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THE WKB METHOD AND GEOMETRIC INSTABILITY FOR NONLINEAR SCHRÖDINGER EQUATIONS ON SURFACES

by Laurent Thomann

Abstract. — In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation on a Riemannian surface which has a stable geodesic. These approximate solutions will lead to some instability properties of the equation.

Résumé (*Méthode WKB et instabilité géométrique pour les équations de Schrödinger non linéaires sur des surfaces*)

À l'aide de la méthode WKB nous construisons des solutions approchées à l'équation de Schrödinger cubique sur une variété qui possède une géodésique stable. Cette construction permet d'obtenir des résultats d'instabilités dans des espaces de Sobolev.

1. Introduction

Let (M, g) be a Riemannian surface (i.e., a Riemannian manifold of dimension 2), orientable or not. We assume that M is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation

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$$
(1) \qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), \quad \varepsilon = \pm 1, \\ u(0,x) = u_0(x) \in H^\sigma(M), \end{cases}
$$

that is, given a small parameter $0 < h < 1$ and an integer N, functions $u_N(h)$ satisfying

$$
(2) \qquad i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h),
$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.
Here h is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.
These approximate solutions to (1) will lead to some instability properties in the following sense (where h^{-1} will play the role of n):

Definition 1.1. — We say that the Cauchy problem (1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1) so that

$$
\begin{aligned}
\|u_{1,n}(0)\|_{H^\sigma(M)},\ \|u_{2,n}(0)\|_{H^\sigma(M)} &\leq C, \\
\|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} &\longrightarrow 0, \\
\limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} &\geq \frac{1}{2}C,
\end{aligned}
$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well-posed, if we refer to the following definition:

Definition 1.2. — Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius R in H^σ. We say that the Cauchy problem (1) is uniformly well-posed in H^σ if the flow map

$$
u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M),
$$

is uniformly continuous for any t.

We now state our instability result:

Proposition 1.3. — *Let $0 < \sigma < \frac{1}{4}$, and assume that M has a stable and non degenerated periodic geodesic (see Assumptions 1 and 2), then the Cauchy problem (1) is not uniformly well-posed.*

This problem is motivated by the following results: Let (M, g) be a riemannian compact surface, then in [5], N. Burq, P. Gérard and N. Tzvetkov prove that (1) is uniformly well-posed in $H^\sigma(M)$ for $\sigma > \frac{1}{2}$. Whereas, in [4], they show that (1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1) of the form

$$
(3) \qquad u_n^\kappa(t,x) = \kappa e^{i\lambda_n^\kappa t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t,x)),
$$

where $0 < \kappa < 1$, $\psi_n = (x_1 + ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n \longrightarrow +\infty$ and where r_n is an error term which is small. To obtain instability, they consider $\kappa_n \longrightarrow \kappa$, then

$$\|u_n^{\kappa}(0) - u_n^{\kappa_n}(0)\|_{H^\sigma(\mathbb{S}^2)} \lesssim |\kappa - \kappa_n| \longrightarrow 0,$$

but

$$\|u_n^{\kappa}(t_n) - u_n^{\kappa_n}(t_n)\|_{H^\sigma(\mathbb{S}^2)} \gtrsim \kappa|\mathrm{e}^{i\lambda_n^{\kappa}t_n} - \mathrm{e}^{i\lambda_n^{\kappa_n}t_n}| \longrightarrow 2\kappa,$$

with a suitable choice of $t_n \longrightarrow 0$.
We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (3).
This result is sharp, because in [6] they show that (1) is uniformly well-posed on $\mathbb{S}^2$ when $\sigma > \frac{1}{4}$.
On the other hand, in [3] J. Bourgain shows that (1) is uniformly well-posed on the rational torus $\mathbb{T}^2$ when $\sigma > 0$.
These results show how the geometry of M can lead to instability for the equation (1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on M:

Assumption 1. — *The manifold M has a periodic geodesic.*

Denote by γ such a geodesic, then there exists a system of coordinates (s, r) near γ, say for $(s, r) \in \mathbb{S}^1 \times]-r_0, r_0[$, called Fermi coordinates such that (see [13], p. 80)

1. The curve $r = 0$ is the geodesic γ parametrized by arclength and
2. The curves $s =$ constant are geodesics parametrized by arclength. The curves $r =$ constant meet these curves perpendicularly.
3. In this system the metric writes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2(s, r) \end{pmatrix}.$$

We set the length of γ equal to 2π. Denote by $R(s, r)$ the Gauss curvature at (s, r), then a is the unique solution of

$$\begin{cases} \dfrac{\partial^2 a}{\partial r^2} + R(s, r)a = 0, \\ a(s, 0) = 1, \ \dfrac{\partial a}{\partial r}(s, 0) = 0. \end{cases} \tag{4}$$

The initial conditions traduce the fact that the curve $r = 0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\det g}}\mathrm{div}(\sqrt{\det g}\, g^{-1}\nabla) = \frac{1}{a}\partial_s(\frac{1}{a}\partial_s) + \frac{1}{a}\partial_r(a\partial_r).$$

A function on M, defined locally near γ, can be identified with a function of $[0, 2\pi]\times] - r_0, r_0[$ such that

$$\forall (s, r) \in [0, 2\pi]\times] - r_0, r_0[\quad f(s + 2\pi, r) = f(s, \omega r)$$

where $\omega = 1$ if M is orientable and $\omega = -1$ if M is not. Define

$$\omega_1 = \frac{1}{2}(\omega - 1) \in \{-1, 0\}. \tag{5}$$

From (4) we deduce that a admits the Taylor expansion

$$a = 1 - \frac{1}{2}R(s)r^2 + R_3(s)r^3 + \cdots + R_p(s)r^p + o(r^p), \tag{6}$$

with $R(s) = R(s, 0)$ and

$$R_k(s) = \frac{1}{k!}\frac{\partial^k a}{\partial r^k}(s, 0), \tag{7}$$

for $k \geq 3$.
As $a(s + 2\pi, r) = a(s, \omega r)$, we deduce $R(s + 2\pi) = R(s)$ and for all $j \geq 3$, $R_j(s + 2\pi) = \omega^j R_j(s)$.
Let $p_2 = \frac{1}{a^2}\sigma^2 + \rho^2$ be the principal symbol of Δ, and

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}s(t) = \frac{\partial p_2}{\partial \sigma} = \frac{2\sigma}{a^2}, \ \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = -\frac{\partial p_2}{\partial s} = -\partial_s(\frac{1}{a^2})\sigma^2, \\ \frac{\mathrm{d}}{\mathrm{d}t}r(t) = \frac{\partial p_2}{\partial \rho} = 2\rho, \ \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{\partial p_2}{\partial r} = -\partial_r(\frac{1}{a^2})\sigma^2, \\ s(0) = s_0, \ \sigma(0) = \sigma_0, r(0) = r_0, \ \rho(0) = \rho_0, \end{cases} \tag{8}$$

its associated hamiltonian system, where $p_2 = p_2(s(t), r(t), \sigma(t), \rho(t))$. The system (8) admits a unique solution and defines the hamiltonian flow

$$\Phi_t : (s_0, \sigma_0, r_0, \rho_0) \longmapsto (s(t), \sigma(t), r(t), \rho(t)).$$

The curve $\Gamma = \{(s(t) = t, \sigma(t) = 1/2, r(t) = 0, \rho(t) = 0), t \in [0, 2\pi]\}$ is solution of (8) and its projection in the (s, r) space is the curve γ. Now denote by ϕ the Poincaré map associated to the trajectory Γ and to the hyperplane $\Sigma = \{s = 0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma = 1/2, r = 0, \rho = 0)$ such that the following makes sense: solve the system (8) with the initial conditions $(0, \sigma_0, r_0, \rho_0) \in \{0\} \times \mathcal{N}$ and let T be such that $s(T) = 2\pi$, then ϕ is the application

$$\phi : (r_0, \rho_0) \longmapsto (r(T), \rho(T)).$$

Moreover, the Poincaré map is continuously differentiable (see [14] p. 193). To obtain its differential $\mathrm{d}\phi(0,0)$ at $(0,0)$, we linearize the system (8) about the orbit Γ, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = 2\sigma, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = 2\rho, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{1}{2}R(s(t))r, \end{cases} \tag{9}$$

then $\sigma = \frac{1}{2}$, $s(t) = t$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -R/2 & 0 \end{pmatrix}\begin{pmatrix} r \\ \rho \end{pmatrix}. \tag{10}$$

Hence the application $\mathrm{d}\phi(0,0)$ is

$$\mathrm{d}\phi(0,0) : (r_0, \rho_0) \longmapsto (r(2\pi), \rho(2\pi)), \tag{11}$$

where (r,ρ) solves (10). As $\mathrm{d}\phi(0,0)$ is symplectic, it admits two eigenvalues Λ and Λ^{-1} that are called the characteristic multipliers of the system (10). We add the following assumption on γ, which can be formulated in terms of the eigenvalues of $\mathrm{d}\phi(0,0)$:

Assumption 2. — *The geodesic γ is stable, i.e., $\mathrm{d}\phi(0,0)$ is a rotation. Then the multipliers take the form $\Lambda = e^{i\lambda}$ and $\Lambda^{-1} = e^{-i\lambda}$ with $\lambda \in \mathbb{R}$. We assume moreover that there exist $\tau, \mu > 0$ such that*

$$\forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad |p - q\frac{\lambda}{\pi}| \geq \frac{\mu}{|(p,q)|^{\tau}}, \tag{12}$$

where $|(p,q)| = |p| + |q|$. When this condition is fulfilled, we say that γ is non degenerated.

Remark 1.4. — Almost every $\lambda \in \mathbb{R}$ satisfies (12) with $\tau > 1$. This is an easy consequence of [1] p. 159, e.g.

Examples 1. — *Let M be a surface which has a periodic geodesic γ. In the general case, the eigenvalues of $\mathrm{d}\phi(0,0)$ defined by (11) are $\Lambda = \rho e^{i\lambda}$ and $\Lambda^{-1} = \rho^{-1}e^{-i\lambda}$, with $\Lambda + \Lambda^{-1} \in \mathbb{R}_+$, i.e.,*

$$(\rho - \rho^{-1})\sin\lambda = 0. \tag{13}$$

Assume that M is a surface of revolution and that $R > 0$ on γ. Then the characteristic multipliers are

$$\Lambda = \rho e^{2\pi i\sqrt{R}} \quad \textit{and} \quad \Lambda^{-1} = \rho^{-1}e^{-2\pi i\sqrt{R}}.$$

i) *If $\lambda = 2\pi\sqrt{R}$ satisfies (12) then $\rho = 1$ and M satisfies the assumptions.*

ii) *Let* $2\sqrt{R} \notin \mathbb{N}$. *Let* $\tilde{M}$ *be a perturbation of* M, *and denote by*

$$\tilde{\Lambda} = \tilde{\rho} e^{i\tilde{\lambda}} \quad \text{and} \quad \tilde{\Lambda}^{-1} = \tilde{\rho}^{-1} e^{-i\tilde{\lambda}},$$

the new characteristic multipliers.
By (13), $\tilde{\rho} = 1$, *and Assumption 2 is satisfied almost surely.*

iii) *Let* $a > 0$, *then the torus* $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/a\mathbb{Z}$ *is not under the hypotheses: in this case* $\mathrm{d}\phi(0,0)$ *is not diagonalizable.*

Notice that the function r which satisfies (10) is solution of

$$\ddot{y}(s) + R(s)y(s) = 0. \tag{14}$$

Consider a_0 the solution of (14) with initial conditions $a_0(0) = 1$ and $\dot{a}_0(0) = i$. Then, from the Floquet theory, there exists a 2π-periodic function P so that

$$a_0(s) = \mathrm{e}^{i\frac{\lambda}{2\pi}s} P(s)$$

(or $a_0(s) = \exp\left(-i\frac{\lambda}{2\pi}s\right)P(s)$, but λ can be replaced with $-\lambda$).
Here, and in all the paper we denote by $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}s}f$ if f is differentiable. This notation is motived by the fact that s will play the role of a time variable (see section 2).

In order to prove Proposition 1.3, we construct stationnary approximate solutions of (1), as stated in the following theorem

Theorem 1.5. — *Assume* 1 *and* 2. *Let* $h \in]0,1]$ *such that* $\frac{1}{h} \in \mathbb{N}$, *let* $\kappa, \sigma > 0$ *and* $k \in \mathbb{N}$. *Let* λ *be given by Assumption* 2 *and* ω_1 *by* (5).
Define $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$.
Then for all $N \in \mathbb{N}$, *there exist* $\lambda_N(k) \in \mathbb{R}$ *and a family* $u_N(h)$ *such that* $C_1 h^\sigma \le \|u_N(h)\|_{L^2(M)} \le C_2 h^\sigma$ *with* $C_1, C_2 > 0$ *independent of* N *and* h, *and*

$$-\Delta u_N(h) = \lambda_N(k) u_N(h) - \varepsilon |u_N(h)|^2 u_N(h) + h^N g_N(h) \tag{15}$$

with for all $N \in \mathbb{N}$

$$\|h^N g_N(h)\|_{H^n(M)} \lesssim h^{N-n}.$$

Moreover

$$\lambda_N(k) = \frac{1}{h^2} - \frac{2}{h}E_0(k) + \frac{1}{\sqrt{h}}\varepsilon\kappa^2 h^{2\sigma} C_0 + \mathcal{O}(1),$$

where $C_0 > 0$ *is independent of* ε, κ *and* σ.

Remark 1.6. — The analog of Theorem 1.5 was proved by J. Ralston in [15] for the linear case ($\varepsilon = 0$), with the same type of assumptions.

REMARK 1.7. — Consider the more general equations

$$i\partial_t u + \Delta u = F(u), \tag{16}$$

where $F : \mathbb{C} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function. The result of Theorem 1.5 is likely to hold with other nonlinearities $F(u)$, for example for $F(z) = z^3$, $F(z) = z^4$ or $F(z) = (1+|z|^2)^\alpha z$ with $\alpha < 1$. However, the instability phenomenon is strongly related to the gauge invariance of the equation (16).

The scheme of the paper is the following: Thanks to a scaling, we reduce the problem (15) to the resolution of linear Schrödinger equations with a harmonic time dependent potential, and we will see, using Assumption 2, that these equations have periodic solutions. To prove Proposition 1.3 we show that the family $u_N(h)$ provides good approximations of (1) in times where instability occurs.

NOTATIONS 1.8. — *In this paper c, C denote constants the value of which may change from line to line. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \leq a \leq Cb$, $a \leq Cb$ respectively. By $\delta_{i,j}$ we mean the Kronecker symbol, i.e., $\delta_{i,j} = 0$ for $i \neq j$ and $\delta_{i,i} = 1$.*

REMARK 1.9. — In the sequel we do not always mention the dependence on h of the functions: we will write u, f, r_i, ... instead of u_h, f_h, $r_{i,h}$, ...

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2. The WKB construction

Consider the equation

$$-\Delta u = \lambda u - \varepsilon |u|^2 u. \tag{17}$$

Given $h > 0$, we are looking for a solution of the form

$$u = \delta h^{-\frac{1}{4}} e^{i\frac{s}{h}} f(s, r, h), \tag{18}$$

where $\delta = \kappa h^\sigma$, with $\kappa > 0$ and $0 \leq \sigma \leq \frac{1}{4}$. In all this section, δ will play the role of a parameter.

We try to find a solution (u, λ) of (17) of the form

$$u \sim \sum_{j \geq 0} h^{j/2} u_j, \quad \lambda \sim h^{-2} \sum_{j \geq 0} h^{j/2} \lambda_j.$$

As we will see, identifying each power of h will lead to a linear equation which can be solved with a suitable choice of λ_j.

Choose h such that $h^{-1} \in \mathbb{N}$, this ensures that $\exp i\frac{s}{h}$ is $2\pi-$periodic. Such a condition on h is natural and is known as a Bohr-Sommerfeld quantification condition.

With the ansatz (18), equation (17) becomes

$$\begin{aligned}&-\frac{1}{a^2}\left(\frac{2i}{h}\partial_s f+\partial_s^2 f-\frac{1}{h^2}f\right)-\frac{1}{a}\partial_s\left(\frac{1}{a}\right)\left(\frac{i}{h}f+\partial_s f\right)\\ &\qquad -\partial_r^2 f-\frac{\partial_r a}{a}\partial_r f=\lambda f-\varepsilon\delta^2 h^{-\frac{1}{2}}|f|^2 f.\end{aligned}\tag{19}$$

We make the change of variables $x=\frac{r}{\sqrt{h}}$ and set $v(s,x,h)=f(s,\sqrt{h}x,h)$. Thus $\partial_r f=\frac{1}{\sqrt{h}}\partial_x v$ and $\partial_r^2 f=\frac{1}{h}\partial_x^2 v$.

Therefore we now have to find $v\sim\sum_{j\geq 0}h^{j/2}v_j$.

Using (6) we obtain the following Taylor expansions in h

$$\frac{1}{a^2}=1+hRx^2-2h^{\frac{3}{2}}R_3x^3+\mathcal{O}(h^2),$$

$$a^{-1}\partial_s(a^{-1})=\mathcal{O}(h)\quad\text{and}\quad a^{-1}\partial_r a=\mathcal{O}(h^{\frac{1}{2}}).$$

Equation (19) can therefore be written, after multiplication by $\frac{1}{2}h$

$$\begin{aligned}&i\partial_s v+\frac{1}{2}\partial_x^2 v-\frac{1}{2}Rx^2 v\\ &\qquad =\frac{1-\lambda h^2}{2h}v+h^{\frac{1}{2}}R_3x^3v+\frac{1}{2}\varepsilon\delta^2h^{\frac{1}{2}}|v|^2v+hPv,\end{aligned}\tag{20}$$

where

$$P=A_1\partial_s^2+A_2\partial_s+A_3\partial_x+A_4\tag{21}$$

is a second order differential operator with coefficients $A_j=A_j(s,x,h)$ satisfying $A_j(s+2\pi,x,h)=A_j(s,\omega x,h)$ for $0\leq j\leq 4$.

Denote by $E=\frac{1-\lambda h^2}{2h}=E_0+h^{\frac{1}{2}}E_1+\cdots+h^{\frac{p}{2}}E_p+o(h^{\frac{p}{2}})$ and write $v=v_0+h^{\frac{1}{2}}v_1+\cdots+h^{\frac{p}{2}}v_p+o(h^{\frac{p}{2}})$ and by identifying the powers of h we obtain the system of equations:

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_0=0,\tag{22}$$

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_1=E_1v_0+R_3x^3v_0+\frac{1}{2}\varepsilon\delta^2|v_0|^2v_0,\tag{23}$$

$$\cdots=\cdots$$

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_p=E_pv_0+Q_p.\tag{24}$$

$$\cdots=\cdots$$

so that the $(j+1)$th equation of unknown (v_j,E_j) corresponds to the annihilation of the coefficient of $h^{\frac{j}{2}}$ in (20).

Here Q_p is a function which only depends on x, s, $(v_j)_{j\leq p-1}$ and $(E_j)_{j\leq p-1}$.

Remark 2.1. — Notice that thanks to the scaling, we have reduced the problem (17) to the resolution of linear equations. However we have to solve them exactly; no smallness assumption on x is possible, as x can be of size $\sim \frac{1}{\sqrt{h}}$.

In this section we will show

Proposition 2.2. — *For all $p \in \mathbb{N}$, there exist $(E_0, \cdots, E_p) \in \mathbb{R}^{p+1}$ and $(v_0, \cdots, v_p) \in (\mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})))^{p+1}$ with $v_0 \neq 0$, which solve the system (22)-(24).*

This permits us to construct approximate solutions of (17); more precisely, we will obtain the following proposition, which is the main result of this section.

Proposition 2.3. — *Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be such that $0 \leq \chi \leq 1$, $\chi = 1$ on $[-r_0/2, r_0/2]$ and suppose moreover that χ is an even function. Let $\delta > 0$. Denote by*

$$u_p(s, r) = \delta h^{-\frac{1}{4}} \chi(r) e^{i\frac{s}{h}} (v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p)(s, \frac{r}{\sqrt{h}}) \tag{25}$$

and by

$$\lambda_p = \frac{1}{h^2} - \frac{2}{h}(E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p). \tag{26}$$

Then u_p satisfies $\|u_p\|_{L^2(M)} \sim \delta$ and

$$-\Delta u_p = \lambda_p u_p - \varepsilon |u_p|^2 u_p + h^{\frac{p-1}{2}} g_p(h) \tag{27}$$

with

$$\forall h \in]0, 1],\ \forall n \in \mathbb{N}, \quad \|h^{\frac{p-1}{2}} g_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \lesssim \delta h^{\frac{p-1}{2} - n}.$$

2.1. Preliminaries: the analysis of the linear equations. — We will solve the system (22)-(24) for $x \in \mathbb{R}$. Notice that the Fermi coordinates are only defined for $|r| \leq r_0$ i.e., for $x \leq \frac{r_0}{\sqrt{h}}$. That's the reason why we need the cutoff which appears in the Proposition 2.3.

We first give an expansion of the operator P defined by (21).

Lemma 2.4. — *Let*

$$P(s, x, h) = A_1(s, x, h)\partial_s^2 + A_2(s, x, h)\partial_s + A_3(s, x, h)\partial_x + A_4(s, x, h),$$

be the differential operator defined by (21). Then for all $p \geq 2$, P can be written

$$P(s, x, h) = \sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k(s, x) + h^{\frac{p}{2}} \tilde{P}_p(s, x, h), \tag{28}$$

so that
i) *For all* $0 \leq k \leq p-1$,

$$P_k(s,x) = A_1^k(s,x)\partial_s^2 + A_2^k(s,x)\partial_s + A_3^k(s,x)\partial_x + A_4^k(s,x),$$

where $A_j^k \in \mathcal{C}^\infty([0,2\pi]\times\mathbb{R})$, *for all* $s \in [0,2\pi]$ *the function* $x \mapsto A_j^k(s,x)$ *is a polynomial and* $A_j^k(s+2\pi,x) = A_j^k(s,\omega x)$.
ii) *Let* $\chi \in \mathcal{C}_0^\infty(]-r_0,r_0[)$ *and* $v \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$, *then for all* $n \in \mathbb{N}$, *there exists* $C = C(p,n)$ *independent of* $h \in]0,1]$ *so that*

$$\|\chi(h^{\frac{1}{2}}x)\tilde{P}_p v(s,x)\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C. \tag{29}$$

Proof. — We first compute the coefficients of P.
By the Taylor formula near $r=0$ we have

$$\frac{1}{a^2}(s,r) = 1 + R(s)r^2 - 2R_3(s)r^3 + \sum_{k=4}^{p+3} r^k R_k(s) + \frac{r^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\left(\frac{1}{a^2}\right)(s,tr)\,\mathrm{d}t,$$

where R_k is given by (7).
Now write $r = \sqrt{h}x$ and obtain

$$\frac{1}{a^2}(s,\sqrt{h}x) = 1 + hR(s)x^2 - 2h^{\frac{3}{2}}R_3(s)x^3 + h^2 I_1(s,x,h), \tag{30}$$

where

$$I_1(s,x,h) = \sum_{k=4}^{p+3} h^{\frac{k-4}{2}} x^k R_k(s) + h^{\frac{p}{2}}\frac{x^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\left(\frac{1}{a^2}\right)(s,\sqrt{h}xt)\,\mathrm{d}t. \tag{31}$$

Similarly

$$\frac{1}{a}\partial_s(\frac{1}{a})(s,\sqrt{h}x) = hI_2(s,x,h), \tag{32}$$

with

$$I_2(s,x,h) = \sum_{k=2}^{p+1} h^{\frac{k-2}{2}}\frac{x^k}{k!}\frac{1}{a}\partial_s(\frac{1}{a})(s,0) + h^{\frac{p}{2}}\frac{x^{p+2}}{(p+1)!}\int_0^1 (1-t)^{p+1}\frac{\partial^{p+2}}{\partial r^{p+2}}\left(\frac{1}{a}\partial_s(\frac{1}{a})\right)(s,\sqrt{h}xt)\,\mathrm{d}t, \tag{33}$$

and

$$\frac{\partial_r a}{a}(s, \sqrt{h}x) = h^{\frac{1}{2}} I_3(s, x, h), \tag{34}$$

where

$$I_3(s,x,h) = \sum_{k=1}^{p} h^{\frac{k-1}{2}} \frac{x^k}{k!} \frac{\partial^k}{\partial r^k}\left(\frac{\partial_r a}{a}\right)(s,0) + \\ + h^{\frac{p}{2}} \frac{x^{p+1}}{p!} \int_0^1 (1-t)^p \frac{\partial^{p+1}}{\partial r^{p+1}}\left(\frac{\partial_r a}{a}\right)(s, \sqrt{h}xt)\,\mathrm{d}t. \tag{35}$$

Plug the expressions (30), (32) and (34) in equation (20), and deduce that coefficients A_j are

$$\begin{aligned} A_1 &= \frac{1}{2}(-1 - hRx^2 + 2h^{\frac{3}{2}} R_3 x^3 - h^2 I_1), \\ A_2 &= -iRx^2 + 2ih^{\frac{1}{2}} R_3 x^3 - ihI_1 - \frac{1}{2} I_2, \\ A_3 &= -\frac{1}{2} I_3, \\ A_4 &= \frac{1}{2}(I_1 - iI_2). \end{aligned}$$

Then with the developments (31), (33) and (35), we see that for all $1 \le j \le 4$ and $0 \le k \le p-1$, $x \mapsto A_j^k(s,x)$ is a polynomial. Moreover as $a(s+2\pi, x) = a(s, \omega x)$, we also have $A_j^k(s+2\pi, x) = A_j^k(s, \omega x)$.

To obtain the bound (29), we now have to control the integral rests which appear in (31),(33) and (35).

Let $q \in \mathbb{N}^*$ and let $(s,r) \mapsto f(s,r)$ be one of the functions a^{-2}, $a^{-1}\partial_s(a^{-1})$ or $a^{-1}\partial_r$. Let $\chi \in \mathcal{C}^\infty(]-r_0, r_0[)$ and define F_q by

$$F_q(s,x) = \chi(\sqrt{h}x) \int_0^1 (1-t)^{q-1} \frac{\partial^q}{\partial r^q} f(s, \sqrt{h}xt)\,\mathrm{d}t.$$

As $f \in \mathcal{C}^\infty([0,2\pi] \times]-r_0, r_0[)$, we deduce that for all $n_1, n_2 \in \mathbb{N}$ there exists $C = C(q, n_1, n_2)$, independent of $h \in]0,1]$ so that

$$\forall (s,x) \in [0,2\pi] \times \mathbb{R}, \qquad |\partial_s^{n_1} \partial_x^{n_2} F_q(s,x)| \le C. \tag{36}$$

Now let $v \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ and $n \in \mathbb{N}$. We can assume that $n \ge 2$, so that H^n is an algebra. Then by (36)

$$\|x^q F_q v\|_{H^n([0,2\pi]\times\mathbb{R})} \le C \|F_q\|_{H^n([0,2\pi]\times\mathbb{R})} \|x^q v\|_{H^n([0,2\pi]\times\mathbb{R})} \le C, \tag{37}$$

and this yields ii). □

Consider the Hilbertian basis of $L^2(\mathbb{R})$ composed of the Hermite functions $(\varphi_k)_{k\geq 0}$ which are the eigenfunctions of the harmonic oscillator $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$, i.e., $H\varphi_k = (k+\frac{1}{2})\varphi_k$. Moreover $\varphi_k(x) = P_k(x)\mathrm{e}^{-x^2/2}$ where P_k is a polynomial of degree k with $P_k(-x) = (-1)^k P_k(x)$. The link between the s-dependent operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$ and H is given by the following result proved by M. Combescure in [11].

THEOREM 2.5. — *Let $a_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be the solution of (14) with $a_0(0) = 1$, $\dot{a}_0(0) = i$. Define*

$$\alpha = \log|a_0|,\ \beta = \frac{1}{2i}\log\frac{a_0}{\overline{a_0}},$$

let the unitary transform $T(s)$ be defined by

$$T(s) = e^{i\dot{\alpha}(s)x^2/2}e^{-i\alpha(s)D},\ \text{where } D = -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x),$$

and let $U(s,\tau)$ be the unitary evolution operator for $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$, i.e., $U(s,\tau)\varphi$ is the unique solution of the problem

$$\begin{cases} \left(i\partial_s + \dfrac{1}{2}\partial_x^2 - \dfrac{1}{2}R(s)x^2\right)u = 0, \\ u(\tau,x) = \varphi(x) \in L^2(\mathbb{R}). \end{cases}$$

Then we have for any $s,\tau \in \mathbb{R}$

$$U(s,\tau) = T(s)e^{-i(\beta(s)-(\beta(\tau))H}T(\tau)^{-1}.$$

REMARK 2.6. — The functions α and β are well defined: suppose that there exists s_0 such that $a_0(s_0) = 0$, then $\operatorname{Re} a_0$ and $\operatorname{Im} a_0$ are linearly dependent, which is impossible with this choice of the initial conditions.

REMARK 2.7. — Define $\theta(s) = \beta(s) - \frac{\lambda}{2\pi}s$ where λ is given by Assumption 2. Then α and θ are 2π-periodic real functions. Moreover $\alpha(0) = \dot{\alpha}(0) = \beta(0) = \theta(0) = 0$.

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space, i.e., the space of smooth functions which are fast decreasing and their derivatives too.

PROPOSITION 2.8. — *Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ and $E \in \mathbb{C}$. Let $f \in \mathcal{C}^\infty([0,2\pi]\times\mathbb{R},\mathbb{R})$ be such that*

$$\forall n \in \mathbb{N},\ \forall s \in [0,2\pi],\quad \partial_s^n f(s,\cdot) \in \mathcal{S}(\mathbb{R}),$$

in other words $f \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.
Let $\psi \in \mathcal{C}^1\big([0,2\pi],L^2(\mathbb{R})\big) \cap \mathcal{C}^0\big([0,2\pi],H^2(\mathbb{R})\big)$ be the solution of

$$
\begin{cases} i\partial_s\psi + \frac{1}{2}\partial_x^2\psi - \frac{1}{2}R(s)x^2\psi - E\psi = f, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{38}
$$

Then $\psi \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$.

Proof. — By replacing ψ with $\mathrm{e}^{iEt}\psi$, we can assume that $E=0$. The solution of equation (38) is given by

$$
\begin{aligned}
\psi(s,\cdot) &= U(s,0)\psi_0 - i\int_0^s U(s,\tau)f(\tau,\cdot)\,\mathrm{d}\tau \\
&= T(s)\mathrm{e}^{-i\beta(s)H}\left(\psi_0 - i\int_0^s e^{i\beta(\tau)H}T(\tau)^{-1}f(\tau,\cdot)\,\mathrm{d}\tau\right).
\end{aligned} \tag{39}
$$

As D is a transport operator, we have

$$
T, T^{-1} : \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right) \longrightarrow \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right),
$$

we only have to show that

$$
\mathrm{e}^{i\beta H} : \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right) \longrightarrow \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right).
$$

This follows from the fact that β is regular and $\mathrm{e}^{iH} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$. □

The description of U given in Theorem 2.5 yields the following representation of $U(s,0)\varphi_k$:

Proposition 2.9. — *For all* $k \in \mathbb{N}$ *and* $s, x \in \mathbb{R}$ *we have*

$$
U(s,0)\varphi_k(x) = e^{i\dot\alpha(s)x^2/2}e^{-i(\frac{1}{2}+k)\beta(s)}e^{-\frac{1}{2}\alpha(s)}\varphi_k\left(xe^{-\alpha(s)}\right). \tag{40}
$$

Proof. — According to Theorem 2.5, and as $H\varphi_k = (k+\frac{1}{2})\varphi_k$,

$$
U(s,0)\varphi_k = \mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(k+\frac{1}{2})\beta(s)}\mathrm{e}^{-i\alpha(s)D}\varphi_k.
$$

Denote by $f(s) = \mathrm{e}^{-i\alpha(s)D}\varphi_k$. Then f is solution of the transport equation

$$
\partial_s f = -\frac{1}{2}\dot\alpha(s)\left(x\partial_x f + \partial_x(xf)\right) = -\frac{1}{2}\dot\alpha(s)\left(f + 2x\partial_x f\right)
$$

with Cauchy data $f(0,x) = \varphi_k(x)$. Make the change of variables $\sigma = \alpha(s)$ and set $g(\sigma) = f(s)$. Therefore g satisfies $\partial_\sigma g = -\frac{1}{2}(g + 2x\partial_x g)$. The equation $x = \dot x$, $x(0) = x_0$ admits the solution $x(\tau) = x_0\mathrm{e}^\tau$ and the characteristics method gives $g(\tau, x(\tau)) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x_0) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x(\tau)\mathrm{e}^{-\tau})$, hence

$$
f(s) = \mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_k(x\mathrm{e}^{-\alpha(s)}).
$$

□

Corollary 2.10. — *Let $k \in \mathbb{N}$, define $\omega_1 = \frac{1}{2}(\omega - 1)$ and $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$. Then*

$$\begin{aligned} w_k &= e^{-isE_0(k)} U(s,0)\varphi_k \\ &= e^{-isE_0(k)} e^{i\dot{\alpha}(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k \left(x e^{-\alpha(s)}\right) \end{aligned} \tag{41}$$

is solution of the equation

$$\left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2 - E_0(k)\right) w_k(s,x) = 0.$$

Proof. — On the one hand, from Proposition 2.9 we deduce

$$\begin{aligned} w_k(s+2\pi, x) &= \mathrm{e}^{-2i\pi E_0(k)} \mathrm{e}^{-i\lambda(\frac{1}{2}+k)} w_k(s,x) = \mathrm{e}^{-ik\omega_1\pi} w_k(s,x) \\ &= (-1)^{k\omega_1} w_k(s,x) = w_k(s,\omega x). \end{aligned}$$

On the other hand, w_k satisfies (22) because of the definition of $U(s,0)$. □

Fix $k_0 \in \mathbb{N}$ and take $v_0 = w_{k_0}$ with the previous choice of $E_0(k_0)$. This choice corresponds to the k_0th level of energy for the harmonic oscillator.

Remark 2.11. — Until now we did not use the restriction (12), but it will be crucial in the following.

Proposition 2.12. — *For all $p \geq 0$, there exist $E_p \in \mathbb{C}$ and $v_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which solve (24).*

Remark 2.13. — As stated in Theorem 1.5, the E_j's are in fact real numbers. This will be proved in Lemma 2.17.

Proof. — We proceed by induction on $p \in \mathbb{N}$.
For $p = 0$ the result was proved in Corollary 2.10.
Let $p \geq 1$, and suppose that for all $j \leq p-1$ there exist $E_j \in \mathbb{C}$ and $v_j \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which solve the $(j+1)$th equation of (22). When $p \geq 2$, set

$$\begin{aligned} \tilde{v}_{p-1} &= h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p-1}{2}} v_{p-1}, \\ \tilde{E}_{p-1} &= h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p-1}{2}} E_{p-1} \end{aligned}$$

and $\tilde{v}_0 = \tilde{E}_0 = 0$. By (28), the function Q_p given by (24) is the coefficient of $h^{\frac{p}{2}}$ in the expansion in h of

$$\tilde{E}_{p-1}\tilde{v}_{p-1} + \frac{1}{2}\varepsilon\delta^2 |v_0 + \tilde{v}_{p-1}|^2 (v_0 + \tilde{v}_{p-1}) + h\left(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k\right)(v_0 + \tilde{v}_{p-1}).$$

Now using the regularity of the v_j's and the fact that for all $0 \leq k \leq p-1$, P_k is an operator

$$P_k : \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})),$$

we obtain $Q_p \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.
Moreover Q_p satisfies, $\forall (s,x) \in [0,2\pi]\times\mathbb{R}$

$$Q_p(s+2\pi,x) = Q_p(s,\omega x)$$

because this property holds for the v_j's, and a.
Define $F_p(s,x) = \mathrm{e}^{-i\dot\alpha(s)\mathrm{e}^{2\alpha(s)}x^2/2}Q_p(s,xe^{\alpha(s)})$, then $F_p \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$ and satisfies $Q_p(s,x) = \mathrm{e}^{i\dot\alpha(s)x^2/2}F_p(s,x\mathrm{e}^{-\alpha(s)})$ and $F_p(s+2\pi,x) = F_p(s,\omega x)$.
Let us decompose F_p on the basis $(\varphi_j)_{j\geq 0}$: there exists a unique family of smooth functions $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ so that

$$F_p(s,y) = \sum_{j\geq 0} g_j^p(s)\varphi_j(y). \tag{42}$$

Then

$$Q_p(s,x) = \sum_{j\geq 0} g_j^p(s)\mathrm{e}^{i\dot\alpha(s)x^2/2}\varphi_j(x\mathrm{e}^{-\alpha(s)}) = \sum_{j\geq 0} h_j^p(s)w_j(s,x), \tag{43}$$

where according to (41)

$$h_j^p(s) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac12+j)\beta(s)}\mathrm{e}^{\frac12\alpha(s)}g_j^p(s). \tag{44}$$

We have

$$Q_p(s,\omega x) = \sum_{j\geq 0} h_j^p(s)w_j(s,\omega x),$$

but also

$$\begin{aligned} Q_p(s,\omega x) = Q_p(s+2\pi,x) &= \sum_{j\geq 0} h_j^p(s+2\pi)w_j(s+2\pi,x) \\ &= \sum_{j\geq 0} h_j^p(s+2\pi)w_j(s,\omega x), \end{aligned}$$

and from the uniqueness of the h_j^p's we deduce $h_j^p(s+2\pi) = h_j^p(s)$.
We are now looking for a solution of (24) of the form

$$v_p(s,x) = \sum_{j\geq 0} e_j^p(s)w_j(s,x) \tag{45}$$

where the e_j^p's are 2π-periodic functions. For all $j \geq 0$, by Corollary 2.10 we have

$$\left(i\partial_s + \frac12\partial_x^2 - \frac12 Rx^2\right)\left(e_j^p w_j\right) = i\dot e_j^p w_j + \left(E_0(k_0) - E_0(j)\right)e_j^p w_j,$$

hence we have to solve the equations

$$i\dot e_j^p + \left(E_0(k_0) - E_0(j)\right)e_j^p = h_j^p + \delta_{j,k_0}E_p. \tag{46}$$

As $E_0(k_0) - E_0(j) = \frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})$, the solutions of (46) take the form

$$e_j^p(s) = \mathrm{e}^{\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\left(C_j^p - i\int_0^s h_j^p(\tau)\mathrm{e}^{-\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}\,\mathrm{d}\tau\right) \tag{47}$$

for $j \neq k_0$, and

$$e_{k_0}^p(s) = C_{k_0}^p - i\int_0^s h_{k_0}^p(\tau)\,\mathrm{d}\tau - iE_p s.$$

The constants $C_j^p \in \mathbb{C}$ and $E_p \in \mathbb{C}$ have to be determined such that $e_j^p(s+2\pi) = e_j^p(s)$.

• Case $j = k_0$:

$$e_{k_0}^p(s+2\pi) = -i\int_0^{2\pi} h_{k_0}^p(\tau)\,\mathrm{d}\tau - 2\pi i E_p + e_{k_0}^p(s),$$

thus $e_{k_0}^p$ is 2π-periodic iff

$$E_p = -\frac{1}{2\pi}\int_0^{2\pi} h_{k_0}^p(\tau)\,\mathrm{d}\tau. \tag{48}$$

• Case $j \neq k_0$:

Denote by $\tilde{h}_j^p : \tau \longmapsto h_j^p(\tau)\mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}$ and by $K = \mathrm{e}^{i(k_0-j)(\pi\omega_1-\lambda)}$. Then

$$\begin{aligned}\int_0^{s+2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau &= \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau + \int_{2\pi}^{s+2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau \\ &= \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau + K^{-1}\int_0^{s} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau,\end{aligned}$$

and by (47)

$$\begin{aligned}e_j^p(s+2\pi) &= K\mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\left(C_j^p - i\int_0^{s+2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau\right) \\ &= \mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\left(KC_j^p - iK\int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau - i\int_0^{s} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau\right).\end{aligned} \tag{49}$$

Notice that $K \neq 1$, as $\lambda \notin \pi\mathbb{Q}$ and choose

$$C_j^p = \frac{iK}{K-1} \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau,$$

then according to (47)and (49), the function e_j^p is $2\pi-$periodic.

Now, we show that the constants C_j^p are uniformly bounded in $j \geq 0$, so that the function v_p given by (45) is well defined. We first need the

LEMMA 2.14. — *Let $(h_j^p)_{j\geq 0} \in l^2(\mathbb{N})$ be the family of $2\pi-$periodic functions defined by (44) and $h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$ its Fourier decomposition. Then for all $n_1, n_2 \in \mathbb{N}$ there exists $C^p > 0$ such that for all $j \in \mathbb{N}$*

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p.$$

Proof. — Consider the function $F_p \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$ which defines the family $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ with (42). Denote by $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$. Let $n_1, n_2 \in \mathbb{N}$ and decompose the function $\partial_s^{n_2} H^{n_1} F_p$ on the basis $(\varphi_j)_{j\geq 0}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0} \tilde{g}_j^p(s)\varphi_j(y)$$

where $(\tilde{g}_j^p)_{j\geq 0}$ is a smooth family of functions in $l^2(\mathbb{N})$.
Using that $H\varphi_j = (j+\frac{1}{2})\varphi_j$ and that $F_p \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$, we have for all $n_1, n_2 \in \mathbb{N}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0} (j+\frac{1}{2})^{n_1} (g_j^p)^{(n_2)}(s)\varphi_j(y).$$

By uniqueness of such a decomposition,

$$\left((j+\frac{1}{2})^{n_1}(g_j^p)^{(n_2)}\right)_{j\geq 0} = (\tilde{g}_j^p)_{j\geq 0} \in l^2(\mathbb{N}).$$

Then by the definition (44) of h_j^p, an easy induction on $n_1, n_2 \in \mathbb{N}$ shows that $\left(j^{n_1}(h_j^p)^{(n_2)}\right)_{j\geq 0} \in l^2(\mathbb{N})$. Write the Fourier decomposition of h_j^p

$$h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$$

and by Parseval

$$\sum_{j\geq 0}\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}(s) |c_{l,j}^p|^2 = \sum_{j\geq 0} j^{2n_1} \int_0^{2\pi} |(h_j^p)^{(n_2)}(s)|^2\,\mathrm{d}s \leq C^p.$$

In particular, for all $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c^p_{l,j}|^2 \leq C^p,$$

hence the result. $\square$

End of the proof of Proposition 2.12: Using the Fourier decomposition of h_j we obtain

$$\begin{aligned} C^p_j &= \frac{iK}{K-1}\int_0^{2\pi} \tilde{h}^p_j(\tau)\,\mathrm{d}\tau \\ &= \frac{iK}{K-1}\sum_{l\in\mathbb{Z}} c^p_{l,j}\int_0^{2\pi} \mathrm{e}^{i(l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi}))\tau} \\ &= -i\sum_{l\in\mathbb{Z}} \frac{c^p_{l,j}}{l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})}. \end{aligned} \tag{50}$$

With Assumption 2 we have

$$\begin{aligned} \big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| &= \frac{1}{2}|(2l-(k_0-j)\omega_1)+(k_0-j)\frac{\lambda}{\pi}| \\ &\geq \frac{1}{2}\frac{\mu}{|(2l-(k_0-j)\omega_1, k_0-j)|^\tau}, \end{aligned}$$

and for $j\geq k_0$, $|2l-(k_0-j)\omega_1|+|k_0-j|\leq 2(|l|+|j|)$, then

$$\big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| \geq \frac{c\mu}{(|l|+|j|)^\tau}. \tag{51}$$

Hence, from (50) and (51) we deduce

$$|C^p_j| \lesssim \sum_{l\in\mathbb{Z}} |c^p_{l,j}|(|j|+|l|)^\tau \lesssim \sum_{l\in\mathbb{Z}} |c^p_{l,j}|(|j|^\tau+|l|^\tau). \tag{52}$$

By Cauchy-Schwarz and Lemma 2.14, from (52) we obtain

$$\begin{aligned} |C^p_j| &\lesssim \sum_{l\in\mathbb{Z}} \frac{1+|l|}{1+|l|}|c^p_{l,j}|(|j|^\tau+|l|^\tau) \\ &\lesssim \Big(\sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^2}\Big)^{\frac{1}{2}} \Big(\sum_{l\in\mathbb{Z}} |c^p_{l,j}|^2(1+|l|)^2(|j|^{2\tau}+|l|^{2\tau})\Big)^{\frac{1}{2}} \\ &\leq C^p. \end{aligned} \tag{53}$$

Set

$$v_p(s,x) = \sum_{j\geq 0} e^p_j(s) w_j(s,x).$$

For all $j \in \mathbb{N}$, $s \longmapsto e_j^p(s)w_j(s,x)$ is continuous and there exists $c > 0$ such that for all $j > k_0$, and for all $s \in [0, 2\pi]$

$$|e_j^p(s)w_j(s,x)| \lesssim |g_j^p(s)||\varphi_j(cx)|$$

and this shows that $v_p \in C\left([0, 2\pi], L^2(\mathbb{R})\right)$. Now using Proposition 2.8 we conclude, by uniqueness of such a solution, that $v_p \in \mathcal{C}^\infty\left([0, 2\pi], \mathcal{S}(\mathbb{R})\right)$. □

2.2. The nonlinear analysis and proof of Proposition 2.3

Lemma 2.15. — *The constant E_1 given by Proposition 2.12 writes $E_1 = -\varepsilon\delta^2 C_0$ where $C_0 > 0$ is independent of ε and δ.*

Proof. — Consider the equation

$$\left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\right)v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0,$$

with

$$v_0(s,x) = \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right).$$

By the definition of Q_p (see (24)),

$$Q_1(s,x) = R_3(s)x^3 v_0(s,x) + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0(s,x),$$

and by (43), Q_1 can be written

$$Q_1(s,x) = \sum_{j\geq 0} h_j^1(s)w_j(s,x).$$

According to formula (48), we only have to compute the term $h_{k_0}^1$ in the previous expansion.
Write the expansion of $|\varphi_{k_0}|^2\varphi_{k_0}$ on the basis $(\varphi_j)_{j\geq 0}$:

$$|\varphi_{k_0}|^2\varphi_{k_0} = \sum_{j\geq 0} p_j\varphi_j, \tag{54}$$

with $p_j \in \mathbb{R}$ and $p_j = 0$ for $j - k_0 = 1$ *mod* 2 as $\varphi_k(-x) = (-1)^k\varphi_k(x)$.
Then by (54) and the expression (41) of w_j

$$\begin{aligned}|v_0|^2 v_0(s,x) &= \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}|\varphi_{k_0}|^2\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right)\\ &= \sum_{j\geq 0} p_j \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}\varphi_j\left(x\mathrm{e}^{-\alpha(s)}\right)\\ &= \sum_{j\geq 0} f_j(s)w_j(s,x)\end{aligned}$$

where

$$\begin{aligned} f_j(s) &= p_j \mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{-\alpha(s)} \\ &= p_j \mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{-\alpha(s)}. \end{aligned}$$

Therefore $f_{k_0}(s) = p_{k_0}\mathrm{e}^{-\alpha(s)}$ with, using (54), $p_{k_0} = \int_{\mathbb{R}} |\phi_{k_0}|^4 > 0$. In the same manner we write

$$x^3\varphi_{k_0}(x) = \sum_{j\geq 0} q_j\varphi_j(x), \tag{55}$$

with $q_j = 0$ when $j - k_0 = 0$ *mod* 2 and by (55) we have

$$\begin{aligned} &R_3(s)x^3v_0(s,x) \\ &\quad = R_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{\frac{5}{2}\alpha(s)}(x\mathrm{e}^{-\alpha(s)})^3\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right) \\ &\quad = \sum_{j\geq 0} q_jR_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{\frac{5}{2}\alpha(s)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_j\left(x\mathrm{e}^{-\alpha(s)}\right). \end{aligned}$$

By (41) we have

$$\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac{1}{2}+j)\beta(s)}\mathrm{e}^{\frac{1}{2}\alpha(s)}.$$

Then

$$R_3(s)x^3v_0(s,x) = \sum_{j\geq 0}\tilde{f}_j(s)w_j(s,x),$$

where

$$\begin{aligned} \tilde{f}_j(s) &= q_jR_3(s)\mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{3\alpha(s)} \\ &= q_jR_3(s)\mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{3\alpha(s)}. \end{aligned}$$

Then $\tilde{f}_{k_0} = 0$ as $q_j = 0$ when $j - k_0 = 0$ *mod* 2. Thus

$$h^1_{k_0}(s) = \frac{1}{2}\varepsilon\delta^2 f_{k_0}(s) = \frac{1}{2}\varepsilon\delta^2 p_{k_0}\mathrm{e}^{-\alpha(s)}.$$

Finally, from (48) we deduce

$$E_1 = -\frac{1}{4\pi}\varepsilon\delta^2 p_{k_0}\int_0^{2\pi}\mathrm{e}^{-\alpha(\tau)}\,\mathrm{d}\tau = -\varepsilon\delta^2 C_0,$$

where $C_0 > 0$ as $p_{k_0} > 0$. □

Lemma 2.16. — *Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi = 0$ near 0, and let $f \in \mathcal{S}(\mathbb{R})$. Then for all $n, N \in \mathbb{N}$, there exists $C = C(n,N)$ so that*

$$\|\psi(h^{\frac{1}{2}}\cdot)f\|_{H^n(\mathbb{R})} \leq Ch^N. \tag{56}$$

Proof. — We only show (56) for $n = 0$, the general case follows from the Leibniz rule. We can assume that $\operatorname{supp}\psi \subset [a,b]$ with $a > 0$. Then as $f \in \mathcal{S}(\mathbb{R})$, for all $N \in \mathbb{N}$, there exists $C_N > 0$ so that

$$|f(x)| \leq C_N \frac{1}{1+|x|^N}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} |\psi(h^{\frac{1}{2}}x)|^2 |f(x)|^2 \,\mathrm{d}x &= h^{-\frac{1}{2}} \int_a^b |\psi(x)|^2 |f(h^{-\frac{1}{2}}x)|^2 \,\mathrm{d}x \\ &\leq C_N h^{N-\frac{1}{2}} \int_a^b |\psi(x)|^2 \frac{1}{h^N + x^{2N}} \,\mathrm{d}x \\ &\leq C_N h^{N-\frac{1}{2}},\end{aligned}$$

hence the result. □

Proof of Proposition 2.3. — Let $p \geq 1$, and consider

$$V_p(s,x) = \left(v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p\right)(s,x),$$

and

$$\tilde{E}_p = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p,$$

where the v_j's and the E_j's are given by Proposition 2.12.
Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be an even function such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $[-r_0/2, r_0/2]$.
We claim that there exists $G_p(h) \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$, so that

$$\forall n \in \mathbb{N}, \quad \|G_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C_{n,p}, \tag{57}$$

where $C_{n,p}$ is independent of $h \in]0,1]$, and such that $G_p(h)$ satisfies

$$\begin{aligned}&\chi(h^{\frac{1}{2}}x)\Big((i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - \tilde{E}_p)V_p \\ &\qquad -h^{\frac{1}{2}} R_3 x^3 V_p - \frac{1}{2}\varepsilon\delta^2 h^{\frac{1}{2}} |V_p|^2 V_p - hPV_p\Big) = h^{\frac{p+1}{2}} G_p(h).\end{aligned} \tag{58}$$

By construction of the v_j's and the E_j's, in the l.h.s. of (58), the coefficient of h^j cancels for $0 \leq j \leq p$.
Then write the expansion in powers of h

$$\frac{1}{2}\varepsilon\delta^2 |V_p|^2 V_p = \sum_{k=0}^{3p+1} h^{\frac{k}{2}} V_p^k,$$

and use (28) to obtain

$$hPV_p = h\left(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k + h^{\frac{p}{2}} \tilde{P}_p\right)\left(\sum_{k=0}^{p} h^{\frac{k}{2}} v_k\right) := \sum_{k=0}^{2p+2} h^{\frac{k}{2}} W_p^k$$

We therefore obtain the explicit formula of $G_p(h)$

$$\begin{aligned}h^{\frac{p+1}{2}}G_p(h) &:= -\chi(h^{\frac12}x)\sum_{k=p+1}^{2p+2}h^{\frac{k}{2}}W_p^k-\chi(h^{\frac12}x)\sum_{k=p+1}^{3p+1}h^{\frac{k}{2}}V_p^k-\chi(h^{\frac12}x)h^{\frac{p+1}{2}}R_3x^3v_p\\ &= -h^{\frac{p+1}{2}}\chi(h^{\frac12}x)\left(\sum_{l=0}^{p+1}h^{\frac{l}{2}}W_p^{l+p+1}\sum_{l=0}^{2p}h^{\frac{l}{2}}V_p^{l+p+1}+R_3x^3v_p\right).\end{aligned}$$

The bound (57) then follows from an application of Lemma 2.4.

Denote by $\tilde V_p=\chi(h^{\frac12}x)V_p$, and write

$$\begin{aligned}P\tilde V_p &= (A_1\partial_s^2+A_2\partial_s+A_3\partial_x+A_4)(\chi(h^{\frac12}x)V_p)\\ &= \chi(h^{\frac12}x)PV_p+h^{\frac12}\chi'(h^{\frac12}x)A_3V_p.\end{aligned}$$

By (58) we deduce that

$$\begin{aligned}&(i\partial_s+\frac12\partial_x^2-\frac12Rx^2-\tilde E_p)\tilde V_p-h^{\frac12}R_3x^3\tilde V_p-\frac12\varepsilon\delta^2h^{\frac12}|\tilde V_p|^2\tilde V_p-hP\tilde V_p\\ &= h^{\frac{p+1}{2}}G_h^p+h^{\frac12}\chi'(h^{\frac12}x)\partial_xV_p+\frac12h\chi''(h^{\frac12}x)V_p\\ &\quad+\frac12\varepsilon\delta^2h^{\frac12}\chi(1-\chi^2)(h^{\frac12}x)|V_p|^2V_p-h^{\frac32}\chi'(h^{\frac12}x)A_3V_p\\ &:= h^{\frac{p+1}{2}}\tilde G_p(h).\end{aligned}$$

Each of the functions χ', χ'' and $\chi(1-\chi^2)$ vanishes near 0, hence by Lemma 2.16 and (57)

$$\forall n\in\mathbb{N},\quad \|\tilde G_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})}\le C_{n,p}. \tag{59}$$

Finally, set

$$u_p=\delta h^{-\frac14}\mathrm{e}^{i\frac{s}{h}}V_p(s,\frac{r}{\sqrt h}),$$

then

$$-\Delta u_p-\lambda_pu_p+\varepsilon|u_p|^2u_p=\frac{2}{h}\mathrm{e}^{i\frac{s}{h}}h^{\frac{p+1}{2}}\tilde G_p(h),$$

and $g_p(h)=2\mathrm{e}^{i\frac{s}{h}}\tilde G_p(h)$ satisfies the conclusion of Proposition 2.3 by (59).

□

LEMMA 2.17. — *Let $p\ge1$ and E_p given by Proposition (2.12). Then $E_p\in\mathbb{R}$.*

Proof. — We already know that $E_0,E_1\in\mathbb{R}$. Let $p\ge3$. Multiply (27) by $\overline{u}_p$, integrate on M and take the imaginary part

$$0=\|u_p\|_{L^2}^2\operatorname{Im}\lambda_p+h^{\frac{p-1}{2}}\operatorname{Im}\int g_p(h)\overline{u}_p.$$

As $\|u_p\|_{L^2} \sim 1$ and $\|g_p\|_{L^2} \lesssim 1$, we obtain the estimate

$$|\mathrm{Im}\,\lambda_p| \lesssim h^{\frac{p-1}{2}} \|g_p\|_{L^2} \|u_p\|_{L^2} \lesssim h^{\frac{p-1}{2}}$$

and as

$$\mathrm{Im}\,\lambda_p = -2(\mathrm{Im}\,E_2 + h^{\frac{1}{2}} \mathrm{Im}\,E_3 + \cdots + h^{\frac{p-1}{2}} \mathrm{Im}\,E_p)$$

it follows that for all $0 \leq j \leq p-1$, $\mathrm{Im}\,E_j = 0$, i.e., $E_j \in \mathbb{R}$. □

3. The instability for the nonlinear Schrödinger equation

3.1. The error estimate

PROPOSITION 3.1. — *Let $\alpha > 0$, $\sigma \in]0, \frac{1}{4}]$ and let $v \in H^2(M)$ be such that*

$$\|v\|_{L^2} \lesssim 1, \quad \|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}, \quad \|\Delta v\|_{L^\infty} \lesssim h^{-\frac{9}{4}+\sigma},$$

and suppose that v satisfies

$$i\partial_t v + \Delta v = \varepsilon |v|^2 v + h^\alpha R(h),$$

with for all $\beta \in [0,2]$, $\|R(h)\|_{H^\beta} \lesssim h^{-\beta}$. Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$

Then, if $\alpha > \frac{1}{4} + 3\sigma$ we have

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof. — Define $w = u - v$ and

$$E(t) = \|w\|_{L^2}^2 + \|h^2 \Delta w\|_{L^2}^2.$$

We have $E(0) = 0$ and the following estimates:

$$\|w\|_{L^2} \leq E^{\frac{1}{2}}, \quad \|\Delta w\|_{L^2} \leq h^{-2} E^{\frac{1}{2}}, \quad \|\nabla w\|_{L^2} \leq h^{-1} E^{\frac{1}{2}}. \tag{60}$$

The function w satisfies the equation

$$i\partial_t w + \Delta w = \varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h). \tag{61}$$

The energy method gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= \mathrm{Im} \int \overline{w}\left(\varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h)\right) \\ &\lesssim h^\alpha \|w\|_{L^2} + \|w\|_{L^4}^4 + \|w\|_{L^2}^2 \|v\|_{L^\infty}^2. \end{aligned}$$

The Gagliardo-Nirenberg inequality gives

$$\|w\|_{L^4}^4 \lesssim \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \lesssim h^{-2} E^2,$$

and as $\|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2. \tag{62}$$

Now, apply Δ to (61)

$$i\partial_t \Delta w + \Delta^2 w = \varepsilon \Delta A - h^\alpha \Delta R(h), \tag{63}$$

with

$$\begin{aligned} A &= |w+v|^2(w+v) - |v|^2 v \\ &= 2w|v|^2 + \overline{w}v^2 + w^2\overline{v} + 2|w|^2 v + |w|^2 w, \end{aligned}$$

then

$$\begin{aligned} |\Delta A| \lesssim{} & |v|^2|\Delta w| + |v||\nabla v||\nabla w| + |\nabla v|^2|w| + |v||\Delta v||w| \\ & + |\Delta v||w|^2 + |w|^2|\Delta w| + |w||\nabla w|^2, \end{aligned}$$

hence

$$\begin{aligned} \|\Delta A\|_{L^2} \lesssim{} & \|v\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}^2\|w\|_{L^2} \\ & + \|v\|_{L^\infty}\|\Delta v\|_{L^\infty}\|w\|_{L^2} + \|\Delta v\|_{L^\infty}\|w\|_{L^4}^2 \\ & + \|w\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^4}^2. \end{aligned} \tag{64}$$

The following inequality holds in dimension 2

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}} \lesssim h^{-1}E^{\frac{1}{2}},$$

and with (60) and (64) we deduce

$$\|\Delta A\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-\frac{13}{4}+\sigma}E + h^{-4}E^{\frac{3}{2}}.$$

But

$$h^{-\frac{13}{4}+\sigma}E = h^{-\frac{5}{4}+\sigma}E^{\frac{1}{4}}h^{-2}E^{\frac{3}{4}} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}},$$

and we obtain

$$\|\Delta(A)\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}. \tag{65}$$

Now, using (65) and $\|\Delta(R(h))\|_{L^2} \lesssim h^{-2}$, the energy method and the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|_{L^2}^2 &= \operatorname{Im}\int \Delta\overline{w}\,(\Delta A - h^\alpha \Delta R(h)) \\ &\lesssim h^{-2}E^{\frac{1}{2}}(h^{\alpha-2} + h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}), \end{aligned} \tag{66}$$

therefore from (62) and (66) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2.$$

Interpolation gives

$$\|w\|_{H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{\dot{H}^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{L^2}^{1-\frac{\sigma}{2}} \|\Delta w\|_{L^2}^{\frac{\sigma}{2}} \lesssim h^{-\sigma} E^{\frac{1}{2}} := F.$$

The function F satisfies $F(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F + h^{-2+2\sigma} F^3. \tag{67}$$

As long as $h^{-2+2\sigma} F^3 \lesssim h^{-\frac{1}{2}+2\sigma} F$, i.e., $F \lesssim h^{\frac{3}{4}}$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F,$$

and the Gronwall inequality yields

$$F \lesssim h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma} t}.$$

The nonlinear term in (67) can be removed with the continuity argument for times such that

$$h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma} t} \lesssim h^{\frac{3}{4}+\eta},$$

with $\eta > 0$ i.e., for $t \lesssim (\alpha - \frac{1}{4} - 3\sigma - \eta) h^{\frac{1}{2}-2\sigma} \log \frac{1}{h}$, which is possible with η small enough as we assume $\alpha > \frac{1}{4} + 3\sigma$. □

Corollary 3.2. — *Let $\kappa > 0$, $0 < \sigma < \frac{1}{4}$ and set $\delta = \kappa h^\sigma$. Denote by $v = e^{-i\lambda_3 t} u_3$ where u_3 and λ_3 are defined by (25) and (26) respectively.*
Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u, \\ u(0, x) = v(0, x). \end{cases}$$

Then $\|v\|_{H^\sigma} \sim 1$ and

$$\|(u - v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof. — The result directly follows from Propositions 2.3 and 3.1, as for all $0 < \sigma < \frac{1}{4}$, we have $\sigma + 1 > \frac{1}{4} + 3\sigma$. □

3.2. The instability argument. — Let $\kappa, \kappa_h > 0$ and consider $v = v^1$ defined in Corollary 3.2 associated with κ and v^2 associated with κ_h. Let u be a solution of

$$\begin{cases} i\partial_t u^j + \Delta u^j = \varepsilon |u^j|^2 u^j, \\ u^j(0,x) = v^j(0,x), \end{cases}$$

and $t_h \sim h^{\frac{1}{2}-2\sigma} \log \frac{1}{h}$. Then

$$\|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \|(v^2 - v^1)(t_h)\|_{H^\sigma} - \|(u^2 - v^2)(t_h)\|_{H^\sigma} - \|(u^1 - v^1)(t_h)\|_{H^\sigma}. \tag{68}$$

From Corollary 3.2 we deduce that for $j = 1, 2$

$$\|(u^j - v^j)(t_h)\|_{H^\sigma} \longrightarrow 0. \tag{69}$$

Observe that

$$\|(v^2 - v^1)(t_h)\|_{H^\sigma} \sim \left| \mathrm{e}^{-i\lambda_3^2 t_h} - \mathrm{e}^{-i\lambda_3^1 t_h} \right| = \left| \mathrm{e}^{i(\lambda_3^2 - \lambda_3^1)t_h} - 1 \right|,$$

from Lemma 2.15 we have

$$(\lambda_3^2 - \lambda_3^1)t_h \sim h^{2\sigma-1}(\kappa - \kappa_h)t_h \sim (\kappa - \kappa_h) \log \frac{1}{h}.$$

It is possible to choose κ_h such that $\kappa_h \longrightarrow \kappa$ and $(\kappa - \kappa_h) \log \frac{1}{h} \longrightarrow \infty$. Then using (68) and (69)

$$\limsup_{h \longrightarrow 0} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \limsup_{h \longrightarrow 0} \|(v^2 - v^1)(t_h)\|_{H^\sigma} \geq 2,$$

even though

$$\|(u^2 - u^1)(0)\|_{H^\sigma} = \|(v^2 - v^1)(0)\|_{H^\sigma} \sim |\kappa - \kappa_h|,$$

which tends to 0 with h. According to Definition 1.1, we have proved Proposition 1.3.

BIBLIOGRAPHY

[1] S. Alinhac & P. Gérard – *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels, InterÉditions, Paris, 1991.

[2] V. Banica – "On the nonlinear Schrödinger dynamics on $\mathbb{S}^2$", *J. Math. Pures Appl. (9)* **83** (2004), p. 77–98.

[3] J. Bourgain – "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations", *Geom. Funct. Anal.* **3** (1993), p. 107–156.

[4] N. Burq, P. Gérad & N. Tzvetkov – "An instability property of the nonlinear Schrödinger equation on S^d", *Math. Res. Lett.* **9** (2002), p. 323–335.

[5] N. Burq, P. Gérard & N. Tzvetkov – "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds", *Amer. J. Math.* **126** (2004), p. 569–605.

[6] ———, "Agmon estimates and nonlinear instability for Schrödinger equations", preprint, 2005.

[7] R. Carles – "Remarks on the Cauchy problem for nonlinear Schrödinger equations with potential", preprint arXiv:math.AP/0609391.

[8] ———, "Geometric optics and instability for semi-classical Schrödinger equations", *Arch. Ration. Mech. Anal.* **183** (2007), p. 525–553.

[9] M. Christ, J. Colliander & T. Tao – "Ill-posedness for nonlinear Schrödinger and wave equation", to appear in *Annales IHP*.

[10] ———, "Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations", *Amer. J. Math.* **125** (2003), p. 1235–1293.

[11] M. Combescure – "The quantum stability problem for some class of time-dependent Hamiltonians", *Ann. Physics* **185** (1988), p. 86–110.

[12] B. Helffer – *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Mathematics, vol. 1336, Springer, 1988.

[13] W. Klingenberg – *Riemannian geometry*, de Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., 1982.

[14] L. Perko – *Differential equations and dynamical systems*, Texts in Applied Mathematics, vol. 7, Springer, 1991.

[15] J. V. Ralston – "Approximate eigenfunctions of the Laplacian", *J. Differential Geometry* **12** (1977), p. 87–100.