebra. Let I be the kernel of the canonical surjection $\mathbb{Z}_p[A] \to A$. We write

$$D(A) = D_I(\mathbb{Z}_p[A])$$

for the divided power algebra of $\mathbb{Z}_p[A]$ with respect to the ideal I. Furthermore, we let $\widehat{D}(A)$ denote its p-adic completion. Recall that any quasi-coherent crystal $\mathcal{E}$ on $\operatorname{Spec} A$ gives rise to a $\widehat{D}(A)$-module $M(\mathcal{E})$ defined by

$$M(\mathcal{E}) = \varprojlim \Gamma\big((\operatorname{Spec} D(A)/p^nD(A), \operatorname{Spec} A, [\ ]), \mathcal{E}\big).$$

The functor $\mathcal{E} \mapsto M(\mathcal{E})$ is faithful.

LEMMA 4.1. — *Suppose that $A \to B$ is a homomorphism of $\mathbb{F}_p$-algebras such that $B = \varinjlim B_\lambda$, with A-algebras B_λ which are faithfully flat and relative complete intersections over A. Then the pull back functor is faithful:*

$$\textit{quasi-coherent crystals}/\operatorname{Spec} A \longrightarrow \textit{quasi-coherent crystals}/\operatorname{Spec} B$$

Proof. — The kernel of $D(A)/p^n D(A) \to A$ is a nil ideal. Hence the homomorphism $A \to B_\lambda$ lifts to a faithfully flat homomorphism $D(A)/p^n D(A) \to \widetilde{B}_\lambda$ in view of the conditions on $A \to B_\lambda$. For example, if

$$B_\lambda = A[x_1, \dots, x_N]/(f_1, \dots, f_c)$$

where $f_1, \dots, f_c$ is a regular sequence, then we choose

$$\widetilde{B}_\lambda = D(A)/p^n D(A)[x_1, \dots, x_N]/(\tilde{f}_1, \dots, \tilde{f}_c)$$

for any lifts $\tilde{f}_i$. By flatness there is a unique divided power structure on $\operatorname{Ker}(\widetilde{B}_\lambda \to B_\lambda)$ compatible with $[\ ]$ on the divided power ideal of $D(A)/p^n D(A)$. Hence we get a factorization

$$D(A)/p^n D(A) \to D(B_\lambda)/p^n D(B_\lambda) \longrightarrow \widetilde{B}_\lambda.$$

We conclude that the functor

$$D(A)/p^n D(A)\text{-modules} \quad \text{to} \quad D(B_\lambda)/p^n D(B_\lambda)\text{-modules}$$

is faithful. The result of the lemma follows as

$$D(B)/p^n D(B) = \varinjlim D(B_\lambda)/p^n D(B_\lambda). \quad \square$$

4.2. — Here is another principle that can be used to study crystals. Let B be a ring, $I \subset B$ an ideal with $p \in I$, and $A = B/I$. Assume that B is written as an inductive limit $B = \varinjlim B_\lambda$ with B_λ flat over $\mathbb{Z}_p$ and such that B_λ/pB_λ has a p-basis for all λ. Let I_λ be the inverse image of I in B_λ, and set $A_\lambda = B_\lambda/I_\lambda$. Set $D_\lambda = D_{I_\lambda}(B_\lambda)$, and $D = D_I(B)$. In this situation the natural functor

$$\text{quasi-coherent crystals of finite presentation}/\operatorname{Spec} A \longrightarrow \widehat{D}\text{-modules}$$

is faithful.

Proof. — Let us write $\mathcal{C}_A$ for the category on the left and $\mathcal{C}(p^n)_A$ for the full subcategory of crystals annihilated by p^n. Since $A = \varinjlim A_\lambda$ we see that for each $n \in \mathbb{N}$ there is an equivalence of categories

$$\mathcal{C}(p^n)_A \longrightarrow \varinjlim \mathcal{C}(p^n)_{A_\lambda}.$$

The assumption that B_λ/pB_λ has a p-basis implies that the natural functor

$$\mathcal{C}(p^n)_{A_\lambda} \longrightarrow \widehat{D}_\lambda\text{-modules}$$

is faithful. This follows from [3, 1.2.6]: If $u: M \to N$ is a morphism of $\mathcal{C}(p^n)_A$, then for any divided power thickening C of A_λ with $p^nC = 0$, we can find a homomorphism $B_\lambda/p^n \to C$ (of thickenings of A_λ). Hence D_λ will map to C. So, if the value of u on D_λ is zero, then so is its value on all such C.

We also have $D = \varinjlim D_\lambda$.

The result follows formally from the assertions above. Indeed, let $\varphi: \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism in $\mathcal{C}_A$ such that $\varphi_{\hat{D}} = 0$. Pick $n \in \mathbb{N}$. There exists a λ and $\varphi_\lambda : \mathcal{E}_{\lambda,1} \to \mathcal{E}_{\lambda,2}$ in $\mathcal{C}(p^n)_{A_\lambda}$ such that φ_λ represents $\varphi \bmod p^n$ in $\mathcal{C}(p^n)_A$. As $D/p^nD = \varinjlim D_\lambda/p^nD_\lambda$ we may assume by increasing λ that $(\varphi_\lambda)_{\widehat{D}_\lambda} = 0$. Hence $\varphi_\lambda = 0$, and thus $\varphi \bmod p^n = 0$. This holds for all n and we are done. ▯

In the proof of the following proposition we are going to use the result of Popescu that any regular ring homomorphism is the limit of smooth homomorphisms. See [8], see also [9]. We recall that a nondegenerate F-crystal $\mathcal{E}$ is a crystal with a Frobenius map $F : \mathcal{E}^\sigma \to \mathcal{E}$ such that there exists an integer n and a map $V : \mathcal{E} \to \mathcal{E}^\sigma$ such that $F \circ V = p^n$ and $V \circ F = p^n$.

PROPOSITION 4.3. — *Let A be an excellent local ring of characteristic p. Put $S = \operatorname{Spec} A$ and let $s \in S$ be the closed point. If $\varphi: \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism of nondegenerate F-crystals with $\varphi_{|\mathrm{CRIS}(s/\Sigma)} = 0$, then φ is torsion.*

Proof. — The proof is in three steps. The first step is to reduce to the case of a complete local ring. Indeed, let $A \to A^\wedge$ be the map of A into its completion. This is a regular ring homomorphism as A is excellent. According to the references above $A^\wedge$ is a limit of smooth A-algebras. Hence by Lemma 4.1 we may replace A by $A^\wedge$.

The second step is to reduce to the case of algebraically closed residue field. We may write $A = \Lambda[[x_1, \dots, x_N]]/I$, where Λ is a Cohen ring for the residue field k of A. By the references given above, the ring $B = \Lambda[[\underline{x}]]$

is the limit of essentially smooth local Λ-algebras B_λ. For each λ, the quotient B_λ/pB_λ has a p-basis, see [6]. Therefore we may apply the principle explained in subsection 4.2. We conclude that the functor from F-crystals to $\widehat{D} = \widehat{D}_I(B)$-modules is faithful.

Let $\overline{\Lambda} = W(\bar{k})$, where $\bar{k}$ is an algebraic closure of k. Choose any map $\Lambda \to \overline{\Lambda}$ reducing to $k \to \bar{k}$ modulo p; this is automatically flat. Let $\bar{I} = I \cdot \overline{\Lambda}[[\underline{x}]]$, and set $\overline{A} = \overline{\Lambda}[[\underline{x}]]/\bar{I}$. Write

$$\overline{D} = D_{\bar{I}}\big(\overline{\Lambda}[[\underline{x}]]\big).$$

Just as in the case of A, there is a faithful functor comparing crystals over $\overline{A}$ to $\widehat{D}_{\bar{I}}(\overline{\Lambda}[[\underline{x}]])$-modules. There is also a commutative diagram of functors:

$$\begin{array}{ccc} \text{crystals on } A & \longrightarrow & \widehat{D}\text{-modules} \\ \big\downarrow & & \big\downarrow \\ \text{crystals on } \overline{A} & \longrightarrow & \widehat{\overline{D}}\text{-modules} \end{array}$$

Thus if the right vertical arrow is faithful on p-adically separated modules, then we have performed our reduction step. For this it suffices to show that the ring maps $D/p^nD \to \overline{D}/p^n\overline{D}$ are faithfully flat. But $\Lambda[[\underline{x}]] \to \overline{\Lambda}[[\underline{x}]]$ is faithfully flat (as the completion of the map of the corresponding polynomial rings localized with respect to the maximal ideal $(p, x_1, \ldots, x_n)$). This implies that

$$\overline{D} \cong D \otimes_{\Lambda[[\underline{x}]]} \overline{\Lambda}[[\underline{x}]]$$

and we are done with the second step.

The third step. Here we write

$$A = k[[x_1, \ldots, x_N]]/I.$$

Let $W = W(k)$ be the ring of Witt vectors of k (k is algebraically closed). Choose $f_1, \ldots, f_r \in W[[\underline{x}]]$ mapping to generators of I in $k[[\underline{x}]] = W[[\underline{x}]]/(p)$. Let $\widehat{D}$ be the p-adic completion of the divided power algebra of $W[[\underline{x}]]$ with respect to the ideal $(p, f_1, \ldots, f_r)$. We write (as usual)

$$\sigma : W[[\underline{x}]] \longrightarrow W[[\underline{x}]]$$

for the Frobenius linear map with $\sigma(x_i) = x_i^p$. It induces a map σ on $\widehat{D}$ as well.

We apply [4, 2.2.2]; in order to do so we need that k has a finite p-basis, which is true since k is algebraically closed. The F-crystals $\mathcal{E}_i$ are given by triples (M_i, ∇_i, F_i), where M_i is a finite free $\widehat{D}$-module, ∇_i is an integrable connection and F_i is a horizontal linear map $M_i \otimes_\sigma \widehat{D} \to M_i$. Nondegeneracy of $\mathcal{E}_i$ means that the kernel and cokernel of F_i are annihilated by a power of p. The morphism φ is given by $\varphi : M_1 \to M_2$ which is horizontal and commutes with the F_i. There is a unique map $\widehat{D} \to W$ which maps x_i to zero and the restriction $\varphi_{|\mathrm{CRIS}(s/\Sigma)}$ corresponds to the induced map $M_1 \otimes W \to M_2 \otimes W$.

Let X be the rigid analytic n-dimensional open unit ball over K, the fraction field of W, with coordinates $x_1, \dots, x_N$. More precisely,

$$X = \mathrm{Spf}\big(W[[\underline{x}]]\big)^{\mathrm{rig}},$$

see [4, Section 7]. The space X may be considered as a "tube" for the scheme $s = \operatorname{Spec} k$. Recall that σ induces a general morphism $\sigma^{\mathrm{rig}} : X \to X$, see [4, 7.2.6].

We introduce some more notation. We set

$$V = \big\{x \in X\ ;\ |f_i(x)| \le |p|\ \big\}.$$

We let B be the quotient of the ring

$$C = W[[\underline{x}]]\{y_1, \dots, y_r\}/(py_i - f_i)$$

by its p-power torsion. Thus $\mathrm{Spf}\, B$ is a formal scheme in FS_W (see [4, Section 7]) to which we may associate a rigid analytic space by Berthelot's construction. The continuous ring homomorphism $W[[\underline{x}]] \to B$ gives rise to a canonical morphism $(\mathrm{Spf}\, B)^{\mathrm{rig}} \to X$, which maps $(\mathrm{Spf}\, B)^{\mathrm{rig}}$ into V. We leave it to the reader to show that this is in fact an isomorphism of $(\mathrm{Spf}\, B)^{\mathrm{rig}}$ with V. In particular, the canonical map $B \to \Gamma(V, \mathcal{O}_V)$ is injective.

There is a $W[[\underline{x}]]$-algebra homomorphism $\rho : \widehat{D} \to B$, given by

$$f_i^{[n]} \longmapsto \frac{f_i^n}{n!} = \Big(\frac{f_i}{p}\Big)^n \cdot \frac{p^n}{n!}.$$

Thus φ induces a map of vector bundles over V by tensoring

$$\varphi_V : M_1 \otimes_{\widehat{D},\rho} \mathcal{O}_V \xrightarrow{\varphi \otimes_\rho 1_{\widehat{D}}} M_2 \otimes_{\widehat{D},\rho} \mathcal{O}_V.$$

These locally free $\mathcal{O}_V$-modules have integrable connections and the map φ_V is horizontal. By assumption, φ_V is zero in the point $\underline{0}$ given by $(x_1,\ldots,x_N) = (0,\ldots,0)$ of V. Let V' be the connected component of V containing the point $\underline{0}$. Since V' is smooth and φ_V horizontal it follows that φ_V is zero on V'.

Note that $\sigma^{\mathrm{rig}}(V) \subset V$ and that F_i induces an isomorphism of $M_{i,V} = M_i \otimes \mathcal{O}_V$ with $(\sigma^{\mathrm{rig}})^*(M_{i,V})$. Furthermore, φ_V commutes with these identifications. There is a small open ball around $\underline{0}$ which is contained in V', as V' is an admissible open of X. This implies that any point $x \in X$ is mapped into V' by some power of σ^{rig}. In particular, any connected component $V'' \subset V$ of V is mapped into V' by some power of σ^{rig}. Now we can use the compatibility of φ_V with F_1 and F_2 to deduce that the restriction of φ_V to V'' is zero as well. We conclude that $\varphi_V = 0$, hence that $\varphi \otimes_\rho 1_B$ is zero (see above).

We want to show this implies that φ is torsion. To do this we will define a homomorphism of rings

$$\tau : B \longrightarrow \widehat{D}/p\text{-power torsion}$$

such that $\tau \circ \rho = \sigma$. Once we have τ, we immediately deduce that $\varphi \otimes_\sigma 1_{\widehat{D}}$ is torsion. However, this implies that φ is p-power torsion as $p^m\varphi$ equals

$$M_1 \xrightarrow{V_1} M_1 \otimes_\sigma \widehat{D} \xrightarrow{\varphi\otimes 1} M_2 \otimes_\sigma \widehat{D} \xrightarrow{F_2} M_2,$$

for a suitable m and V_1 such that $F_1 \circ V_1 = p^m$ (such a Verschiebung V_1 exists as $\mathcal{E}_1$ is a nondegenerate F-crystal).

We come to the construction of τ. Recall that B is the quotient of

$$C = W[[\underline{x}]]\{y_1,\ldots,y_r\}/(py_i - f_i)$$

by its p-power torsion. Thus it suffices to define a suitable map from C to $\widehat{D}$. Let us write

$$\sigma(f_i) = f_i^p + pu_i$$

for certain $u_i = u(f_i) \in W[[\underline{x}]]$. Computing formally, we should have

$$\tau\left(\frac{f_i}{p}\right) = \frac{\sigma(f_i)}{p} = \frac{f_i^p + pu_i}{p} = (p-1)!\, f_i^{[p]} + u_i.$$

Hence we would like to map y_i to $(p-1)!f_i^{[p]} + u_i$. (We remark that in [4, 5.5] there is a mistake in the formula for $\tau(i/p)$: the correction term $u(i)$ was omitted.) A general element of C can be written in the form

$$f = \sum_I a_I y_1^{i_1} \cdots y_r^{i_r},$$

where $a_I \to 0$ in the $(p, \underline{x})$-adic topology of $W[[\underline{x}]]$. By the above, the image of f should be the formal expression

$$\sum_I \sigma(a_I)\Big(\sum_{j_1,\dots,j_r} \tbinom{i_1}{j_1}\cdots\tbinom{i_r}{j_r}\frac{(j_1p)!\cdots(j_rp)!}{p^{j_1}\cdots p^{j_r}} f_1^{[j_1p]}\cdots f_r^{[j_rp]} u_1^{i_1-j_1}\cdots u_r^{i_r-j_r}\Big).$$

Let us make sense out of this. For each fixed multi-index $J=(j_1,\dots,j_r)$ we let

$$b_J = \sum_I \sigma(a_I)\tbinom{i_1}{j_1}\cdots\tbinom{i_r}{j_r}u_1^{i_1-j_1}\cdots u_r^{i_r-j_r}.$$

This is a well defined element of $W[[\underline{x}]]$ as $\sigma(a_I)\to 0$ in the complete ring $W[[\underline{x}]]$. Then we form

$$\tau(f) := \sum_J b_J \frac{(j_1p)!\cdots(j_rp)!}{p^{j_1}\cdots p^{j_r}} f_1^{[j_1p]}\cdots f_r^{[j_rp]}.$$

This converges p-adically as one sees by looking at the p-adic valuations of $(jp)!/p^j$.

Here is an observation which allows us to conclude that the construction above is well defined and is a homomorphism. Modulo p^n we are mapping into the $W[[\underline{x}]]$-submodule of $\widehat{D}/p^n$ generated by the elements $f_1^{[j_1p]}\cdots f_r^{[j_rp]}$, with $0\le j_i\le p^n$. Since this is a finite module, it is separated and complete in the $(p,\underline{x})$-topology and our map is continuous for this topology (by construction). Thus we need only to check that our map is well defined and a homomorphism on the elements of $W[[\underline{x}]][y_1,\dots,y_r]/(py_i - f_i)$. This we leave to the reader. The property $\tau\circ\rho=\sigma$ was built into the construction. This ends the proof of Proposition 4.3. ∎

PROPOSITION 4.4. — *Let A be an excellent local ring of characteristic p. The functor $\mathbb{D}$ from the category of Barsotti-Tate groups over A to the category of Dieudonné crystals over A is fully faithful up to isogeny.*

Proof. — We will argue by induction on the dimension of A. We remark that the statement for A is equivalent to the statement for $A_{\mathrm{red}} = A/\operatorname{Nil}(A)$, see [4, proof of 5.1.2]. The case of $\dim A = 0$, *i.e.*, the case of a field, is [3, 4.1.1].

Let us assume that A is reduced. Let A' be the product of the normalizations of $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through the minimal primes of A. The morphism $\operatorname{Spec} A'\to\operatorname{Spec} A$ is finite and is an isomorphism outside a nowhere dense closed subset. The annihilator I, of A'/A, defines this closed subset, and

$$A = A'\times_{A'/IA'} A/I.$$

If we have the result of the proposition for the rings A', A/I and A'/IA', then the result for A follows. Indeed, let G_1 and G_2 be Barsotti-Tate groups over $\operatorname{Spec} A$ and let $\varphi : \mathbb{D}(G_2) \to \mathbb{D}(G_1)$ be a homomorphism of Dieudonné crystals. By our assumption we get $\psi_{A'}$, $\psi_{A/I}$. Again by assumption, there will be $n, m \in \mathbb{N}$ such that

$$p^n \psi_{A'|A'/IA'} = p^m \psi_{A/I|A'/IA'}.$$

The first displayed formula shows that this implies there exists a $\psi : G_1 \to G_2$ such that

$$p^n \psi_{A'} = \psi_{|A'} \quad \text{and} \quad p^m \psi_{A/I} = \psi_{|A/I}.$$

We still have to check that $\mathbb{D}(\psi) - \varphi$ is torsion. This follows from Proposition 4.3, as we have the desired equality over $\operatorname{Spec} A/\mathfrak{m}_A$ by definition of $\psi_{A/I}$. Finally, observe that $\dim A/I < \dim A$ and $\dim A'/IA' < \dim A$, hence we reduce to the case where A is a normal domain.

The functor of the proposition is faithful as A is reduced, see Section 1. Let G_1 and G_2 be Barsotti-Tate groups over $\operatorname{Spec} A$ and let $\varphi : \mathbb{D}(G_2) \to \mathbb{D}(G_1)$ be a homomorphism of Dieudonné crystals. By fully faithfulness over fields we get a morphism $\psi_K : G_{1,K} \to G_{2,K}$ over the fraction field K of A such that $\mathbb{D}(\psi_K) = \varphi_{|\mathrm{CRIS}(\operatorname{Spec} K/\Sigma)}$. By [5, Introduction] we can extend ψ_K to a homomorphism $\psi : G_1 \longrightarrow G_2$ over $\operatorname{Spec} A$. Note that $\delta = \mathbb{D}(\psi) - \varphi$ is a map of F-crystals such that $\delta_K = 0$. We want to prove that δ is torsion.

Let A' be the perfect closure of A (*i.e.*, the integral closure of A in the perfect closure of K). The fact that $\delta_K = 0$ implies that the action of δ on the values of $\mathbb{D}(G_i)$ over $(\operatorname{Spec} A', \operatorname{Spec} W_n(A'), [\ ])$ are zero. Hence, $\delta_{|\mathrm{CRIS}(s'/\Sigma)} = 0$, where s' is the unique closed point of $\operatorname{Spec} A'$. This implies $\delta_{|\mathrm{CRIS}(s/\Sigma)} = 0$, as the morphism $s' \to s$ induces a faithful pull-back functor on crystals (by Lemma 4.1 for example). We conclude by the previous proposition. □

REMARK 4.5. — What is the obstruction to proving this for a general excellent scheme? Over any normal excellent scheme one can argue by restricting to the generic points and then extending using [5]. However, it is problematic to characterize locally morphisms of F-crystals that are torsion.

THEOREM 4.6. — *Let S be an excellent scheme all of whose local rings are complete intersections. Then the crystalline Dieudonné functor is fully faithful over S.*

Proof. — Let $\varphi : \mathbb{D}(G_2) \to \mathbb{D}(G_1)$ be a morphism of Dieudonné crystals of two p-divisible groups over S. By Proposition 4.4 we get for all points $s \in S$ a morphism

$$\psi_s : G_{1,\operatorname{Spec}\mathcal{O}_{S,s}} \longrightarrow G_{2,\operatorname{Spec}\mathcal{O}_{S,s}}$$

such that $\mathbb{D}(\psi_s) = p^{n_s}\varphi_{|\mathrm{CRIS}(\mathcal{O}_{S,s}/\Sigma)}$ for some integers n_s. By the arguments preceding the proof of Theorem 3 and using Lemma 4.7 below, we see that we may actually choose ψ_s such that $n_s = 0$. The morphisms ψ_s and $\psi_{s'}$ agree over

$$\operatorname{Spec}\mathcal{O}_{S,s} \times_S \operatorname{Spec}\mathcal{O}_{S,s'}$$

by faithfulness of $\mathbb{D}$ over this scheme (Proposition 1.2). The usual descent gives $\psi : G_1 \to G_2$ over S. Finally, we have to show that $\mathbb{D}(\psi)$ equals φ. For this it suffices to show that the functor

$$\text{quasi-coherent crystals over } S \longrightarrow \prod_{s\in S} \text{quasi-coherent crystals over } \operatorname{Spec}\mathcal{O}_{S,s}$$

is faithful. To see this it suffices to work with crystals annihilated by p^n (all n). If S is affine, say $\operatorname{Spec} A$, then $\operatorname{Spec} D(A)/p^n \to \operatorname{Spec} A$ is bijective. If $\mathfrak{p}$ is a prime ideal of A, and $\mathfrak{p}'$ is the corresponding prime ideal of $D(A)/p^n$, then a morphism u of quasi-coherent crystals over S whose restriction to $\operatorname{Spec} A_{\mathfrak{p}}$ is zero, gives a map of $D(A)/p^n$-modules whose localization at $\mathfrak{p}'$ is zero. If this holds for all primes $\mathfrak{p} \in \operatorname{Spec} A$, then the map of $D(A)/p^n$-modules is zero, *i.e.* $u = 0$. $\square$

LEMMA 4.7. — *Let A be an excellent local ring of characteristic p which is a complete intersection (see Section 1). Then the p-adically completed universal divided power algebra $\widehat{D}(A)$ (see beginning of this section) is p-torsion free.*

Proof. — We leave some of the details to the reader. Let $A \to A^\wedge$ be the map of A into its completion. We know that $A^\wedge$ is the limit of smooth A algebras B_λ. Therefore $D(A^\wedge)/p^n$ is the limit of the faithfully flat $D(A)/p^n$-algebras $D(B_\lambda)/p^n$. (Exercise: $A \to B$ syntomic implies $D(A)/p^n \to D(B)/p^n$ is flat.) This implies that $D(A)/p^n \to D(A^\wedge)/p^n$ is a faithfully flat map. Thus we may replace A by $A^\wedge$.

Write $A^\wedge = k[[x_1,\dots,x_N]]/(f_1,\dots,f_c)$, where $f_1,\dots,f_c$ is a regular sequence. The map $k[x_1,\dots,x_n] \to k[[x_1,\dots,x_n]]$ is regular. Hence we can write $k[[x_1,\dots,x_n]]$ as the limit, $\varinjlim A_\lambda$ of smooth $k[x_1,\dots,x_n]$-algebras. We may also assume that for each λ there are $f_{i,\lambda} \in A_\lambda$ mapping to f_i in $k[[x_1, \ldots, x_n]]$. Of course, we may localize each A_λ and assume that A_λ is an essentially smooth local $k[x_i]$-algebra with residue field k. Obviously we have

$$A^\wedge = \varinjlim A_\lambda/(f_{1,\lambda}, \ldots, f_{c,\lambda}).$$

We will show that $f_{1,\lambda}, \ldots, f_{c,\lambda}$ is a regular sequence in A_λ. Thus we have written $A^\wedge$ as a limit of complete intersection rings B_λ essentially of finite type over k. If B is a complete intersection ring essentially of finite type over k, then $D(B)/p^n$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ (by the above exercise). Hence $D(A^\wedge)/p^n = \varinjlim D(B_\lambda)/p^n$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$. This implies that $\widehat{D}(A^\wedge)$ is flat over $\mathbb{Z}_p$.

To check that $f_{1,\lambda}, \ldots, f_{c,\lambda}$ is a regular sequence in A_λ we may pass to the completion. The reader checks immediately that

$$A_\lambda^\wedge \cong k[[x_1, \ldots, x_n, y_1, \ldots, y_N]],$$

where N depends on λ, and where the y_i are chosen in the kernel of the canonical map $A_\lambda^\wedge \to k[[x_i]]$. Thus $A_\lambda^\wedge/(y_1, \ldots, y_N, f_{1,\lambda}, \ldots, f_{c,\lambda})$ is isomorphic to $k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_c)$. We conclude by a dimension count that $y_1, \ldots, y_N, f_{1,\lambda}, \ldots, f_{c,\lambda}$ is a regular sequence in $A_\lambda^\wedge$. Hence $f_{1,\lambda}, \ldots, f_{c,\lambda}$ is a regular sequence in $A_\lambda^\wedge$ as desired. $\square$

BIBLIOGRAPHIE

[1] BERTHELOT (P.). — *Cohomologie rigide et cohomologie rigide à support propre*, version provisoire du 9-08-1991.

[2] BERTHELOT (P.), BREEN (L.), MESSING (W.). — *Théorie de Dieudonné cristalline*, II, Lecture Notes in Mathematics **930**, Springer Verlag, 1982.

[3] BERTHELOT (P.), MESSING (W.). — *Théorie de Dieudonné cristalline*, III, in The Grothendieck Festschrift I, Progress in Math. **86**, Birkhäuser, 1990, p. 171–247.

[4] DE JONG (A.J.). — *Crystalline Dieudonné module theory via formal and rigid geometry*, Publications Math., t. **82**, 1995, p. 5–96.

[5] DE JONG (A.J.). — *Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic*, Invent. Math., t. **134**, 1998.

[6] KIMURA (T.), NIITSUMA (N.). — *Regular local rings of characteristic p and p-basis*, J. Math. Soc. Japan, t. **32**, 1980, p. 363–371.

[7] OORT (F.). — *Commutative group schemes*, Lecture Notes in Mathematics **15**, Springer, 1966.

[8] POPESCU (D.). — *General Néron desingularization*, Nagoya Math. J., t. **100**, 1985, p. 97–126.

[9] SWAN (R.G.). — *Néron-Popescu desingularization*, Lecture Notes of the University of Chicago, 1995; to appear in Proceedings of the International Conference on Algebra and Geometry, Taipei (Taiwan, 1995), p. 1–44.

[10] DEMAZURE (M.), GROTHENDIECK (A.). — *SGA3, Schémas en groupes*, I, Lecture Notes in Math. **151**, Springer Verlag, 1970.