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UNIT VECTOR FIELDS ON ANTIPODALLY PUNCTURED SPHERES: BIG INDEX, BIG VOLUME

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ABSTRACT. — We establish in this paper a lower bound for the volume of a unit vector field $\vec{v}$ defined on $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. This lower bound is related to the sum of the absolute values of the indices of $\vec{v}$ at x and $-x$.

RÉSUMÉ (*Champs unitaires dans les sphères antipodalement trouées : grand indice entraîne grand volume*)

Nous établissons une borne inférieure pour le volume d'un champ de vecteurs $\vec{v}$ défini dans $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. Cette borne inférieure dépend de la somme des valeurs absolues des indices de $\vec{v}$ en x et en $-x$.

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1. Introduction

The volume of a unit vector field $\vec{v}$ on a closed Riemannian manifold M is defined [10] as the volume of the section $\vec{v} : M \to T^1M$, where the Sasakian metric is considered in T^1M. The volume of $\vec{v}$ can be computed from the Levi-Civita connection ∇ of M. If we denote by ν the volume form, for an orthonormal local frame $\{e_a\}_{a=1}^n$, we have

$$\text{(1)} \qquad \operatorname{vol}(\vec{v}) = \int_M \Big(1 + \sum_{a=1}^{n} \|\nabla_{e_a}\vec{v}\|^2 + \sum_{a_1<a_2} \|\nabla_{e_{a_1}}\vec{v} \wedge \nabla_{e_{a_2}}\vec{v}\|^2 + \cdots + \sum_{a_1<\cdots<a_{n-1}} \|\nabla_{e_{a_1}}\vec{v} \wedge \cdots \wedge \nabla_{e_{a_{n-1}}}\vec{v}\|^2\Big)^{\frac{1}{2}} \nu.$$

Note that $\operatorname{vol}(\vec{v}) \geq \operatorname{vol}(M)$ and also that only parallel fields attain the trivial minimum.

For odd-dimensional spheres, vector fields homologous to the Hopf fibration $\vec{v}_H$ have been studied, see [10], [3], [9] and [2]. In [5], a non-trivial lower bound of the volume of unit vector fields on spaces of constant curvature was obtained. In $\mathbb{S}^{2k+1}$, only the vector field $\vec{n}$ tangent to the geodesics from a fixed point (with two singularities) attains the volume of that bound. We call this field $\vec{n}$ north-south or radial vector field. We notice that unit vector fields with singularities show up in a natural way, see also [12].

For manifolds of dimension 5, a theorem showing how the topology of a vector field influences its volume appears in [4]. More precisely, the result in [4] is an inequality relating the volume of $\vec{v}$ and the Euler form of the orthogonal distribution to $\vec{v}$.

The purpose of this paper is to establish a relationship between the volume of unit vector fields and the indices of those fields around isolated singularities.

We consider these notes to be a preliminary effort to understand this phenomenon. For this reason, we have chosen a simple model where such a relationship is found. We hope this could serve as inspiration for more complex situations to be treated in a near future.

Precisely, we prove here:

THEOREM 1.1. — *Let $W = \mathbb{S}^n \setminus \{N, S\}$, $n = 2$ or 3, be the standard Euclidean sphere where two antipodal points N and S are removed. Let $\vec{v}$ be a unit smooth vector field defined on W. Then,*

$$\text{for } n = 2, \quad \operatorname{vol}(\vec{v}) \geq \tfrac{1}{2}\big(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\big)\operatorname{vol}(\mathbb{S}^2);$$

$$\text{for } n = 3, \quad \operatorname{vol}(\vec{v}) \geq \big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big)\operatorname{vol}(\mathbb{S}^3),$$

where $I_{\vec{v}}(P)$ stands the Poincaré index of $\vec{v}$ around P.

It is easy to verify that the north-south field $\vec{n}$ achieves the equalities in the theorem. In fact, the volume of $\vec{n}$ in $\mathbb{S}^2$ is equal to $\frac{1}{2}\pi\mathrm{vol}(\mathbb{S}^2)$, and in $\mathbb{S}^3$ is $2\mathrm{vol}(\mathbb{S}^3)$. We have to point out that $\mathrm{vol}(\vec{n}) = \mathrm{vol}(\vec{v}_H)$ in $\mathbb{S}^3$.

The lower bound in $\mathbb{S}^3$ when the singularities are trivial (i.e. $I_{\vec{v}}(N) = I_{\vec{v}}(S) = 0$) has no special meaning.

We will comment briefly some possible extensions for this result in Section 3 of this paper.

2. Proof of the theorem

A key ingredient in the proof of the theorem is the application of the following result of Chern [7]. The second part of this statement is a special case of the result of Section 3 of that article.

PROPOSITION 2.1 (see Chern [7]). — *Let M^n be an orientable Riemannian manifold of dimension n, with Riemannian connection 1-form ω and curvature form Ω. Then, there is an $(n-1)$-form Π on the unit tangent bundle T^1M with $\pi : T^1M \to M$ the bundle projection, so that:*

$$\mathrm{d}\Pi = \begin{cases} e(\Omega) & \textit{if } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

In addition, $\int_{\pi^{-1}(x)} \Pi = 1$ for any $x \in M$, that is, $\Pi_{|\pi^{-1}(x)}$ is the induced volume form of the fiber $\pi^{-1}(x)$, normalized to have volume 1.

The form Π as described by Chern is somewhat complicated. First, define forms ϕ_k for $k \in \{0, \dots, [\frac{1}{2}n] - 1\}$, by choosing a frame $\{e_1, \dots, e_n\}$ of TM, so that $\{e_1, \dots, e_{n-1}\}$ frame $\pi^{-1}(x)$ at $e_n \in \pi^{-1}(x)$. Then, at $e_n \in T^1M$,

$$\phi_k = \sum_{1\le\alpha_1,\dots,\alpha_{n-1}\le n-1} \epsilon_{\alpha_1\dots\alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \wedge \dots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1}n} \wedge \dots \wedge \omega_{\alpha_{n-1}n},$$

where $\epsilon_{\alpha_1\dots\alpha_{n-1}}$ is the sign of the permutation, and from this

$$\Pi = \begin{cases} \dfrac{1}{\pi^{\frac{1}{2}n}} \displaystyle\sum_{k=0}^{\frac{1}{2}n-1} \frac{(-1)^k}{1\cdot 3\cdots(n-2k-1)\cdot 2^{k+\frac{1}{2}n} k!}\,\phi_k & \text{if } n \text{ is even,} \\ \dfrac{1}{2^n \pi^{\frac{1}{2}(n-1)} (\frac{1}{2}(n-1))!} \displaystyle\sum_{k=0}^{\frac{1}{2}(n-1)} (-1)^k \binom{\frac{1}{2}(n-1)}{k} \phi_k & \text{if } n \text{ is odd.} \end{cases}$$

Subsequent treatments of this general theory [8], [11] use more elegant formulations of forms similar to this, but usually only for the bundle of frames, and avoid the case where M is odd-dimensional.

The cases relevant to this research are for $n = 2$ and $n = 3$, where these formulas simplify to

$$\Pi = \begin{cases} \frac{1}{2\pi}\omega_{12} & \text{if } n = 2, \\ \frac{1}{4\pi}\left(\omega_{13} \wedge \omega_{23} - \Omega_{12}\right) & \text{if } n = 3. \end{cases}$$

Even though there is a common line of reasoning in the proof of both parts of the theorem, each dimension has its special features. For that reason, we provide separate proofs for dimensions 2 and 3.

2.1. Case $n = 2$. — Denote by g the usual metric on $\mathbb{S}^2$ induced from $\mathbb{R}^3$. Without loss of generality we take $N = (0, 0, 1)$ and $S = (0, 0, -1)$. On W we consider an oriented orthonormal local frame $\{e_1, e_2 = \vec{v}\}$. Its dual basis is denoted by $\{\theta_1, \theta_2\}$ and the connection 1-forms of ∇ are $\omega_{ij}(X) = g(\nabla_X e_j, e_i)$ for $i, j = 1, 2$ where X is a vector in the corresponding tangent space. In dimension 2, the volume (1) reduces to:

$$\operatorname{vol}(\vec{v}) = \int_{\mathbb{S}^2} \sqrt{1 + k^2 + \tau^2}\,\nu,$$

where $k = g(\nabla_{\vec{v}}\vec{v}, e_1)$ is the geodesic curvature of the integral curves of $\vec{v}$ and $\tau = g(\nabla_{e_1}\vec{v}, e_1)$ is the geodesic curvature of the curves orthogonal to $\vec{v}$. Also,

$$\omega_{12} = \tau\theta_1 + k\theta_2.$$

The first goal is to relate the integrand of the volume with the connection form ω_{12}. If S^1_φ is the parallel of $\mathbb{S}^2$ at latitude $\varphi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ consider the unit field $\vec{u}$ on S^1_φ such that $\{\vec{u}, \vec{n}\}$ is positively oriented where $\vec{n}$ is the field pointing toward N. Let $\alpha \in [0, 2\pi]$ be the oriented angle from $\vec{u}$ to $\vec{v}$. Then $\vec{u} = \sin\alpha\, e_1 + \cos\alpha\,\vec{v}$. If $i : S^1_\varphi \to \mathbb{S}^2$ is the inclusion map, we have

$$i^*\omega_{12}(\vec{u}) = \tau\theta_1(\vec{u}) + k\theta_2(\vec{u}) = \tau\sin\alpha + k\cos\alpha. \tag{2}$$

We split the domain of the integral in northern and southern hemisphere, H^+ and H^- respectively. First we consider the northern hemisphere H^+. From the general inequality $\sqrt{a^2 + b^2} \geq |a\cos\beta + b\sin\beta| \geq a\cos\beta + b\sin\beta$, for any $a, b, \beta \in \mathbb{R}$, we have:

$$\begin{aligned} \sqrt{1 + k^2 + \tau^2} &\geq \cos\varphi + \sqrt{k^2 + \tau^2}\sin\varphi \\ &\geq \cos\varphi + \left|k\cos\alpha + \tau\sin\alpha\right|\sin\varphi = \cos\varphi + \left|i^*\omega_{12}(\vec{u})\right|\sin\varphi. \end{aligned} \tag{3}$$

Denote by ν' the induced volume form to S^1_φ. From (2) and (3) we get

$$\text{(4)}\qquad \begin{aligned}\operatorname{vol}(\vec{v})_{|H^+} &\geq \int_{H^+}\big(\cos\varphi + \big|i^*\omega_{12}(\vec{u})\big|\sin\varphi\big)\nu \\ &= \int_0^{\frac12\pi}\int_{S^1_\varphi}\cos\varphi\nu'\mathrm{d}\varphi + \int_0^{\frac12\pi}\int_{S^1_\varphi}\big|i^*\omega_{12}(\vec{u})\big|\sin\varphi\nu'\mathrm{d}\varphi \\ &\geq \int_0^{\frac12\pi} 2\pi\cos^2\varphi\mathrm{d}\varphi + \int_0^{\frac12\pi}\sin\varphi\Big|\int_{S^1_\varphi} i^*\omega_{12}\Big|\mathrm{d}\varphi.\end{aligned}$$

The connection form ω_{12} satisfies $\mathrm{d}\omega_{12} = \theta_1\wedge\theta_2$. Therefore, the area of the annulus region

$$A\big(\varphi, \tfrac12\pi - \epsilon\big) = \big\{(x_1,x_2,x_3)\in\mathbb{S}^2 \mid \sin\varphi \leq x_3 \leq \sin(\tfrac12\pi-\epsilon)\big\}$$

provides the equality

$$\text{(5)}\qquad \int_{A(\varphi,\frac12\pi-\epsilon)}\mathrm{d}\omega_{12} = \text{area of } A = \int_\varphi^{\frac12\pi-\epsilon} 2\pi\cos t\mathrm{d}t = 2\pi\big(\sin(\tfrac12\pi-\epsilon) - \sin\varphi\big).$$

The boundary of $A(\varphi,\frac12\pi-\epsilon)$ is $\partial A = S^1_\varphi \cup S^1_{\frac12\pi-\epsilon}$ (with the appropriate orientation), so by (5) and Stokes' Theorem

$$\text{(6)}\qquad \begin{aligned}\int_{S^1_\varphi} i^*\omega_{12} &= \int_{A(\varphi,\frac12\pi-\epsilon)}\mathrm{d}\omega_{12} + \int_{S^1_{\frac12\pi-\epsilon}} i^*\omega_{12} \\ &= 2\pi\big(\sin\big(\tfrac12\pi-\epsilon\big) - \sin\varphi\big) + \int_{S^1_{\frac12\pi-\epsilon}} i^*\omega_{12}.\end{aligned}$$

If ω is the Riemannian connection form of the standard metric on $\mathbb{S}^2$, since the limit as ϵ goes to 0 of $\vec{v}_{|S^1_{\frac12\pi-\epsilon}}$ maps $S^1_{\frac12\pi-\epsilon}$ onto the fiber $I_{\vec{v}}(N)$ times, from Proposition 2.1 we have

$$\lim_{\epsilon\to0}\int_{S^1_{\frac12\pi-\epsilon}} i^*\omega_{12} = 2\pi\lim_{\epsilon\to0}\int_{S^1_{\frac12\pi-\epsilon}} i^*\vec{v}^*\Pi = 2\pi I_{\vec{v}}(N)\int_{\pi^{-1}(N)}\Pi = 2\pi I_{\vec{v}}(N).$$

Thus, from (6)

$$\text{(7)}\qquad \int_{S^1_\varphi} i^*\omega_{12} = 2\pi(1-\sin\varphi) + 2\pi I_{\vec{v}}(N).$$

Following from (4) with (7) we have:

$$
\begin{aligned}
(8)\quad \operatorname{vol}(\vec{v})_{|H^+} &\geq \frac{\pi^2}{2} + \int_0^{\frac{1}{2}\pi} \sin\varphi \cdot \big|2\pi(1-\sin\varphi) + 2\pi I_{\vec{v}}(N)\big| \mathrm{d}\varphi \\
&= \frac{\pi^2}{2} + \int_0^{\frac{1}{2}\pi} \big|2\pi \sin\varphi\, I_{\vec{v}}(N) - 2\pi\sin\varphi(\sin\varphi - 1)\big| \mathrm{d}\varphi \\
&\geq \frac{\pi^2}{2} + \int_0^{\frac{1}{2}\pi} \Big|\big|2\pi \sin\varphi\, I_{\vec{v}}(N)\big| - \big|2\pi\sin\varphi(\sin\varphi - 1)\big|\Big| \mathrm{d}\varphi \\
&\geq \frac{\pi^2}{2} + \Big|\int_0^{\frac{1}{2}\pi} \big(|2\pi \sin\varphi\, I_{\vec{v}}(N)| - |2\pi\sin\varphi(\sin\varphi - 1)|\big) \mathrm{d}\varphi\Big| \\
&= \frac{\pi^2}{2} + \Big|2\pi\big|I_{\vec{v}}(N)\big| \int_0^{\frac{1}{2}\pi} \sin\varphi \mathrm{d}\varphi - 2\pi \int_0^{\frac{1}{2}\pi} (\sin\varphi - \sin^2\varphi) \mathrm{d}\varphi\Big| \\
&= \frac{\pi^2}{2} + \Big|2\pi\big|I_{\vec{v}}(N)\big| - 2\pi + \frac{\pi^2}{2}\Big|.
\end{aligned}
$$

For the southern hemisphere, the index of $\vec{v}$ at S is obtained by

$$
\lim_{\epsilon \to 0} \int_{S^1_{-\frac{1}{2}\pi+\epsilon}} i^*\omega_{12} = \operatorname{vol}(\mathbb{S}^1)\, I_{\vec{v}}(S).
$$

Therefore, if $-\frac{1}{2}\pi < \varphi \leq 0$ we have

$$
(9)\qquad \int_{S^1_\varphi} i^*\omega_{12} = 2\pi I_{\vec{v}}(S) - 2\pi(\sin\varphi + 1).
$$

In order to obtain a similar equation to (3) we take $\beta = -\varphi$, and together with (2) we have

$$
\begin{aligned}
(10)\qquad \operatorname{vol}(\vec{v})_{|H^-} &\geq \int_{H^-} \big(\cos\varphi - \big|i^*\omega_{12}(\vec{u})\big| \sin\varphi\big)\nu \\
&\geq \int_{-\frac{1}{2}\pi}^0 2\pi\cos^2\varphi \mathrm{d}\varphi - \int_{-\frac{1}{2}\pi}^0 \Big|\int_{S^1_\varphi} i^*\omega_{12}\Big| \sin\varphi \mathrm{d}\varphi.
\end{aligned}
$$

From (9) and (10):

$$
\begin{aligned}
(11)\quad \operatorname{vol}(\vec{v})\big|_{H^-} &\geq \frac{\pi^2}{2} - \int_{-\frac{1}{2}\pi}^0 \big|2\pi I_{\vec{v}}(S) - 2\pi(\sin\varphi + 1)\big| \sin\varphi \mathrm{d}\varphi \\
&\geq \frac{\pi^2}{2} + \Big|2\pi\big|I_{\vec{v}}(S)\big| \int_{-\frac{1}{2}\pi}^0 |\sin\varphi| \mathrm{d}\varphi - 2\pi \int_{-\frac{1}{2}\pi}^0 |\sin^2\varphi + \sin\varphi| \mathrm{d}\varphi\Big| \\
&= \frac{\pi^2}{2} + \Big|2\pi\big|I_{\vec{v}}(S)\big| - 2\pi + \frac{\pi^2}{2}\Big|.
\end{aligned}
$$

Finally, recall that the sum of the indices of a field in $\mathbb{S}^2$ must be 2, therefore the sum of the absolute values of the indices must be greater or equal than 2. So, from (8) and (11), the volume of $\vec{v}$ is bounded by

$$\begin{aligned}\operatorname{vol}(\vec{v}) &\geq \pi^2 + \Big|2\pi\big|I_{\vec{v}}(N)\big| - 2\pi + \frac{\pi^2}{2}\Big| + \Big|2\pi\big|I_{\vec{v}}(S)\big| - 2\pi + \frac{\pi^2}{2}\Big| \\ &\geq \pi^2 + \big|2\pi\big|I_{\vec{v}}(N)\big| + 2\pi\big|I_{\vec{v}}(S)\big| - 4\pi + \pi^2\big| \\ &= \pi^2 + \big|2\pi\big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\big) + \pi^2\big| \\ &= 2\pi^2 + 2\pi\big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\big) = \big(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\big)\frac{\operatorname{vol}(\mathbb{S}^2)}{2}\cdot\end{aligned}$$

2.2. Case $n = 3$. — As before, denote by g the metric in $\mathbb{S}^3$ and consider a general situation where $N = (0,0,0,1)$, $S = (0,0,0,-1)$ and $I_{\vec{v}}(N) \geq 0$ (and therefore $I_{\vec{v}}(S) \leq 0$).

If $\vec{v}$ is a unit vector field on W, consider on W an oriented orthonormal local frame such that $\{e_1, e_2, e_3 = \vec{v}\}$. The dual basis will be denoted by $\{\theta_1, \theta_2, \theta_3\}$. The coefficients of the second fundamental form of the orthogonal distribution to $\vec{v}$, possibly non-integrable, are $h_{ij} = \omega_{i3}(e_j) = g(\nabla_{e_j}\vec{v}, e_i)$. The coefficients of the acceleration of $\vec{v}$ are given by $\nabla_{\vec{v}}\vec{v} = a_1 e_1 + a_2 e_2$. Finishing the notation, we will use J for the integrand of the volume (1) and

$$\sigma_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{vmatrix}, \quad \sigma_{2,1} = \begin{vmatrix} h_{11} & a_1 \\ h_{21} & a_2\end{vmatrix}, \quad \sigma_{2,2} = \begin{vmatrix} a_1 & h_{12} \\ a_2 & h_{22}\end{vmatrix}.$$

It is easy to see that

$$J = \Big(1 + \sum_{i,j=1}^{2} h_{ij}^2 + a_1^2 + a_2^2 + \sigma_2^2 + (\sigma_{2,1})^2 + (\sigma_{2,2})^2\Big)^{\frac{1}{2}}.$$

Note that $(1 + |\sigma_2|)^2 = 1 + 2|\sigma_2| + \sigma_2^2 \leq 1 + \sum_{i,j=1}^{2} h_{ij}^2 + \sigma_2^2$. Therefore

$$J \geq \sqrt{(1 + |\sigma_2|)^2 + |\sigma_{2,1}|^2}, \tag{12}$$

where equality holds if and only if $a_1 = a_2 = 0$ and we have either $h_{11} = h_{22}$ and $h_{12} = -h_{21}$, or $h_{11} = -h_{22}$ and $h_{12} = h_{21}$.

Now we want to identify the last term in (12) with the evaluation of certain forms.

In the frame $\{e_1, e_2, \vec{v}\}$ we can demand that e_1 will be tangent to S^2_φ, the parallel of $\mathbb{S}^3$ with latitude $\varphi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. We complete a frame in S^2_φ with $\vec{u}$ in such a way $\{e_1, \vec{u}\}$ is an oriented local frame compatible with the normal field $\vec{n}$ that points toward the North Pole. That is, in such a way that $\{e_1, \vec{u}, \vec{n}\}$

is a positively oriented local frame of $\mathbb{S}^3$. Let $\alpha \in [0, 2\pi]$ be the oriented angle from TS^2_φ to $\vec{v}$ and $i : S^2_\varphi \to \mathbb{S}^3$ the inclusion map. In this way, $\vec{u} = \cos\alpha\vec{v} + \sin\alpha e_2$ and

$$\begin{aligned} i^*(\theta_1 \wedge \theta_2)(e_1, \vec{u}) &= \sin\alpha, \\ i^*(\theta_1 \wedge \theta_3)(e_1, \vec{u}) &= \cos\alpha, \\ i^*(\theta_2 \wedge \theta_3)(e_1, \vec{u}) &= 0. \end{aligned}$$

In order to evaluate $i^*(\omega_{13} \wedge \omega_{23})$, first we note that

$$\omega_{13} \wedge \omega_{23} = \sigma_2\theta_1 \wedge \theta_2 + \sigma_{2,1}\theta_1 \wedge \theta_3 - \sigma_{2,2}\theta_2 \wedge \theta_3.$$

So, $i^*(\omega_{13} \wedge \omega_{23})(e_1, \vec{u}) = \sin\alpha\sigma_2 + \cos\alpha\sigma_{2,1}$.

As in (3) with $\beta \in [0, \frac{1}{2}\pi]$ such that $\sin\beta = |\sin\alpha|$ and $\cos\beta = |\cos\alpha|$, from (12) we get

$$\begin{aligned} J &\geq \sin\beta\bigl(1 + |\sigma_2|\bigr) + \cos\beta|\sigma_{2,1}| \\ &= |\sin\alpha| + |\sin\alpha| \cdot |\sigma_2| + |\cos\alpha| \cdot |\sigma_{2,1}| \\ &\geq |\sin\alpha| + |\sin\alpha\,\sigma_2 + \cos\alpha\,\sigma_{2,1}| \\ &= \bigl|i^*(\theta_1 \wedge \theta_2)(e_1, \vec{u})\bigr| + \bigl|i^*(\omega_{13} \wedge \omega_{23})(e_1, \vec{u})\bigr| \\ &\geq \bigl|\bigl(i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23})\bigr)(e_1, \vec{u})\bigr|. \end{aligned} \tag{13}$$

We split W in northern and southern hemisphere, H^+ and H^- respectively. Then, from (13)

$$\begin{aligned} \operatorname{vol}(\vec{v})|_{H^+} &\geq \int_{H^+} \bigl|\bigl(i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23})\bigr)(e_1, \vec{u})\bigr|\nu \\ &\geq \int_0^{\frac{1}{2}\pi} \Bigl|\int_{S^2_\varphi} \bigl(i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23})\bigr)\Bigr| \mathrm{d}\varphi. \end{aligned} \tag{14}$$

We know that $\mathrm{d}\omega_{12} = \omega_{13} \wedge \omega_{23} + \theta_1 \wedge \theta_2$. If $A(\varphi, \frac{1}{2}\pi - \epsilon)$ is the annulus region between the parallels S^2_φ and $S^2_{\frac{1}{2}\pi-\epsilon}$, $0 \leq \varphi < \frac{1}{2}\pi - \epsilon < \frac{1}{2}\pi$, we have by Stokes' Theorem

$$\int_{S^2_\varphi} i^*(\omega_{13} \wedge \omega_{23}) + i^*(\theta_1 \wedge \theta_2) = \int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*(\omega_{13} \wedge \omega_{23}) + \int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*(\theta_1 \wedge \theta_2). \tag{15}$$

We bound $i^*(\theta_1 \wedge \theta_2)(e_1, \vec{u}) = \sin\alpha \geq -1$ on $S^2_{\frac{1}{2}\pi-\epsilon}$ and consequently

$$\int_{S^2_\varphi} i^*(\omega_{13} \wedge \omega_{23}) + i^*(\theta_1 \wedge \theta_2) \geq \int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*(\omega_{13} \wedge \omega_{23}) - 4\pi\cos^2\bigl(\tfrac{1}{2}\pi - \epsilon\bigr).$$

Applying Proposition 2.1, since as before, the limit as ϵ goes to 0 of $\vec{v}_{|S^1_{\frac{1}{2}\pi-\epsilon}}$ maps $S^2_{\frac{1}{2}\pi-\epsilon}$ onto the fiber $I_{\vec{v}}(N)$ times and noting that the curvature term is horizontal so goes to 0 in the limit,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) &= \lim_{\epsilon\to 0}\int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*(\omega_{13}\wedge\omega_{23}-\Omega_{12})\\ &= 4\pi\lim_{\epsilon\to 0}\int_{S^2_{\frac{1}{2}\pi-\epsilon}} i^*\vec{v}^*\,\Pi = 4\pi I_{\vec{v}}(N)\int_{\pi^{-1}(N)}\Pi = 4\pi I_{\vec{v}}(N).\end{aligned}$$

So,

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \geq 4\pi I_{\vec{v}}(N) \geq 0. \tag{16}$$

From (14) and (16) we get

$$\mathrm{vol}(\vec{v})_{|H^+} \geq \int_0^{\frac{1}{2}\pi} 4\pi\big|I_{\vec{v}}(N)\big|\,\mathrm{d}\varphi = 2\pi^2\big|I_{\vec{v}}(N)\big|. \tag{17}$$

In a similar way for the southern hemisphere, the integral of $d\omega_{12}$ over the annulus region $A(-\frac{1}{2}\pi+\epsilon,\varphi)$, $-\frac{1}{2}\pi < -\frac{1}{2}\pi+\epsilon < \varphi \leq 0$ provides exactly (15) but now we bound $\sin\alpha \leq 1$ to obtain

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \leq \int_{S^2_{-\frac{1}{2}\pi+\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) + 4\pi\cos^2\big(-\tfrac{1}{2}\pi+\epsilon\big).$$

The index of $\vec{v}$ at S can be calculated as

$$\lim_{\epsilon\to 0}\int_{S^2_{-\frac{1}{2}\pi+\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) = \mathrm{vol}(\mathbb{S}^2)I_{\vec{v}}(S).$$

So,

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \leq 4\pi I_{\vec{v}}(S) \leq 0.$$

Therefore,

$$\begin{aligned}\mathrm{vol}(\vec{v})|_{H^-} &\geq \int_{-\frac{1}{2}\pi}^0 \Big|\int_{S^2_\varphi} i^*(\theta_1\wedge\theta_2) + i^*(\omega_{13}\wedge\omega_{23})\Big|\,\mathrm{d}\varphi\\ &\geq \int_{-\frac{1}{2}\pi}^0 4\pi\big|I_{\vec{v}}(S)\big|\,\mathrm{d}\varphi = 2\pi^2\big|I_{\vec{v}}(S)\big|.\end{aligned} \tag{18}$$

Thus, from (17) and (18) we have

$$\mathrm{vol}(\vec{v}) \geq 2\pi^2\big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big) = \big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big)\mathrm{vol}(\mathbb{S}^3).$$

3. Concluding remarks

These results should extend to higher dimensions if one makes use of some rather complicated inequalities involving the volume integrand in (1) of a unit vector field and some symmetric functions coming from the second fundamental form of the orthogonal distribution (which is generally non integrable). Some of these inequalities can be found in [6] or [5].

Index results should exist also for the case when the spheres are punctured differently. In other words, if we have two singularities which are not antipodal points of $\mathbb{S}^2$ or $\mathbb{S}^3$ or if we have more than two singularities, what could be said? We believe that some results relating indices and positions of the singularities to the volume of a unit vector field may be found.

For singular vector fields on $\mathbb{S}^2$ another natural situation is the one of unit vector fields defined on $\mathbb{S}^2 \setminus \{x\}$. In a recent paper [1], see also [12], a unit vector field $\vec{p}$ is defined on $\mathbb{S}^2 \setminus \{x\}$ by parallel translation of a given tangent vector at $-x$ along the minimizing geodesics to x. It has been proved in [1] that $\vec{p}$ minimizes the volume of unit vector fields defined on $\mathbb{S}^2 \setminus \{x\}$. By a direct calculation, we obtain the inequality $\mathrm{vol}(\vec{p}) > \mathrm{vol}(\vec{n})$, where $\vec{n}$ is the north-south vector field tangent to the longitudes of W.

Now, new questions arise about minimality on specific topological-geometrical configurations on the punctured spheres.

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