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FORMALITY OF THE FUNCTION SPACE OF FREE MAPS INTO AN ELLIPTIC SPACE

BY TOSHIHIRO YAMAGUCHI (*)

ABSTRACT. — Let X be an n-connected elliptic space and Y a non rationally contractible, finite-type, q-dimensional CW complex, where $q \leq n$. We show that the function space X^Y of free maps from Y into X is formal if and only if the rational cohomology algebra $H^*(X;Q)$ is free, that is, X has the rational homotopy type of a product of odd dimensional spheres.

RÉSUMÉ. — FORMALITÉ DES ESPACES DE FONCTIONS LIBRES DANS UN ESPACE ELLIPTIQUE. — Soient X un espace elliptique n-connexe et Y un CW complexe non rationnellement contractile, de type fini et de dimension $q \leq n$. Nous montrons que l'espace X^Y des fonctions libres de Y dans X est formel si et seulement si l'algèbre $H^*(X,Q)$ est libre, *i.e.* X a le type d'homotopie rationnelle d'un produit de sphères de dimensions impaires.

1. Introduction

D. Sullivan's minimal model $(\Lambda V, d)$ satisfies a nilpotence condition on d, *i.e.*, there is a well ordered basis $\{v_i\}_{i\in I}$ of V such that, $i<j$ if $\deg v_i < \deg v_j$ for each $i,j \in I$ and $d(v_i) \in \Lambda V_{<i}$. Here $V_{<i}$ denotes the subspace of V generated by basis elements $\{v_j \,;\, j \in I,\ j<i\}$. According to [9, Def. 1.2], $(\Lambda V, d)$ is called *normal* if $\operatorname{Ker}[d_{|V}] = \operatorname{Ker}(d_{|V})$ where

$$\operatorname{Ker}\big[d_{|V}\big] := \big\{v_i \in V \,;\ i \in I,\ d(v_i) \text{ is cohomologus to zero in } (\Lambda V_{<i}, d)\big\}.$$

Let F, E and B be connected nilpotent spaces and let $\mathcal{M}(B)$ be a normal minimal model. In this paper, we say that a rational fibration [7, p. 200]

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$F \xrightarrow{i} E \xrightarrow{\pi} B$ is *M.N* if there is a KS-extension:

$$
\begin{array}{ccccc}
\mathcal{M}(B) & \xrightarrow{\text{inclusion}} & \big(\mathcal{M}(B)\otimes\wedge V, D\big) & \xrightarrow{\text{projection}} & (\wedge V, \bar{D}) \\
\big\downarrow \simeq & & \big\downarrow \simeq & & \big\downarrow \simeq \\
A^*(B) & \xrightarrow{\quad \pi^* \quad} & A^*(E) & \xrightarrow{\quad i^* \quad} & A^*(F)
\end{array}
\tag{1.1}
$$

in which $(\mathcal{M}(B)\otimes\wedge V, D)$ is minimal (*i.e.*, D is decomposable) and normal by a suitable change of KS-basis. Here $A^*(X)$ denotes the rational de-Rham complex of a space X, $\mathcal{M}(F)\cong(\wedge V,\bar{D})$ and "$\simeq$" means quasi-isomorphic, *i.e.*, the map induces an isomorphism in cohomology. We remark that "M.N" is a characteristic of the rational fibration but not of the total space.

Many rational fibrations are M.N. For example, the rational fibration given by a KS-extension:

$$(\wedge(x,y),0) \longrightarrow \big(\wedge(x,y,z),D\big) \longrightarrow (\wedge z,0)$$

with $|x|=3$ (where $|v|$ means $\deg(v)$ for $v\in V$), $|y|=3$, $|z|=5$ and $D(z)=xy$ is M.N. Of course, any rationally trivial fibration is M.N. On the other hand, many rational fibrations are not M.N. For example, in the KS-model of the Hopf fibration $S^3\to S^7\to S^4$, the model of the total space $(\mathcal{M}(S^4)\otimes\wedge(x_3),D)$ with $|x_3|=3$ is not even minimal. The rational fibration given by a KS-extension:

$$\big(\wedge(x,y),d\big) \longrightarrow \big(\wedge(x,y,z),D\big) \longrightarrow (\wedge z,0)$$

with $|x|=2$, $|y|=5$, $|z|=3$, $D(x)=d(x)=0$, $D(y)=d(y)=x^3$ and $D(z)=x^2$ is minimal but can not be normal by any change of KS-basis.

In the following, a fibration means a rational fibration. A nilpotent space X or the minimal model $\mathcal{M}(X)$ is called (rationally) *formal* if there is a quasi-isomorphism from $\mathcal{M}(X)$ to $(H^*(X;Q),0)$ (see [3]). The reason we consider M.N-type fibrations is that we can then state a necessary (but perhaps not sufficient) condition for the formality of the total space as in [3, Thm 4.1] when the base space is formal (see Lemma 2.3).

A fibration $F\to E\to B$ is called:

- *σ-F* if it has a rational section;
- *W.H.T* if $\pi_*(E)\otimes Q=(\pi_*(B)\otimes Q)\oplus(\pi_*(F)\otimes Q)$ for the rational number field Q and
- *H.T* if it is rationally trivial (see [11]).

The following lemma expresses the relations among these different types of fibrations.

LEMMA 1.1.

1) *"M.N" is embedded in the sequence of implications:*

$$\sigma \cdot F \Longrightarrow M.N \Longrightarrow W.H.T,$$

where the reversed implications are false in general.

2) *If a fibration $F \to E \to B$ is σ·F and E is formal, then B is formal* (*compare* [4, Lemme 2])

Our object of interest is the function space X^Y of free, continuous maps from a connected space Y into a connected space X, endowed with the compact-open topology. Observe that X^Y is infinite dimensional and is connected if X is n-connected and Y is a q-dimensional CW-complex, where $q \leq n$. Furthermore, X^Y is the total space of the fibration:

$$(X,*)^{(Y,*)} \longrightarrow X^Y \xrightarrow{\ \pi\ } X, \tag{*}$$

where $(X,*)^{(Y,*)}$ is the function space of pointed maps, and π is the evaluation at the base point. We know that (*) has a section s, where $s(x)$ is the constant map at x. Therefore (*) is σ·F. When $Y = S^1$, N. Dupont and M. Vigué-Poirrier proved the following formality result.

THEOREM (see [4, Théorème]). — *Let X be a simply connected space where $H^*(X;Q)$ is finitely generated. Then X^{S^1} is formal iff $H^*(X;Q)$ is free, i.e., X has the rational homotopy type of a product of Eilenberg Maclane spaces.*

Our goal in this article is to generalize the theorem of Dupont and Vigué-Poirrier to X^Y, when Y is of finite-type, *i.e.*, $\pi_i(Y) \otimes Q$ is finite-dimensional for all i, provided that X is elliptic, *i.e.*, the total dimensions of $H^*(X;Q)$ and $\pi_*(X) \otimes Q$ are finite. More precisely, we prove the following theorem.

THEOREM 1.2. — *Let X be an n-connected elliptic space, and let Y be a non rationally contractible, finite-type, q-dimensional CW complex, where $q \leq n$. Then X^Y is formal iff $H^*(X;Q)$ is free, i.e., X has the rational homotopy type of a product of odd dimensional spheres.*

In proving Theorem 1.2, we use a model due to Brown and Szczarba [2] for the connected component in X^Y of a map $f: Y \to X$, which is constructed from minimal models of X, Y and f. We remark that, under the hypotheses of Theorem 1.2, this *non-formalizing tendency* of X^Y does not depend on the rational homotopy type of Y. We cannot easily relax the connectivity hypothesis.

For example, when $X = \mathbb{CP}^2$ and $Y = S^3$, we can see $X_{(0)}^Y \simeq (\mathbb{CP}^2 \times K(Q,2))_{(0)}$ by the calculation in [2]. In particular, ${X_{(0)}}^Y$ is formal even though X does not have the rational homotopy type of a product of odd dimensional spheres. Also we must consider each connected component of X^Y in the general case.

In the following sections, our category is CDGA, that is, the objects are commutative differential graded algebras (cdga) over Q, and the morphisms are maps of differential graded algebra. Also, $H^*(\ )$ means $H^*(\ ;Q)$ and $I(S)$ denotes the ideal in the algebra A generated by a basis of a subspace S in A. When B is a subalgebra of A and both A and B contain S, then $I(S)$ denotes the ideal in the algebra A and $I_B(S)$ the ideal in the algebra B, unless otherwise noted.

2. Two changes of KS-basis

When a cdga $\mathcal{A}$ is formal, we can choose a minimal model $\mathcal{M} = (\Lambda V, d)$ of $\mathcal{A}$ such that $V = \mathrm{Ker}(d_{|V}) \oplus \mathrm{Ker}(\psi_{|V})$ for a quasi-isomorphism $\psi: \mathcal{M} \to (H^*(\mathcal{A}), 0)$. Therefore, according to [3, Thm 4.1], $\mathcal{A}$ is formal iff there is a complement N to $\mathrm{Ker}(d_{|V})$, $V = \mathrm{Ker}(d_{|V}) \oplus N$, such that any d-cocycle of $I(N)$ is d-exact. We remark this '$\mathcal{M}$' must be a normal minimal model. Conversely, if $\mathcal{M} = (\Lambda V, d)$ is a normal minimal model and formal, $H^*(\mathcal{M})$ is generated by $\mathrm{Ker}(d_{|V})$ as an algebra (see [9, Lemma 1.8]). Therefore for any quasi-isomorphism $\psi: \mathcal{M} \to (H^*(\mathcal{A}), 0)$, we have $V = \mathrm{Ker}(d_{|V}) \oplus \mathrm{Ker}(\psi_{|V})$.

Following [8, p. 5], we use the term "change of KS-basis" in this paper as follows. Suppose that

$$(B^*, d_B) \longrightarrow (B^* \otimes \Lambda V, \delta) \longrightarrow (\Lambda V, \bar{\delta})$$

is a KS-extension with KS-basis $\{v_i\}_{i \in I}$, *i.e.*, a well-ordered basis of V such that $i < j$ if $|v_i| < |v_j|$ for each $i, j \in I$ and $\delta(v_i) \in B^* \otimes \Lambda V_{<i}$. Define a map of algebras $\phi: B^* \otimes \Lambda V \to B^* \otimes \Lambda V$ by setting

$$\phi_{|B} = \mathrm{id}_B \quad \text{and} \quad \phi(v_i) = v_i + \chi_i$$

on basis elements of V, where $\chi_i \in B^* \otimes \Lambda V_{<i}$ (To be exact, this is different from the definition of "KS-change of basis" of [8, p. 5] since χ_i may not be contained in $B^+ \otimes \Lambda V$.) Finally, define a new differential D on $B^* \otimes \Lambda V$ by

$$D = \phi^{-1} \circ \delta \circ \phi.$$

Then we have an isomorphism of KS-extensions

$$\begin{array}{ccccc} (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \Lambda V, D) & \xrightarrow{\text{proj.}} & (\Lambda V, \bar{D}) \\ \Big\downarrow{=} & & \phi\Big\downarrow{\cong} & & \bar{\phi}\Big\downarrow{\cong} \\ (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \Lambda V, \delta) & \xrightarrow{\text{proj.}} & (\Lambda V, \bar{\delta}), \end{array} \tag{2.1}$$

where $D_{|B^*} = \delta_{|B^*} = d_B$.

In this section we introduce two changes of KS-basis. If the fibration (1.1) is M.N, the normal minimal model $\mathcal{M}(E) \cong (\mathcal{M}(B) \otimes \wedge V, D)$ is given by a change of KS-basis that we denote ϕ_1, one of two basis changes studied in this section.

Proof of Lemma 1.1.

1) The implication ($\sigma \cdot F \Rightarrow M.N$) is given in terminology of (1.1) with KS-basis $\{v_i\}_{i\in I}$ as follows. We know that (1.1) is $\sigma \cdot F$ iff $Dv - \overline{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$ for $v \in V$ (see [10, VI.6.(1)]). Therefore the minimality follows. Suppose there are $\{v_i\}_{i\in J}$ with $J \subset I$ such that Dv_i is cohomologous to 0. For $i \in J$, we can change KS-basis inductively, as $\phi_1(v_i) = v_i - \chi_i$ if $Dv_i = D(\chi_i)$ where $\chi_i \in B^* \otimes \wedge V_{<i}$ and $\phi_1(v_i) = v_i$ for $i \in I - J$. Put $\widetilde{D} = \phi_1^{-1} \circ D \circ \phi_1$ and then we have $\widetilde{D}(v_i) = 0$ for $i \in J$. Thus we have $\mathrm{Ker}[\widetilde{D}_{|V}] = \mathrm{Ker}(\widetilde{D}_{|V})$. We put again $D = \widetilde{D}$. Since again $Dv - \overline{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$ for $v \in V$, we have for $\mathcal{M}(B) = (\wedge V_B, d_B)$

$$\mathrm{Ker}[D_{|V_B}] = \mathrm{Ker}[d_{B|V_B}] = \mathrm{Ker}(d_{B|V_B}) = \mathrm{Ker}(D_{|V_B}).$$

The implication ($M.N \Rightarrow W.H.T$) is clear from the decomposability of D. On the other hand, the first and last examples in Section 1 provide counter-examples to the first and second converses, respectively.

2) From 1), we can assume $(\wedge(V_B \oplus V), D)$ of (1.1) is minimal and normal. Since E is formal, there is a complement N to $\mathrm{Ker}(D_{|V_B \oplus V})$ in $V_B \oplus V$ such that any D-cocycle of $I(N)$ is D-exact since $H^*(E)$ is generated by $\mathrm{Ker}(D_{|V_B \oplus V})$ (see [3], [9]). Then $N \cap V_B$ is a complement to $\mathrm{Ker}(D_{|V_B}) = \mathrm{Ker}(d_{B|V_B})$ in V_B. From $(D - \overline{D})(V) \subset \wedge^+ V_B \otimes \wedge^+ V$ and $D_{|V_B} = d_B$, we see any d_B-cocycle of $I_B(N \cap V_B)$ is d_B-exact, where $I_B(S)$ is the ideal of $\wedge V_B$ generated by a subset S of V_B. The formality of B follows again from [3, Thm 4.1]. $\square$

As in the proof of Lemma 1.1, part 1), given any KS-extension (1.1), we can change KS-basis

$$\phi_1 \colon \big(\wedge(V_B \oplus V), \widetilde{D}\big) \cong \big(\wedge(V_B \oplus V), D\big)$$

so that $\mathrm{Ker}[\widetilde{D}_{|V}] = \mathrm{Ker}(\widetilde{D}_{|V})$. We put again $D = \widetilde{D}$.

Next we introduce the second type change of KS-basis, which we denote ϕ_2.

LEMMA 2.1. — *Let E be a formal space and $F \to E \to B$ a W.H.T fibration, with KS-model $(\wedge V_B, d_B) \to (\wedge(V_B \oplus V), D) \to (\wedge V, \overline{D})$. Then, for any complement N to $\mathrm{Ker}(D_{|V})$ in V, there is a change KS-basis*

$$\phi_2 \colon \big(\wedge(V_B \oplus V), \widetilde{D}\big) \cong \big(\wedge(V_B \oplus V), D\big)$$

such that any $\widetilde{D}$-cocycle of $I(N)$ is $\widetilde{D}$-exact.

Proof. — Since the fibration is W.H.T, $(\Lambda(V_B \oplus V), D)$ is a minimal model of E with a KS-basis $\{v_i\}_{i \in I}$ of V. Let $\psi: (\Lambda(V_B \oplus V), D) \to (H^*(E), 0)$ be a quasi-isomorphism and $K = \operatorname{Ker}(\psi_{|V})$ with the sub-KS-basis $\{v_i\}_{i \in I_1}$. Then K is a complement of $\operatorname{Ker}(D_{|V})$ in V such that any D-cocycle of $I(K)$ is D-exact (see [3], [9]). Let $\{v_j\}_{j \in I_2}$ be the sub-KS-basis of $\operatorname{Ker}(D_{|V})$ where we assume that I is indexed by $i > j$ if $|v_i| = |v_j|$ for $i \in I_1$ and $j \in I_2$. Here $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \phi$. Then we can choose a basis of the given complement N to $\operatorname{Ker}(D_{|V})$ as $\{v_i + \sum_{j \in J_i} a_{ij} v_j\}_{i \in I_1}$ with some $a_{ij} \in Q$. Here $J_i = \{j \in I_2; |v_j| = |v_i|\}$. There is a regular liner transformation $\phi_2: V \to V$ given by

$$\phi_2(v_i) = v_i - \sum_{j \in J_i} a_{ij} v_j \text{ for } i \in I_1 \quad \text{and} \quad \phi_2(v_j) = v_j \text{ for } j \in I_2.$$

Extend it to an algebra map

$$\phi_2: \Lambda(V_B \oplus V) \longrightarrow \Lambda(V_B \oplus V)$$

by $\phi_{2|V_B} = \mathrm{id}_{V_B}$ and define $\widetilde{D} = \phi_2^{-1} D \phi_2$. Then N is also a complement to $\operatorname{Ker}(\widetilde{D}_{|V})$ in V and equals $\operatorname{Ker}(\psi\phi_{2|V})$. If an element w of $I(N)$ is a $\widetilde{D}$-cocycle, then $[w] = \psi\phi_2(w) = 0$ and w is $\widetilde{D}$-exact since $\psi^*\phi_2^*$ is an isomorphism on cohomology and $\operatorname{Ker}[\widetilde{D}_{|V}] = \operatorname{Ker}(\widetilde{D}_{|V})$. □

COROLLARY 2.2. — *Let $\mathcal{M} = (\Lambda V, d)$ be a normal minimal model. If $\mathcal{M}$ is formal, for any complement N to $\operatorname{Ker}(d_{|V})$ there is a change of basis $(\Lambda V, \tilde{d}) \cong (\Lambda V, d)$ of V so that any $\tilde{d}$-cocycle of $I(N)$ is $\tilde{d}$-exact.*

Proof. — It follows by applying Lemma 2.1 when the base space is the one-point space. □

Let a fibration $F \to E \to B$ be M.N. Let $\mathcal{M}(B) = (\Lambda V_B, d_B)$ be normal and $(\Lambda V_B, d_B) \xrightarrow{i} (\Lambda(V_B \oplus V), D)$ a KS-extension, which is normal by a suitable KS-basis change ϕ_1. Let B be formal. Then there is a quasi-isomorphism $\rho_B: (\Lambda V_B, d_B) \to (H^*(B), 0)$ embedded in the commutative diagram:

$$\begin{array}{ccc} (\Lambda V_B, d_B) & \xrightarrow{\;\;i\;\;} & (\Lambda(V_B \oplus V), D) \\ {\scriptstyle \rho_B}\downarrow & & \downarrow{\scriptstyle \rho} \\ (H^*(B), 0) & \xrightarrow{\;\;i'\;\;} & (H^*(B) \otimes \Lambda V, D'), \end{array} \tag{2.2}$$

which is push out in CDGA, *i.e.*, $D'_{|V} := (\rho_B \otimes 1) \circ D_{|V}$. Here i and i' are inclusions and ρ is a quasi-isomorphism since ρ_B is [1]. There is a complement N_B to $\operatorname{Ker}(d_{B|V_B})$ in V_B such that any d_B-cocycle of $I_B(N_B)$ is d_B-exact (see [3], [9]). We remark that

$$N_B = \operatorname{Ker}(\rho_{B|V_B}) = \operatorname{Ker}(\rho_{|V_B \oplus V}).$$

LEMMA 2.3. — *Let E and B be formal spaces and $F \to E \to B$ be an M.N fibration, with KS-model $(\wedge V_B, d_B) \to (\wedge(V_B \oplus V), D) \to (\wedge V, \bar{D})$. Then, for any complement N to $\mathrm{Ker}(D_{|V})$ in V, there is a change of KS-basis*

$$\phi_2 \colon \big(\wedge(V_B \oplus V), \widetilde{D}\big) \cong \big(\wedge(V_B \oplus V), D\big)$$

such that any $\widetilde{D}$-cocycle of $I(N_B \oplus N)$ is $\widetilde{D}$-exact.

Proof. — Let $\psi \colon (\wedge(V_B \oplus V), D) \to (H^*(E), 0)$ be a quasi-isomorphism and $\rho \colon (\wedge(V_B \oplus V), D) \to (H^*(B) \otimes \wedge V, D')$ a quasi-isomorphism as in (2.2). If $b \in N_B$, $\rho(b) = 0$ since $\mathrm{Ker}(\rho_{B|V_B}) = \mathrm{Ker}(\rho_{|V_B})$. Then

$$0 = \psi^* {\rho^*}^{-1}\big[\rho(b)\big] = \big[\psi(b)\big] = \psi(b)$$

in $H^*(E)$. Hence $N_B \subset \mathrm{Ker}(\psi_{|V_B})$. We can change KS-basis by some ϕ_2 for a given complement N to $\mathrm{Ker}(D_{|V})$ in V as in Lemma 2.1, so that $N = \mathrm{Ker}(\psi\phi_{2|V})$. Then $\phi_{2|V_B} = \mathrm{id}_{V_B}$ and therefore we have

$$\mathrm{Ker}(\psi\phi_{2|V_B \oplus V}) = \mathrm{Ker}(\psi_{|V_B}) \oplus \mathrm{Ker}(\psi\phi_{2|V}) \supset N_B \oplus \mathrm{Ker}(\psi\phi_{2|V}) = N_B \oplus N.$$

Thus we have that any $\widetilde{D}$-cocycle of $I(N_B \oplus N)$ is $\widetilde{D}$-exact since $\psi\phi_2$ is a quasi-isomorphism and

$$\begin{aligned}\mathrm{Ker}[\widetilde{D}_{|V_B \oplus V}] &= \mathrm{Ker}[D\phi_{2|V_B \oplus V}] = \mathrm{Ker}[D_{|V_B \oplus V}] \\ &= \mathrm{Ker}(D_{|V_B \oplus V}) = \mathrm{Ker}(D\phi_{2|V_B \oplus V}) = \mathrm{Ker}(\widetilde{D}_{|V_B \oplus V}). \quad \square\end{aligned}$$

3. Proof of Theorem 1.2

We begin this section by recalling the construction of the model of X^Y due to Brown and Szczarba [2]. Let $(\wedge V, d)$ a free cdga and (B, d_B) a finite-type cdga. Let (B_*, d_*) be the differential graded coalgebra with $B_q = \mathrm{Hom}(B^{-q}, Q)$. The differential d_* on B_* is the dual of d_B and the coproduct $\partial \colon B_* \to B_* \otimes B_*$ is the dual of multiplication. Let $\wedge(\wedge V \otimes B_*)$ be the free cdga generated by the vector space $\wedge V \otimes B_*$ with the differential induced by the tensor product differential $\tilde{d}$ on $\wedge V \otimes B_*$, and let I be the ideal in $\wedge(\wedge V \otimes B_*)$ generated by $1 \otimes 1 - 1$ and by the all elements of the form

$$v_1 v_2 \otimes \beta - \sum_i (-1)^{|v_2| \cdot |\beta_i|} (v_1 \otimes \beta_i)(v_2 \otimes \beta_i')$$

with $v_1, v_2 \in \wedge V$, $\beta_i, \beta_i' \in B_*$ and $\partial\beta = \sum_i \beta_i \otimes \beta_i'$. Then there is a natural isomorphism

$$\kappa \colon \wedge(V \otimes B_*) \cong \wedge(\wedge V \otimes B_*)/I$$

as graded algebras, induced by the inclusion $V\otimes B_* \to \wedge V\otimes B_*$ (see [2, Thm 3.5]). Note that $\tilde{d}(I) \subset I$. Define δ on $\wedge(V \otimes B_*)$ by $\delta = \kappa^{-1}\tilde{d}\kappa$. For example, if $dv = v_1v_2$ where $v_1, v_2 \in V$ and $\partial\beta = \sum_i \beta_i \otimes \beta_i'$ (see [2, p. 6]),

$$\delta(v\otimes\beta) = \sum_i (-1)^{|v_2|\cdot|\beta_i|}(v_1\otimes\beta_i)\cdot(v_2\otimes\beta_i') + (-1)^{|v|}v\otimes d_*(\beta).$$

In the following, we suppose that X is n-connected and finite-type, with $\mathcal{M}(X) = (\wedge V, d)$ (so that $V = \bigoplus_{i>n} V^i$), and Y is a non rationally contractible, finite-type, q-dimensional CW complex, where $q \le n$. Let $\mathcal{M}(Y) = (\wedge V_Y, d_Y)$ and $\mathcal{M}_*(Y) = \mathrm{Hom}(\mathcal{M}(Y), Q)$. Then

$$(\wedge(\wedge V\otimes\mathcal{M}_*(Y)), \tilde{d})/I \cong (\wedge(V\otimes\mathcal{M}_*(Y)), \delta) \cong (\wedge W, \delta)\otimes\mathcal{C} \simeq (\wedge W, \delta),$$

where $W \subset V\otimes\mathcal{M}_*(Y)$ with

$$W \equiv V\otimes\{\text{the cohomology classes of } d_Y\text{-cocycles}\}_* \cong V\otimes H_*(Y)$$

as vector spaces and $\mathcal{C}$ a contractible cdga. Here a basis of W is inductively constructed so that $V\otimes y_* \subset W$ for $y \in V_Y$ with $d_Y(y) = 0$ and $\delta(W) \subset \wedge W$ (see [2]). According to [2, Thm 1.5], the minimal model of X^Y is given by $\mathcal{M}(X^Y) \cong (\wedge W, \delta)$.

Write $W = V\oplus W_+$, where $W_+ := W\cap(V\otimes\mathcal{M}_+(Y))$. Then a KS-model of the fibration (*) (see Section 1) for a normal minimal model $\mathcal{M}(X) = (\wedge V, d)$ is given as

$$(**)\qquad (\wedge V, d) \longrightarrow (\wedge V\otimes\wedge W_+, \delta) \longrightarrow (\wedge W_+, \overline{\delta}),$$

where $(\wedge W, \delta) = (\wedge V\otimes\wedge W_+, \delta)$ is minimal but may not be normal. Since (*) has a section, it is M.N by Lemma 1.1, part 1). Then there is a KS-basis change of $(**)$ that can be given as follows:

$$(3.1)\qquad \begin{tikzcd} (\wedge V, d) \arrow[r, "i"] \arrow[d, "="] & (\wedge V\otimes\wedge W_+, D) \arrow[r] \arrow[d, "\phi_1", "\cong"'] & (\wedge W_+, \overline{D}) \arrow[d, "\overline{\phi}_1", "\cong"'] \\ (\wedge V, d) \arrow[r] & (\wedge V\otimes\wedge W_+, \delta) \arrow[r] & (\wedge W_+, \overline{\delta}), \end{tikzcd}$$

where $(\wedge W, D) = (\wedge V\otimes\wedge W_+, D)$ is normal with $D = \phi_1^{-1}\delta\phi_1$.

In the following, we suppose that X is elliptic (*i.e.*, $\dim_Q V < \infty$) and X^Y is formal, which implies that X is formal by Lemma 1.1, part 2). If X is elliptic, it is known that $H^*(X)$ is a Poincaré algebra (see [6]). Furthermore, if X is elliptic and formal, it is known that $(\wedge V, d)$ is two stage, *i.e.*, $V = V_0 \oplus V_1$ with $dV_0 = 0$ and $dV_1 \subset \wedge V_0$ (see [5]). Therefore we can put

$$W_{(0)} = \{V_0 \otimes \mathcal{M}_*(Y)\} \cap W \quad \text{and} \quad W_{(1)} = \{V_1 \otimes \mathcal{M}_*(Y)\} \cap W.$$

Then $W = W_{(0)} \oplus W_{(1)}$, $V_0 \subset W_{(0)}$, $V_1 \subset W_{(1)}$, $\delta W_{(0)} = 0$, and $\delta W_{(1)} \subset \wedge W_{(0)}$. Then $\phi_{1|\mathrm{Ker}(\delta_{|W})} = \mathrm{id}_{\mathrm{Ker}(\delta_{|W})}$ and especially $\phi_{1|W_{(0)}} = \mathrm{id}_{W_{(0)}}$.

For the quasi-isomorphism $\rho_X : (\wedge V, d) \to (H^*(X), 0)$ with $\mathrm{Ker}(\rho_{X|V}) = V_1$, there are the push outs:

$$(3.2)\qquad \begin{array}{ccc} (\wedge V, d) & \longrightarrow & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \eta} \\ \big(H^*(X), 0\big) & \longrightarrow & \big(H^*(X) \otimes \wedge W_+, \delta'\big) \end{array}$$

and for the KS-basis change $\phi_2 : (\wedge V \otimes \wedge W_+, \widetilde{D}) \cong (\wedge V \otimes \wedge W_+, D)$ corresponding to a certain complement N to $\mathrm{Ker}(D_{|V})$ in V as in Lemma 2.1,

$$(3.3)\qquad \begin{array}{ccc} (\wedge V, d) & \xrightarrow{\ i\ } & (\wedge V \otimes \wedge W_+, \widetilde{D}) \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \tilde{\rho}} \\ (H^*(X), 0) & \xrightarrow{\ i'\ } & (H^*(X) \otimes \wedge W_+, \widetilde{D}'), \end{array}$$

where ρ_X, η and $\tilde{\rho}$ are quasi-isomorphisms.

Claim. — *$(H^*(X) \otimes \wedge W_+, \widetilde{D}') \cong (H^*(X) \otimes \wedge W_+, \delta')$ as cdgas.*

Proof of Claim. — Since (3.3) is a push out, there is a map $(\phi_1\phi_2)'$ such that the following commutes:

$$(3.4)\qquad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \widetilde{D}) & \xrightarrow{\ \phi_1\phi_2\ } & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \eta} \\ (H^*(X) \otimes \wedge W_+, \widetilde{D}') & \xrightarrow{\ (\phi_1\phi_2)'\ } & (H^*(X) \otimes \wedge W_+, \delta'). \end{array}$$

On the other hand, since (3.2) is a push out, there is a map $(\phi_2^{-1}\phi_1^{-1})'$ such that the following commutes:

$$(3.5)\qquad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \delta) & \xrightarrow{\ \phi_2^{-1}\phi_1^{-1}\ } & (\wedge V \otimes \wedge W_+, \widetilde{D}) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\rho}} \\ (H^*(X) \otimes \wedge W_+, \delta') & \xrightarrow{\ (\phi_2^{-1}\phi_1^{-1})'\ } & (H^*(X) \otimes \wedge W_+, \widetilde{D}'). \end{array}$$

Then $(\phi_2^{-1}\phi_1^{-1})' \circ (\phi_1\phi_2)' = \mathrm{id}$ and $(\phi_1\phi_2)' \circ (\phi_2^{-1}\phi_1^{-1})' = id$ by universality. Hence $(\phi_1\phi_2)'$ is an isomorphism in (3.4). $\square$

Proof of Theorem 1.2. — The *if* part is obvious since $\delta = 0$ if $d = 0$. The *only if* part is shown as follows. Suppose $V_1 \neq 0$. Let $v = v_i$ be a non-zero basis element in a basis $\{v_j\}_{j\in I}$ of V_1, where $i = \max\{j \in I\,;\ |v_j| = |v|\}$, and let y be a non-zero basis element of V_Y with $d_Y(y) = 0$ for $\mathcal{M}(Y) = (\wedge V_Y, d_Y)$. Such an element y surely exists since Y is not rationally contractible and since V_Y has a well-ordered basis $\{y_i\}_i$ such that $d_Y(y_i) \in \wedge(V_{Y<i})$. Then we can regard $v \otimes y_*$ as a basis element of $W_{(1)}$ with the index k of the basis for some k and $\{v_j \otimes y_*\}_{j<i} \subset W_{<k}$ from the construction of W.

Suppose $\delta(v\otimes y_*) = \delta(\chi)$ for some $\chi \in \wedge W_{<k}$. We can uniquely write

$$\chi = \sum_{j<i} \theta_j (v_j \otimes y_*) + \mu$$

for $\theta_j \in \wedge V_{<i}$ and $\mu \notin \wedge V \otimes (V \otimes y_*)$. Then

$$0 = \delta(v\otimes y_*) - \delta\chi = \kappa^{-1}\tilde{d}\kappa\big((v-\theta)\otimes y_*\big) = \kappa^{-1}\big(d(v-\theta)\otimes y_*\big)$$

for $\theta = \sum_{j<i}\theta_j v_j$ and $\delta(\mu) = 0$ since

(a) $\tilde{d}(V\otimes z_*) \subset \wedge V\cdot(V\otimes z_*)$ for any $z\in V_Y$, since $d_*(z_*) = z_*\circ d_Y = 0$ due to the decomposability of d_Y, and

(b) $\tilde{d}(V\otimes z_*) \subset \wedge V\cdot(V\otimes(\wedge^{>1}V_Y)_*) \oplus \wedge V\cdot\wedge^{>1}(V\otimes\mathcal{M}_+(Y))$ for any $z\in\wedge^{>1}V_Y$.

Since the derivation $(\ )\otimes y_*\colon \wedge V \to \wedge W$ is injective, $d(v) = d(\theta)$ for $\theta\in\wedge V_{<i}$, which contradicts the normality of $(\wedge V, d)$. Thus $\delta(v\otimes y_*)$ is not cohomologus to zero.

We see therefore

$$\phi_1(v\otimes y_*) = v\otimes y_*$$

in (3.1) from the definition of change of KS-basis in the proof of Lemma 1.1 (1). Also

$$D(v\otimes y_*) = \phi_1^{-1}\delta\phi_1(v\otimes y_*) = \phi_1^{-1}\delta(v\otimes y_*) = \delta(v\otimes y_*) \neq 0$$

since $\delta(W_{(1)}) \subset \wedge W_{(0)}$ and $\phi_{1|W_{(0)}} = \mathrm{id}_{W_{(0)}}$. Hence $v\otimes y_* \notin \mathrm{Ker}(D_{|W})$. Then, from Lemma 2.1, we can change KS-basis $\phi_2\colon(\wedge W, \widetilde{D}) \cong (\wedge W, D)$, so that any $\widetilde{D}$-cocycle of $I(N)$ is $\widetilde{D}$-exact, for some subspace N of $W_{(1)}\cap W_+$ with $v\otimes y_*\in N$. We fix a particular N.

Let $[w]$ be the fundamental class of $H^*(X)$. Then $[w]\cdot(v\otimes y_*)$ is a δ'-cocycle of $H^*(X)\otimes\wedge W_+$. In fact, if $dv = \sum_i a_i v_{i_1}\cdots v_{i_{n_i}}$ for $v_{i_.}\in V_0$ and $a_i\in Q$,

$$\begin{aligned}\delta'\big([w]\cdot(v\otimes y_*)\big) &= [w]\cdot(\rho_X\otimes 1)\delta(v\otimes y_*)\\ &= \sum_i\sum_{1\le j\le n_i} \pm a_i[wv_{i_1}\cdots\hat{v}_{i_j}\cdots v_{i_{n_i}}]\cdot(v_{i_j}\otimes y_*)\end{aligned}$$

must be zero since the degree of $wv_{i_1}\cdots\hat{v}_{i_j}\cdots v_{i_{n_i}}$ is always greater than the formal dimension of X.

Let $\phi_2(v \otimes y_*) = v \otimes y_* + c$ with c a D-cocycle. Since $\phi_{1|W_{(0)}} = \mathrm{id}_{W_{(0)}}$, we obtain

$$0 = Dc = \phi_1^{-1}\delta\phi_1(c) = \delta\phi_1(c),$$

i.e., $\phi_1(c)$ is a δ-cocycle. Then $[w](v \otimes y_* + \phi_1(c))$ is a δ'-cocycle but cannot be δ'-exact since $v \otimes y_* + \phi_1(c)$ contains a non-zero element of $W_{(1)} \cap W_+$ due to the definition of change of KS-basis and since

(a) $\delta'(W_+) \subset H^*(X) \otimes (W_{(0)} \cap W_+)$, and

(b) $\delta'(\Lambda^{>1}W_+) \subset H^*(X) \otimes \Lambda^{>1}W_+$.

Then, according to the Claim above, $[w](v \otimes y_*)$ is a non-exact $\widetilde{D}'$-cocycle, since

$$\big(\phi_1\phi_2)'([w](v \otimes y_*)\big) = [w]\big(\phi_1(v \otimes y_*) + \phi_1(c)\big) = [w]\big((v \otimes y_*) + \phi_1(c)\big)$$

in (3.4). Since $\tilde{\rho}$ is a quasi-isomorphism in (3.3), there exists a non-exact $\widetilde{D}$-cocycle $w \cdot (v \otimes y_*) + \xi$ in ΛW, such that $\tilde{\rho}(\xi) = 0$. Since

$$\mathrm{Ker}(\tilde{\rho}_{|V \oplus W_+}) = \mathrm{Ker}(\rho_{X\,|V \oplus W_+}) = V_1,$$

we obtain that $\xi \in I(V_1)$ and thus $w \cdot (v \otimes y_*) + \xi \in I(V_1 \oplus N)$. This contradicts Lemma 2.3. Hence $V_1 = 0$.

Since $\dim_Q H^*(X) < \infty$, this means $V_0 = V_0^{\mathrm{odd}}$ and $H^*(X) = \Lambda(V_0^{\mathrm{odd}})$, *i.e.*, X has the rational homotopy type of a product of odd dimensional spheres if $V_0 \neq 0$ and is rationally contractible if $V_0 = 0$. □

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