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APPENDIX TO THE ARTICLE OF T. PETERNELL: THE KODAIRA DIMENSION OF KUMMER THREEFOLDS

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ABSTRACT. — We prove that Kummer threefolds T/G with algebraic dimension 0 have Kodaira dimension 0.

RÉSUMÉ (*La dimension de Kodaira des variétés de Kummer*). — Nous montrons que les variétés de Kummer T/G de dimension 3 et de dimension algébrique 0 sont de dimension de Kodaira nulle.

In this appendix we prove the following

THEOREM. — *Let T be a 3-dimensional torus with algebraic dimension $a(T) = 0$. Let G be a finite group acting on T. Let $h : X \to X' = T/G$ be a desingularisation. Then $\kappa(X) = 0$.*

Proof. — 1) We first treat the case that T is not simple, therefore covered by elliptic curves. In this case we have a holomorphic map $g : T \to B$ to a 2-dimensional torus (of course with $a(B) = 0$). Then G acts on B and we obtain a map

$$\bar{g} : T/G \longrightarrow B/G.$$

Since $\kappa(B/G)$ (= κ of a desingularisation, by definition) vanishes (otherwise B/G would be uniruled!), we obtain $\kappa(X) = 0$. In fact, the fibers of $\bar{g}$ are

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elliptic; they cannot be rational, because g has no multi-sections, so that $h : T \to T/G$ is étale on the general fiber of g. So $\kappa(X) = 0$ follows from $C_{3,1}$.

2) So from now on we shall assume T simple, hence there are no curves in T at all. In particular X' has only isolated singularities. Moreover X' is $\mathbb{Q}$-Gorenstein with only log-terminal singularities (see *e.g.* [3]), and

$$h^*(mK_{X'}) = mK_T = \mathcal{O}_T,$$

if m is chosen such that $mK_{X'}$ is Cartier. It will be sufficient to show that X' has only canonical singularities. Since $\kappa(h^*(L)) = \kappa(L)$ for any line bundle L (*e.g.* [7, 5.13]), it follows that $\kappa(mK_{X'}) = 0$, *i.e.* $nK_{X'} = \mathcal{O}_{X'}$ for some $n \in \mathbb{N}$. We choose $r \in \mathbb{N}$ minimal with $rK_{X'} = \mathcal{O}_{X'}$.

Let $p : Y \to X'$ be the associated canonical cover (see [3], [6]). Then $K_Y = \mathcal{O}_Y$ and Y has only canonical singularities [3].

Let $\tau : \hat{Y} \to Y$ be a crepant partial resolutions so that $\hat{Y}$ has only terminal singularities, in particular $K_{\hat{Y}} = \mathcal{O}_{\hat{Y}}$ (see [6]). Reid's construction being local, it works also in the analytic category (notice that isolated singularities are algebraic by Artin). Since $\hat{Y}$ is Gorenstein, we have the following Riemann-Roch formula:

$$24\chi(\mathcal{O}_{\hat{Y}}) = -K_{\hat{Y}} \cdot c_2(\hat{Y}),$$

see [4], [2], [6] (the arguments *e.g.* in [4] work as well in the analytic category). Thus

$$\chi(\mathcal{O}_{\hat{Y}}) = 0.$$

On the other hand $h^2(\mathcal{O}_{\hat{Y}}) \neq 0$, since $\hat{Y}$ is not algebraic. Hence $q(\hat{Y}) = h^1(\mathcal{O}_{\hat{Y}}) \neq 0$ and we can consider the Albanese map of $\hat{Y}$ (or of a desingularisation of $\hat{Y}$). Since $\hat{Y}$ is simple, it follows $q(\hat{Y}) = 3$ and the Albanese is finite étale. Hence $\hat{Y}$ is a torus, and therefore $\hat{Y} = Y$.

The upshot is that we may assume from the beginning that $h : T \to X'$ is the canonical cover, in particular $G \cong \mathbb{Z}_r$. Notice also that instead using Riemann-Roch one can use [5] to construct a 1-form on $\hat{Y}$ from the 2-form, so that $q(\hat{Y}) > 0$.

3) So from now on h is the canonical cover determined by $rK_{X'} \simeq \mathcal{O}_{X'}$. Since $H^2(\mathcal{O}_{X'}) \neq 0$, Serre duality (see *e.g.* [6, p. 348]) gives

$$H^1(K_{X'}) \simeq H^2(X', \mathcal{O}_{X'}) \neq 0.$$

We claim that

$$(*) \qquad H^1(\lambda K_{X'}) \neq 0$$

for $2 \leq \lambda \leq r-1$. By Riemann's extension theorem (see *e.g.* [1, II.3.6])

$$H^1(\lambda K_{X'}) = H^1(X'_{\mathrm{reg}}, \lambda K_{X'})$$

since $\operatorname{codim} \operatorname{Sing} X' = 3$. Analogously

$$H^1(\lambda K_T) = H^1(h^{-1}(X'_{\mathrm{reg}}), h^*\lambda K_{X'}).$$

Therefore we can compute H^1- cohomology on the "étale part" of h.

By [7, 5.12], $H^1(K_{X'}) \cong H^1(K_T)^G$. Now

$$h_*(\mathcal{O}_T) = \mathcal{O}_X \oplus K_{X'} \oplus \cdots \oplus (r-1)K_{X'},$$

at least on X'_{reg}. Since $K_T = h_*(K_{X'})$ over X'_{reg}, the projection formula yields

$$h_*(\lambda K_T) = \lambda K_{X'} \oplus (\lambda+1)K_{X'} \oplus \cdots \oplus (r+1)K_{X'} \oplus \cdots \oplus (r+\lambda-1)K_{X'}$$

over X'_{reg}. Now $(r+1)K_{X'} = K_{X'}$, since $rK_{X'} = \mathcal{O}_{X'}$. Thus $H^1(\lambda K_T)$ contains a G-invariant part, namely $H^1(K_T)^G \cong H^1(K_{X'})$. Hence $H^1(\lambda K_T)^G \neq 0$, and therefore we get $H^1(\lambda K_{X'}) \neq 0$. By (*) and $h^1(\mathcal{O}_T) = 3$, we conclude $r \leq 4$, since

$$h^1(\mathcal{O}_T) = \sum_{\lambda=0}^{r-1} h^1(\lambda K_{X'}).$$

In case $r = 2$ or $r = 3$ Reid [6, p. 376] shows that the singularities are canonical. So suppose $r = 4$. If a singularity $x_i \in X'$ is not canonical, [6] implies that it is of type $\frac{1}{4}(1,1,1)$ in Reid's notation which is just multiplication by i in every component. Now a holomorphic 2-form on $\mathbb{C}^3$ which is invariant under multiplication with i, must have a zero 0. Therefore no 2-form on the torus T can survive this action. □

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