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LAGRANGIAN FIBRATIONS ON GENERALIZED KUMMER VARIETIES

BY MARTIN G. GULBRANDSEN

ABSTRACT. — We investigate the existence of Lagrangian fibrations on the generalized Kummer varieties of Beauville. For a principally polarized abelian surface A of Picard number one we find the following: The Kummer variety K^nA is birationally equivalent to another irreducible symplectic variety admitting a Lagrangian fibration, if and only if n is a perfect square. And this is the case if and only if K^nA carries a divisor with vanishing Beauville-Bogomolov square.

RÉSUMÉ (*Fibrations lagrangiennes sur les variétés de Kummer généralisées*)

Nous étudions l'existence de fibrations lagrangiennes sur les variétés de Kummer généralisées ăde Beauville. Pour une surface abélienne principalement polarisée dont le nombre de Picard égale 1 nous prouvons le résultat suivant : la variété de Kummer K^nA est birationnellement équivalente à une ăvariété symplectique irréductible admettant une fibration lagrangienne si et seulement si n est un carré parfait. Et cela est le cas si et seulement si K^nA supporte un diviseur dont le carré de Beauville-Bogomolov s'annule.

1. Introduction

Let X denote a projective irreducible symplectic variety of dimension $2n$. We refer the reader to Huybrechts' exposition [6] for definitions and general

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background material. Matsushita [8], [9], [10], [11] studied *fibrations* of X, that is, proper maps

$$f\colon X \longrightarrow B, \tag{1}$$

such that a generic fibre is connected and has positive dimension. Assuming B to be projective and nonsingular, Matsushita showed that every component of every fibre of f is a Lagrangian subvariety of X of dimension n, and every nonsingular fibre is an abelian variety. Furthermore, the base B is n-dimensional Fano and its Hodge numbers agree with those of $\mathbb{P}^n$. It is a conjecture that B is in fact isomorphic to $\mathbb{P}^n$.

The setup can be generalized slightly:

DEFINITION 1.1. — With X and B as above, a rational map

$$f\colon X \dashrightarrow B$$

is a *rational fibration* of X over B if there exist another projective irreducible symplectic variety X' and a birational equivalence $\phi\colon X' \xrightarrow{\sim} X$ such that the composition $f \circ \phi$ is a (regular) fibration of X' over B.

A basic tool in the study of irreducible symplectic varieties is the *Beauville-Bogomolov form*, which is an integral quadratic form q on $H^2(X, \mathbb{Z})$, satisfying

$$q(\alpha)^n = c \deg(\alpha^{2n})$$

for a positive real constant c. A birational map between irreducible symplectic varieties induces an isomorphism on $H^2(-, \mathbb{Z})$, compatible with the Beauville-Bogomolov forms. It follows that in the situation of Definition 1.1, the pullback $D = f^*H$ of any divisor H on B satisfies $q(D) = 0$. Conversely, one may ask:

QUESTION 1.2. — *Suppose X carries a nontrivial divisor D with vanishing Beauville-Bogomolov square. Does X admit a rational fibration over $\mathbb{P}^n$?*

One may try to answer the question for the known examples of projective irreducible symplectic varieties. There are two standard series of examples, both due to Beauville [1]: The first is the Hilbert scheme $S^{[n]}$ (of dimension $2n$) parametrizing finite subschemes of length n of a K3 surface S. The second is the (generalized) *Kummer variety* K^nA (of dimension $2n-2$) associated to an abelian surface A, defined as the fibre of the map

$$\sigma\colon A^{[n]} \longrightarrow A \tag{2}$$

induced by the group law on A. The map σ is locally trivial in the étale topology, and in particular all fibres are isomorphic. So there is no ambiguity in this definition. Recently, Sawon [15] and Markushevich [7] answered Question 1.2 in the affirmative for the Hilbert scheme $S^{[n]}$ of a generic K3 surface. In this text, we consider the case of the Kummer varieties.

To state our result, we need the notion of the dual divisor class: If $C \in \mathrm{Pic}(A)$ is an ample divisor class, then there is a canonically defined dual divisor class $\widehat{C} \in \mathrm{Pic}(\widehat{A})$, which is also ample, and the two divisors C and $\widehat{C}$ have the same self intersections. A precise definition is given in Example 2.4. With this notation, our result is the following:

THEOREM 1.3. — *Let A be an abelian surface carrying an effective divisor $C \subset A$ with self intersection $2n$, where $n > 2$, and assume there exist nonsingular irreducible curves in the linear system $|\widehat{C}|$ on $\widehat{A}$. Then the Kummer variety $K^n A$ admits a rational fibration*

$$f : K^n A \dashrightarrow |\widehat{C}| \cong \mathbb{P}^{n-1}. \tag{3}$$

REMARK 1.4. — The assumption that $|\widehat{C}|$ contains nonsingular irreducible curves is only used to verify that a generic fibre of f is connected. We will see in Example 2.4 that this assumption is satisfied whenever A is indecomposable, i.e. not a product of elliptic curves, and also whenever C is nonprimitive, i.e. divisible in the Néron-Severi group of A.

The theorem is proved in Section 3. We have the following corollary, which answers Question 1.2 in the affirmative for the Kummer varieties associated to a generic principally polarized abelian surface, and which is proved in Section 3.5:

COROLLARY 1.5. — *If the abelian surface A has Picard number one and admits a principal polarization, then the following are equivalent, for each $n > 2$:*

1) *The Kummer variety $K^n A$ admits a rational fibration over $\mathbb{P}^{n-1}$.*
2) *$K^n A$ carries a divisor with vanishing Beauville-Bogomolov square.*
3) *n is a perfect square.*

The present work has been carried out independently of the works of Sawon and Markushevich, but the construction is similar. However, Sawon and Markushevich are able to answer Question 1.2 for the Hilbert scheme of *any* (generic) K3 surface, and their construction involves a certain moduli space of twisted sheaves. In this text, we avoid twisted sheaves, but are only able to answer Question 1.2 for (generic) *principally* polarized abelian surfaces. It might be possible to extend the construction to arbitrary polarizations by adapting the use of twisted sheaves in the construction of Sawon and Markushevich. [(1)]

(1) After this paper was written, K. Yoshioka (arXiv:math.AG/0605190) answered Question 1.2 affirmatively for Kummer varieties of arbitrarily polarized abelian surfaces. The proof uses twisted sheaves.

I would like to thank Geir Ellingsrud for numerous fruitful discussions, and Manfred Lehn for introducing me to the question of existence of Lagrangian fibrations.

2. Preparation

We work in the category of noetherian schemes over $\mathbb{C}$. By a map of schemes we mean a morphism in this category. By a sheaf on a scheme X we mean a coherent $\mathcal{O}_X$-module.

If A is an abelian variety, we denote the identity element for the group law on A by 0, and if a is a point on A, we write $T_a\colon A \to A$ for translation by a. We write $\widehat{A}$ for the dual abelian variety. We denote by $\mathcal{P}_x$ the homogeneous line bundle on A corresponding to a point $x \in \widehat{A}$. If D is a divisor on A, we denote by

$$\phi_D\colon A \longrightarrow \widehat{A}$$

the map that takes a point $a \in A$ to the (invertible sheaf associated to the) divisor $T_a^*D - D$.

We use the same symbol to denote a divisor on a variety, its class in the Picard group and its class in the second cohomology group.

In this section, we recall a few results from the literature on sheaves on abelian surfaces.

2.1. The Fourier-Mukai transform. — Let $X \to T$ be an abelian scheme over T, and let $\widehat{X} \to T$ denote its dual abelian scheme. Let $\mathcal{P}$ be the Poincaré line bundle on $X \times_T \widehat{X}$, normalized such that the restrictions of $\mathcal{P}$ to $X \times 0$ and $0 \times \widehat{X}$ are trivial. Let

$$X \xleftarrow{p} X \times_T \widehat{X} \xrightarrow{q} \widehat{X}$$

denote the two projections.

Following Mukai [12], [13], we define a functor $\widehat{S}$ from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_{\widehat{X}}$-modules by

$$\widehat{S}(\mathcal{E}) = q_*\bigl(p^*(\mathcal{E}) \otimes \mathcal{P}\bigr).$$

Reversing the roles of X an $\widehat{X}$, we get a functor S taking an $\mathcal{O}_{\widehat{X}}$-module $\mathcal{F}$ to the $\mathcal{O}_X$-module

$$S(\mathcal{F}) = p_*\bigl(q^*(\mathcal{F}) \otimes \mathcal{P}\bigr).$$

DEFINITION 2.1. — An $\mathcal{O}_X$-module $\mathcal{E}$ satisfies the *weak index theorem* (WIT) with index i if

$$R^p\widehat{S}(\mathcal{E}) = 0 \quad \text{for all} \quad p \neq i.$$

The *Fourier-Mukai transform* of such a sheaf $\mathcal{E}$ is the $\mathcal{O}_{\widehat{X}}$-module

$$\widehat{\mathcal{E}} = R^i\widehat{S}(\mathcal{E}).$$

For each $t \in T$, we may view $\mathcal{E} \otimes k(t)$ as a sheaf on the fibre X_t, which is an abelian variety. We have the following base change result:

THEOREM 2.2 (see Mukai [13]). — *Let $\mathcal{E}$ be a sheaf on $X \to T$, flat over T. The locus of points $t \in T$ such that $\mathcal{E} \otimes k(t)$ satisfies* WIT *is open. If $\mathcal{E} \otimes k(t)$ satisfies* WIT *with index i for all $t \in T$, then $\mathcal{E}$ also satisfies* WIT *with index i, $\widehat{\mathcal{E}}$ is flat over T and we have for all $t \in T$*

$$\widehat{\mathcal{E} \otimes k(t)} \cong \widehat{\mathcal{E}} \otimes k(t).$$

We will apply this only in the case $X = A \times T$, where A is an abelian surface, and view $\mathcal{E}$ as a family of sheaves on A parametrized by T.

Mukai's discovery was the following:

THEOREM 2.3 (see Mukai [12]). — *Let A be an abelian variety of dimension g. The functor $\widehat{S}$ induces an equivalence of derived categories*

$$R\widehat{S} \colon D(A) \longrightarrow D(\widehat{A})$$

with quasi-inverse taking a complex $\mathcal{K}^\bullet$ to $(-1)_A^* RS(\mathcal{K}^\bullet)[-g]$. In particular, if $\mathcal{E}$ is a sheaf on A satisfying WIT *with index i, then $\widehat{\mathcal{E}}$ also satisfies* WIT*, with index $g - i$, and we have a natural isomorphism*

$$\widehat{\widehat{\mathcal{E}}} \cong (-1)_A^*\mathcal{E}.$$

A similar statement holds when A is replaced with an arbitrary abelian scheme [13], but we will not need this.

Mukai [13] also calculated the Chern character in $H^*(\widehat{A}, \mathbb{Z})$ of $R\widehat{S}(\mathcal{K}^\bullet)$ in terms of the Chern character of the complex $\mathcal{K}^\bullet$: To state the result, recall that there is a canonical duality between the cohomology groups of A and those of $\widehat{A}$. Thus, using Poincaré duality, we may identify

$$H^p(\widehat{A}, \mathbb{Z}) \cong H^{2g-p}(A, \mathbb{Z}). \tag{4}$$

Writing ch^p for the $2p$'th component of the Chern character, and suppressing the isomorphism (4), Mukai found

$$\mathrm{ch}^p\left(R\widehat{S}(\mathcal{K}^\bullet)\right) = (-1)^p\,\mathrm{ch}^{g-p}(\mathcal{K}^\bullet). \tag{5}$$

In particular, whenever $\mathcal{E}$ satisfies WIT with index i, we have

$$\operatorname{ch}^p(\widehat{\mathcal{E}}) = (-1)^{i+p} \operatorname{ch}^{g-p}(\mathcal{E}). \tag{6}$$

We remark that, on an abelian surface, the components of the Chern character are the rank, the first Chern class and the Euler characteristic:

$$\operatorname{ch}^0(\mathcal{E}) = r(\mathcal{E}), \quad \operatorname{ch}^1(\mathcal{E}) = c_1(\mathcal{E}), \quad \operatorname{ch}^2(\mathcal{E}) = \chi(\mathcal{E})$$

EXAMPLE 2.4. — Let $C \subset A$ be an effective divisor with positive self intersection on an abelian surface. By a theorem of Mumford [14, §16], we have

$$H^p\big(A, \mathcal{P}_x(C)\big) = 0, \quad \text{for all } p > 0 \text{ and all } x \in \widehat{A}.$$

Hence, $\mathcal{O}_A(C)$ satisfies WIT with index 0, and $\widehat{\mathcal{O}_A(C)}$ is locally free on $\widehat{A}$. Applying formula (6), we see that $\widehat{\mathcal{O}_A(C)}$ has rank equal to $\chi(\mathcal{O}_A(C)) = \frac{1}{2}C^2$ and Euler characteristic 1, whereas under the isomorphism (4), we have

$$c_1\big(\widehat{\mathcal{O}_A(C)}\big) = -C.$$

Thus, defining $\widehat{C} \in \operatorname{Pic}(\widehat{A})$ to be the divisor class such that

$$\mathcal{O}_{\widehat{A}}(-\widehat{C}) \cong \det \widehat{\mathcal{O}_A(C)},$$

we see that the classes of C and $\widehat{C}$ in $H^2(-, \mathbb{Z})$ correspond under the isomorphism (4).

Note that the pullback of $\widehat{\mathcal{O}_A(C)}$ by the map $\phi_C \colon A \to \widehat{A}$ is [12]

$$\phi_C^*\big(\widehat{\mathcal{O}_A(C)}\big) = \mathcal{O}_A(-C)^{\oplus d}$$

where $d = \frac{1}{2}C^2$, which is also the degree of ϕ_C. It follows that

$$\phi_C^*(\widehat{C}) = dC.$$

Consequently, $\widehat{C}$ is ample, and its self intersection is $\widehat{C}^2 = C^2$.

We also note that, by Bertini's theorem, the assumption in Theorem 1.3, that $|\widehat{C}|$ contains nonsingular irreducible curves, is automatically satisfied unless $|\widehat{C}|$ has base points. This can only happen if both $\widehat{C}$ is indivisible in $H^2(\widehat{A}, \mathbb{Z})$ [14, §6, §16] and $\widehat{A}$ is a product of elliptic curves [2, §10.1]. This proves the claim in Remark 1.4, since C is divisible if and only if $\widehat{C}$ is, and A is a product if and only if $\widehat{A}$ is.

2.2. Moduli of sheaves on an abelian surface. — Let A be an abelian surface and fix a polarization H. By a (semi-) stable sheaf on A we will mean a Gieseker (semi-) stable sheaf with respect to H. Fixing a rank $r \geq 0$, first Chern class $c_1 \in \mathrm{NS}(A)$ and Euler characteristic χ, we denote by $M_A(r, c_1, \chi)$ the Simpson moduli space of stable sheaves with the given invariants. In the cases of interest to us, stability and semi-stability will be equivalent, so that $M_A(r, c_1, \chi)$ is going to be projective.

We will in fact only consider sheaves of rank one or zero, so we note that every torsion free sheaf of rank one is stable, whereas in the rank zero case, Riemann-Roch gives the following condition: A pure one-dimensional sheaf $\mathcal{E}$ on A is stable if and only if we have

$$\frac{\chi(\mathcal{F})}{\deg_H(\mathcal{F})} < \frac{\chi(\mathcal{E})}{\deg_H(\mathcal{E})} \tag{7}$$

for every nontrivial proper subsheaf $\mathcal{F} \subset \mathcal{E}$.

Yoshioka [16] defines a (regular) map

$$\alpha \colon M_A(r, c_1, \chi) \longrightarrow A \times \widehat{A} \tag{8}$$

that can be described at the level of sets as follows, except that we take the liberty to make a sign change: Choose a representative $\mathcal{L} \in \mathrm{Pic}(A)$ in the class c_1, and also a representative $\mathcal{L}' \in \mathrm{Pic}(\widehat{A})$ in the class corresponding to c_1 via Poincaré duality (4). Then define $\alpha = (\delta, \widehat{\delta})$, where

$$\delta(\mathcal{F}) = \det\left(R\widehat{S}(\mathcal{F})\right)^{-1} \otimes \mathcal{L}'^{-1}, \quad \widehat{\delta}(\mathcal{F}) = \det(\mathcal{F}) \otimes \mathcal{L}^{-1}.$$

Note that $\widehat{\delta}(\mathcal{F})$ is an element of $\mathrm{Pic}^0(A) = \widehat{A}$ and, by equation (5), $\delta(\mathcal{F})$ is an element of $\mathrm{Pic}^0(\widehat{A}) = A$.

THEOREM 2.5 (see Yoshioka [16]). — *Assume the triple (r, c_1, χ) is primitive in the even cohomology $\bigoplus_i H^{2i}(A, \mathbb{Z})$ and that semi-stability and stability are equivalent conditions on a sheaf with these invariants. Furthermore assume the polarization H is generic. If the dimension of $M_A(r, c_1, \chi)$ is at least 8, then*

1) *$M_A(r, c_1, \chi)$ is deformation equivalent to $A^{[n]} \times \widehat{A}$ for suitable n.*
2) *The map α in (8) is locally trivial in the étale topology.*
3) *A fibre $K_A(r, c_1, \chi)$ of the map α is deformation equivalent to the Kummer variety $K^n A$. In particular, $K_A(r, c_1, \chi)$ is an irreducible symplectic variety.*

As we will be free to choose the polarization H arbitrarily, the genericity hypothesis will not be of importance to us. We remark, however, that in the case where A has Picard number one, every polarization is generic.

2.3. The Beauville-Bogomolov form on Kummer varieties. — Beauville [1] has determined explicitly the second cohomology group of a Kummer variety, together with the Beauville-Bogomolov form on it. See also Britze [4, Proposition 1] or Yoshioka [16, Section 4.3.1] for the calculation of the Beauville-Bogomolov form.

Firstly, there is a canonical monomorphism

$$H^2(A,\mathbb{C}) \longrightarrow H^2(K^nA,\mathbb{C}) \tag{9}$$

which is compatible with the Hodge structure. Secondly, there is a primitive integral class $\epsilon \in H^2(K^nA,\mathbb{C})$ such that 2ϵ is the fundamental class of the locus $E \subset K^nA$ consisting of nonreduced subschemes. Thus ϵ is a $(1,1)$-class. Together, $H^2(A,\mathbb{C})$ and ϵ generate $H^2(K^nA,\mathbb{C})$. In fact, we have:

PROPOSITION 2.6. — *There is a direct sum decomposition*

$$H^2(K^nA,\mathbb{C}) \cong H^2(A,\mathbb{C}) \oplus \mathbb{C}\epsilon$$

which is orthogonal with respect to the Beauville-Bogomolov form q. Furthermore, the restriction of q to $H^2(A,\mathbb{C})$ is the intersection form on A, whereas

$$q(\epsilon) = -2n.$$

We are interested in classes in $H^2(K^nA,\mathbb{C})$ coming from divisors, that is, the Néron-Severi group $\mathrm{NS}(K^nA)$. Since the inclusion (9) is compatible with the Hodge structure, and ϵ is a primitive $(1,1)$-class, we find

$$\mathrm{NS}(K^nA) \cong \mathrm{NS}(A) \oplus \mathbb{Z}\epsilon, \tag{10}$$

by the Lefschetz theorem on $(1,1)$-classes.

3. Construction

Consider the setup of Theorem 1.3, that is, we have a curve $C \subset A$ with self intersection $2n$ on an abelian surface A.

To construct the fibration in Theorem 1.3, we want to associate to each $\xi \in A^{[n]}$ a curve in a certain linear system. As a first try, one might ask whether there exists a curve in the linear system $|C|$ containing ξ. This turns out to be too restrictive:

LEMMA 3.1. — *A generic element $\xi \in A^{[n]}$ is not contained in any curve in the linear system $|C|$.*

Proof. — As we have seen in Example 2.4, we have

$$H^p\big(A, \mathcal{O}_A(C)\big) = 0 \quad \text{for all } p > 0$$

and thus, by Riemann Roch,

$$\dim H^0\big(A, \mathcal{O}_A(C)\big) = \chi\big(\mathcal{O}_A(C)\big) = n.$$

Thus the complete linear system $|C|$ has dimension $n-1$. It follows that the set of subschemes $\xi \in A^{[n]}$ contained in a curve in $|C|$ forms a family of dimension $2n-1$. (If $|C|$ contains singular or nonreduced curves, this is not entirely obvious, but follows from Briançon's result [3] that the family of length k subschemes supported at a fixed point on a surface has dimension $k-1$. In any case, for our purpose it is sufficient that the family of *reduced* subschemes ξ contained in a curve in $|C|$ has dimension $2n-1$, which is clear.) On the other hand, $A^{[n]}$ has dimension $2n$. □

Let us, starting from the observation in the lemma, sketch our construction: By allowing not only curves in $|C|$, but in the linear systems associated to $\mathcal{P}_x(C)$ for any $x \in \widehat{A}$, we see that we "win" two more degrees of freedom: The set of length n subschemes contained in a curve in $|\mathcal{P}_x(C)|$, for some $x \in \widehat{A}$, forms a family of dimension $2n+1$. Since, again, $A^{[n]}$ has dimension $2n$, we expect the locus

$$D_\xi = \big\{x \in \widehat{A} \mid H^0(A, \mathcal{I}_\xi \otimes \mathcal{P}_x(C)) \neq 0\big\} \tag{11}$$

to be a curve. We will see that this is indeed true for generic ξ, and furthermore, when ξ is a generic element of the Kummer variety K^nA, the curve D_ξ belongs to the linear system $|\widehat{C}|$. The fibration f in Theorem 1.3 is given by sending ξ to D_ξ.

More precisely we will see that, for generic $\xi \in K^nA$, the sheaf $\mathcal{I}_\xi(C)$ satisfies WIT with index 1. Sending ξ to the Fourier-Mukai transform $\widehat{\mathcal{I}_\xi(C)}$ induces a birational equivalence

$$K^nA \dashrightarrow K_{\widehat{A}}(0, \widehat{C}, -1) \tag{12}$$

where the target space is the symplectic variety introduced in Yoshioka's Theorem 2.5. The sheaves parametrized by $K_{\widehat{A}}(0, \widehat{C}, -1)$ are supported on curves in the linear system $|\widehat{C}|$, and sending a sheaf to its support defines a map

$$K_{\widehat{A}}(0, \widehat{C}, -1) \longrightarrow |\widehat{C}|. \tag{13}$$

The composition of the two maps (12) and (13) again gives us the fibration of Theorem 1.3. We remark that the support of $\widehat{\mathcal{I}_\xi(C)}$ is precisely the curve D_ξ in (11). In fact, the fibres of $\widehat{\mathcal{I}_\xi(C)}$ are the vector spaces

$$\widehat{\mathcal{I}_\xi(C)} \otimes k(x) \cong H^1\big(A, \mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big)$$

which vanish precisely when $H^0(A, \mathcal{I}_\xi \otimes \mathcal{P}_x(C))$ vanish, since both the Euler characteristic and the second cohomology of $\mathcal{I}_\xi \otimes \mathcal{P}_x(C)$ is zero.

It turns out to be convenient to extend the setup as follows: We will first see that there is a natural identification $A^{[n]} \times \widehat{A} \cong M_A(1, C, 0)$ in such a way that the Kummer variety is recovered as the fibres of the map

$$\alpha \colon M_A(1, C, 0) \to A \times \widehat{A}$$

introduced in Section 2.2. Then we will construct a commutative diagram

$$\begin{array}{ccccc} M_A(1,C,0) & \overset{\Psi}{\dashrightarrow} & M_{\widehat{A}}(0,\widehat{C},-1) & \xrightarrow{\quad F \quad} & P \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ A \times \widehat{A} & \xrightarrow{\eta} & \widehat{A} \times A & \xrightarrow{\quad q \quad} & A \end{array} \tag{14}$$

where Ψ is a birational map induced by the Fourier-Mukai transform, η is an isomorphism, q denotes second projection and $P \to A$ is a projective bundle with the complete linear system associated to $\mathcal{P}_a(\widehat{C})$ as fibre over a. Choosing compatible base points in the lower row, and restricting the upper row to the respective fibres, we recover the maps (12) and (13).

3.1. Rank one sheaves and the Hilbert scheme. — As usual, $A^{[n]}$ can be regarded as a moduli space of rank one sheaves on A. More precisely, there is an isomorphism

$$A^{[n]} \times \widehat{A} \cong M_A(1, 0, -n) \tag{15}$$

which, on the level of sets, is given by the map

$$(\xi, x) \longmapsto \mathcal{I}_\xi \otimes \mathcal{P}_x.$$

By twisting with C, we can furthermore identify $M_A(1, 0, -n)$ with $M_A(1, C, 0)$. Including the isomorphism (15), we can thus identify

$$A^{[n]} \times \widehat{A} \cong M_A(1, C, 0).$$

We want to describe the composition

$$A^{[n]} \times \widehat{A} \cong M_A(1, C, 0) \xrightarrow{\alpha} A \times \widehat{A}$$

where α is the map (8) of Yoshioka. Recall that to define α, we must choose invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ representing $c_1 = C$ on A and on $\widehat{A}$, respectively. By Example 2.4, we have the natural choices

$$\mathcal{L} = \mathcal{O}_A(C), \quad \mathcal{L}' = \mathcal{O}_{\widehat{A}}(\widehat{C}),$$

and then we have:

LEMMA 3.2. — *The diagram*

$$\begin{array}{ccc} A^{[n]} \times \widehat{A} & \cong & M_A(1,C,0) \\ \downarrow{\scriptstyle \sigma\times 1_{\widehat{A}}} & & \downarrow{\scriptstyle \alpha} \\ A \times \widehat{A} & \xrightarrow{\ \theta\ } & A \times \widehat{A} \end{array}$$

is commutative, where θ is the isomorphism

$$\theta(a,x) = \big(a + \phi_{\widehat{C}}(x), x\big).$$

In particular, the fibres $K^n A$ on the left are taken isomorphically to the fibres $K_A(1,C,0)$ on the right.

Proof. — Let us, for the sake of readability, use additive notation in the Picard groups. Firstly, we have

$$\widehat{\delta}(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)) = \det\big(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big) + \mathcal{O}_A(-C) = \mathcal{P}_x.$$

Secondly, applying the Fourier-Mukai functor to the short exact sequence

$$0 \to \mathcal{I}_\xi \otimes \mathcal{P}_x(C) \longrightarrow \mathcal{P}_x(C) \longrightarrow \mathcal{O}_\xi \to 0 \tag{16}$$

we obtain an exact sequence

$$0 \to \widehat{S}\big(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big) \longrightarrow \widehat{S}\big(\mathcal{P}_x(C)\big) \longrightarrow \widehat{S}(\mathcal{O}_\xi) \to R^1\widehat{S}\big(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big) \to 0,$$

since $\mathcal{P}_x(C)$ satisfies WIT with index 0, as in Example 2.4. Thus we have

$$\delta\big(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big) = -\det\widehat{S}\big(\mathcal{P}_x(C)\big) + \det\widehat{S}(\mathcal{O}_\xi) + \mathcal{O}_{\widehat{A}}(-\widehat{C}).$$

To determine $\det\widehat{S}(\mathcal{P}_x(C))$, apply the fact [12, §3] that tensoring with $\mathcal{P}_x$ before applying $\widehat{S}$ is the same thing as translating with x after applying $\widehat{S}$. Hence

$$\det\widehat{S}\big(\mathcal{P}_x(C)\big) = \mathcal{O}_{\widehat{A}}(-T_x^*\widehat{C})$$

by the definition of $\widehat{C}$ in Example 2.4.

To calculate $\det\widehat{S}(\mathcal{O}_\xi)$, note that, whenever $\eta \subset \xi$ is a subscheme of length $n-1$, we have an exact sequence

$$0 \to k(a) \longrightarrow \mathcal{O}_\xi \longrightarrow \mathcal{O}_\eta \to 0$$

where $\xi = \eta + a$ as cycles on A. The induced exact sequence

$$0 \to \mathcal{P}_a \longrightarrow \widehat{S}(\mathcal{O}_\xi) \longrightarrow \widehat{S}(\mathcal{O}_\eta) \to 0$$

shows that $\det\widehat{S}(\mathcal{O}_\xi) = \det\widehat{S}(\mathcal{O}_\eta) + \mathcal{P}_a$. By induction on the length of ξ, we find

$$\det\widehat{S}(\mathcal{O}_\xi) = \mathcal{P}_{\sigma(\xi)}$$

where σ is the summation map (2).

We have thus shown that

$$\delta\big(\mathcal{I}_\xi \otimes \mathcal{P}_{\sigma(\xi)}(C)\big) = \mathcal{P}_{\sigma(\xi)} + \mathcal{O}_{\widehat{A}}(T_x^*\widehat{C} - \widehat{C}).$$

More concisely, we may write this as

$$\alpha\big(\mathcal{I}_\xi \otimes \mathcal{P}_x(C)\big) = \big(\sigma(\xi) + \phi_{\widehat{C}}(x), x\big).$$

which is what we wanted to prove. □

3.2. The weak index property

LEMMA 3.3. — *The (open) locus of sheaves $\mathcal{E} \in M_A(1, C, 0)$ satisfying* WIT *with index* 1 *is nonempty. In fact, there exist* WIT*-sheaves in every fibre $K_A(1, C, 0)$ of α.*

Proof. — The operations of translation and twisting by a homogeneous line bundle

$$\mathcal{E} \longmapsto T_a^*\mathcal{E}, \quad \mathcal{E} \longmapsto \mathcal{E} \otimes \mathcal{P}_x$$

are essentially exchanged by the Fourier-Mukai functor [12, §3], and hence do not affect the WITness of a sheaf $\mathcal{E}$. Thus, it is enough to prove the existence of a WIT-sheaf in $M_A(1, C, 0)$, since we can move such a sheaf to any fibre of α by translating and twisting.

Let $\mathcal{E} = \mathcal{I}_\xi(C)$. We have

$$H^2\big(A, \mathcal{I}_\xi(C) \otimes \mathcal{P}_x\big) = 0$$

for all $x \in \widehat{A}$, for instance by the short exact sequence (16), so $R^2\widehat{S}(\mathcal{I}_\xi(C)) = 0$. Furthermore, by Lemma 3.1, we have

$$H^0\big(A, \mathcal{I}_\xi(C)\big) = 0$$

for generic ξ. But $\widehat{S}(\mathcal{I}_\xi(C))$ is torsion free, hence we conclude that $\widehat{S}(\mathcal{I}_\xi(C)) = 0$ for generic ξ. Thus $\mathcal{I}_\xi(C)$ satisfies WIT with index 1. □

3.3. Stability

LEMMA 3.4. — *Let $\mathcal{E}$ be a sheaf in $M_A(1, C, 0)$ satisfying* WIT *with index* 1*. Then the Fourier-Mukai transform $\widehat{\mathcal{E}}$ is stable with respect to any polarization of $\widehat{A}$.*

Proof. — We first show that $\widehat{\mathcal{E}}$ is pure. Being the Fourier-Mukai transform of a WIT-sheaf with index 1, $\widehat{\mathcal{E}}$ itself satisfies WIT with index 1. It has rank zero and first Chern class $\widehat{C} \neq 0$, hence it is one-dimensional. If $\mathcal{T} \subset \widehat{\mathcal{E}}$ is a zero-dimensional subsheaf, then $\mathcal{T}$ satisfies WIT with index 0, but

$$S(\mathcal{T}) \subseteq S(\widehat{\mathcal{E}}) = 0$$

and hence $\mathcal{T} = 0$. Thus $\widehat{\mathcal{E}}$ is pure of dimension 1.

Suppose $\mathcal{F} \subset \widehat{\mathcal{E}}$ were a destabilizing subsheaf. Then $\mathcal{F}$ also satisfies WIT with index 1.

As $\widehat{\mathcal{E}}/\mathcal{F}$ is torsion, its degree is nonnegative, so we have

$$\deg(\mathcal{F}) \leq \deg(\widehat{\mathcal{E}})$$

with respect to any polarization of $\widehat{A}$. On the other hand, since $\mathcal{F}$ is destabilizing, we have by (7)

$$\frac{\chi(\mathcal{F})}{\deg(\mathcal{F})} > \frac{\chi(\widehat{\mathcal{E}})}{\deg(\widehat{\mathcal{E}})}$$

and thus

$$\chi(\mathcal{F}) > \chi(\widehat{\mathcal{E}}) = -1.$$

Since the Fourier-Mukai transform $\widehat{\mathcal{F}}$ has rank $-\chi(\mathcal{F}) < 1$ by equation (6), it must be a torsion sheaf. Now, applying the Fourier-Mukai functor to the exact sequence

$$0 \to \mathcal{F} \longrightarrow \widehat{\mathcal{E}} \longrightarrow \widehat{\mathcal{E}}/\mathcal{F} \to 0$$

we obtain a left exact sequence

$$0 \to S(\widehat{\mathcal{E}}/\mathcal{F}) \longrightarrow \widehat{\mathcal{F}} \longrightarrow \widehat{\widehat{\mathcal{E}}} \cong (-1)^*\mathcal{E}$$

where Theorem 2.3 is applied to obtain the isomorphism on the right. But both $S(\widehat{\mathcal{E}}/\mathcal{F})$ and $(-1)^*\mathcal{E}$ are torsion free, hence it is impossible for the middle term $\widehat{\mathcal{F}}$ to be torsion. Thus we have reached a contradiction. □

We are now ready to construct the leftmost square in diagram (14): Let $U \subset M_A(1, C, 0)$ denote the set of sheaves satisfying WIT with index 1. Then U is open and nonempty, by Theorem 2.2 and Lemma 3.3. Let $\mathcal{U}$ denote the restriction of the universal family on $M_A(1, C, 0)$ to U. Applying Theorem 2.2 again, $\mathcal{U}$ satisfies WIT with index 1, and its Fourier-Mukai transform $\widehat{\mathcal{U}}$ is a flat family of sheaves on $\widehat{A}$ parametrized by U. The fibres of $\widehat{\mathcal{U}}$ are stable by Lemma 3.4, and by equation (6) they have rank one, first Chern class $\widehat{C}$ and Euler characteristic -1. Thus there is an induced rational map

$$\Psi \colon M_A(1, C, 0) \dashrightarrow M_{\widehat{A}}(0, \widehat{C}, -1)$$

which is regular on U. In fact, by Theorem 2.3, the restriction of Ψ to U is an open immersion. It follows that Ψ is birational, as $M_{\widehat{A}}(0, \widehat{C}, -1)$ is irreducible by Theorem 2.5. Let us verify that Ψ fits into the diagram (14), i.e. we check the commutativity of the leftmost square. So let $\mathcal{E}$ be a sheaf in $M_A(1, C, 0)$ satisfying WIT with index 1. Then

$$\delta(\mathcal{E}) = \det(\widehat{\mathcal{E}}) \otimes \mathcal{O}_{\widehat{A}}(-\widehat{C}), \quad \widehat{\delta}(\mathcal{E}) = \det(\mathcal{E}) \otimes \mathcal{O}_A(-C)$$

whereas

$$\delta(\widehat{\mathcal{E}}) = \det(\widehat{\widehat{\mathcal{E}}}) \otimes \mathcal{O}_A(-C) = (-1)_A^* \det(\mathcal{E}) \otimes \mathcal{O}_A(-C),$$
$$\widehat{\delta}(\widehat{\mathcal{E}}) = \det(\widehat{\mathcal{E}}) \otimes \mathcal{O}_{\widehat{A}}(-\widehat{C}).$$

Thus we see that, defining the map η in diagram (14) by

$$\eta(a,x) = (-x,a) + \big((-1)^*C - C, 0\big),$$

the left square in that diagram commutes. Since, by Lemma 3.3, no fibre $K_A(1,C,0)$ of α is contained in the base locus of Ψ, we conclude that Ψ restricts to a birational equivalence

$$\Phi\colon K_A(1,C,0) \dashrightarrow K_{\widehat{A}}(0,\widehat{C},-1). \tag{17}$$

3.4. The fibration. — Let $\mathcal{G}$ denote the Fourier-Mukai transform of $\mathcal{O}_{\widehat{A}}(\widehat{C})$. By the base change theorem in cohomology, the fibre of $\mathcal{G}$ over $a \in A$ is canonically isomorphic to $H^0(\widehat{A}, \mathcal{P}_a(\widehat{C}))$. Thus, the associated projective bundle

$$P = \mathbb{P}(\mathcal{G}^\vee) \longrightarrow A \tag{18}$$

has the complete linear systems associated to $\mathcal{P}_a(\widehat{C})$ as fibres.

The Fitting ideal of a sheaf $\mathcal{F}$ in $M_{\widehat{A}}(0,\widehat{C},-1)$ defines a curve representing the first Chern class of $\mathcal{F}$, and hence a point in the bundle P. The map of sets

$$F\colon M_{\widehat{A}}(0,\widehat{C},-1) \longrightarrow P \tag{19}$$

thus obtained is in fact a (regular) map of varieties, since formation of the Fitting ideal commutes with base change. Clearly, F fits into diagram (14), making its rightmost square commute. Thus, restricting F to the fibre $K_{\widehat{A}}(0,\widehat{C},-1)$ above zero in $\widehat{A} \times A$, we find a map

$$f\colon K_{\widehat{A}}(0,\widehat{C},-1) \longrightarrow |\widehat{C}|. \tag{20}$$

We claim that f is a fibration, i.e. a generic fibre is connected. For this, let $D \in |\widehat{C}|$ be a nonsingular curve. Viewing D as a point in P, the fibre $F^{-1}(D)$ is just the Jacobian J^{n-1} of D, parametrizing invertible sheaves of degree $n-1$ on D. The restriction of α to J^{n-1} can be identified with the summation map

$$J^{n-1} \longrightarrow A \tag{21}$$

sending a divisor $\sum n_i p_i$ on D to the point $\sum n_i p_i$ on A, using the group law on A. Thus, the fibre of f above D equals a fibre of the map (21). It follows from D being ample that such a fibre is connected. This concludes the proof of Theorem 1.3.

3.5. Principally polarized surfaces. — Let us prove Corollary 1.5. Thus we assume (A, H) is a principally polarized abelian surface with Picard number one.

The implication 1) $\Rightarrow$ 2) is automatic, as explained in the introduction. For the implication 2) $\Rightarrow$ 3), suppose $K^n A$ admits a divisor D with vanishing Beauville-Bogomolov square, corresponding to $rH + s\epsilon$ under the isomorphism (10), where r and s denote integers. Then

$$0 = q(D) = (rH)^2 + s^2 q(\epsilon) = 2r^2 - 2s^2 n$$

from which it is immediate that n is a perfect square.

Finally, the implication 3) $\Rightarrow$ 1) follows from Theorem 1.3: If $n = m^2$ is a perfect square, the effective curve $C = mH$ has self intersection $2n$, and hence the theorem applies. The corollary is proved.

4. On the base locus

Again let (A, H) be a principally polarized abelian surface, and let $C = mH$ and $n = m^2$. Then there does exist $\xi \in A^{[n]}$ such that $\mathcal{I}_\xi(C)$ fails WIT: It is easy to check that this is the case whenever $\xi \in A^{[n]}$ is contained in some translate $T_a^{-1}(H)$ of the polarization.

In the first nontrivial case $n = 4$, assuming the Picard number of A is one, the author has checked [5] that the base locus of the map Φ in equation (17) is exactly the locus of sheaves failing WIT. Furthermore, this locus has the structure of a $\mathbb{P}^2$-bundle Q over A. By a careful study of the map Φ one can show that the base locus of the fibration in Theorem 1.3 is the same locus Q. It seems likely that Φ is in fact the Mukai elementary transform along Q.

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