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THE JACOBIAN MAP, THE JACOBIAN GROUP AND THE GROUP OF AUTOMORPHISMS OF THE GRASSMANN ALGEBRA

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THE JACOBIAN MAP, THE JACOBIAN GROUP AND THE GROUP OF AUTOMORPHISMS OF THE GRASSMANN ALGEBRA

BY VLADIMIR V. BAVULA

ABSTRACT. — There are nontrivial dualities and parallels between polynomial algebras and the Grassmann algebras (e.g., the Grassmann algebras are dual of polynomial algebras as quadratic algebras). This paper is an attempt to look at the Grassmann algebras at the angle of the Jacobian conjecture for polynomial algebras (which is the question/conjecture about the *Jacobian set* – the set of all algebra endomorphisms of a polynomial algebra with the Jacobian 1 – the Jacobian conjecture claims that the Jacobian set is a *group*). In this paper, we study in detail the Jacobian set for the Grassmann algebra which turns out to be a *group* – the *Jacobian group* Σ – a sophisticated (and large) part of the group of automorphisms of the Grassmann algebra Λ_n. It is proved that the Jacobian group Σ is a rational unipotent algebraic group. A (minimal) set of generators for the algebraic group Σ, its dimension and coordinates are found explicitly. In particular, for $n \geq 4$,

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

The same is done for the Jacobian ascents - some natural algebraic overgroups of Σ. It is proved that the Jacobian map $\sigma \mapsto \det(\frac{\partial \sigma(x_i)}{\partial x_j})$ is surjective for odd n, and is *not* for even n though, in this case, the image of the Jacobian map is an algebraic subvariety of codimension 1 given by a single equation.

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Résumé (*L'application de Jacobi, le groupe de Jacobi et le groupe des automorphismes de l'algèbre grassmanienne*)

Il existe des dualités et des parallélismes non-triviaux entre les algèbres polynomiales et les algèbres grassmaniennes (par ex., les algèbres grassmaniennes sont duales des algèbres polynomiales en tant qu'algèbres quadratiques). Cet article est une tentative d'étude des algèbres grassmaniennes du point de vue de la conjecture de Jacobi sur les algèbres polynomiales (qui est la question/conjecture sur l'*ensemble de Jacobi* — l'ensemble de tous les endomorphismes d'algèbre d'une algèbre polynomiale avec jacobien 1 —, la conjecture de Jacobi affirme que l'ensemble de Jacobi est un *groupe*. Dans cet article nous étudions en détail l'ensemble de Jacobi pour l'algèbre grassmanienne qui s'avère être un *groupe* — le *groupe de Jacobi* Σ —, une partie grande et sophistiquée du groupe d'automorphismes de l'algèbre grassmanienne Λ_n. Nous démontrons que le groupe de Jacobi Σ est un groupe algébrique rationnel unipotent. Nous calculons explicitement un ensemble (minimal) de générateurs pour le groupe algébrique Σ, sa dimension et ses coordonnées. En particulier, pour $n \geq 4$,

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{si } n \text{ est pair,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{si } n \text{ est impair.} \end{cases}$$

Nous faisons de même pour les ascendants jacobiens — certains surgroupes algébriques naturels de Σ. Nous démontrons que l'application de Jacobi $\sigma \mapsto \det(\frac{\partial\sigma(x_i)}{\partial x_j})$ est surjective pour n impair, et ne l'est *pas* pour n pair, néanmoins, dans ce cas, l'image d'une application de Jacobi est une sous-variété algébrique de codimension 1, donnée par une seule équation.

1. Introduction

Throughout, ring means an associative ring with 1. Let K be an arbitrary ring (not necessarily commutative). The *Grassmann algebra* (the *exterior algebra*) $\Lambda_n = \Lambda_n(K) = K\lfloor x_1, \dots, x_n \rfloor$ is generated freely over K by elements $x_1, \dots, x_n$ that satisfy the defining relations:

$$x_1^2 = \cdots = x_n^2 = 0 \ \text{ and } \ x_i x_j = -x_j x_i \ \text{ for all } \ i \neq j.$$

What is the paper about? Motivation. — Briefly, for the Grassmann algebra Λ_n over a commutative ring K we study in detail the *Jacobian map*

$$\mathcal{J}(\sigma) := \det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right)$$

which is a 'straightforward' generalization of the usual Jacobian map $\mathcal{J}(\sigma) := \det(\frac{\partial\sigma(x_i)}{\partial x_j})$ for a polynomial algebra $P_n = K[x_1, \dots, x_n]$, $\sigma \in \mathrm{End}_{K-alg}(P_n)$. The polynomial Jacobian map is not yet a well-understood map, one of the open questions about this map is the Jacobian conjecture (JC) which claims *that* $\mathcal{J}(\sigma) = 1$ *implies* $\sigma \in \mathrm{Aut}_K(P_n)$ (where K is a field of characteristic zero). Obviously, one can reformulate the Jacobian conjecture as the question of whether the Jacobian monoid $\Sigma(P_n) := \{\sigma \in \mathrm{End}_{K-alg}(P_n) \,|\, \mathscr{J}(\sigma) = 1\}$ is a *group*? The analogous Jacobian monoid $\Sigma = \Sigma(\Lambda_n)$ for the Grassmann algebra Λ_n *is*, by a trivial reason, a *group*, it is a subgroup of the group $\mathrm{Aut}_K(\Lambda_n)$ of automorphisms of the Grassman algebra Λ_n. It turns out that properties of the Jacobian map $\mathscr{J}$ are closely related to properties of the Jacobian group Σ which should be treated as the 'kernel' of the Jacobian map $\mathscr{J}$ despite the fact that $\mathscr{J}$ is *not* a homomorphism.

For a polynomial algebra $P_n = K[x_1, \dots, x_n]$, $n \geq 2$, over a field of characteristic zero K, the group $\mathrm{Aut}_K(P_n)$ of algebra automorphisms is an infinite algebraic group. We know little about this group for $n \geq 3$. There are three old open questions about the group $\mathrm{Aut}_K(P_n)$.

Question 1. What are the defining relations of the algebraic group $\mathrm{Aut}_K(P_n)$ *(as an infinite dimensional algebraic variety)?*

Question 2. What are generators for $\mathrm{Aut}_K(P_n)$*?*

Question 3. What is a 'minimal' set of generators for $\mathrm{Aut}_K(P_n)$*?*

The Jacobian Conjecture (if true) gives an answer to the first question. For the last two questions there are no even reasonable conjectures. In this paper, answers for 'analogous' questions are given for the Grassmann algebras.

It turns out that the Jacobian group Σ is a large subgroup of $\mathrm{Aut}_K(\Lambda_n)$, so we start the paper considering the structure of the group $\mathrm{Aut}_K(\Lambda_n)$ and its subgroups. The Jacobian map and the Jacobian group are not transparent objects to deal with. Therefore, several (important) subgroups of $\mathrm{Aut}_K(\Lambda_n)$ are studied first. Some of them are given by explicit generators, another are defined via certain 'geometric' properties. That is why we study these subgroups in detail. They are building blocks in understanding the structure of the Jacobian map and the Jacobian group. Let us describe main results of the paper.

In the Introduction, K is a *reduced commutative* ring with $\frac{1}{2} \in K$, $n \geq 2$ (though many results of the paper are true under milder assumptions, see in the text), $\Lambda_n = \Lambda_n(K) = K\lfloor x_1, \dots, x_n \rfloor$ be the Grassmann K-algebra and $\mathfrak{m} := (x_1, \dots, x_n)$ be its augmentation ideal. The algebra Λ_n is endowed with the $\mathbb{Z}$-grading $\Lambda_n = \oplus_{i=0}^n \Lambda_{n,i}$ and $\mathbb{Z}_2$-grading $\Lambda_n = \Lambda_n^{\mathrm{ev}} \oplus \Lambda_n^{\mathrm{od}}$, and so each element $a \in \Lambda_n$ is a unique sum $a = a^{\mathrm{ev}} + a^{\mathrm{od}}$ where $a^{\mathrm{ev}} \in \Lambda_n^{\mathrm{ev}}$ and $a^{\mathrm{od}} \in \Lambda_n^{\mathrm{od}}$. For each $s \geq 2$, the algebra Λ_n is also a $\mathbb{Z}_s$-graded algebra ($\mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$).

The structure of the group of automorphisms of the Grassmann algebra and its subgroups. — In Sections 2 and 9, we study the group $G := \mathrm{Aut}_K(\Lambda_n(K))$ of K-algebra automorphisms of Λ_n and various its subgroups (and their relations):

- G_{gr}, the subgroup of G elements of which respect $\mathbb{Z}$-grading,
- $G_{\mathbb{Z}_2-gr}$, the subgroup of G elements of which respect $\mathbb{Z}_2$-grading,
- $G_{\mathbb{Z}_s-gr}$, the subgroup of G elements of which respect $\mathbb{Z}_s$-grading,

- $U := \{\sigma \in G \,|\, \sigma(x_i) = x_i + \cdots$ for all $i\}$ where the three dots mean bigger terms with respect to the $\mathbb{Z}$-grading,
- $G^{\rm od} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm od}$ for all $i\}$ and $G^{\rm ev} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm ev}$ for all $i\}$,
- $\mathrm{Inn}(\Lambda_n) := \{\omega_u : x \mapsto uxu^{-1}\}$ and $\mathrm{Out}(\Lambda_n) := G/\mathrm{Inn}(\Lambda_n)$, the groups of inner and outer automorphisms,
- $\Omega := \{\omega_{1+a} \,|\, a \in \Lambda_n^{\rm od}\}$,
- For each odd number s such that $1 \leq s \leq n$, $\Omega(s) := \{\omega_{1+a} \,|\, a \in \sum_{1\leq j \text{ is odd}} \Lambda_{n,js}\}$,
- $\Gamma := \{\gamma_b \,|\, \gamma_b(x_i) = x_i + b_i,\ b_i \in \Lambda_n^{\rm od} \cap \mathfrak{m}^3, i = 1, \ldots, n\}$, $b = (b_1, \ldots, b_n)$,
- For each even number s such that $3 \leq s \leq n$, $\Gamma(s) := \{\gamma_b \,|\, \text{all } b_i \in \sum_{j\geq 1} \Lambda_{n,1+js}\}$,
- $U^n := \{\tau_\lambda \,|\, \tau_\lambda(x_i) = x_i + \lambda_i x_1 \cdots x_n,\ \lambda = (\lambda_1, \ldots, \lambda_n) \in K^n\} \simeq K^n$, $\tau_\lambda \leftrightarrow \lambda$,
- $\mathrm{GL}_n(K)^{op} := \{\sigma_A \,|\, \sigma_A(x_i) = \sum_{j=1}^n a_{ij}x_j,\ A = (a_{ij}) \in \mathrm{GL}_n(K)\}$,
- $\Phi := \{\sigma : x_i \mapsto x_i(1 + a_i) \,|\, a_i \in \Lambda_n^{\rm ev} \cap \mathfrak{m}^2,\ i = 1, \ldots, n\}$.

If $K = \mathbb{C}$ the group $\Gamma\mathrm{GL}_n(\mathbb{C})^{op}$ was considered in [2]. If $K = k$ is a field of characteristic $\neq 2$ it was proved in [4] that G is a semidirect product $\mathrm{Inn}(\Lambda_k(k)) \rtimes G_{\mathbb{Z}_2-gr}$. One can find a lot of information about the Grassmann algebra (i.e. the exterior algebra) in [3].

- (Lemma 2.8.(5)) Ω *is an abelian group canonically isomorphic to the additive group* $\Lambda_n^{\rm od}/\Lambda_n^{\rm od} \cap Kx_1 \cdots x_n$ via $\omega_{1+a} \mapsto a$.
- (Lemma 2.9, Corollary 2.15.(3)) $\mathrm{Inn}(\Lambda_n) = \Omega$ *and* $\mathrm{Out}(\Lambda_n) \simeq G_{\mathbb{Z}_2-gr}$.
- (Theorem 2.14) $U = \Omega \rtimes \Gamma$.
- (Theorem 2.17) Ω *is a maximal abelian subgroup of* U *if* n *is even* ($\Omega \supseteq U^n$)*; and* $\Omega U^n = \Omega \times U^n$ *is a maximal abelian subgroup of* U *if* n *is odd* ($\Omega \cap U^n = \{e\}$).
- (Theorem 2.14, Corollary 2.15, Lemma 2.16)
  1. $G = U \rtimes \mathrm{GL}_n(K)^{op} = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$,
  2. $G = \Omega \rtimes G_{\mathbb{Z}_2-gr}$, *and*
  3. $G = G^{\rm ev}G^{\rm od} = G^{\rm od}G^{\rm ev}$.
- (Lemma 2.16.(1)) $G^{\rm od} = G_{\mathbb{Z}_2-gr} = \Gamma \rtimes \mathrm{GL}_n(K)^{op}$.
- (Lemma 3.6) *Let* $s = 2, \ldots, n$. *Then*

$$G_{\mathbb{Z}_s-gr} = \begin{cases} \Gamma(s) \rtimes \mathrm{GL}_n(K)^{op}, & \text{if } s \text{ is even,} \\ \Omega(s) \rtimes \mathrm{GL}_n(K)^{op}, & \text{if } s \text{ is odd.} \end{cases}$$

The Jacobian matrix and an analogue of the Jacobian Conjecture for Λ_n. — The even subalgebra $\Lambda_n^{\rm ev}$ of Λ_n belongs to the centre of the algebra Λ_n. A K-linear map $\delta : \Lambda_n \to \Lambda_n$ is called a *left skew derivation* if $\delta(a_ia_j) = \delta(a_i)a_j + (-1)^i a_i\delta(a_j)$ for all homogeneous elements a_i and a_j of graded degree i and j

respectively. Surprisingly, skew derivations rather than ordinary derivations are more important in study of the Grassmann algebras. This and the forthcoming paper [1] illustrate this phenomenon.

The 'partial derivatives' $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are *left skew K-derivations* of $\Lambda_n(K)$ ($\partial_i(x_j) = \delta_{ij}$, the Kronecker delta; $\partial_k(a_i a_j) = \partial_k(a_i)a_j + (-1)^i a_i \partial_k(a_j)$). Let $\mathscr{E}^{\rm od} := {\rm End}_{K-alg}(\Lambda_n)^{\rm od} := \{\sigma \in {\rm End}_{K-alg}(\Lambda_n) \,|$ all $\sigma(x_i) \in \Lambda_n^{\rm od}\}$. For each endomorphism $\sigma \in \mathscr{E}^{\rm od}$, its *Jacobian matrix* $\frac{\partial\sigma}{\partial x} := (\frac{\partial\sigma(x_i)}{\partial x_j}) \in M_n(\Lambda_n^{\rm ev})$ has even (hence central) entries, and so, the *Jacobian* of σ,

$$\mathscr{J}(\sigma) := \det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right),$$

is a well-defined element of the even subalgebra $\Lambda_n^{\rm ev}$. For $\sigma, \tau \in \mathscr{E}^{\rm od}$, the 'chain rule' holds

$$\frac{\partial(\sigma\tau)}{\partial x} = \sigma\left(\frac{\partial\tau}{\partial x}\right) \cdot \frac{\partial\sigma}{\partial x},$$

which implies, $\mathscr{J}(\sigma\tau) = \mathscr{J}(\sigma) \cdot \sigma(\mathscr{J}(\tau))$, i.e. the Jacobian map is almost a homomorphism of monoids (with zeros). It follows that the sets $\Sigma \subset \widetilde{\Sigma} \subset \mathscr{O}$ are monoids where

$$\begin{aligned} \mathscr{O} &:= \{\sigma \in \mathscr{E}^{\rm od} \,|\, \mathscr{J}(\sigma) \ \text{ is a unit}\}, \\ \widetilde{\Sigma} &:= \{\sigma \in \mathscr{E}^{\rm od} \,|\, \mathscr{J}(\sigma) = 1\}, \\ \Sigma &:= \{\sigma \in \mathscr{E}^{\rm od} \cap \Gamma = \Gamma \,|\, \mathscr{J}(\sigma) = 1\}. \end{aligned}$$

If $\sigma \in \mathscr{E}^{\rm od} \cap {\rm Aut}_K(\Lambda_n) = G_{\mathbb{Z}_2-gr} = \Gamma{\rm GL}_n(K)^{op}$ (Lemma 2.16.(1)) then $\sigma^{-1} \in \mathscr{E}^{\rm od} \cap {\rm Aut}_K(\Lambda_n)$ and

$$1 = \mathscr{J}({\rm id}_{\Lambda_n}) = \mathscr{J}(\sigma\sigma^{-1}) = \mathscr{J}(\sigma)\sigma(\mathscr{J}(\sigma^{-1})),$$

and so $\mathscr{J}(\sigma)$ is a unit in Λ_n.

An analogue of the Jacobian Conjecture for the Grassmann algebra Λ_n, i.e. *$\sigma \in \mathscr{O}$ implies σ is an automorphism (i.e. $\mathscr{O} \subseteq G := {\rm Aut}_K(\Lambda_n)$)*, is trivially true.

- $\mathscr{O} \subseteq G$.

Proof. — Let $\sigma \in \mathscr{O}$. Then $\sigma(x) = Ax + \cdots$ where $x := (x_1, \ldots, x_n)^t$, $A \in M_n(K)$, and the three dots mean higher terms with respect to the $\mathbb{Z}$-grading. Since

$$\mathscr{J}(\sigma) \equiv \det(A) \mod \mathfrak{m}$$

and $\mathscr{J}(\sigma)$ is a unit, the determinant must be a unit in K. Changing σ for $\sigma\sigma_{A^{-1}}$ where $\sigma_{A^{-1}}(x) = A^{-1}x$ ($\sigma\sigma_{A^{-1}}(x) = A^{-1}Ax + \cdots = x + \cdots$) one can assume that, for each $i = 1, \ldots, n$, $\sigma(x_i) = x_i - a_i$ where $a_i \in \mathfrak{m}^3 \cap \Lambda_n^{\rm od}$. Let us denote $\sigma(x_i)$ by x_i', then $x_i = x_i' + a_i(x)$. After repeating several times ($\leq n$

times) these substitutions simultaneously in the tail of each element x_i, i.e. elements of degree ≥ 3, it is easy to see that $x_i = x'_i + b_i(x')$ for some element $b_i(x') \in \mathfrak{m}^3 \cap \Lambda_n^{\rm od}$. This gives the *inverse* map for σ, i.e. $\sigma \in G$. The elements b_i can be found even explicitly using the inversion formula (Theorem 3.1). □

The next two facts follow directly from the inclusion $\mathcal{O} \subseteq G$ and the formula $\mathcal{J}(\sigma^{-1}) = \sigma^{-1}(\mathcal{J}(\sigma)^{-1})$, for all $\sigma \in G$.

- Σ, $\widetilde{\Sigma}$, *and* $\mathcal{O}$ *are groups.*
- $\mathcal{O} = G_{\mathbb{Z}_2-gr} = \Gamma \rtimes {\rm GL}_n(K)^{op}$ *and* $\widetilde{\Sigma} = \Sigma \rtimes {\rm SL}_n(K)^{op}$.

The Jacobian map and the Jacobian group. — The set $E_n := K^* + \sum_{m\geq 1} \Lambda_{n,2m}$ is the group of units of the even subalgebra $\Lambda_n^{\rm ev} := \oplus_{m\geq 0}\Lambda_{n,2m}$ of Λ_n where K^* is the group of units of the ring K; and $E'_n := 1+\sum_{m\geq 1} \Lambda_{n,2m}$ is the subgroup of E_n. Due to the equality $\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))$, to study the *Jacobian map*

$$\mathcal{J} : \Gamma{\rm GL}_n(K)^{op} \to E_n, \quad \sigma \mapsto \mathcal{J}(\sigma),$$

is the same as to study its restriction to Γ:

$$\mathcal{J} : \Gamma \to E'_n, \quad \sigma \mapsto \mathcal{J}(\sigma).$$

When we mention the Jacobian map it means as a rule this map. The Jacobian group $\Sigma = \{\sigma \in \Gamma \,|\, \mathcal{J}(\sigma) = 1\}$ is trivial iff $n \leq 3$. So, we always assume that $n \geq 4$ in the results on the Jacobian group Σ and its subgroups.

An algebraic group A over K is called *affine* if its algebra of regular functions is a polynomial algebra $K[t_1,\dots,t_d]$ with coefficients in K where $d := \dim(A)$ is called the *dimension* of A (i.e. A is an *affine space*). If K is a field then $\dim(A)$ is the usual dimension of the algebraic group A over the field K.

- (Theorem 6.3) *The Jacobian group Σ is an affine group over K of dimension*
$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$
- *The coordinate functions on Σ are given explicitly by* (93) *and Corollary 4.11.(5).*

A subgroup of an algebraic group A over K is called a 1-*parameter subgroup* if it is isomorphic to the algebraic group $(K,+)$. *A minimal set of generators* for an affine algebraic group A over K is a set of 1-parameter subgroups that generate the group A as an abstract group but each smaller subset does not generate A.

- (Theorem 6.1) *A (minimal) set of generators for Σ is given explicitly.*
- (Corollary 4.13) *The Jacobian group Σ is not a normal subgroup of Γ iff* $n \geq 5$.

– (Theorem 7.9) *The Jacobian map* $\mathscr{J}:\Gamma\to E'_n$, $\sigma\mapsto\mathscr{J}(\sigma)$, *is surjective if* n *is odd, and it is not surjective if* n *is even but in this case its image is a closed affine subvariety of* E'_n *of codimension* 1 *which is given by a single equation.*

The subgroups Σ' and Σ'' of the Jacobian group Σ. — To prove the (above) results about the Jacobian group Σ, we, first, study in detail two of its subgroups:

$$\Sigma' := \Sigma\cap\Phi = \{\sigma : x_i\mapsto x_i(1+a_i)\,|\,\mathscr{J}(\sigma)=1,\ a_i\in\Lambda_n^{\mathrm{ev}}\cap\mathfrak{m}^2,\ 1\leq i\leq n\}$$

and the subgroup Σ'' which is generated by the explicit automorphisms of Σ (see (64)):

$$\xi_{i,b_i}: x_i\mapsto x_i+b_i,\quad x_j\mapsto x_j,\quad j\neq i,$$

where $b_i\in K\lfloor x_1,\dots,\widehat{x}_i,\dots,x_n\rfloor^{\mathrm{od}}_{\geq 3}$ and $i=1,\dots,n$.

The importance of these subgroups is demonstrated by the following two facts.

– (Corollary 4.11.(1)) $\Sigma=\Sigma'\Sigma''$.
– (Theorem 4.9.(1)) $\Gamma=\Phi\Sigma''$.

Note that each element x_i is a *normal* element of Λ_n: $x_i\Lambda_n=\Lambda_n x_i$. Therefore, the ideal (x_i) of Λ_n generated by the element x_i determines a coordinate 'hyperplane.' The groups Σ' and Σ'' have the following geometric interpretation: the group Σ' preserves the coordinate 'hyperplanes' and elements of the group Σ'' can be seen as 'rotations.'

By the definition, the group Σ' is a closed subgroup of Σ, it is not a normal subgroup of Σ unless $n\leq 5$. It is not obvious from the outset whether the subgroup Σ'' is closed or normal. In fact, it is.

– (Theorem 6.4.(2)) Σ'' *is the closed normal subgroup of* Σ, Σ'' *is an affine group of dimension*

$$\dim(\Sigma'')=\begin{cases}(n-1)2^{n-1}-n^2+2-(n-3)\binom{n}{2} & \text{if } n \text{ is even,}\\ (n-1)2^{n-1}-n^2+1-(n-3)\binom{n}{2} & \text{if } n \text{ is odd,}\end{cases}$$

and the factor group $\Sigma/\Sigma''\simeq\Sigma'/\Sigma'^5$ *is an abelian affine group of dimension* $\dim(\Sigma/\Sigma'')=n\binom{n-1}{2}-\binom{n}{2}=(n-3)\binom{n}{2}$.
– (Corollary 5.6) *The group* Σ' *is an affine group over* K *of dimension*

$$\dim(\Sigma')=\begin{cases}(n-2)2^{n-2}-n+2 & \text{if } n \text{ is even,}\\ (n-2)2^{n-2}-n+1 & \text{if } n \text{ is odd.}\end{cases}$$

- (Lemma 6.2) *The intersection* $\Sigma'\cap\Sigma''$ *is a closed subgroup of* Σ, *it is an affine group over* K *of dimension*

$$\dim(\Sigma'\cap\Sigma'')=\begin{cases}(n-2)2^{n-2}-n+2-(n-3)\binom{n}{2} & \text{if } n \text{ is even,}\\ (n-2)2^{n-2}-n+1-(n-3)\binom{n}{2} & \text{if } n \text{ is odd.}\end{cases}$$

- *The coordinates on* Σ' *and* Σ'' *are given explicitly by* (93) *and* (98).

To find coordinates for the groups Σ, Σ', and Σ'' explicitly, we introduce avoidance functions and a series of subgroups $\{\Phi'^{2s+1}\}$, $s=1,2,\dots,[\frac{n-1}{2}]$, of Φ that are given explicitly (see Section 5). They are too technical to explain in the introduction.

- (Theorem 5.4) *This theorem is a key result in finding coordinates for the groups* Σ, Σ', Σ'', *etc.*

The Jacobian ascents Γ_{2s}. — In order to study the image of the Jacobian map $\mathscr{J}:\Gamma\to E_n'$, $\sigma\mapsto\mathscr{J}(\sigma)$, certain overgroups of the Jacobian group Σ are introduced. They are called the *Jacobian ascents*. The problem of finding the image $\mathrm{im}(\mathscr{J})$ is equal to the problem of finding generators for these groups. Let us give some details. The Grassmann algebra Λ_n has the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^i\}$. Therefore, the group E_n' has the induced $\mathfrak{m}$-adic filtration:

$$E_n'=E_{n,2}'\supset E_{n,4}'\supset\cdots\supset E_{n,2m}'\supset\cdots\supset E_{n,2[\frac{n}{2}]}'\supset E_{n,2[\frac{n}{2}]+2}'=\{1\},$$

where $E_{n,2m}':=E_n'\cap(1+\mathfrak{m}^{2m})$. Correspondingly, the group Γ has the *Jacobian filtration*:

$$\Gamma=\Gamma_2\supseteq\Gamma_4\supseteq\cdots\supseteq\Gamma_{2m}\supseteq\cdots\supseteq\Gamma_{2[\frac{n}{2}]}\supseteq\Gamma_{2[\frac{n}{2}]+2}=\Sigma,$$

where $\Gamma_{2m}:=\Gamma_{n,2m}:=\mathscr{J}^{-1}(E_{n,2m}')=\{\sigma\in\Gamma\,|\,\mathscr{J}(\sigma)\in E_{n,2m}'\}$. It follows from the equality $\mathscr{J}(\sigma\tau)=\mathscr{J}(\sigma)\sigma(\mathscr{J}(\tau))$ that all Γ_{2m} are subgroups of Γ, they are called, the *Jacobian ascents* of the Jacobian group Σ.

The Jacobian ascents are *distinct* groups with a *single* exception when two groups coincide. This is a subtle fact, it explains (partly) why formulae for various dimensions differ by 1 in odd and even cases.

- (Corollary 7.7) *Let* K *be a commutative ring and* $n\geq 4$.
  1. *If* n *is an odd number then the Jacobian ascents*
  $$\Gamma=\Gamma_2\supset\Gamma_4\supset\cdots\supset\Gamma_{2s}\supset\cdots\supset\Gamma_{2[\frac{n}{2}]}\supset\Gamma_{2[\frac{n}{2}]+2}=\Sigma$$
  are distinct groups.
  2. *If* n *is an even number then the Jacobian ascents*
  $$\Gamma=\Gamma_2\supset\Gamma_4\supset\cdots\supset\Gamma_{2s}\supset\cdots\supset\Gamma_{2[\frac{n}{2}]-2}\supset\Gamma_{2[\frac{n}{2}]}=\Gamma_{2[\frac{n}{2}]+2}=\Sigma$$
  are distinct groups except the last two groups, i.e. $\Gamma_{2[\frac{n}{2}]}=\Gamma_{2[\frac{n}{2}]+2}$.

The subgroups $\{\Gamma^{2s+1}\}$ of Γ are given explicitly,

$$\Gamma^{2s+1} := \{\sigma : x_i \mapsto x_i + a_i \mid a_i \in \Lambda_n^{\mathrm{od}} \cap \mathfrak{m}^{2s+1},\ 1 \leq i \leq n\},\quad s \geq 1,$$

they have clear structure. The next result explains that the Jacobian ascents $\{\Gamma_{2s}\}$ have clear structure too, $\Gamma_{2s} = \Gamma^{2s+1}\Sigma$, and so the structure of the Jacobian ascents is completely determined by the structure of the Jacobian group Σ.

– (Theorem 7.1) *Let K be a commutative ring and $n \geq 4$. Then*
  1. $\Gamma_{2s} = \Gamma^{2s+1}\Sigma = \Phi^{2s+1}\Sigma = \Phi'^{2s+1}\Sigma$ *for each* $s = 1, 2, \ldots, [\frac{n-1}{2}]$.
  2. *If n is an even number then $\Gamma_n = \Sigma$, i.e. $\Gamma_n = \Gamma_{n+2} = \Sigma$.*

The next theorem introduces an isomorphic affine structure on the algebraic group Γ.

– (Theorem 7.2) *Let K be a commutative ring, $n \geq 4$, and $s = 1, \ldots, [\frac{n-1}{2}]$. Then each automorphism $\sigma \in \Gamma$ is a unique product $\sigma = \phi'_{a(2)}\phi'_{a(4)}\cdots\phi'_{a(2[\frac{n-1}{2}])}\gamma$ for unique elements $a(2s) \in \Lambda_{n,2s}$ and $\gamma \in \Gamma_{2[\frac{n-1}{2}]+2} = \Sigma$ (by (100)). Moreover,*

$$a(2) \equiv \mathscr{J}(\sigma) - 1 \mod E'_{n,4},$$

$$a(2t) \equiv \mathscr{J}(\phi'^{-1}_{a(2t-2)}\cdots\phi'^{-1}_{a(2)}\sigma) - 1 \mod E'_{n,2t+2},\ t = 2, \ldots, [\frac{n-1}{2}],$$

$$\gamma = (\phi'_{a(2)}\phi'_{a(4)}\cdots\phi'_{a(2[\frac{n-1}{2}])})^{-1}\sigma.$$

The automorphisms $\phi'_{a(2s)}$ are given explicitly (see Section 7 for details), they are too technical to explain here. The theorem above is a key result in proving that various quotient spaces, like Γ_{2s}/Γ_{2t} $(s < t)$, are *affine*, and in finding their dimensions. An algebraic variety V over K is called *affine* (i.e. an *affine space* over K) if its algebra of regular functions $\mathscr{O}(V)$ is a polynomial algebra $K[v_1, \ldots, v_d]$ over K where $d := \dim(V)$ is called the *dimension* of V over K. In this paper, all algebraic groups and varieties will turn out to be affine (i.e. affine spaces), and so the word 'affine' is used only in this sense. This fact strengthen relations between the Grassmann algebras and polynomial algebras even more.

– (Corollary 7.5) *Let K be a commutative ring, $n \geq 4$. Then all the Jacobian ascents are affine groups over K and closed subgroups of Γ, and*

$$\dim(\Gamma_{2s}) = \dim(\Sigma) + \sum_{i=s}^{[\frac{n-1}{2}]} \binom{n}{2i},\quad s = 1, \ldots, [\frac{n-1}{2}].$$

– (Corollary 7.8) *Let K be a commutative ring and $n \geq 4$. Then the quotient space $\Gamma/\Sigma := \{\sigma\Sigma \mid \sigma \in \Gamma\}$ is an affine variety.*

1. *If n is odd then the Jacobian map $\Gamma/\Sigma \to E'_n$, $\sigma\Sigma \mapsto \mathcal{J}(\sigma)$, is an isomorphism of the affine varieties over K, and* $\dim(\Gamma/\Sigma) = 2^{n-1}-1$.
2. *If n is even then the Jacobian map $\Gamma/\Sigma \to E'_n/E'_{n,n}$, $\sigma\Sigma \mapsto \mathcal{J}(\sigma)E'_{n,n}$, is an isomorphism of the affine varieties over K (where $E'_{n,n} = 1+Kx_1\cdots x_n$), and* $\dim(\Gamma/\Sigma) = 2^{n-1}-2$.

– (Theorem 7.9) *Let K be a commutative ring, $n \geq 4$, $\mathcal{J} : \Gamma \to E'_n$, $\sigma \mapsto \mathcal{J}(\sigma)$, be the Jacobian map, and $s = 1, 2, \dots, [\frac{n-1}{2}]$. Then,*
  1. *for an odd number n, the Jacobian map $\mathcal{J}$ is surjective, and*
  2. *for an even number n, the Jacobian map $\mathcal{J}$ is not surjective. In more detail, the image* $\mathrm{im}(\mathcal{J})$ *is a closed algebraic variety of E'_n of codimension 1.*

The unique presentation $\sigma = \omega_{1+a}\gamma_b\sigma_A$ for $\sigma \in \mathrm{Aut}_K(\Lambda_n)$. — Each automorphism $\sigma \in G = \Omega\Gamma\mathrm{GL}_n(K)^{op}$ is a *unique* product (Theorem 2.14)

$$\sigma = \omega_{1+a}\gamma_b\sigma_A$$

where $\omega_{1+a} \in \Omega$ ($a \in \Lambda'^{\mathrm{od}}_n$), $\gamma_b \in \Gamma$, and $\sigma_A \in \mathrm{GL}_n(K)^{op}$ where $\Lambda'^{\mathrm{od}}_n := \oplus_i \Lambda_{n,i}$ and i runs through *odd* natural numbers such $1 \leq i \leq n-1$. The next theorem determines explicitly the elements a, b, and A via the vector-column $\sigma(x) := (\sigma(x_1), \dots, \sigma(x_n))^t$ (for, only one needs to know explicitly the inverse γ_b^{-1} for each $\gamma_b \in \Gamma$ which is given by the inversion formula below, Theorem 3.1).

– (Theorem 9.1) *Each element $\sigma \in G$ is a unique product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ (Theorem 2.14.(3)) where $a \in \Lambda'^{\mathrm{od}}_n$ and*
  1. $\sigma(x) = Ax + \cdots$ *(i.e. $\sigma(x) \equiv Ax \mod \mathfrak{m}$) for some* $A \in \mathrm{GL}_n(K)$,
  2. $b = A^{-1}\sigma(x)^{\mathrm{od}} - x$, *and*
  3. $a = \frac{1}{2}(-1 + x_1\cdots x_n\partial_n\cdots\partial_1)\gamma_b(\sum_{i=1}^{n-1} x_1\cdots x_i\partial_i\cdots\partial_1\partial_{i+1}(a'_{i+1}) + \partial_1(a'_1))$ *where $a'_i := (A^{-1}\gamma_b^{-1}(\sigma(x)^{\mathrm{ev}}))_i$, the i'th component of the column-vector $A^{-1}\gamma_b^{-1}(\sigma(x)^{\mathrm{ev}})$,*

  where $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are left skew K-derivations of $\Lambda_n(K)$ ($\partial_i(x_j) = \delta_{ij}$, the Kronecker delta).

The inversion formula for automorphisms. — The formula (22) for multiplication of elements of G shows that the most non-trivial (difficult) part of the group G is the group Γ. Elements of the group Γ should be seen as n-tuples of noncommutative polynomials in anti-commuting variables $x_1, \dots, x_n$, and the multiplication of two n-tuples is the composition of functions. The group Γ (and $\Gamma\mathrm{GL}_n(K)^{op}$) is the part of the group G that 'behaves' in a similar fashion as polynomial automorphisms. This very observation we will explore in the paper. The analogy between the group Γ and the group of polynomial automorphisms is far more reaching than one may expect.

– (Theorem 3.1) (The Inversion Formula) *Let K be a commutative ring, $\sigma \in \Gamma\mathrm{GL}_n(K)^{op}$ and $a \in \Lambda_n(K)$. Then $\sigma^{-1}(a) = \sum_{\alpha\in\mathscr{B}_n} \lambda_\alpha x^\alpha$ where*

$$\lambda_\alpha := (1-\sigma(x_n)\partial'_n)(1-\sigma(x_{n-1})\partial'_{n-1})\cdots(1-\sigma(x_1)\partial'_1)\partial'^\alpha(a) \in K,$$
$$\partial'^\alpha := \partial'^{\alpha_n}_n \partial'^{\alpha_{n-1}}_{n-1}\cdots\partial'^{\alpha_1}_1,$$
$$\partial'_i(\cdot) := \frac{1}{\det(\frac{\partial\sigma(x_\nu)}{\partial x_\mu})}\det\begin{pmatrix} \frac{\partial\sigma(x_1)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_1)}{\partial x_n}\\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_1}(\cdot) & \cdots & \frac{\partial}{\partial x_n}(\cdot)\\ \vdots & \vdots & \vdots \\ \frac{\partial\sigma(x_n)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_n)}{\partial x_n}\end{pmatrix}, \quad i=1,\dots,n,$$

where $\partial_1 := \frac{\partial}{\partial x_1},\dots,\partial_n := \frac{\partial}{\partial x_n}$ are left skew K-derivations of $\Lambda_n(K)$.

Then, for any automorphism $\sigma = \omega_{1+a}\gamma_b\sigma_A \in G$, one can write explicitly the formula for the inverse σ^{-1}, (23). Note that $\omega_{1+a}^{-1} = \omega_{1-a}$ and $\sigma_A^{-1} = \sigma_{A^{-1}}$. So, γ_b^{-1} is the most difficult part of the inverse map σ^{-1}. The formula for γ_b^{-1} is written via *skew differential operators* (i.e. linear combinations of products of powers of skew derivations). That is why we start the paper with various properties of skew derivations. Detailed study of skew derivations is continued in [1].

Analogues of the Poincaré Lemma. — The crucial step in finding the b (in $\sigma = \omega_{1+a}\gamma_b\sigma_A$) is an analogue of the Poincaré Lemma for Λ_n (Theorem 8.2) where the solutions (as well as necessary and sufficient conditions for existence of solutions) are given explicitly for the following system of equations in Λ_n where $a \in \Lambda_n$ is unknown, and $u_i \in \Lambda_n$:

$$\begin{cases} x_1 a = u_1,\\ x_2 a = u_2,\\ \quad\vdots \\ x_n a = u_n.\end{cases}$$

– (Theorem 8.2) *Let K be an arbitrary ring. The system above has a solution iff (i) $u_1 \in (x_1),\dots,u_n \in (x_n)$, and (ii) $x_iu_j = -x_ju_i$ for all $i \neq j$. Then*

$$a = x_1\cdots x_n a_n + \sum_{i=1}^{n-1} x_1\cdots x_i\partial_i\cdots\partial_1\partial_{i+1}(u_{i+1}) + \partial_1(u_1), \quad a_n \in K,$$

are all the solutions.

Note that the left multiplication on x_i is, up to the scalar $\frac{1}{2}$, a skew derivation in Λ_n: $x_i(a_ja_k) = \frac{1}{2}((x_ia_j)a_k + (-1)^j a_j(x_ia_k))$ for all homogeneous elements a_j and a_k of $\mathbb{Z}$-graded degree j and k respectively.

Another version of the Poincaré Lemma for Λ_n is Theorem 8.3 where the solutions are given explicitly for the system (of first order partial skew differential operators):

$$\begin{cases} \partial_1(a) = u_1, \\ \partial_2(a) = u_2, \\ \quad\vdots \\ \partial_n(a) = u_n, \end{cases}$$

where $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the left partial skew K-derivatives of Λ_n.

– (Theorem 8.3) *Let K be an arbitrary ring. The system above has a solution iff (i) $u_i \in K\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n\rangle$ for all i; and (ii) $\partial_i(u_j) = -\partial_j(u_i)$ for all $i \neq j$. Then*

$$a = \lambda + \sum_{0\neq \alpha \in \mathscr{B}_n} (1 - x_n\partial_n)(1 - x_{n-1}\partial_{n-1}) \cdots (1 - x_1\partial_1)(u_\alpha)x^\alpha, \quad \lambda \in K,$$

are all the solutions where for $\alpha = \{i_1 < \cdots < i_k\}$ we have $u_\alpha := \partial_{i_k}\partial_{i_{k-1}} \cdots \partial_{i_2}(u_{i_1})$.

Minimal set of generators for the group Γ and some of its subgroups. — For each $i = 1, \dots, n$; $\lambda \in K$; and $j < k < l$, let us consider the automorphism $\sigma_{i,\lambda x_jx_kx_l} \in \Gamma$: $x_i \mapsto x_i + \lambda x_jx_kx_l$, $x_m \mapsto x_m$, for all $m \neq i$. Then

$$\sigma_{i,\lambda x_jx_kx_l}\sigma_{i,\mu x_jx_kx_l} = \sigma_{i,(\lambda+\mu)x_jx_kx_l}, \quad \sigma^{-1}_{i,\lambda x_jx_kx_l} = \sigma^{-1}_{i,-\lambda x_jx_kx_l}.$$

So, the group $\{\sigma_{i,\lambda x_jx_kx_l} \,|\, \lambda \in K\}$ is isomorphic to the algebraic group $(K, +)$ via $\sigma_{i,\lambda x_jx_kx_l} \mapsto \lambda$.

– (Theorem 3.11.(1)) *The group Γ is generated by all the automorphisms $\sigma_{i,\lambda x_jx_kx_l}$, i.e. $\Gamma = \langle \sigma_{i,\lambda x_jx_kx_l} \,|\, i = 1, \dots, n; \lambda \in K; j < k < l\rangle$. The subgroups $\{\sigma_{i,\lambda x_jx_kx_l}\}_{\lambda\in K}$ form a minimal set of generators for Γ.*
– (Theorem 3.11.(3)) *The group U is generated by all the automorphisms $\sigma_{i,\lambda x_jx_kx_l}$ and all the automorphisms $\omega_{1+\lambda x_i}$, i.e. $U = \langle \sigma_{i,\lambda x_jx_kx_l}, \omega_{1+\lambda x_i} \,|, i = 1, \dots, n; \lambda \in K; j < k < l\rangle$. The subgroups $\{\sigma_{i,\lambda x_jx_kx_l}\}_{\lambda\in K}$, $\{\omega_{1+\lambda x_i}\}_{\lambda\in K}$ form a minimal set of generators for U.*
– (Corollary 3.12) *The group Φ is generated by all the automorphisms $\sigma_{i,\lambda x_ix_kx_l}$, i.e. $\Phi = \langle \sigma_{i,\lambda x_ix_kx_l} \,|\, i = 1, \dots, n; \lambda \in K; k < l; i \notin \{k, l\}\rangle$. The subgroups $\{\sigma_{i,\lambda x_ix_kx_l}\}_{\lambda\in K}$ form a minimal set of generators for Φ.*

2. The group of automorphisms of the Grassmann ring

For reader's convenience, at the beginning of this section some elementary results on Grassmann rings and their left skew derivations are collected. Later in the paper they are used in proofs of many explicit formulae. In the second part of this section, all the results on the group of automorphisms of the Grassmann ring and its subgroups (from the Introduction) are proved.

The Grassmann algebra and its gradings. — Let K be an *arbitrary* ring (not necessarily commutative). The *Grassmann algebra* (the *exterior algebra*) $\Lambda_n = \Lambda_n(K) = K\lfloor x_1, \dots, x_n\rfloor$ is generated freely over K by elements $x_1, \dots, x_n$ that satisfy the defining relations:

$$x_1^2 = \dots = x_n^2 = 0 \text{ and } x_i x_j = -x_j x_i \text{ for all } i \neq j.$$

Let $\mathscr{B}_n$ be the set of all subsets of the set of indices $\{1, \dots, n\}$. We may identify the set $\mathscr{B}_n$ with the direct product $\{0,1\}^n$ of n copies of the two-element set $\{0,1\}$ by the rule $\{i_1, \dots, i_k\} \mapsto (0, \dots, 1, \dots, 1, \dots, 0)$ where 1's are on $i_1, \dots, i_k$ places and 0's elsewhere. So, the set $\{0,1\}^n$ is the set of all the characteristic functions on the set $\{1, \dots, n\}$.

$$\Lambda_n = \bigoplus_{\alpha \in \mathscr{B}_n} Kx^\alpha = \bigoplus_{\alpha \in \mathscr{B}_n} x^\alpha K, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n = \mathscr{B}_n$. Note that the order in the product x^α is fixed. So, Λ_n is a free left and right K-module of rank 2^n. The ring $\Lambda_n(K)$ is commutative iff K is commutative and either $n = 1$ or $-1 = 1$. Note that $(x_i) := x_i\Lambda_n = \Lambda_n x_i$ is an ideal of Λ_n. Each element $a \in \Lambda_n$ is a unique sum $a = \sum a_\alpha x^\alpha$, $a_\alpha \in K$. One can view each element a of Λ_n as a 'function' $a = a(x_1, \dots, x_n)$ in the non-commutative variables x_i. The K-algebra epimorphism

$$\Lambda_n \to \Lambda_n/(x_{i_1}, \dots, x_{i_k}) = K\lfloor x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_k}}, \dots, x_n\rfloor,$$
$$a \mapsto a|_{x_{i_1}=0, \dots, x_{i_k}=0} := a + (x_{i_1}, \dots, x_{i_k}),$$

may be seen as the operation of taking value of the function $a(x_1, \dots, x_n)$ at the point $x_{i_1} = \dots = x_{i_k} = 0$ where here and later the hat over a symbol means that it is missed.

For each $\alpha \in \mathscr{B}_n$, let $|\alpha| := \alpha_1 + \dots + \alpha_n$. The ring $\Lambda_n = \oplus_{i=0}^n \Lambda_{n,i}$ is a $\mathbb{Z}$-*graded* ring ($\Lambda_{n,i}\Lambda_{n,j} \subseteq \Lambda_{n,i+j}$ for all i,j) where $\Lambda_{n,i} := \oplus_{|\alpha|=i} Kx^\alpha$. The ideal $\mathfrak{m} := \oplus_{i\geq 1}\Lambda_{n,i}$ of Λ_n is called the *augmentation* ideal. Clearly, $K \simeq \Lambda_n/\mathfrak{m}$, $\mathfrak{m}^n = Kx_1 \cdots x_n$ and $\mathfrak{m}^{n+1} = 0$. We say that an element α of $\mathscr{B}_n$ is *even* (resp. *odd*) if the set α contains even (resp. odd) number of elements. By definition, the empty set is even. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. The ring $\Lambda_n = \Lambda_{n,\bar{0}} \oplus \Lambda_{n,\bar{1}}$ is a $\mathbb{Z}_2$-*graded* ring where $\Lambda_{n,\bar{0}} := \Lambda_n^{\mathrm{ev}} := \oplus_{\alpha \text{ is even}} Kx^\alpha$ is the subring of even elements of Λ_n and $\Lambda_{n,\bar{1}} := \Lambda_n^{\text{od}} := \oplus_{\alpha \text{ is odd}} Kx^\alpha$ is the Λ_n^{ev}-module of odd elements of Λ_n. The ring Λ_n has the $\mathfrak{m}$-*adic* filtration $\{\mathfrak{m}^i\}_{i\geq 0}$. The even subring Λ_n^{ev} has the induced $\mathfrak{m}$-*adic* filtration $\{\Lambda_{n,\geq i}^{\text{ev}} := \Lambda_n^{\text{ev}} \cap \mathfrak{m}^i\}$. The Λ_n^{ev}-module Λ_n^{od} has the induced $\mathfrak{m}$-*adic* filtration $\{\Lambda_{n,\geq i}^{\text{od}} := \Lambda_n^{\text{od}} \cap \mathfrak{m}^i\}$.

The K-linear map $a \mapsto \overline{a}$ from Λ_n to itself which is given by the rule

$$\overline{a} := \begin{cases} a, & \text{if } a \in \Lambda_{n,\bar{0}}, \\ -a, & \text{if } a \in \Lambda_{n,\bar{1}}, \end{cases}$$

is a ring automorphism such that $\overline{\overline{a}} = a$ for all $a \in \Lambda_n$. For all $a \in \Lambda_n$ and $i = 1, \dots, n$,

$$x_i a = \overline{a} x_i \text{ and } a x_i = x_i \overline{a}. \tag{1}$$

So, each element x_i of Λ_n is a *normal* element, i.e. the two-sided ideal (x_i) generated by the element x_i coincides with both left and right ideals generated by x_i: $(x_i) = \Lambda_n x_i = x_i \Lambda_n$.

For an arbitrary $\mathbb{Z}$-graded ring $A = \oplus_{i\in\mathbb{Z}} A_i$, an additive map $\delta : A \to A$ is called a *left skew derivation* if

$$\delta(a_i a_j) = \delta(a_i) a_j + (-1)^i a_i \delta(a_j) \text{ for all } a_i \in A_i,\ a_j \in A_j. \tag{2}$$

In this paper, a skew derivation means a *left* skew derivation. Clearly, $1 \in \ker(\delta)$ ($\delta(1) = \delta(1\cdot 1) = 2\delta(1)$ and so $\delta(1) = 0$). The restriction of the left skew derivation δ to the even subring $A^{\text{ev}} := \oplus_{i\in 2\mathbb{Z}} A_i$ of A is an ordinary derivation. Recall that an additive subgroup B of A is called a homogeneous subgroup if $B = \oplus_{i\in\mathbb{Z}} B \cap A_i$. If the kernel $\ker(\delta)$ of δ is a homogeneous additive subgroup of A then $\ker(\delta)$ is a subring of A, by (2).

Definition. For the ring $\Lambda_n(K)$, consider the set of *left* skew K-derivations:

$$\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$$

given by the rule $\partial_i(x_j) = \delta_{ij}$, the Kronecker delta. Informally, these skew K-derivations will be called (left) *partial skew derivatives*.

Example. $\partial_i(x_1 \cdots x_i \cdots x_k) = (-1)^{i-1} x_1 \cdots x_{i-1} x_{i+1} \cdots x_k$.

The Taylor formula and its generalization. — In this paper, $\partial_1, \dots, \partial_n$ mean *left* partial skew derivatives (if it is not stated otherwise). Note that

$$\partial_1^2 = \cdots = \partial_n^2 = 0 \text{ and } \partial_i\partial_j = -\partial_j\partial_i \text{ for all } i \neq j,$$

and $K_i := \ker(\partial_i) = K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor$.

LEMMA 2.1. — *(The Taylor Formula) For each* $a = \sum_{\alpha \in \mathscr{B}_n} a_\alpha x^\alpha \in \Lambda_n(K)$,

$$a = \sum_{\alpha \in \mathscr{B}_n} \partial^\alpha(a)(0)\, x^\alpha$$

where $\partial^\alpha := \partial_n^{\alpha_n} \partial_{n-1}^{\alpha_{n-1}} \cdots \partial_1^{\alpha_1}$, *in the reverse order here and everywhere.*

Proof. — It is obvious since $a_\alpha \equiv \partial^\alpha(a) \mod \mathfrak{m}$. □

The operation of taking value at 0 in the Taylor Formula is rather 'annoying'. Later, we will give an 'improved' (more economical) version of the Taylor Formula without the operation of taking value at 0 (Theorem 2.3).

LEMMA 2.2. — *Recall that* $K_i := \ker(\partial_i) = K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor$ *and* $\Lambda_n(K) = K_i \oplus x_i K_i$. *Then*

1. *for each* $i = 1, \dots, n$, *the map* $\phi_i := 1 - x_i \partial_i : \Lambda_n \to \Lambda_n$ *is the projection onto* K_i.
2. *The composition of the maps* $\phi := \phi_n \phi_{n-1} \cdots \phi_1 : \Lambda_n \to \Lambda_n$ *is the projection onto* K *in* $\Lambda_n = K \oplus \mathfrak{m}$.
3. $\phi = (1 - x_n\partial_n)(1 - x_{n-1}\partial_{n-1}) \cdots (1 - x_1\partial_1) = \sum_{\alpha \in \mathscr{B}_n} (-1)^{|\alpha|} x^\alpha \partial^\alpha$ *where* $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *and* $\partial^\alpha := \partial_n^{\alpha_n} \partial_{n-1}^{\alpha_{n-1}} \cdots \partial_1^{\alpha_1}$, *in the reverse order.*

Proof. — 1. By the very definition, ϕ_i is a right K_i-module endomorphism of Λ_n with $\phi_i(x_i) = x_i - x_i = 0$, hence ϕ_i is the projection onto K_i since $\Lambda_n = K_i \oplus x_i K_i$.

2. This follows from statement 1 and the decomposition $\Lambda_n = \oplus_{\alpha \in \mathscr{B}_n} K x^\alpha$.

3.

$$\begin{aligned}
\phi &= \sum_{i_1 > \cdots > i_k} (-1)^k x_{i_1} \partial_{i_1} x_{i_2} \partial_{i_2} \cdots x_{i_k} \partial_{i_k} \\
&= \sum_{i_1 > \cdots > i_k} (-1)^k (-1)^{1+2+\cdots+k-1} x_{i_1} \cdots x_{i_k} \partial_{i_1} \cdots \partial_{i_k} \\
&= \sum_{i_1 > \cdots > i_k} (-1)^k x_{i_1} \cdots x_{i_k} \partial_{i_k} \cdots \partial_{i_1} = \sum_{\alpha \in \mathscr{B}_n} (-1)^{|\alpha|} x^\alpha \partial^\alpha.
\end{aligned}$$
□

The next theorem gives a kind of the Taylor Formula which is more economical then the original Taylor Formula (no evaluation at 0).

THEOREM 2.3. — *For each* $a = \sum_{\alpha \in \mathscr{B}_n} a_\alpha x^\alpha \in \Lambda_n(K)$,

1. $a = \sum_{\alpha \in \mathscr{B}_n} \phi(\partial^\alpha(a)) x^\alpha$.
2. $a = \sum_{\alpha \in \mathscr{B}_n} \left(\sum_{\beta \in \mathscr{B}_n} (-1)^{|\beta|} x^\beta \partial^\beta \partial^\alpha(a)\right) x^\alpha$.

Proof. — 1. Note that $\partial^\alpha(a) \equiv a_\alpha \mod \mathfrak{m}$, hence $a_\alpha = \phi(\partial^\alpha(a))$, and so the result.

2. This follows from statement 1 and Lemma 2.2.(3). □

The Grassmann ring $\Lambda_n(K) = \oplus_{\alpha\in\mathscr{B}_n} x^\alpha K$ is a free right K-module of rank 2^n. The ring $\mathrm{End}_K(\Lambda_n)$ of right K-module endomorphisms of Λ_n is canonically isomorphic to the ring $M_{2^n}(K)$ of all $2^n \times 2^n$ matrices with entries from K by taking the matrix of map with respect to the canonical basis $\{x^\alpha, \alpha \in \mathscr{B}_n\}$ of Λ_n as the right K-module. We often identify these two rings. Let $\{E_{\alpha\beta} \mid \alpha, \beta \in \mathscr{B}_n\}$ be the matrix units of $M_{2^n}(K) = \oplus_{\alpha,\beta\in\mathscr{B}_n} KE_{\alpha\beta}$ (i.e. $E_{\alpha\beta}(x^\gamma) = \delta_{\beta\gamma}x^\alpha$). One can identify the ring Λ_n with its isomorphic copy in $\mathrm{End}_K(\Lambda_n)$ via the ring monomorphism $a \mapsto (x \mapsto ax)$.

Theorem 2.4. — *Recall that* $\phi := (1 - x_n\partial_n)(1 - x_{n-1}\partial_{n-1}) \cdots (1 - x_1\partial_1)$. *Then*

1. *for each* $\alpha, \beta \in \mathscr{B}_n$, $E_{\alpha\beta} = x^\alpha\phi\partial^\beta$.
2. $\mathrm{End}_K(\Lambda_n) = \oplus_{\alpha\in\mathscr{B}_n} \Lambda_n\partial^\alpha = \oplus_{\alpha\in\mathscr{B}_n} \partial^\alpha\Lambda_n$.

Proof. — 1. Each element $a = \sum_{\gamma\in\mathscr{B}_n} x^\gamma a_\gamma \in \Lambda_n$, $a_\gamma \in K$, can also be written as $a = \sum_{\gamma\in\mathscr{B}_n} a_\gamma x^\gamma$. As we have seen in the proof of Theorem 2.3, $a_\gamma = \phi\partial^\gamma(a)$, hence $x^\alpha\phi\partial^\beta(a) = x^\alpha a_\beta$ which means that $E_{\alpha\beta} = x^\alpha\phi\partial^\beta$.

2. Since $\partial_1, \dots, \partial_n$ are skew commuting skew derivations (i.e. $\partial_i\partial_j = -\partial_j\partial_i$), the following equality is obvious

$$\sum_{\alpha\in\mathscr{B}_n} \Lambda_n\partial^\alpha = \sum_{\alpha\in\mathscr{B}_n} \partial^\alpha\Lambda_n.$$

Now, $\mathrm{End}_K(\Lambda_n) = S := \sum_{\alpha\in\mathscr{B}_n} \Lambda_n\partial^\alpha$ since $E_{\alpha\beta} = x^\alpha\phi\partial^\beta \in S$. The sum S is a direct sum: suppose that $\sum u_\alpha\partial^\alpha = 0$ for some elements $u_\alpha \in \Lambda_n$ not all of which are equal to zero, we seek a contradiction. Let u_β be a nonzero element with $|\beta| = \beta_1 + \cdots + \beta_n$ the least possible. Then $0 = \sum u_\alpha\partial^\alpha(x^\beta) = u_\beta$, a contradiction. □

Let A be a ring. For $a, b \in A$, $\{a, b\} := ab + ba$ is called the *anti-commutator* of elements a and b.

Corollary 2.5. — *The ring* $\mathrm{End}_K(\Lambda_n)$ *is generated over* K *by elements* $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ *that satisfy the following defining relations: for all* i, j,

$$x_i^2 = 0, \qquad x_ix_j = -x_jx_i,$$
$$\partial_i^2 = 0, \qquad \partial_i\partial_j = -\partial_j\partial_i,$$
$$\partial_ix_j + x_j\partial_i = \delta_{ij}, \text{ the Kronecker delta.}$$

Proof. — It is straightforward that these relations hold. Theorem 2.4.(2) implies that these relations are defining. □

By Corollary 2.5, we have the duality K-automorphism of the ring $\mathrm{End}_K(\Lambda_n)$:

$$\Delta : x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad i = 1, \dots, n.$$

Clearly, $\Delta^2 = \mathrm{id}$. Similarly, by Corollary 2.5, we have the duality K-anti-automorphism of the ring $\mathrm{End}_K(\Lambda_n)$:

$$\nabla : x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad i = 1, \dots, n.$$

Clearly, the anti-automorphism ∇ is an *involution*, i.e. $\nabla^2 = \mathrm{id}$, $\nabla(ab) = \nabla(b)\nabla(a)$.

We say that the element $2 := 1 + 1 \in K$ is *regular* if $2\lambda = 0$ for some $\lambda \in K$ implies $\lambda = 0$. If K contains a field then 2 is regular iff the characteristic of K is not 2. If K is a commutative ring then the ring $\Lambda_n(K)$ is non-commutative iff $n \geq 2$ and $2 \neq 0$ in K. A commutative ring is called *reduced* if 0 is the only nilpotent element of the ring.

Lemma 2.6. — *If the ring K is commutative, $2 \in K$ is regular, and $n \geq 2$, then the centre of $\Lambda_n(K)$ is equal to*

$$Z(\Lambda_n) = \begin{cases} \Lambda_{n,\bar{0}}, & \text{if } n \text{ is even}, \\ \Lambda_{n,\bar{0}} \oplus K x_1 \cdots x_n, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. — Since $x_1, \dots, x_n$ are K-algebra generators for Λ_n, an element $u \in \Lambda_n$ is central iff it commutes with all x_i. Now, the result is obvious due to (1) and the fact that $x_i \cdot (x_1 \cdots x_n) = 0$, $i \geq 1$. □

Let $\Lambda'_{n,\bar{0}} := \oplus K x^\alpha$ where α runs through all even subsets of $\mathscr{B}_n$ distinct from $\{1, 2, \dots, n\}$. So, $\Lambda'_{n,\bar{0}} \subseteq \Lambda_{n,\bar{0}} \subseteq Z(\Lambda_n)$, and $\Lambda'_{n,\bar{0}} = \Lambda_{n,\bar{0}}$ iff n is odd.

Let $G := \mathrm{Aut}_K(\Lambda_n(K))$ be the group of K-automorphisms of the ring $\Lambda_n(K)$. Each K-automorphism $\sigma \in G$ is a uniquely determined by the images of the canonical generators:

$$x'_1 := \sigma(x_1), \dots, x'_n := \sigma(x_n).$$

Note that $x'_1 \dots, x'_n$ is another set of canonical generators for Λ_n.

Till the end of this section, let K be a *commutative* ring. Consider the subgroup G_{gr} of G, elements of which preserve the $\mathbb{Z}$-grading of Λ_n:

$$G_{gr} := \{\sigma \in G \,|\, \sigma(\Lambda_{n,i}) = \Lambda_{n,i} \text{ for all } i \in \mathbb{Z}\} = \{\sigma \in G \,|\, \sigma(\Lambda_{n,1}) = \Lambda_{n,1}\}.$$

The last equality is due to the fact that the $\Lambda_{n,1}$ generates Λ_n over K. Clearly, $\sigma \in G_{gr}$ iff $\sigma = \sigma_A$ where $\sigma_A(x_i) = \sum_{j=1}^n a_{ij} x_j$ for some $A = (a_{ij}) \in \mathrm{GL}_n(K)$. This can be written in the matrix form as $\sigma(x) = Ax$ where $x := (x_1, \dots, x_n)^t$ is the vector-column of indeterminates. Since $\sigma_A \sigma_B = \sigma_{BA}$, the group G_{gr} is canonically isomorphic to the group $\mathrm{GL}_n(K)^{op}$ *opposite* to $\mathrm{GL}_n(K)$ via the map $\mathrm{GL}_n(K)^{op} \to G_{gr}$, $A \mapsto \sigma_A$. We identify the group G_{gr} with $\mathrm{GL}_n(K)^{op}$

via this isomorphism. Note that $\mathrm{GL}_n(K) \to \mathrm{GL}_n(K)^{op}$, $A \mapsto A^{-1}$, is the isomorphism of groups. One can write

$$G_{gr} = \{\sigma_A \mid A \in \mathrm{GL}_n(K)\}.$$

From this moment and till the end of this section, K is a *reduced commutative* ring (if it is not stated otherwise). For $\sigma \in G$, let $x_i' := \sigma(x_i)$. Then $x_i'^2 = \sigma(x_i^2) = \sigma(0) = 0$. If $\lambda_i \equiv x_i' \mod \mathfrak{m}$ for some $\lambda_i \in K$ then $\lambda_i^2 = 0$, hence $\lambda_i = 0$ since K is reduced. Therefore, $\sigma(\mathfrak{m}) = \mathfrak{m}$, and so

$$\sigma(\mathfrak{m}^i) = \mathfrak{m}^i \text{ for all } i \geq 1. \tag{3}$$

This proves the next lemma.

LEMMA 2.7. — *If K is a reduced commutative ring and $\sigma \in G$ then $\sigma(x) = Ax + b$ for some $A \in \mathrm{GL}_n(K)$ and $b := (b_1, \ldots, b_n)^t$ where all $b_i \in \mathfrak{m}^2$.*

Consider the following subgroup of G,

$$U := \{\sigma \in G \mid \sigma(x) = x + b \text{ for some } b \in (\mathfrak{m}^2)^{\times n}\} = \{\sigma \in G \mid (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^2\}.$$

For each $\sigma \in G$ written as $\sigma(x) = Ax + b$ and each $\tau \in U$, we have $\sigma\sigma_{A^{-1}}, \sigma_{A^{-1}}\sigma \in U$, and $\sigma\tau\sigma^{-1} \in U$. These mean that U is a *normal* subgroup of G such that

$$G = G_{gr}U = UG_{gr}, \quad G_{gr} \cap U = \{e\}. \tag{4}$$

Therefore, G is a *skew product* of the groups G_{gr} and U:

$$G = G_{gr} \ltimes U = \mathrm{GL}_n(K)^{op} \ltimes U \simeq \mathrm{GL}_n(K) \ltimes U. \tag{5}$$

For each $i \geq 2$, consider the subgroup U^i of U:

$$U^i := \{\sigma \in U \mid (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^i\}. \tag{6}$$

By the very definition, $U = U^2 \supset U^3 \supset \cdots \supset U^n \supset U^{n+1} = \{e\}$. Note that $\sigma \in U^i$ iff, for all j, $\sigma(x_j) = x_j + m_j$ for some $m_j \in \mathfrak{m}^i$. Recall that $\sigma(\mathfrak{m}^i) = \mathfrak{m}^i$ for all $\sigma \in G$ and $i \geq 1$ (since K is a reduced commutative ring). For any $\tau \in G$, $\sigma \in U^i$, and x_j,

$$\tau\sigma\tau^{-1}(x_j) \equiv \tau(1 + \sigma - 1)\tau^{-1}(x_j) \equiv \tau\tau^{-1}(x_j) \equiv x_j \mod \mathfrak{m}^i.$$

Hence, each U^i is a *normal* subgroup of G. Each factor group U^i/U^{i+1} is abelian: Let $\sigma, \tau \in U^i$, $\sigma(x_j) = x_j + a_j + \cdots$ and $\tau(x_j) = x_j + b_j + \cdots$ for some elements $a_j, b_j \in \Lambda_{n,i}$ and three dots denote elements of $\mathfrak{m}^{i+1}$. Then $\sigma\tau(x_j) = x_j + a_j + b_j + \cdots$ and $\tau\sigma(x_j) = x_j + b_j + a_j + \cdots$. Therefore, $\sigma\tau U^{i+1} = \tau\sigma U^{i+1}$, and U is a nilpotent group.

Let K^n be the direct sum of n copies of the additive group $(K, +)$. Let $\theta := x_1 \cdots x_n$. The map

$$K^n \to U^n, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto (\sigma_\lambda : x_i \mapsto x_i + \lambda_i\theta), \tag{7}$$

is an isomorphism of groups ($\sigma_{\lambda+\mu} = \sigma_\lambda\sigma_\mu$). This follows directly from the fact that $\mathfrak{m}\theta = \theta\mathfrak{m} = 0$. So, $U^n = \{\sigma_\lambda \mid \lambda \in K^n\}$. One can easily verify that for any $\sigma \in G$ with $\sigma(x) = Ax + b$, $A = (a_{ij}) \in \mathrm{GL}_n(K)$,

$$\sigma^{-1}\sigma_\lambda\sigma = \sigma_{\frac{A\lambda}{\det(A)}}. \tag{8}$$

Indeed,

$$\begin{aligned}\sigma^{-1}\sigma_\lambda\sigma(x_i) &= \sigma^{-1}\sigma_\lambda(\sum_{j=1}^n a_{ij}x_j + b_i) = \sigma^{-1}(\sum_{j=1}^n a_{ij}x_j + b_i + \sum_{j=1}^n a_{ij}\lambda_j\theta)\\ &= \sigma^{-1}(\sigma(x_i) + \sum_{j=1}^n a_{ij}\lambda_j\theta) = x_i + \frac{\sum_{j=1}^n a_{ij}\lambda_j}{\det(A)}\theta.\end{aligned}$$

The equality (8) describes completely the group structure of $\mathrm{Aut}_K(\Lambda_2)$ (where K is a reduced commutative ring since $U = U^2$):

$$\mathrm{Aut}_K(\Lambda_2) = \mathrm{GL}_2(K)^{op}U = \{\sigma_A\sigma_\lambda \mid \sigma_A \in \mathrm{GL}_2(K)^{op}, \sigma_\lambda \in U\},$$
$$\sigma_A\sigma_\lambda \cdot \sigma_B\sigma_\mu = \sigma_{BA}\sigma_{\frac{B\lambda}{\det(B)}+\mu}.$$

An element $a(x_1, \dots, x_n) = \lambda + \cdots \in \Lambda_n$ is a unit iff $a(0, \dots, 0) = \lambda \in K$ is a unit. For each unit $a \in \Lambda_n$, the map $\omega_a(x) := axa^{-1}$ is called an *inner* automorphism of Λ_n. Since $\omega_a(x_i) = (\lambda + \cdots)x(\lambda + \cdots)^{-1} = \lambda x_i\lambda^{-1} + \cdots = x_i + \cdots$ we have $\omega_a \in U$, i.e. the group $\mathrm{Inn}(\Lambda_n)$ of all the inner automorphisms of Λ_n is a subgroup of U,

$$\mathrm{Inn}(\Lambda_n) \subseteq U. \tag{9}$$

Let us denote the automorphism $a \mapsto \overline{a}$ of Λ_n by s. Recall that $\Lambda_{n,\overline{0}} = \{a \in \Lambda_n \mid s(a) = a\}$ and $\Lambda_{n,\overline{1}} = \{a \in \Lambda_n \mid s(a) = -a\}$. Consider the subgroup $G_{\mathbb{Z}_2-gr}$ of G elements of which respect $\mathbb{Z}_2$-grading on Λ_n:

$$G_{\mathbb{Z}_2-gr} := \{\sigma \in G \mid \sigma(\Lambda_{n,\overline{0}}) = \Lambda_{n,\overline{0}},\ \sigma(\Lambda_{n,\overline{1}}) = \Lambda_{n,\overline{1}}\}.$$

Clearly,

$$G_{\mathbb{Z}_2-gr} = \{\sigma \in G \mid \sigma s = s\sigma\}. \tag{10}$$

Then

$$\Gamma := U \cap G_{\mathbb{Z}_2-gr} = \{\sigma \in U \mid \sigma s = s\sigma\} = \{\sigma \in U \mid \sigma(x_i) \in \Lambda_{n,\overline{1}}, 1 \leq i \leq n\} \tag{11}$$

is the subgroup of U (the last equality follows easily from the fact that the set $\Lambda_{n,\overline{1}}$ generates the K-algebra Λ_n and that $\Lambda_n = \Lambda_{n,\overline{0}} \oplus \Lambda_{n,\overline{1}}$ is a $\mathbb{Z}_2$-graded K-algebra). So, $\sigma \in \Gamma$ iff, for each $i = 1, \dots, n$,

$$\sigma(x_i) = x_i + a_{i,3} + a_{i,5} + \cdots + a_{i,2j+1} + \cdots,\ a_{i,2j+1} \in \Lambda_{n,2j+1}, \tag{12}$$

all summands are odd. Note that for arbitrary commutative ring K (not necessarily reduced) the set of automorphisms σ from (12) is a group Γ. Clearly, $\mathrm{GL}_n(K)^{op} \subseteq G_{\mathbb{Z}_2-gr}$ and $\mathrm{GL}_n(K)^{op} \cap \Gamma = \{e\}$ since $\mathrm{GL}_n(K)^{op} \cap U = \{e\}$ and $\Gamma \subseteq U$. The group Γ (over an arbitrary commutative ring K) can be defined as

$$\Gamma = \{\sigma \in G \,|\, \sigma(x_i) - x_i \in \Lambda_{n,\bar{1}} \cap \mathfrak{m}^3,\ i \geq 1\}. \tag{13}$$

The group Γ is endowed with the descending chain of its normal subgroups:

$$\Gamma = \Gamma^2 \supseteq \Gamma^3 \supseteq \cdots \supseteq \Gamma^i := \Gamma \cap U^i \supseteq \cdots \supseteq \Gamma^n \supseteq \Gamma^{n+1} = \{e\}.$$

Since $\Gamma^i = \{\sigma \in \Gamma \,|\, \sigma(x_j) - x_j \in \Lambda^{\mathrm{od}}_{n,\geq i},\ j = 1, \dots, n\}$, it is obvious that

$$\Gamma = \Gamma^2 = \Gamma^3 \supset \Gamma^4 = \Gamma^5 \supset \cdots \supset \Gamma^{2m} = \Gamma^{2m+1} \supset \cdots .$$

Recall that $[x, y] := xy - yx$ is the *commutator* of elements x and y, and $\omega_s : t \mapsto sts^{-1}$ is an inner automorphism.

Lemma 2.8. — *Let K be a commutative ring.*

1. *For each $a \in \Lambda^{\mathrm{od}}_n$, $a^2 = 0$.*
2. *$[\Lambda^{\mathrm{od}}_n, \Lambda_n] \subseteq \Lambda^{\mathrm{ev}}_n \subseteq Z(\Lambda_n)$ and $[\Lambda^{\mathrm{od}}_n, [\Lambda^{\mathrm{od}}_n, \Lambda_n]] = 0$.*
3. *For each $a \in \Lambda^{\mathrm{od}}_n$ and $x \in \Lambda_n$, $\omega_{1+a}(x) = x + [a, x]$.*
4. *For each $a, b \in \Lambda^{\mathrm{od}}_n$, $\omega_{1+a}\omega_{1+b} = \omega_{1+a+b} = \omega_{1+b}\omega_{1+a}$ and $\omega^{-1}_{1+a} = \omega_{1-a}$.*
5. *The map $\omega : \Lambda^{\mathrm{od}}_n \to U$, $a \mapsto \omega_{1+a}$, is a homomorphism of groups. It is a monomorphism if n is even and has the kernel $\ker(\omega) = Kx_1 \cdots x_n$ if n is odd.*
6. *For each $a \in \Lambda_n$, $a\bar{a} = \bar{a}a = a_0^2$ where $a = a_0 + a_1$, $a_0 \in \Lambda^{\mathrm{od}}_n$, $a_1 \in \Lambda^{\mathrm{ev}}_n$, $\bar{a} := a_0 - a_1$.*

Proof. — 1. For $a \in \Lambda^{\mathrm{od}}_n$, $a = \sum \lambda_\alpha x^\alpha$ where α runs through non-empty *odd* subsets of the set $\{1, \dots, n\}$. For any two such subsets α and β, $x^\alpha x^\beta = -x^\beta x^\alpha$ and $(x^\alpha)^2 = 0$. Now, $a^2 = \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta (x^\alpha x^\beta + x^\beta x^\alpha) + \sum_\alpha \lambda_\alpha^2 (x^\alpha)^2 = 0$.

2. $[\Lambda^{\mathrm{od}}_n, \Lambda_n] = [\Lambda^{\mathrm{od}}_n, \Lambda^{\mathrm{od}}_n + Z(\Lambda_n)] \subseteq [\Lambda^{\mathrm{od}}_n, \Lambda^{\mathrm{od}}_n] \subseteq \Lambda^{\mathrm{ev}}_n \subseteq Z(\Lambda_n)$. Then the second equality is obvious.

3. It suffices to prove the equality for monomials $x = x^\alpha$. If x^α is even, hence central, the equality is obvious. If x^α is odd then

$$\begin{aligned}\omega_{1+a}(x^\alpha) &= (1 + a)x^\alpha(1 + a)^{-1} \\ &= (1 + a)x^\alpha(1 - a) \qquad \text{(by statement 1)} \\ &= x^\alpha + [a, x^\alpha] - ax^\alpha a = x^\alpha + [a, x^\alpha] + a^2 x^\alpha \\ &= x^\alpha + [a, x^\alpha] \qquad \text{(by statement 1)}.\end{aligned}$$

4. For each $a, b \in \Lambda_n^{\text{od}}$ and $x \in \Lambda_n$,

$$\begin{aligned}\omega_{1+a}\omega_{1+b}(x) &= x + [a+b, x] + [a, [b, x]] = x + [a+b, x] \quad \text{(by statement 2)}\\ &= \omega_{1+a+b}(x) = \omega_{1+b+a}(x) = \omega_{1+b}\omega_{1+a}(x).\end{aligned}$$

$\omega_{1+a}^{-1} = \omega_{(1+a)^{-1}} = \omega_{1-a}$ since $a^2 = 0$, by statement 1.

5. By statement 4, the map is a group homomorphism. By statement 3, an element $a \in \Lambda_n^{\text{od}}$ belongs to the kernel iff $[a, x_i] = 0$ for all i iff $a \in Z(\Lambda_n) = \Lambda_n^{\text{ev}} + Kx_1 \cdots x_n$. Now, the result is obvious since a is odd.

6. $a\overline{a} = a_0^2 - a_1^2 = a_0^2$ and $\overline{a}a = a_0^2 - a_1^2 = a_0^2$ since $a_1^2 = 0$, by statement 1. □

Let Ω be the image of the group homomorphism ω in Lemma 2.8.(5),

$$\Omega := \operatorname{im}(\omega) = \{\omega_{1+a} \mid a \in \Lambda_n^{\text{od}}\}. \tag{14}$$

Lemma 2.9. — *Let K be a commutative ring. Then $\Omega = \operatorname{Inn}(\Lambda_n)$. In particular, the group $\operatorname{Inn}(\Lambda_n)$ of inner automorphisms of the Grassmann algebra $\Lambda_n(K)$ is an abelian group.*

Proof. — Let u be a unit of Λ_n. Then $u = \lambda + a + b$ for some $\lambda \in K^*$, an odd element $a \in \mathfrak{m}$, and an even element $b \in \mathfrak{m}$. Note that $\lambda + b$ is a central element and that the element $a' := \frac{a}{\lambda+b} \in \mathfrak{m}$ is odd. Now, $\omega_u = \omega_{(\lambda+b)(1+a')} = \omega_{\lambda+b}\omega_{1+a'} = \omega_{1+a'} \in \Omega$. Therefore, $\Omega = \operatorname{Inn}(\Lambda_n)$. □

So, Ω is a *normal abelian* subgroup of G. Since $\Omega = \operatorname{Inn}(\Lambda_n) \subseteq U$, by (9), the group Ω is endowed with the induced filtration

$$\Omega = \Omega^2 \supseteq \Omega^3 \supseteq \cdots \supseteq \Omega^i := \Omega \cap U^i \supseteq \cdots \supseteq \Omega^{n-1} \supseteq \Omega^n = \{e\}.$$

Note that $\Omega = \Omega^2 \supset \Omega^3 = \Omega^4 \supset \Omega^5 = \Omega^6 \supset \cdots \supset \Omega^{2i+1} = \Omega^{2i+2} \supset \cdots$. By Lemma 2.8.(5), the group Ω is canonically isomorphic to the factor group $\Lambda_n^{\text{od}}/\Lambda_n^{\text{od}} \cap Kx_1 \cdots x_n$. Under this isomorphism the filtration $\{\Omega^i\}$ coincides with the filtration on $\Lambda_n^{\text{od}}/\Lambda_n^{\text{od}} \cap Kx_1 \cdots x_n$ shifted by -1 that is the induced filtration from the $\mathfrak{m}$-adic filtration of the ring Λ_n, i.e. $\Omega^{2m} = \{\omega_{1+a} \mid a \in \Lambda_n^{\text{od}} \cap \mathfrak{m}^{2m-1}\}, m \geq 1$. Note that (Lemma 2.8.(3))

$$\omega_{1+a}(x_i) - x_i = [a, x_i] \in \Lambda_n^{\text{ev}} \text{ for all } a \in \Lambda_n^{\text{od}} \text{ and } i. \tag{15}$$

Generators for and dimension of the algebraic group U. — Define the function $\delta_{\cdot,ev}:\mathbb{N}\to\{0,1\}$ by the rule

$$\delta_{n,ev}=\begin{cases}1, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

For each $n\geq 2$, $[\frac{n}{2}]-\delta_{n,ev}$ (resp. $[\frac{n}{2}]$) is the number of odd (resp. even) numbers m such that $2\leq m\leq n$.

THEOREM 2.10. — *Let K be a commutative ring in statement 2, and let K be a reduced commutative ring with $\frac{1}{2}\in K$ in statements 1, 3, and 4. Let $\Lambda_n=\Lambda_n(K)$, $U=U(\Lambda_n)$, and $n\geq 2$. Then*

1. *the associated graded group $\prod_{i\geq 2}U^i/U^{i+1}$ is isomorphic to the direct product of d_n copies of the additive group K where*

$$d_n=n\sum_{m=1}^{[\frac{n}{2}]-\delta_{n,ev}}\binom{n}{2m+1}+\sum_{m=1}^{[\frac{n}{2}]}\binom{n}{2m-1}.$$

   In more detail,

2. *for each $m=1,\dots,[\frac{n}{2}]-\delta_{n,ev}$, the map*

$$(\Lambda_{n,2m+1})^n\to U^{2m+1}/U^{2m+2},\ a=(a_1,\dots,a_n)\mapsto\sigma_aU^{2m+2},$$

   is a group isomorphism where $\sigma_a\in U^{2m+1}:x_i\mapsto x_i+a_i$;

3. *for each $m=1,\dots,[\frac{n}{2}]$, the map*

$$\Lambda_{n,2m-1}\to U^{2m}/U^{2m+1},\ a\mapsto\omega_{1+a}U^{2m+1},$$

   is a group isomorphism where $\omega_{1+a}\in U^{2m}$ is the inner automorphism of $\Lambda_n:x\mapsto(1+a)x(1+a)^{-1}$.

4. *All the elements σ_a and ω_{1+b} from statements 2 and 3 are generators for the group U.*

Proof. — 1. The first statement follows directly from statements 2 and 3 since $\Lambda_{n,i}\simeq K^{\binom{n}{i}}$ for all $i=0,1,\dots,n$.

4. This statement follows from statements 1–3.

2. One can easily see that $\sigma_a\in U^{2m+1}$ for any $a\in(\Lambda_{n,2m+1})^n$; and $\sigma_a\sigma_bU^{2m+2}=\sigma_{a+b}U^{2m+2}$ for any a and b. By the very definition, the map $a\mapsto\sigma_aU^{2m+2}$ is an injection. It suffices to show that it is a surjection. Let σ be an arbitrary element of U^{2m+1}. Then $\sigma(x_i)=x_i+a_i+\cdots$ for some $a_i\in\Lambda_{n,2m+1}$ where the the dots mean bigger terms with respect to the $\mathbb{Z}$-grading on Λ_n. Then

$$\sigma_{-a}\sigma(x_i)=\sigma_{-a}(x_i+a_i+\cdots)=x_i-a_i+a_i+\cdots=x_i+\cdots,$$

hence $\sigma_{-a}\sigma\in U^{2m+2}$, i.e. $\sigma U^{2m+2}=\sigma_aU^{2m+2}$, and we are done.

3. Clearly, $\omega_{1+a} \in U^{2m}$ since $\omega_{1+a}(x_i) = x_i + [a, x_i]$ (Lemma 2.8.(3)). By Lemma 2.8.(5), the map $a \mapsto \omega_{1+a}U^{2m+1}$ is a group homomorphism. By the very definition, this map is injective. It suffices to show that it is surjective. This will be done in the next lemma in the proof of which an algorithm is given of how, for a given element σ of U^{2m}/U^{2m+1}, to find an element $a \in \Lambda_{n,2m-1}$ such that $\sigma \equiv \omega_{1+a}U^{2m+1}$. □

LEMMA 2.11. — *We keep the assumptions of Theorem 2.10.(3). For each* $m = 1, \dots, [\frac{n}{2}]$, *and each* $\sigma \in U^{2m}$, *there exist elements* $c_{i+1} \in K\lfloor x_{i+1}, \dots, x_n \rfloor_{2m-i}$, $i = 1, \dots, 2m$, *such that* $\omega\sigma \in U^{2m+1}$ *where* $\omega := \omega^{-1}_{1+x_1\cdots x_{2m-1}c_{2m+1}} \omega^{-1}_{1+x_1\cdots x_{2m-2}c_{2m}} \cdots \omega^{-1}_{1+x_1\cdots x_{i-1}c_{i+1}} \cdots \omega^{-1}_{1+x_1c_3}\omega^{-1}_{1+c_2}$.

Remark. If $n = 2m$ then $c_{2m+1} \in K$.

Proof. — Since each element $x_1 \cdots x_{i-1}c_{i+1}$ is a homogeneous element of Λ_n of degree $i - 1 + 2m - i = 2m - 1$, each automorphism $\omega_{1+x_1\cdots x_{i-1}c_{i+1}}$ belongs to the group U^{2m}, and so does their product ω. For each $i = 1, \dots, n$, let $x'_i := \sigma(x_i) = x_i + a_i + \cdots$ for some $a_i \in \Lambda_{n,2m}$. We prove the lemma in several steps.

Step 1. For each $i = 1, \dots, n$, $a_i = x_ib_i$ *for some element* $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor_{2m-1}$. Note that each element $a_i \in \Lambda_{n,2m} \subseteq Z(\Lambda_n)$ is central. Then $x'^2_i = \sigma(x^2_i) = 0$, and so

$$0 = x'^2_i = (x_i + a_i + \cdots)^2 = 2x_ia_i + \cdots$$

hence $x_ia_i = 0$ since $\frac{1}{2} \in K$; and so $a_i = x_ib_i$ for some element $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor_{2m-1}$.

Step 2. Let us prove that for each pair $i \neq j$,

$$b_i|_{x_j=0} = b_j|_{x_i=0}. \tag{16}$$

By Step 1, $x'_i = x_i(1 + b_i) + \cdots$ for all i. Since $b_i, b_j \in \Lambda_{n,2m-1}$ and $2m - 1 \geq 2 - 1 \geq 1$, we see that the homogeneous elements $b_i - b_j$ and b_ib_j have distinct degrees (if the elements are nonzero). Computing separately both sides of the equality $x'_ix'_j = -x'_jx'_i$ we have

$$x'_ix'_j = (x_i(1 + b_i) + \cdots)(x_j(1 + b_j) + \cdots) = x_ix_j(1 - b_i + b_j) + \cdots,$$
$$-x'_jx'_j = -x_jx_i(1 - b_j + b_i) + \cdots = x_ix_j(1 + b_i - b_j) + \cdots.$$

Since the degree of the elements $b_i - b_j$ is $2m - 1 \geq 2 - 1 \geq 1$ and $\frac{1}{2} \in K$, we must have

$$x_ix_j(b_i - b_j) = 0. \tag{17}$$

This equality is equivalent to the equality $b_i|_{x_i=0,x_j=0} = b_j|_{x_i=0,x_j=0}$ which is obviously equivalent to (16) (since each b_k does not depend on x_k, by Step 1).

Step 3. We are going to prove that *one can choose elements* $c_{i+1} \in K\lfloor x_{i+1}, \dots, x_n\rfloor_{2m-i}$, $i = 1, \dots, 2m$, *such that for each* $i = 1, \dots, 2m$,

$$\omega^{-1}_{1+x_1\cdots x_{i-1}c_{i+1}} \cdots \omega^{-1}_{1+x_1c_3}\omega^{-1}_{1+c_2}(x'_j) \equiv x_j \mod \mathfrak{m}^{2m+1}, \quad j = 1, \dots, i. \tag{18}$$

We use induction on i. Briefly, (18) will follow from (16). Note that $b_1 \in K\lfloor x_2, \dots, x_n\rfloor_{2m-1}$. Then $c_2 := -\frac{1}{2}b_1 \in K\lfloor x_2, \dots, x_n\rfloor_{2m-1}$, and

$$x'_1 = x_1 + x_1b_1 + \cdots = x_1 - b_1x_1 + \cdots = x_1 + [-\frac{1}{2}b_1, x_1] + \cdots = \omega_{1+c_2}(x_1) + \cdots,$$

hence $\omega^{-1}_{1+c_2}(x_1) \equiv x_1 \mod \mathfrak{m}^{2m+1}$.

Changing σ to $\omega^{-1}_{1+c_2}\sigma$, one can assume that $x'_1 = x_1 + \cdots$, i.e. $b_1 = 0$. Applying (16) in the case $j = 1$ and $i \geq 2$, we have $b_i|_{x_1=0} = 0$, hence $b_i = -2x_1c_{i+1}$ for unique $c_{i+1} \in K\lfloor x_2, \dots \widehat{x_i}, \dots, x_n\rfloor_{2m-2} \subseteq Z(\Lambda_n)$. Now,

$$x'_2 = x_2 + x_2(-2x_1c_3) + \cdots = x_2 + [x_1c_3, x_2] + \cdots = \omega_{1+x_1c_3}(x_2) + \cdots,$$

hence $\omega^{-1}_{1+x_1c_3}(x'_2) = x_2 + \cdots$. Note that $\omega^{-1}_{1+x_1c_3}(x'_1) = x_1 + \cdots$ since

$$\omega_{1+x_1c_3}(x'_1) = \omega_{1+x_1c_3}(x_1) + \cdots = x_1 + [x_1c_3, x_1] + \cdots = x_1 + \cdots.$$

This proves (18) for $i = 2$.

Let $i \geq 3$, and suppose that we have found already elements $c_{k+1} \in K\lfloor x_{k+1}, \dots x_n\rfloor_{2m-k}$, $k = 1, \dots, i$, that satisfy (18). We have to find c_{i+2}. Changing σ for $\omega^{-1}_{1+x_1\cdots x_{i-1}c_{i+1}} \cdots \omega^{-1}_{1+x_1c_3}\omega^{-1}_{1+c_2}\sigma$, if necessary, one can assume that $x'_k = x_k + \cdots$ for $k = 1, \dots, i$, i.e. $b_k = 0$ for $k = 1, \dots, i$. By (16),

$$b_{i+1}|_{x_k=0} = b_k|_{x_{i+1}=0} = 0, \quad k = 1, \dots, i,$$

hence $b_{i+1} = -2x_1 \cdots x_ic_{i+2}$ for a unique element $c_{i+2} \in K\lfloor x_{i+2}, \dots, x_n\rfloor_{2m-i-1}$. Now,

$$x'_{i+1} = x_{i+1} + x_{i+1}(-2x_1 \cdots x_ic_{i+2}) + \cdots = x_{i+1} + [x_1 \cdots x_ic_{i+2}, x_{i+1}] + \cdots$$
$$= \omega_{1+x_1\cdots x_ic_{i+2}}(x_{i+1}) + \cdots,$$

hence $\omega^{-1}_{1+x_1\cdots x_ic_{i+2}}(x'_{i+1}) = x_{i+1} + \cdots$. Note that $\omega^{-1}_{1+x_1\cdots x_ic_{i+2}}(x'_k) = x_k + \cdots$, $k = 1, \dots, i$, since

$$\omega^{-1}_{1+x_1\cdots x_ic_{i+2}}(x'_k) = \omega^{-1}_{1+x_1\cdots x_ic_{i+2}}(x_k + \cdots) = x_k + [x_1 \cdots x_ic_{i+2}, x_k] + \cdots = x_k + \cdots.$$

So, (18) holds for $i + 1$. By induction, (18) holds for all $i = 1, \dots, 2m$. In particular, it does for $i = 2m$. Then changing, if necessary, σ for $\omega^{-1}_{1+x_1\cdots x_{2m-1}c_{2m+1}} \cdots \omega^{-1}_{1+x_1c_3}\omega^{-1}_{1+c_2}\sigma$ one can assume that $x'_k = x_k + \cdots$ for $k = 1, \dots, 2m$, i.e. $b_k = 0$ for $k = 1, \dots, 2m$. Note that in order to prove Lemma 2.11, we have to show that $b_k = 0$ for $k = 1, \dots, n$. If $n = 2m$ we are done. If $n > 2m$ then by (16) for each $i > 2m$ and $j = 1, \dots, 2m$: $b_i|_{x_j=0} = b_j|_{x_i=0} = 0$. Hence, $b_i \in (x_1 \cdots x_{2m})$, but $b_i \in \Lambda_{n,2m-1}$, therefore $b_i = 0$. This proves Lemma 2.11 and Theorem 2.10. □

In Step 3, it was, in fact, proved that the conditions (18) *uniquely* determines the elements $c_2, \dots, c_{2m+1}$ (the idea of finding the elements c_i is to kill the 'leading term', this determines uniquely c_i by the expression given in the proof above). So, Lemma 2.11 can be strengthened as follows.

COROLLARY 2.12. — *The elements $c_2, \dots, c_{2m+1}$ from Lemma 2.11 are unique provided (18) holds for all $1 \le j \le i \le 2m$.*

Proof. — This fact also follows at once from Lemma 2.8 and Theorem 8.1.(1). □

COROLLARY 2.13. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, and $\sigma \in U$. Then σ is a (unique) product $\sigma = \cdots \omega_{1+a_5}\sigma_{b_5}\omega_{1+a_3}\sigma_{b_3}\omega_{1+a_1}$ for unique elements $a_i \in \Lambda_{n,i}$ and $b_j = (b_{j1}, \dots, b_{jn}) \in \Lambda^n_{n,j}$.*

By Corollary 2.13, the coefficients of the elements $\dots, a_5, b_5, a_3, b_3, a_1$ are coordinate functions for the algebraic group U over K. Therefore, the K-algebra of (regular) functions of the algebraic group U is a polynomial algebra over K in d_n variables where d_n is defined in Theorem 2.10.

It follows from Corollary 2.13 that

$$U^i = \Omega^i \rtimes \Gamma^i, \ i \ge 2. \tag{19}$$

The group structure of $G := \mathrm{Aut}_K(\Lambda_n(K))$

THEOREM 2.14. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then*

1. *The group $U = \Omega \rtimes \Gamma$ is the semi-direct product of its subgroups (Ω is a normal subgroup of U, $\Omega \cap \Gamma = \{e\}$, and $U = \Omega\Gamma$).*
2. *$G = U \rtimes \mathrm{GL}_n(K)^{op}$ (U is a normal subgroup of U, $U \cap \mathrm{GL}_n(K)^{op} = \{e\}$, and $G = U\mathrm{GL}_n(K)^{op}$).*
3. *$G = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$.*

Proof. — 1. By (13) and (15), $\Omega \cap \Gamma = \{e\}$. For any $\gamma \in \Gamma$ and $\omega_{1+a} \in \Omega$ (resp. $\omega_{1+a} \in \Omega^i := \Omega \cap U^i$, $i \ge 2$)

$$\gamma\omega_{1+a}\gamma^{-1} = \omega_{\gamma(1+a)} \in \Omega \quad (\text{resp. } \gamma\omega_{1+a}\gamma^{-1} = \omega_{\gamma(1+a)} \in \Omega^i, \ \ i \ge 2), \tag{20}$$

since $\Gamma\Lambda_n^{\mathrm{od}} \subseteq \Lambda_n^{\mathrm{od}}$ and $1 + a \in \Lambda_n^{\mathrm{od}}$ (resp. and $\Gamma\mathfrak{m}^i \subseteq \mathfrak{m}^i$, $i \ge 1$). So, in order to finish the proof of statement 1 it is enough to show that $U = \Gamma\Omega$. This equality follows from Corollary 2.13 and (20).

2. See (5).

3. This follows from statements 1 and 2. □

Let $B = B_n$ be the set of all the n-tuples (columns) $b = (b_1, \dots, b_n)^t$ where all b_i are arbitrary odd elements of Λ_n of the form $b_1 = x_1 + \cdots, \dots, b_n = x_n + \cdots$ where the three dots mean bigger terms. Then

$$\Gamma = \{\gamma_b \,|\, b \in B, \ \gamma_b(x_i) = b_i, \ i = 1, \dots, n\}$$

and the map $B \to \Gamma$, $b \mapsto \gamma_b$ is a bijection. The product of two elements $\gamma_b, \gamma_c \in \Gamma$ is given by the rule

$$\gamma_b\gamma_c = \gamma_{c\circ b}$$

where $c \circ b$ is the composition of functions; namely, the i'th coordinate $(c\circ b)_i$ of the n-tuple $c \circ b$ is equal to $c_i(b_1, \dots, b_n)$ where $c_i = c_i(x_1, \dots, x_n)$ (we have substituted elements b_i for x_i in the function $c_i = c_i(x_1, \dots, x_n)$).

By Theorem 2.14, each element σ of the group $G = \Omega\Gamma\mathrm{GL}_n(K)^{op}$ is the *unique* product

$$\sigma = \omega_{1+a}\gamma_b\sigma_A, \quad \omega_{1+a} \in \Omega, \ \gamma_b \in \Gamma, \ \sigma_A \in \mathrm{GL}_n(K)^{op}. \tag{21}$$

The product of two elements of G is given by the rule

$$\omega_{1+a}\gamma_b\sigma_A \cdot \omega_{1+a'}\gamma_{b'}\sigma_{A'} = \omega_{1+a+\gamma_b\sigma_A(a')}\, \gamma_{A^{-1}\sigma_A(b')\circ b}\, \sigma_{A'A} \tag{22}$$

where $\sigma_A(b') := (\sigma_A(b_1'), \dots, \sigma_A(b_n'))^t$ and $\sigma_A(b') \circ b := (\sigma_A(b_1') \circ b, \dots, \sigma_A(b_n') \circ b)$. This formula shows that the most sophisticated part of the group G is the group Γ. To prove (22), note first that $\sigma_A\gamma_{b'}\sigma_A^{-1} = \gamma_{A^{-1}\sigma_A(b')}$, then

$$\begin{aligned}
\omega_{1+a}\gamma_b\sigma_A \cdot \omega_{1+a'}\gamma_{b'}\sigma_{A'} &= \omega_{1+a} \cdot \gamma_b\sigma_A\omega_{1+a'}(\gamma_b\sigma_A)^{-1} \cdot \gamma_b \cdot \sigma_A\gamma_{b'}\sigma_A^{-1} \cdot \sigma_A\sigma_{A'} \\
&= \omega_{1+a}\omega_{1+\gamma_b\sigma_A(a')} \cdot \gamma_b\gamma_{A^{-1}\sigma_A(b')} \cdot \sigma_{A'A} \\
&= \omega_{1+a+\gamma_b\sigma_A(a')}\, \gamma_{A^{-1}\sigma_A(b')\circ b}\, \sigma_{A'A}.
\end{aligned}$$

We know how to find inverse elements for the group Ω ($\omega_{1+a}^{-1} = \omega_{1-a}$) and for the group $\mathrm{GL}_n(K)^{op}$ ($\sigma_A^{-1} = \sigma_{A^{-1}}$). The inversion formula for elements γ_b of the group Γ is given explicitly by Theorem 3.1. So, one can find explicitly an element b' such that $\gamma_b^{-1} = \gamma_{b'}$ by applying Theorem 3.1: $b_i' := \gamma_b^{-1}(x_i)$. Now, one can write down the explicit formula for the inverse of any element $\sigma = \omega_{1+a}\gamma_b\sigma_A \in G$,

$$(\omega_{1+a}\gamma_b\sigma_A)^{-1} = \omega_{1-\sigma_{A^{-1}}\,\gamma_{b'}(a)}\gamma_{A\sigma_{A^{-1}}(b')}\,\sigma_{A^{-1}}. \tag{23}$$

In more detail, by (22), we have $(\omega_{1+a}\gamma_b\sigma_A)^{-1} = \sigma_{A^{-1}}\gamma_{b'}\omega_{1-a} = \omega_{1-\sigma_{A^{-1}}\,\gamma_{b'}(a)}\gamma_{A\sigma_{A^{-1}}(b')}\,\sigma_{A^{-1}}$. □

COROLLARY 2.15. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then*

1. $G_{\mathbb{Z}_2-gr} = \Gamma \rtimes \mathrm{GL}_n(K)^{op}$.
2. $G = \Omega \rtimes G_{\mathbb{Z}_2-gr}$.
3. $\mathrm{Out}(\Lambda_n) \simeq G_{\mathbb{Z}_2-gr}$.

Proof. — By (22), $G' := \Gamma\mathrm{GL}_n(K)^{op} = \Gamma \rtimes \mathrm{GL}_n(K)^{op}$ is the subgroup of G such that $G = \Omega \rtimes G'$ (Theorem 2.14) and $G' \subseteq G_{\mathbb{Z}_2-gr}$. By (15), $G_{\mathbb{Z}_2-gr} \cap \Omega = \{e\}$, hence $G_{\mathbb{Z}_2-gr} = G'$ and $G = \Omega \rtimes G' = \Omega \rtimes G_{\mathbb{Z}_2-gr}$. Then $\mathrm{Out}(\Lambda_n) \simeq \Omega \rtimes G_{\mathbb{Z}_2-gr}/\Omega \simeq G_{\mathbb{Z}_2-gr}$. □

Let K be a commutative ring. Consider the sets of *even* and *odd* automorphisms of Λ_n:

$$G^{\rm ev} := \{\sigma \in G \mid \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm ev}, \forall i\},$$
$$G^{\rm od} := \{\sigma \in G \mid \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm od}, \forall i\}.$$

One can easily verify that $G^{\rm od}$ is a subgroup of G. It is not obvious from the outset that $G^{\rm ev}$ is also a subgroup of G.

LEMMA 2.16. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then*

1. $G^{\rm od} = G_{\mathbb{Z}_2-gr} = \Gamma \rtimes {\rm GL}_n(K)^{op}$.
2. $G^{\rm ev} = \Omega \rtimes {\rm GL}_n(K)^{op}$.
3. $G^{\rm od} \cap G^{\rm ev} = {\rm GL}_n(K)^{op}$.
4. $G = G^{\rm od}G^{\rm ev} = G^{\rm ev}G^{\rm od}$.

Proof. — 3. $G^{\rm od} \cap G^{\rm ev} = \{\sigma \in G \mid \sigma(x_i) \in \Lambda_{n,1}, \forall i\} = {\rm GL}_n(K)^{op}$.

4. This follows from statements 1 and 2 since $G = \Omega\Gamma{\rm GL}_n(K)^{op}$ (Theorem 2.14).

1 and 2. Recall that $G_{\mathbb{Z}_2-gr} = \Gamma{\rm GL}_n(K)^{op}$ (Corollary 2.15) and $G = \Omega\Gamma{\rm GL}_n(K)^{op} = \Omega G_{\mathbb{Z}_2-gr}$ (Theorem 2.14). Clearly, $G_{\mathbb{Z}_2-gr} \subseteq G^{\rm od}$ and $\Omega \cap G^{\rm od} = \{e\}$, it follows that

$$G^{\rm od} = (G^{\rm od} \cap \Omega)G_{\mathbb{Z}_2-gr} = G_{\mathbb{Z}_2-gr}.$$

Similarly, $\Omega{\rm GL}_n(K)^{op} \subseteq G^{\rm ev}$ and $\Gamma\cap G^{\rm ev} = \{e\}$ give the equality $\Omega{\rm GL}_n(K)^{op} = G^{\rm ev}$: $G^{\rm ev} = \Omega(\Gamma \cap G^{\rm ev}){\rm GL}_n(K)^{op} = \Omega{\rm GL}_n(K)^{op}$. □

Example. For $n = 2$, $G_2 = \Omega_2{\rm GL}_2(K)^{op} = \{\omega_{1+a}\sigma_A \mid a = \lambda_1x_1 + \lambda_2x_2, \lambda_i \in K, A \in {\rm GL}_n(K)\}$. The group Ω_2 is canonically isomorphic to K^2 via $\omega_{1+\lambda_1x_1+\lambda_2x_2} \mapsto \lambda := (\lambda_1, \lambda_2)^t$. Then $G_2 \simeq K^2{\rm GL}_2(K)^{op} = \{(\lambda, A) \mid \lambda \in K^2, A \in {\rm GL}_2(K)\}$ and

$$(\lambda, A) \cdot (\lambda', A') = (\lambda + A^t\lambda', A'A) \quad \text{and} \quad (\lambda, A)^{-1} = (-(A^t)^{-1}\lambda, A^{-1}).$$

Example. For $n = 3$, $G_3 = \Omega_3\Gamma_3{\rm GL}_3(K)^{op}$. The group Ω_3 is canonically isomorphic to K^3 via $\omega_{1+\lambda_1x_1+\lambda_2x_2+\lambda_3x_3} \mapsto \lambda := (\lambda_1, \lambda_2, \lambda_3)^t$. Similarly, the group Γ_3 is canonically isomorphic to K^3 via $\gamma_b = \gamma_{(x_1+\mu_1\theta, x_2+\mu_2\theta, x_3+\mu_3\theta)} \mapsto \mu = (\mu_1, \mu_2, \mu_3)^t$ where $\theta := x_1x_2x_3$. Then $G_3 \simeq K^3K^3{\rm GL}_3(K)^{op} = \{(\lambda, \mu, A) \mid \lambda, \mu \in K^3, A \in {\rm GL}_3(K)\}$ and

$$(\lambda, \mu, A) \cdot (\lambda', \mu', A') = (\lambda + A^t\lambda', \mu + \det(A)A^{-1}\mu', A'A),$$
$$(\lambda, \mu, A)^{-1} = (-(A^t)^{-1}\lambda, -\det(A^{-1}) \cdot A\mu, A^{-1}).$$

For $n = 2, 3$, the group $U := U_n = \Omega_n\Gamma_n$ is abelian: $U_2 = \Omega_2 \simeq K^2$ and $U_3 = \Omega_3\Gamma_3 = \Omega_3U_3^3 \simeq K^3 \times K^3$. The next result shows that these are the only cases where U is an abelian group.

Maximal abelian subgroups of U

THEOREM 2.17. — *Let K be a reduced commutative ring with $\frac{1}{2}\in K$. Then*

1. *The group Ω is a maximal abelian subgroup of U if n is even ($\Omega\supseteq U^n$).*
2. *The group $\Omega U^n=\Omega\times U^n$ is a maximal abelian subgroup of U if n is odd ($\Omega\cap U^n=\{e\}$).*

Proof. — Recall that Ω is an abelian subgroup of U (Theorem 2.8.(5)) and that $U=\Omega\Gamma(=\Omega\rtimes\Gamma)$ is the semidirect product of the groups Ω and Γ (Theorem 2.14.(1)). Suppose that Ω' is an abelian subgroup of U such that $\Omega\subseteq\Omega'$. Then $\Omega'=\Omega(\Gamma\cap\Omega')$. Each element $\gamma_b\in\Gamma\cap\Omega'$ must commute with all the elements of Ω: $\omega_{1+a}\gamma_b=\gamma_b\omega_{1+a}=\omega_{1+\gamma_b(a)}\gamma_b$ for all a iff $\omega_{1+\gamma_b(a)-a}=e$ iff $[\gamma_b(a)-a,x]=0$ for all $x\in\Lambda_n$ (since $\omega_{1+a'}(x)=x+[a',x]$, Lemma 2.8.(3)) iff $\gamma_b(a)-a\in Z(\Lambda_n)$, the centre of Λ_n, iff $\gamma_b(a)=a$ for all a if n is even; and $\gamma_b(a)-a\in Kx_1\cdots x_n$ if n is odd, iff $\gamma_b=e$ if n is even; and $\gamma_b\in U^n$ if n is odd. Since the group U^n is abelian and all elements of Ω commute with elements of U^n if n is odd, the result follows. □

3. The group Γ, its subgroups, and the Inversion Formula

In this section, the *inversion formula* (Theorem 3.1) is given for any automorphism $\sigma\in\Gamma\mathrm{GL}_n(K)^{op}$; the groups $\Gamma_{\mathbb{Z}_s-gr}$, $s\geq 2$, are found (Lemma 3.6); minimal sets of generators are given for the groups Γ and U (Theorem 3.11) and the commutator series are found for them; several important subgroups are introduced: Φ, Φ', $\Phi(i)$, $\mathscr{E}_{n,i}$, $\mathscr{E}'_{n,i}$, $\mathscr{E}''_{n,i}$.

The Jacobian and the inversion formula for automorphism. — Let K be a commutative ring and $\partial_1:=\frac{\partial}{\partial x_1},\dots,\partial_n:=\frac{\partial}{\partial x_n}$ be the left skew partial derivatives for Λ_n. For each automorphism $\sigma\in\Gamma\mathrm{GL}_n(K)^{op}$, the matrix of left skew partial derivatives $\frac{\partial\sigma}{\partial x}:=(\frac{\partial\sigma(x_i)}{\partial x_j})$ is called the *Jacobian* matrix for the automorphism σ. Note that the entries of the Jacobian matrix are even elements, hence central elements. The determinant $\mathscr{J}(\sigma):=\det(\frac{\partial\sigma(x_i)}{\partial x_j})$ is called the *Jacobian* of σ. One can easily verify that the '*chain rule*' holds for automorphisms $\sigma,\tau\in\Gamma\mathrm{GL}_n(K)^{op}$:

$$\frac{\partial(\sigma\tau)}{\partial x}=\sigma(\frac{\partial\tau}{\partial x})\cdot\frac{\partial\sigma}{\partial x}\tag{24}$$

where $\sigma(\frac{\partial\tau}{\partial x}):=(\sigma(\frac{\partial\tau(x_i)}{\partial x_j}))$. By taking the determinant of both sides we have the equality

$$\mathscr{J}(\sigma\tau)=\sigma(\mathscr{J}(\tau))\mathscr{J}(\sigma).\tag{25}$$

Then, for each $\sigma \in \Gamma\mathrm{GL}_n(K)^{op}$,

$$\mathcal{J}(\sigma^{-1}) = \sigma^{-1}(\mathcal{J}(\sigma)^{-1}). \tag{26}$$

Let $\sigma \in \Gamma\mathrm{GL}_n(K)^{op}$ and $x_1' := \sigma(x_1), \ldots, x_n' := \sigma(x_n)$. The elements $x_1', \ldots, x_n'$ are another set of canonical generators for Λ_n: $x_i'x_j' = -x_j'x_i'$ and $x_i'^2 = 0$. The corresponding left skew derivations $\partial_1' := \frac{\partial}{\partial x_1'}, \ldots, \partial_n' := \frac{\partial}{\partial x_n'}$ are equal to

$$\partial_i'(\cdot) := \frac{1}{\mathcal{J}(\sigma)} \det \begin{pmatrix} \frac{\partial \sigma(x_1)}{\partial x_1} & \cdots & \frac{\partial \sigma(x_1)}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_1}(\cdot) & \cdots & \frac{\partial}{\partial x_n}(\cdot) \\ \vdots & \vdots & \vdots \\ \frac{\partial \sigma(x_n)}{\partial x_1} & \cdots & \frac{\partial \sigma(x_n)}{\partial x_n} \end{pmatrix}, \quad i = 1, \ldots, n, \tag{27}$$

where we 'drop' $\sigma(x_i)$ in the determinant $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j})$ and $(\cdot)$ stands for the argument of function.

For each $i = 1, \ldots, n$, let

$$\phi_i' := 1 - x_i'\partial_i' : \Lambda_n \to \Lambda_n, \tag{28}$$

and (the order is important)

$$\phi_\sigma := \phi_n'\phi_{n-1}' \cdots \phi_1' = (1 - x_n'\partial_n')(1 - x_{n-1}'\partial_{n-1}') \cdots (1 - x_1'\partial_1') : \Lambda_n \to \Lambda_n. \tag{29}$$

The next theorem gives the inversion formula for automorphisms of the group $\Gamma\mathrm{GL}_n(K)^{op}$.

Theorem 3.1. — (The Inversion Formula) *Let K be a commutative ring, $\sigma \in \Gamma\mathrm{GL}_n(K)^{op}$ and $a \in \Lambda_n(K)$. Then*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathcal{B}_n} \phi_\sigma(\partial'^\alpha(a)) x^\alpha$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial'^\alpha := \partial_n'^{\alpha_n} \cdots \partial_1'^{\alpha_1}$.

Proof. — By Theorem 2.3.(1), $a = \sum_{\alpha \in \mathcal{B}_n} \phi_\sigma(\partial'^\alpha(a)) x'^\alpha$ where $x'^\alpha := x_1'^{\alpha_1} \cdots x_n'^{\alpha_n}$. Applying σ^{-1} we have the result

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathcal{B}_n} \phi_\sigma(\partial'^\alpha(a)) \sigma^{-1}(x'^\alpha) = \sum_{\alpha \in \mathcal{B}_n} \phi_\sigma(\partial'^\alpha(a)) x^\alpha. \qquad \square$$

The abelian groups of units E_n and E'_n. — Let K be a commutative ring and E_n be the group of units of the commutative algebra $\Lambda_n^{\rm ev}$. So, $E_n = K^* + \Lambda_{n,\geq 2}^{\rm ev}$ where K^* is the group of units of the ring K and $\Lambda_{n,\geq 2}^{\rm ev} := \mathfrak{m}^2 \cap \Lambda_n^{\rm ev} = \oplus_{m=1}^{[\frac{n}{2}]} \Lambda_{n,2m}$. There is the natural descending chain of subgroups of E_n determined by the $\mathfrak{m}$-adic filtration of the Grassmann algebra Λ_n:

$$(30) \qquad E_n = E_{n,2} \supset E_{n,4} \supset \cdots \supset E_{n,[\frac{n}{2}]} \supset E_{n,[\frac{n}{2}]+2} = K^*$$

where $E_{n,2m} := K^* + \mathfrak{m}^{2m} \cap \Lambda_n^{\rm ev} = K^* + \sum_{i=m}^{[\frac{n}{2}]} \Lambda_{n,2i}$.

Each element $e \in E_n$ is a unique sum $e = \lambda + e^+$ for some $\lambda \in K^*$ and $e^+ \in \mathfrak{m}^2 \cap \Lambda_n^{\rm ev}$. The map

$$v : E_n \to 2\mathbb{Z}, \quad e \mapsto v(e) = \max\{2m \,|\, e^+ \in E_{n,2m}\},$$

satisfies the following properties: for $e, f \in E_n$,

1. $v(ef) \geq \min\{v(e), v(f)\}$,
2. $v(e^{-1}) = v(e)$.

The group $E_n = K^* E'_n = K^* \times E'_n$ is the direct product of its subgroups K^* and $E'_n := 1 + \Lambda_{n,\geq 2}^{\rm ev}$ ($E_n = K^* E'_n$ and $K^* \cap E'_n = \{1\}$). The chain (30) induces the chain of subgroups in E'_n:

$$(31) \qquad E'_n = E'_{n,2} \supset E'_{n,4} \supset \cdots \supset E'_{n,[\frac{n}{2}]} \supset E'_{n,[\frac{n}{2}]+2} = \{1\}$$

where $E'_{n,2m} := E'_n \cap E_{n,2m} = 1 + \mathfrak{m}^{2m} \cap \Lambda_n^{\rm ev} = 1 + \sum_{i=m}^{[\frac{n}{2}]} \Lambda_{n,2i}$. If K is a reduced commutative ring then, by (3),

$$(32) \quad \sigma(E_{n,2m}) = E_{n,2m} \text{ and } \sigma(E'_{n,2m}) = E'_{n,2m} \text{ for all } \sigma \in G_{\mathbb{Z}_2-gr},\ m \geq 1.$$

It follows that (where K is reduced)

$$(33) \qquad v(\sigma(e)) = v(e) \text{ for all } e \in E_n,\ \sigma \in G_{\mathbb{Z}_2-gr}.$$

LEMMA 3.2. — *Let K be a commutative ring. Then*

1. *each element $\sigma \in \Gamma$ is a (unique) product $\sigma = \cdots \sigma_{b_7}\sigma_{b_5}\sigma_{b_3}$ for unique elements $b_i = (b_{i1}, \dots, b_{in}) \in \Lambda_{n,i}^n$ (see Corollary 2.13).*
2. *The elements $\{\gamma_{i,\lambda x^\alpha} \,|\, 1 \leq i \leq n, \lambda \in K, \alpha \in \mathcal{B}_n, 3 \leq |\alpha| \text{ is odd}\}$ are generators for the group Γ where $\gamma_{i,\lambda x^\alpha}(x_i) := x_i + \lambda x^\alpha$ and $\gamma_{i,\lambda x^\alpha}(x_j) := x_j$ for $i \neq j$.*

Proof. — 1. This follows from Theorem 2.10.(2).

2. This statement follows from statement 1 and Theorem 2.10.(2). □

The dimension of the algebraic group Γ. — Let K be a commutative ring and $\Lambda_n = \Lambda_n(K)$. A typical element of Γ is an automorphism $x_1 \mapsto x_1 + a_1, \dots, x_n \mapsto x_n + a_n$ where all $a_i \in \Lambda^{\mathrm{od}}_{n,\geq 3}$. The group Γ is a unipotent algebraic group over K where the coefficients of the elements a_i are the affine coordinates for the algebraic group Γ over K, and the algebra of (regular) functions $\mathscr{O}(\Gamma)$ of the algebraic group Γ is a polynomial algebra in

$$\dim(\Gamma) = n(2^{n-1} - n) \tag{34}$$

variables since

$$\dim(\Gamma) = \mathrm{rk}_K((\Lambda^{\mathrm{od}}_{n,\geq 3})^n) = n \sum_{i=1}^{[\frac{n-1}{2}]} \binom{n}{2i+1} = n(\sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} - n) = n(2^{n-1} - n).$$

If K is a field then $\dim(\Gamma)$ is the usual dimension of the algebraic group Γ.

A noncommutative analogue of the Taylor expansion

THEOREM 3.3. — (An analogue of the Taylor expansion) *Let K be a commutative ring, $f = f(x_1, \dots, x_n) = \sum x^\alpha \lambda_\alpha \in \Lambda_n$ where the coefficients $\lambda_\alpha \in K$ of f are written on the right, and $\sigma \in \Gamma$. Let $x_1' := \sigma(x_1) = x_1 + a_1, \dots, x_n' := \sigma(x_n) = x_n + a_n$ where $a_1, \dots, a_n$ are odd elements of $\mathfrak{m}^3$. Then*

$$(\sigma(f))(x_1, \dots, x_n) = f(x_1 + a_n, \dots, x_n + a_n) = \sum_{\alpha \in \mathscr{B}_n} a^\alpha \partial^\alpha(f)$$

where $a^\alpha := a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ and $\partial^\alpha = \partial_n^{\alpha_n} \cdots \partial_1^{\alpha_1}$, $\partial_i := \frac{\partial}{\partial x_i}$ are the left partial skew derivatives of Λ_n.

Proof. — It suffices to prove the statement for $f = x_1 \cdots x_m \lambda$ where $\lambda \in K$. Then

$$\begin{aligned}\sigma(f) = (x_1 + a_1) \cdots (x_m + a_m)\lambda &= \sum_{i_1 < \cdots < i_s} x_1 \cdots a_{i_1} \cdots a_{i_2} \cdots a_{i_s} \cdots x_m \lambda \\ &= \sum_{i_1 < \cdots < i_s} a_{i_1} \cdots a_{i_s} \partial_{i_s} \cdots \partial_{i_1}(f) = \sum_{\alpha \in \mathscr{B}_n} a^\alpha \partial^\alpha(f).\end{aligned}$$ □

The groups $\Gamma(s)$ and $\Omega(s)$. — The next Lemma introduces subgroups determined by even subgroups of $\mathbb{Z}$ (the subgroups of type $2m\mathbb{Z}$).

LEMMA 3.4. — *Let K be a commutative ring and $\Lambda_n = \Lambda_n(K)$.*

1. *For each even number $s = 2, 4, \dots, 2[\frac{n}{2}]$, the subset of Γ, $\Gamma(s) := \{\sigma \in \Gamma \mid \sigma(x_i) \in x_i + \sum_{j \geq 1} \Lambda_{n, 1+js}$ for all $i\}$ is a subgroup of Γ.*
2. *If $s | s'$ (s divides s') then $\Gamma(s') \subseteq \Gamma(s)$.*

Proof. — 1. It is easy to see that the set $\Gamma(s)$ is closed under multiplication and that it contains the identity. The fact that the set $\Gamma(s)$ is closed under the operation of taking inverse is a consequence of repeated use of Theorem 3.3. Let $\sigma\in\Gamma(s)$ and $x_i':=\sigma(x_i)=x_i-a_i$ for some odd element $a_i\in\sum_{j\geq 1}\Lambda_{n,1+js}$. Now,

$$\begin{aligned}
x_i &= x_i'+a_i(x_1,\dots,x_n)=x_i'+a_i(x_1'+a_1(x),\dots,x_n'+a_n(x))\\
&= x_i'+a_i(x_1',\dots,x_n')+\sum_{0\neq\alpha\in\mathscr{B}_n} a^\alpha(x)\partial^\alpha(a_i)(x')\\
&= x_i'+a_i(x')+\sum_{0\neq\alpha\in\mathscr{B}_n} a^\alpha(x_1'+a_1(x),\dots,x_n'+a_n(x))\partial^\alpha(a_i)(x')\\
&= x_i'+a_i(x')+\sum_{0\neq\alpha\in\mathscr{B}_n} a^\alpha(x')\partial^\alpha(a_i)(x')+\\
&\qquad \sum_{0\neq\alpha\in\mathscr{B}_n}(\sum_{0\neq\beta\in\mathscr{B}_n} a^\beta(x)\partial^\beta(a^\alpha)(x'))\partial^\alpha(a_i)(x')\\
&= \cdots,
\end{aligned}$$

keep going making substitutions $x_i=x_i'+a_i$ and then using Theorem 3.3 we get the result (in no more than $[\frac{n}{s}]+1$ steps).

2. This is obvious. □

Lemma 3.5. — *Let K be a commutative ring, and $s=2,4,\dots,2[\frac{n}{2}]$. Then*

1. *each element $\sigma\in\Gamma(s)$ is a (unique) product $s=\cdots\sigma_{b_{1+3s}}\sigma_{b_{1+2s}}\sigma_{b_{1+s}}$ for unique elements $b_i=(b_{i,1},\dots,b_{i,n})\in\Lambda_{n,i}^n$ (see Lemma 3.2).*
2. *The elements $\{\gamma_{i,\lambda x^\alpha}\,|\,1\leq i\leq n,\lambda\in K^*,\alpha\in\mathscr{B}_n$ such that $\alpha\in 1+s\mathbb{N}\}$ are generators for the group $\Gamma(s)$ (see Lemma 3.2).*

Proof. — These statements follow from Lemma 3.2. □

For each odd number s such that $1\leq s\leq n$, the set

$$\Omega(s):=\{\omega_{1+a}\,|\,a\in\sum_{j\geq 1}\Lambda_{n,js}\}=\{\omega_{1+a}\,|\,a\in\sum_{1\leq j\text{ is odd}}\Lambda_{n,js}\} \tag{35}$$

is a subgroup of Ω (Lemma 2.8.(4)). Note that $\Omega(1)=\Omega$ and $\Omega(n)=\{e\}$.

The groups $G_{\mathbb{Z}_s-gr}$. — Recall that $G_{\mathbb{Z}_s-gr}$ is the group of all K-automorphisms of the Grassmann algebra $\Lambda_n(K)$ that respect its $\mathbb{Z}_s$-grading ($\sigma\in G_{\mathbb{Z}_s-gr}$ iff $\sigma(\Lambda_{n,is})=\Lambda_{n,is}$, $i\geq 0$). The next result describes the groups $G_{\mathbb{Z}_s-gr}$.

Lemma 3.6. — *Let K be a reduced commutative ring with $\frac{1}{2}\in K$. Then*

1. *if s is even then $G_{\mathbb{Z}_s-gr}=\Gamma(s)\mathrm{GL}_n(K)^{op}=\Gamma(s)\rtimes\mathrm{GL}_n(K)^{op}$ and $\Gamma(s)=\Gamma\cap G_{\mathbb{Z}_s-gr}$;*

2. *if* $s \geq 3$ *is odd then* $G_{\mathbb{Z}_s-gr} = \Omega(s)\mathrm{GL}_n(K)^{op} = \Omega(s) \rtimes \mathrm{GL}_n(K)^{op}$ *and* $\Omega(s) = \Omega \cap G_{\mathbb{Z}_s-gr} = \{\omega_{1+a} \,|\, a \in \sum_{1\leq j \text{ is odd}} \Lambda_{n,sj}\}$.

Proof. — 1. The number s is even, hence $G_{\mathbb{Z}_s-gr} \subseteq G_{\mathbb{Z}_2-gr} = \Gamma\mathrm{GL}_n(K)^{op}$ (Lemma 2.15.(1)). Then it follows from the inclusion $\mathrm{GL}_n(K)^{op} \subseteq G_{\mathbb{Z}_s-gr}$ that $G_{\mathbb{Z}_s-gr} = (\Gamma \cap G_{\mathbb{Z}_s-gr})\mathrm{GL}_n(K)^{op}$. So, to finish the proof of statement 1 it suffices to show that $\Gamma(s) = \Gamma \cap G_{\mathbb{Z}_s-gr}$. The inclusion $\Gamma(s) \subseteq \Gamma \cap G_{\mathbb{Z}_s-gr}$ is obvious. If $e \neq \gamma \in \Gamma \cap G_{\mathbb{Z}_s-gr}$ then $\gamma = \cdots \sigma_{b_{2k+3}}\sigma_{b_{2k+1}}$ (Lemma 3.2) where $b_{2k+1} = (b_{2k+1,1}, \dots, b_{2k+1,n}) \neq 0$, $\gamma(x_i) = x_i + b_{2k+1,i} + \cdots$ for all i. Hence $2k+1 \in 1 + s\mathbb{Z}$, i.e. $\sigma_{b_{2k+1}} \in \Gamma(s)$. Applying the same argument to $\gamma\sigma^{-1}_{b_{2k+1}} = \cdots \sigma_{b_{2k+3}} \in \Gamma \cap G_{\mathbb{Z}_s-gr}$ and using induction on k we see that all $\sigma_{b_{2k+i}}$ in the product for γ belong to $\Gamma(s)$. Therefore, $\Gamma(s) = \Gamma \cap G_{\mathbb{Z}_s-gr}$.

2. By Lemma 2.8.(3), $\Omega \cap G_{\mathbb{Z}_s-gr} = \{\omega_{1+a} \,|\, a \in \sum_{1\leq j \text{ is odd}} \Lambda_{n,sj}\} = \Omega(s)$. Considering the action of automorphisms from the intersection $G_{\mathbb{Z}_s-gr} \cap \Omega\Gamma$ on the generators $x_1, \dots, x_n$ (with help of Corollary 2.13) it is easy to show that $G_{\mathbb{Z}_s-gr} \cap \Omega\Gamma = G_{\mathbb{Z}_s-gr} \cap \Omega$. Recall that $G = \Omega\Gamma\mathrm{GL}_n(K)^{op}$ and $\mathrm{GL}_n(K)^{op} \subseteq G_{\mathbb{Z}_s-gr}$ hence

$$G_{\mathbb{Z}_s-gr} = (G_{\mathbb{Z}_s-gr} \cap \Omega\Gamma)\mathrm{GL}_n(K)^{op} = \Omega(s)\mathrm{GL}_n(K)^{op} = \Omega(s) \rtimes \mathrm{GL}_n(K)^{op}. \quad \square$$

The groups Φ, $\Phi(i)$, and Φ'. — Let K be a commutative ring. Clearly,

$$\Gamma = \{\sigma : x_i \mapsto x_i(1+a_i) + b_i,\ i = 1, \dots, n\} \tag{36}$$

where $a_i, b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor$, $a_i \in \Lambda_n^{\mathrm{ev}} \cap \mathfrak{m}^2$ and $b_i \in \Lambda_n^{\mathrm{od}} \cap \mathfrak{m}^3$. Consider the subset Φ of Γ where all $b_i = 0$,

$$\Phi := \{\sigma : x_i \mapsto x_i(1+a_i),\ i = 1, \dots, n\}. \tag{37}$$

The set Φ can be characterized as

$$\Phi = \{\sigma \in \Gamma \,|\, \sigma(x_1) \in (x_1), \dots, \sigma(x_n) \in (x_n)\}.$$

Then it is obvious that $\Phi\Phi \subseteq \Phi$ and $\mathrm{id}_{\Lambda_n} \in \Phi$.

Lemma 3.7. — *Let K be a commutative ring. Then* $\Phi = \{\sigma \in \Gamma \,|\, \sigma((x_1)) = (x_1), \dots, \sigma((x_n)) = (x_n)\}$ *is a subgroup of* Γ.

Proof. — It remains to show that, for each $\sigma \in \Phi$, $\sigma^{-1}(x_i) \in (x_i)$ for all i. The equation $x_i' := \sigma(x_i) := x_i(1-a_i)$ can be written as $x_i = x_i' + x_i a_i(x_1, \dots, x_n)$ (we have changed the sign of the a_i for computational reason). Our aim is to show that $x_i = x_i'(1+a_i')$, then the result will follow as $x_i' \in (x_i)$. We use in turn, first, the substitution $x_i = x_i' + x_i a_i(x)$ and then the Taylor expansion (Theorem 3.3). After repeating these no more than $n+1$ times we will get the result (since all elements are nilpotent and any product of $n+1$ of them is zero).

$$\begin{aligned} x_i &= x'_i + x_i a_i(x) = x'_i + (x'_i + x_i a_i(x)) a_i(x'_1 + x_1 a_1(x), \dots, x'_n + x_n a_n(x)) \\ &= x'_i + (x'_i + x_i a_i(x))(a_i(x'_1, \dots, x'_n) \\ &\quad + \sum_{0 \neq \alpha \in \mathscr{B}_n} (xa(x))^\alpha \partial^\alpha(a_i)(x'_1, \dots, x'_n)) = \cdots . \end{aligned}$$ □

The group Φ is the solutions of the polynomial equations in coefficients of the elements a_i and b_j: $b_1 = 0, \dots, b_n = 0$. So, Φ is a closed subgroup of Γ with respect to the Zariski topology. The group Φ is an algebraic group, the algebra of functions on Φ is a polynomial algebra over K in $n \cdot \mathrm{rk}_K(\Lambda^{\mathrm{ev}}_{n-1,\geq 2}) = n(2^{n-2}-1)$ variables. So, the algebraic group Φ is affine and

$$\dim(\Phi) = n(2^{n-2}-1). \tag{38}$$

Note that, in general, the set $\{\sigma \in \Gamma \,|\, \sigma(x_1) = x_1 + b_1, \dots, \sigma(x_n) = x_n + b_n\}$ is not a subgroup of Γ where each $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor \cap \mathfrak{m}^3$ is odd.

For each $i = 1, \dots, n$, let

$$\Phi(i) := \{\sigma \in \Gamma \,|\, \sigma(x_i) \in (x_i)\} = \{\sigma \in \Gamma \,|\, \sigma(x_i) = x_i(1 + a_i)\}$$

where $a_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor^{\mathrm{ev}}_{\geq 2}$. Clearly, $\Phi(i)\Phi(i) \subseteq \Phi(i)$ and the set $\Phi(i)$ contains the identity map.

Lemma 3.8. — *Let K be a commutative ring. Then $\Phi(i) = \{\sigma \in \Gamma \,|\, \sigma((x_i)) = (x_i)\}$ is a subgroup of Γ.*

Proof. — It remains to prove that, given $\sigma \in \Phi(i)$, $\sigma^{-1} \in \Phi(i)$. Repeat word for word the proof of Lemma 3.7. Note that if K is a field the result is obvious since $\dim_K((x_i)) = \dim_K(\sigma((x_i)))$: it follows from $\sigma((x_i)) \subseteq (x_i)$ that $\sigma((x_i)) = (x_i)$, hence $\sigma^{-1}((x_i)) = (x_i)$, as required. □

It is obvious that $\Phi = \cap_{i=1}^n \Phi(i)$, and

$$\begin{aligned} \Phi(i_1, \dots, i_s) := \cap_{\nu=1}^s \Phi(i_\nu) &= \{\sigma \in \Gamma \,|\, \sigma(x_{i_1}) \in (x_{i_1}), \dots, \sigma(x_{i_s}) \in (x_{i_s})\} \\ &= \{\sigma \in \Gamma \,|\, \sigma((x_{i_1})) = (x_{i_1}), \dots, \sigma((x_{i_s})) = (x_{i_s})\} \end{aligned}$$

is a subgroup of Γ. The submonoid of G,

$$\Phi' := \{\sigma \in G \,|\, \sigma(x_1) \in (x_1), \dots \sigma(x_n) \in (x_n)\},$$

is a subgroup, as the next Lemma shows. Let $\mathbb{T}^n$ be the subgroup of all the diagonal matrices of $\mathrm{GL}_n(K)^{op}$, the n-dimensional torus.

Lemma 3.9. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then $\Phi' = (\Omega \rtimes \Phi) \rtimes \mathbb{T}^n$ and $\Phi' = \{\sigma \in G \,|\, \sigma((x_1)) = (x_1), \dots, \sigma((x_n)) = (x_n)\}$.*

Proof. — It is obvious that $\Omega \subseteq \Phi'$, $\mathbb{T}^n \subseteq \Phi'$, and $\Phi' \cap \Gamma\mathrm{GL}_n(K)^{op} = \Phi\mathbb{T}^n$. Since $G = \Omega\Gamma\mathrm{GL}_n(K)^{op}$ (Theorem 2.14.(3)),

$$\Phi' = \Omega(\Phi' \cap \Gamma\mathrm{GL}_n(K)^{op}) = \Omega\Phi\mathbb{T}^n = (\Omega \rtimes \Phi) \rtimes \mathbb{T}^n.$$

Then by Lemma 3.7, $\Phi' = \{\sigma \in G \mid \sigma((x_1)) = (x_1), \dots, \sigma((x_n)) = (x_n)\}$. □

The groups $\mathcal{E}_{n,i}$ and its subgroups. — Let K be a commutative ring. For each $i = 1, \dots, n$, the stabilizer of the elements x_j, $j \neq i$, in Γ,

$$\mathcal{E}_{n,i} := \{\gamma \in \Gamma \mid \gamma(x_j) = x_j, \forall j \neq i\},$$

is a subgroup of Γ. Clearly,

$$\mathcal{E}_{n,i} = \{\gamma_{1+a,b} : x_i \mapsto x_i(1+a) + b,\ x_j \mapsto x_j, \forall j \neq i\} \tag{39}$$

where $a \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{ev}}_{\geq 2}$ and $b \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3}$, and

$$\gamma_{1+a,b}\gamma_{1+a',b'} = \gamma_{(1+a)(1+a'),b(1+a')+b'}, \quad \gamma^{-1}_{1+a,b} = \gamma_{(1+a)^{-1},-(1+a)^{-1}b},$$

$$\gamma_{1+a,b}\gamma_{1+a',b'}\gamma^{-1}_{1+a,b} = \gamma_{1+a',(1+a)^{-1}(ba'+b')}, \qquad \gamma_{1+a,b} = \gamma_{1,(1+a)^{-1}b}\gamma_{1+a,0}.$$

Below, the equality (42) explains importance of these small subgroups. So, $\mathcal{E}'_{n,i} = \{\gamma_{1+a,0}\}$ and $\mathcal{E}''_{n,i} := \{\gamma_{1,b}\}$ are abelian subgroups of $\mathcal{E}_{n,i}$ such that $\mathcal{E}'_{n,i} \cap \mathcal{E}''_{n,i} = \{e\}$, $\mathcal{E}_{n,i} = \mathcal{E}''_{n,i}\mathcal{E}'_{n,i}$, and $\mathcal{E}''_{n,i}$ is a normal subgroup of $\mathcal{E}_{n,i}$ since

$$\gamma_{1+a,0}\gamma_{1,b}\gamma^{-1}_{1+a,0} = \gamma_{1,(1+a)^{-1}b}. \tag{40}$$

Therefore, $\mathcal{E}_{n,i} = \mathcal{E}''_{n,i} \rtimes \mathcal{E}'_{n,i}$. Clearly, $\mathcal{E}'_{n,i} = \mathcal{E}_{n,i} \cap \Phi$.

Let $E'_{n,\widehat{i}}$ be the group of units E'_{n-1} in the case of the Grassmann algebra $K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor$, i.e.

$$E'_{n,\widehat{i}} := \{1 + a \mid a \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{ev}}_{\geq 2}\}.$$

Lemma 3.10. — *Let K be a commutative ring. Then*

1. $\mathcal{E}_{n,i} = \mathcal{E}''_{n,i} \rtimes \mathcal{E}'_{n,i}$.
2. *The map* $E'_{n,\widehat{i}} \to \mathcal{E}'_{n,i}$, $1 + a \mapsto \gamma_{1+a,0}$, *is a group isomorphism.*
3. *The map* $K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3} \to \mathcal{E}''_{n,i}$, $b \mapsto \gamma_{1,b}$, *is a group isomorphism.*
4. $\mathcal{E}_{n,i} = \langle \gamma_{1+\lambda x^\alpha,0}, \gamma_{1,\lambda x_j x_k x_l} \mid \lambda \in K, |\alpha| = 2,\ j < k < l, \alpha \cup \{j,k,l\} \subseteq \{1, \dots, \widehat{i}, \dots, n\}\rangle$.

Proof. — Statements 1–3 are obvious. Statement 4 follows from statement 1 and the two facts: (i) $\{\gamma_{1+\lambda x^\alpha,0}\}$ are generators for the group $\mathcal{E}'_{n,i}$ since $\{1 + \lambda x^\alpha\}$ are generators for the group $E'_{n,\widehat{i}}$, and (ii)

(41)
$$\gamma_{1+a,0}\gamma_{1,b}\gamma^{-1}_{1+a,0}\gamma^{-1}_{1,b} = \gamma_{1,(1+a)^{-1}b-b} = \gamma_{1,-ab+a^2b-a^3b+\cdots} = \gamma_{1,-ab}\gamma_{1,a^2b-a^3b+\cdots}.$$

The proof of the Lemma is complete. □

For each $j \geq 2$, let $\{\mathscr{E}_{n,i}^{j} := \mathscr{E}_{n,i} \cap U^j\}$ be the induced (descending) filtration on the group $\mathscr{E}_{n,i}$. Each subgroup $\mathscr{E}_{n,i}^{j} = \{\sigma \in \mathscr{E}_{n,i} \,|\, (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^j\}$ is a normal subgroup of $\mathscr{E}_{n,i}$. By Lemma 3.2 and Theorem 2.10.(2), the group Γ is a *finite* product

$$\Gamma = \cdots \prod_{i=1}^{n} \mathscr{E}_{n,i}^{[2m+1]} \cdots \prod_{i=1}^{n} \mathscr{E}_{n,i}^{[5]} \cdot \prod_{i=1}^{n} \mathscr{E}_{n,i}^{[3]}, \tag{42}$$

where

$$\mathscr{E}_{n,i}^{[2m+1]} := \{\gamma_{1+a,b} \in \mathscr{E}_{n,i} \,|\, a \in K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_n \rfloor_{2m}, \\ b \in K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_n \rfloor_{2m+1}\}.$$

Clearly, $\mathscr{E}_{n,i}^{[2m+1]} \subseteq \mathscr{E}_{n,i}^{2m+1}$.

Minimal sets of generators for the groups Γ, U, and Φ. — For each $i = 1, \ldots, n$; $\lambda \in K$, and $\alpha \subseteq \{1, \ldots, n\}$, $3 \leq |\alpha|$ is odd, let us consider the automorphism of Γ,

$$\sigma_{i,\lambda x^\alpha} : x_i \mapsto x_i + \lambda x^\alpha, \; x_j \mapsto x_j, \; \forall j \neq i.$$

Then

$$\sigma_{i,\lambda x^\alpha}^{-1} = \sigma_{i,-\lambda x^\alpha}. \tag{43}$$

For two elements a and b of a group A, the *group commutator* of the elements a and b is defined as $[a,b] := aba^{-1}b^{-1} \in A$. A direct (rather lengthy) calculation shows that

$$[\sigma_{i,\lambda x_i x_j x^\alpha}, \sigma_{j,\mu x_j x^\beta}] = \sigma_{i,-\lambda\mu x_i x_j x^\beta x^\alpha} \tag{44}$$

for all $\lambda, \mu \in K$; $i \neq j$; α and β are subsets of $\{1, \ldots, n\} \backslash \{i,j\}$ such that $\alpha \cap \beta = \varnothing$, α is odd and β is even, $|\alpha| \geq 1$ and $|\beta| \geq 2$. Similarly,

$$[\sigma_{i,\lambda x_i x^\alpha}, \sigma_{i,\mu x^\beta}] = \sigma_{i,-\lambda\mu x^\beta x^\alpha} \tag{45}$$

for all $\lambda, \mu \in K$; $i = 1, \ldots, n$; and $\alpha, \beta \subseteq \{1, \ldots, n\} \backslash \{i\}$ such that $\alpha \cap \beta = \varnothing$, α is even and β is odd, $|\alpha| \geq 2$ and $|\beta| \geq 3$.

For a group A, let us consider its series of commutators:

$$A^{(0)} := A, \; A^{(i)} := [A, A^{(i-1)}], \; i \geq 1.$$

For each $i = 1, \ldots, n$; $\lambda \in K$; and $j < k < l$, let us consider the automorphism $\sigma_{i,\lambda x_j x_k x_l} \in \Gamma$: $x_i \mapsto x_i + \lambda x_j x_k x_l$, $x_m \mapsto x_m$, for all $m \neq i$. Then

$$\sigma_{i,\lambda x_j x_k x_l} \sigma_{i,\mu x_j x_k x_l} = \sigma_{i,(\lambda+\mu) x_j x_k x_l}, \quad \sigma_{i,\lambda x_j x_k x_l}^{-1} = \sigma_{i,-\lambda x_j x_k x_l}.$$

So, the set $\{\sigma_{i,\lambda x_j x_k x_l} \,|\, \lambda \in K\}$ is isomorphic to the additive abelian group K, $\sigma_{i,\lambda x_j x_k x_l} \mapsto \lambda$.

THEOREM 3.11. — *Let K be a commutative ring in statements 1 and 2; and let K be a reduced commutative ring with $\frac{1}{2} \in K$ in statements 3 and 4. Then*

1. *the group Γ is generated by all the automorphisms $\sigma_{i,\lambda x_j x_k x_l}$, i.e. $\Gamma = \langle \sigma_{i,\lambda x_j x_k x_l} \mid i = 1,\dots,n; \lambda \in K; j < k < l\rangle$. The subgroups $\{\sigma_{i,\lambda x_j x_k x_l}\}_{\lambda\in K}$ of Γ form a minimal set of generators for Γ.*
2. $\Gamma^{(i)} = \Gamma^{2i+3} := \{\sigma \in \Gamma \mid (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^{2i+3}\}$, $i \geq 0$.
3. *The group U is generated by all the automorphisms $\sigma_{i,\lambda x_j x_k x_l}$ and all the automorphisms $\omega_{1+\lambda x_i}$, i.e. $U = \langle \sigma_{i,\lambda x_j x_k x_l}, \omega_{1+\lambda x_i} \mid, i = 1,\dots,n; \lambda \in K; j < k < l\rangle$. The subgroups $\{\sigma_{i,\lambda x_j x_k x_l}\}_{\lambda\in K}$, $\{\omega_{1+\lambda x_i}\}_{\lambda\in K}$ of U form a minimal set of generators for U.*
4. $U^{(i)} = U^{i+2} := \{\sigma \in U \mid (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^{i+2}\}$, $i \geq 0$.

Proof. — 1. In view of (42) and Lemma 3.10.(4), it suffices to show that each automorphism $\gamma_{1+\lambda x^\alpha,0}$ from Lemma 3.10.(4) is a product of the generators from statement 1. In the σ-notation, the automorphism $\gamma_{1+\lambda x^\alpha,0}$ is of the form

$$\sigma := \sigma_{i_1,\lambda x_{i_1} x_{i_2}(x_{i_3}x_{i_4})\cdots(x_{i_{2m-1}}x_{i_{2m}})(x_{i_{2m+1}}x_{i_{2m+2}})x_{i_{2m+3}}} \tag{46}$$

for some *distinct* elements $i_1, i_2, \dots, i_{2m+3}$, $m \geq 0$, and $\lambda \in K$. The result is obvious for $m = 0$. So, let $m \geq 1$. Then, applying (44) m times we have

$$\sigma = [\dots[\sigma_{i_1,\lambda x_{i_1}x_{i_2}x_{i_{2m+3}}}, \sigma_{i_2,-x_{i_2}x_{i_{2m+1}}x_{i_{2m+2}}}], \sigma_{i_2,-x_{i_2}x_{i_{2m-1}}x_{i_{2m}}}], \dots], \sigma_{i_2,-x_{i_2}x_{i_3}x_{i_4}}]. \tag{47}$$

The claim that the 'one dimensional' abelian subgroups $\{\sigma_{i,\lambda x_j x_k x_l}\}$ of Γ form a minimal set of generators is obvious due to the isomorphism in Theorem 2.10.(2) in the case $m = 1$ there.

2. By Theorem 2.10.(2), $\Gamma^{(m)} \subseteq \Gamma^{2m+3}$ for all $m \geq 0$. Clearly, $\Gamma^{(n)} = \{e\} = \Gamma^{2n+3}$. Now, using downward induction on m (starting with $m = n$), in view of Theorem 2.10.(2) and (47), in order to prove the equality $\Gamma^{(m-1)} = \Gamma^{2m+1}$, it suffices to show that each automorphism of the type

$$\sigma = \sigma_{i,\lambda x_{i_1}x_{i_2}x_{i_3}(x_{i_4}x_{i_5})\cdots(x_{i_{2m-2}}x_{i_{2m-1}})(x_{i_{2m}}x_{i_{2m+1}})}$$

(where the elements $i, i_1, i_2, \dots, i_{2m+1}$ are *distinct*, and $\lambda \in K$) can be expressed as $(m-1)$-commutator of the generators from statement 1 (i.e. $m-1$ brackets are involved). Below is such a presentation (apply (45))

$$\sigma = [\sigma_{i,-\lambda x_i x_{i_{2m}} x_{i_{2m+1}}}, [\sigma_{i,-x_i x_{i_{2m-2}} x_{i_{2m-1}}} [\dots [\sigma_{i,-x_i x_{i_4} x_{i_5}}, \sigma_{i,x_{i_1}x_{i_2}x_{i_3}}]\dots]. \tag{48}$$

3. By Theorem 2.10 and statement 1, it suffices to prove that any inner automorphism $\omega_{1+\lambda x_{i_1}x_{i_2}\cdots x_{i_{2m+1}}}$, $i_1 < \cdots < i_{2m+1}$, $m \geq 0$, $\lambda \in K$, is a product of the generators from statement 3. For any automorphism $\sigma \in G$ and an odd element $a \in \Lambda_n$ (Lemma 2.8.(2,4)):

$$[\sigma, \omega_{1+a}] = \sigma\omega_{1+a}\sigma^{-1}\omega_{1+a}^{-1} = \omega_{1+\sigma(a)-a}. \tag{49}$$

Applying this formula m times we have

(50)

$$\omega_{1+\lambda x_{i_1}x_{i_2}\cdots x_{i_{2m+1}}} = [\sigma_{i_1,x_{i_1}x_{i_2}x_{i_3}}, [\sigma_{i_1,x_{i_1}x_{i_4}x_{i_5}}[\ldots[\sigma_{i_1,x_{i_1}x_{i_{2m}}x_{i_{2m+1}}}, \omega_{1+\lambda x_{i_1}}]\ldots].$$

The statement that generators in statement 3 are minimal follows from the isomorphisms in Theorem 2.10.(2,3) for $m=1$ there.

4. By Theorem 2.10.(2,3), $U^{(m)}\subseteq U^{m+2}$, $m\geq 0$. Clearly, $U^{(n)}=\{e\}=U^{n+2}$. Now, using downward induction on m (starting with $m=n$), in view of Theorem 2.10.(2,3), statement 2 and (50), we have $U^{(m)}=U^{m+2}$ for all $m\geq 0$. □

Corollary 3.12. — *Let K be a commutative ring. Then*

1. *the group Φ is generated by all the automorphisms $\sigma_{i,\lambda x_i x_k x_l}$, i.e. $\Phi=\langle \sigma_{i,\lambda x_i x_k x_l}\,|\, i=1,\ldots,n; \lambda\in K; k<l; i\notin\{k,l\}\rangle$. The subgroups $\{\sigma_{i,\lambda x_i x_k x_l}\}_{\lambda\in K}$ of Φ form a minimal set of generators for Φ.*
2. $\Phi^{(i)}=\Phi^{2i+3}:=\{\sigma\in\Phi\,|\,(\sigma-1)(\mathfrak{m})\subseteq\mathfrak{m}^{2i+3}\}$, $i\geq 0$.
3. *Each element $\sigma\in\Phi$ is a unique finite product $\sigma=\cdots\sigma_{b_7}\sigma_{b_5}\sigma_{b_3}$ for unique elements $b_i:=(b_{i1},\ldots,b_{in})\in\Lambda^n_{n,i}$ (see Corollary 2.13) such that $b_{ij}\in(x_j)$ for all j.*

Proof. — 3. Statement 3 follows from Lemma 3.2 and Theorem 2.10.(2).

1. By statement 3, the elements of the type (46) are generators for the group Φ. Then the result follows from (47).

2. By Theorem 2.10.(2), $\Phi^{(i)}\subseteq\Phi^{2i+3}$ for all $i\geq 0$. The reverse inclusion follows from (47). □

4. The Jacobian group Σ and the equality $\Sigma=\Sigma'\Sigma''$

The Jacobian map. — Let K be a commutative ring. Recall that the group Γ consists of all automorphisms $\gamma_b: x_1\mapsto x_1+b_1,\ldots,x_n\mapsto x_n+b_n$, where $b:=(b_1,\ldots,b_n)$ is an n-tuple of odd elements of $\mathfrak{m}^3$. Consider the matrix $B:=\frac{\partial b}{\partial x}:=(\frac{\partial b_i}{\partial x_j})$ of the *skew gradients* $\mathrm{grad}(b_i):=(\frac{\partial b_i}{\partial x_1},\ldots,\frac{\partial b_i}{\partial x_n})$ for the element $b=(b_1,\ldots,b_n)$ (where $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ are the left partial skew K-derivatives of $\Lambda_n(K)$), and its *characteristic polynomial*

$$\det(t+B)=t^n+\sum_{i=1}^n \mathrm{tr}_i(B)t^{n-i}.$$

Clearly, $\mathrm{tr}_1(B)=\mathrm{tr}(B)=\sum_{i=1}^n\frac{\partial b_i}{\partial x_i}$ is the trace of the matrix B, $\mathrm{tr}_n(B)=\det(B)$ is its determinant, and $\mathrm{tr}_i(B)=\sum_{1\leq j_1<\cdots<j_i\leq n}\det(\frac{\partial b_{j_\mu}}{\partial x_{j_\nu}})_{\mu,\nu=1,\ldots,i}$.

Now, the jacobian of the automorphism γ_b is given by the rule

$$\mathscr{J}(\gamma_b) = \det(t+B)|_{t=1} = 1 + \sum_{i=1}^{n} \mathrm{tr}_i(B). \tag{51}$$

Note that the sum of the traces above is an element of $\mathfrak{m}^2$ since $\mathrm{tr}_i(B) \in \mathfrak{m}^{2i}$, $i \geq 1$. So, the *Jacobian map* is given by the rule

$$\mathscr{J} : \Gamma \to E'_n, \gamma_b \mapsto \mathscr{J}(\gamma_b) = \det(t+B)|_{t=1} = 1 + \sum_{i=1}^{n} \mathrm{tr}_i(B). \tag{52}$$

It is a polynomial map in the coefficients of the elements $b_1, \dots, b_n$. Recall that the abelian multiplicative group of units E'_n is equal to $E'_n = 1 + \sum_{i \geq 1} \Lambda_{n,2i}$.

The Jacobian group Σ. — Let K be a commutative ring. Despite the fact that the jacobian map $\mathscr{J} : \Gamma \to E'_n$ is *not* a group homomorphism, its 'kernel'

$$\Sigma := \{\sigma \in \Gamma \,|\, \mathscr{J}(\sigma) = 1\}$$

is a *subgroup* of Γ as it easily follows from (25) and (26). We call Σ the *Jacobian group*. This is a sophisticated subgroup of Γ. By (52), the elements of the group Σ are solutions to the system of polynomial equations over K

$$\Sigma = \{\gamma_b \in \Gamma \,|\, \sum_{i=1}^{n} \mathrm{tr}_i(B) = 0\}. \tag{53}$$

So, Σ is a unipotent algebraic group over the ring K. If K is a field then Σ is an algebraic group over K (in the usual sense). By Theorem 2.3.(1), the system of polynomial equations in (53) can be made explicit

$$\Sigma = \{\gamma_b \in \Gamma \,|\, \phi(\partial^\alpha(\sum_{i=1}^{n} \mathrm{tr}_i(B))) = 0, \quad \text{for all even} \quad 0 \neq \alpha \in \mathscr{B}_n\}. \tag{54}$$

The Jacobian group Σ is the solution to the non-linear system (53) of skew differential operators equations. It looks like this is the first example of a group of this kind. The Jacobian group Σ is a closed subgroup of Γ in the Zariski topology. The group Γ contains the descending chain of its normal subgroups

$$\Gamma = \Gamma^3 \supset \Gamma^5 \supset \cdots \supset \Gamma^{2m+1} := \Gamma \cap U^{2m+1} \supset \cdots \supset \Gamma^{2[\frac{n}{2}]+1} \supseteq \Gamma^{2[\frac{n}{2}]+3} = \{e\},$$

with the abelian factors $\Gamma^{2m+1}/\Gamma^{2m+3}$ where $\Gamma^{2m+1} = \{\sigma \in \Gamma \,|\, (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^{2m+1}\}$.

The group Σ contains the descending chain of its normal subgroups

$$\Sigma = \Sigma^3 \supseteq \Sigma^5 \supseteq \cdots \supseteq \Sigma^{2m+1} := \Sigma \cap \Gamma^{2m+1} \supseteq \cdots \supseteq \Sigma^{2[\frac{n}{2}]+1} \supseteq \Sigma^{2[\frac{n}{2}]+3} = \{e\}$$

with the abelian factors $\{\Sigma^{2m+1}/\Sigma^{2m+3}\}$ since $\Sigma^{2m+1}/\Sigma^{2m+3}\subseteq \Gamma^{2m+1}/\Gamma^{2m+3}$, the abelian group.

The Jacobian group Σ and the image of the Jacobian map for $n=3$. — For $n=3$,

$$\Gamma=\{\sigma_\lambda\,|\,\sigma(x_1)=x_1(1+\lambda_1x_2x_3),\sigma(x_2)=x_2(1+\lambda_2x_1x_3),\sigma(x_3)=x_3(1+\lambda_3x_1x_2),\lambda\in K^3\}$$

where $\lambda:=(\lambda_1,\lambda_2,\lambda_3)$, and $\Gamma\to K^3$, $\sigma_\lambda\mapsto\lambda$, is the group isomorphism. Since $\mathcal{J}(\sigma_\lambda)=1+\lambda_1x_2x_3+\lambda_2x_1x_3+\lambda_3x_1x_2$, the Jacobian group $\Sigma=\{e\}$ is trivial, and $\mathrm{im}(\mathcal{J})=E_3'$, i.e. the Jacobian map $\mathcal{J}:\Gamma\to E_3'$ is surjective.

Lemma 4.1. — *Let K be a commutative ring and $\sigma,\tau\in\Gamma$. Then*

1. $\mathcal{J}(\sigma)=\mathcal{J}(\tau)$ *iff* $\tau\in\sigma\Sigma$.
2. $\mathcal{J}(\sigma^{-1})=\mathcal{J}(\tau^{-1})$ *iff* $\tau\in\Sigma\sigma$.

Proof. — 1. Note that $\mathcal{J}(\sigma)=\sigma(\mathcal{J}(\sigma^{-1})^{-1})$ as it follows from the equality $1=\mathcal{J}(\sigma\sigma^{-1})=\mathcal{J}(\sigma)\sigma(\mathcal{J}(\sigma^{-1}))$. Now, $\tau\in\sigma\Sigma$ iff $\sigma^{-1}\tau\in\Sigma$ iff $1=\mathcal{J}(\sigma^{-1}\tau)=\mathcal{J}(\sigma^{-1})\sigma^{-1}(\mathcal{J}(\tau))$ iff $\mathcal{J}(\tau)=\sigma(\mathcal{J}(\sigma^{-1})^{-1})=\mathcal{J}(\sigma)$.

2. By statement 1, $\mathcal{J}(\sigma^{-1})=\mathcal{J}(\tau^{-1})$ iff $\tau^{-1}\in\sigma^{-1}\Sigma$ iff $\tau\in\Sigma\sigma$. □

Remark. Lemma 4.1 explains 'intuitively' why, in general, the Jacobian group Σ is *not* a normal subgroup of Γ: note first that $\mathcal{J}(\sigma^{-1})=\sigma^{-1}(\mathcal{J}(\sigma)^{-1})$ and $\mathcal{J}(\tau^{-1})=\tau^{-1}(\mathcal{J}(\tau)^{-1})$. Suppose that Σ is a normal subgroup of Γ, then $\sigma\Sigma=\Sigma\sigma$ for all $\sigma\in\Sigma$, and so the two statements of Lemma 4.1 are equivalent, i.e. $\mathcal{J}(\sigma)=\mathcal{J}(\tau)=:u$ iff $\mathcal{J}(\sigma^{-1})=\mathcal{J}(\tau^{-1})$ iff $\sigma^{-1}(u)=\tau^{-1}(u)$. Since the image of $\mathcal{J}$ is 'big' there is no reason to believe that the automorphisms σ^{-1} and τ^{-1} acts always identically on u.

The image $\mathrm{im}(\mathcal{J})$ and Γ/Σ. — By Lemma 4.1, the map

$$\mathcal{J}:\Gamma/\Sigma\to\mathrm{im}(\mathcal{J}),\ \ \sigma\Sigma\mapsto\mathcal{J}(\sigma), \tag{55}$$

is a bijection.

The groups Γ_{2m}, the Jacobian ascents. — By (31), the abelian group E_n' contains the descending chain of subgroups

$$E_n'=E_{n,2}'\supset E_{n,4}'\supset\cdots\supset E'_{n,2[\frac{n}{2}]}\supset E'_{n,2[\frac{n}{2}]+2}=\{1\}.$$

If K is a commutative ring then, for each $m=1,2,\dots,[\frac{n}{2}]+1$, the preimage

$$\Gamma_{2m}:=\Gamma_{n,2m}:=\mathcal{J}^{-1}(E_{n,2m}')=\{\sigma\in\Gamma\,|\,\mathcal{J}(\sigma)\in E_{n,2m}'\} \tag{56}$$

is, in fact, a subgroup of Γ: let $\sigma,\tau\in\Gamma_{2m}$, then

$$\mathcal{J}(\sigma\tau)=\mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))\subseteq E_{n,2m}'\sigma(E_{n,2m}')\subseteq E_{n,2m}'E_{n,2m}'\subseteq E_{n,2m}',$$

i.e. $\Gamma_{2m}\Gamma_{2m}\subseteq\Gamma_{2m}$; and

$$\mathcal{J}(\sigma^{-1})=(\sigma^{-1}(\mathcal{J}(\sigma)))^{-1}\subseteq(\sigma^{-1}(E_{n,2m}'))^{-1}\subseteq(E_{n,2m}')^{-1}=E_{n,2m}',$$

i.e. $\Gamma_{2m}^{-1} \subseteq \Gamma_{2m}$.

Note that

$$\Sigma = \Gamma_{2[\frac{n}{2}]+2}. \tag{57}$$

We call the groups $\Gamma_{2m} = \Gamma_{n,2m}$ the *Jacobian ascents.* We have the descending chain of subgroups in Γ, the *Jacobian filtration*:

$$\Gamma = \Gamma_2 \supseteq \Gamma_4 \supseteq \cdots \supseteq \Gamma_{2[\frac{n}{2}]} \supseteq \Gamma_{2[\frac{n}{2}]+2} = \Sigma. \tag{58}$$

We will see later that all these groups are distinct except the last two if n is even (Corollary 7.7); each group Γ_{2m+2} is a normal subgroup of Γ_{2m} such that the factor group $\Gamma_{2m}/\Gamma_{2m+2}$ is abelian (Lemma 4.2).

Lemma 4.2. — *Let K be a commutative ring. For each natural number $m \geq 1$, the group $\Gamma_{n,2m+2}$ is a normal subgroup of $\Gamma_{n,2m}$ such that the factor group $\Gamma_{n,2m}/\Gamma_{n,2m+2}$ is abelian.*

Proof. — Recall that the groups $\{E'_{n,2m}\}$ are Γ-invariant. The result is an immediate consequence of the following obvious fact: for each $m \geq 1$, $a \in E'_{n,2m}$, and $\sigma \in \Gamma$,

$$\sigma(a) \equiv a \mod E'_{n,2m+2}. \tag{59}$$

Indeed, to prove that $\Gamma_{n,2m+2}$ is a normal subgroup of $\Gamma_{n,2m}$ we have to show that, for any $\sigma \in \Gamma_{n,2m}$ and $\tau \in \Gamma_{n,2m+2}$, $\sigma\tau\sigma^{-1} \in \Gamma_{n,2m+2}$, i.e. $\mathcal{J}(\sigma\tau\sigma^{-1}) \in E'_{n,2m+2}$. By (25) and (26),

$$\mathcal{J}(\sigma\tau\sigma^{-1}) = \mathcal{J}(\sigma)\,\sigma(\mathcal{J}(\tau))\,\sigma\tau(\mathcal{J}(\sigma^{-1})) = \mathcal{J}(\sigma)\,\sigma(\mathcal{J}(\tau))\,\sigma\tau\sigma^{-1}(\mathcal{J}(\sigma)^{-1}).$$

Note that $\sigma(\mathcal{J}(\tau)) \in E'_{n,2m+2}$ since $\mathcal{J}(\tau) \in E'_{n,2m+2}$; and $\sigma\tau\sigma^{-1}(\mathcal{J}(\sigma)^{-1}) \equiv \mathcal{J}(\sigma)^{-1} \mod E'_{n,2m+2}$, by (59). Now,

$$\mathcal{J}(\sigma\tau\sigma^{-1}) \equiv \mathcal{J}(\sigma)\mathcal{J}(\sigma)^{-1} \equiv 1 \mod E'_{n,2m+2},$$

i.e. $\mathcal{J}(\sigma\tau\sigma^{-1}) \in E'_{n,2m+2}$, as required.

To prove that the factor group $\Gamma_{n,2m}/\Gamma_{n,2m+2}$ is abelian, we have to show that, for any $\sigma, \tau \in \Gamma_{n,2m}$, $\sigma\tau\sigma^{-1}\tau^{-1} \in \Gamma_{n,2m+2}$, that is $\mathcal{J}(\sigma\tau\sigma^{-1}\tau^{-1}) \in E'_{n,2m+2}$. By (25), (26), and (59), we have

$$\begin{aligned}\mathcal{J}(\sigma\tau\sigma^{-1}\tau^{-1}) &\equiv \mathcal{J}(\sigma)\,\sigma(\mathcal{J}(\tau))\,\sigma\tau(\mathcal{J}(\sigma^{-1}))\,\sigma\tau\sigma^{-1}(\mathcal{J}(\tau^{-1}))\\ &\equiv \mathcal{J}(\sigma)\,\sigma(\mathcal{J}(\tau))\,\sigma\tau\sigma^{-1}(\mathcal{J}(\sigma)^{-1})\,\sigma\tau\sigma^{-1}\tau^{-1}(\mathcal{J}(\tau)^{-1})\\ &\equiv \mathcal{J}(\sigma)\mathcal{J}(\tau)\mathcal{J}(\sigma)^{-1}\mathcal{J}(\tau)^{-1} \equiv 1 \mod E'_{n,2m+2}. \qquad \square\end{aligned}$$

The following result which is a part of Lemma 4.2 has a more short and direct proof.

Corollary 4.3. — *The Jacobian group Σ is a normal subgroup of $\Gamma_{2[\frac{n}{2}]}$ such that the factor group $\Gamma_{2[\frac{n}{2}]}/\Sigma$ is abelian.*

Proof. — Since each automorphism $\sigma \in \Gamma$ acts trivially (i.e. as the identity map) on $E'_{n,2[\frac{n}{2}]}$, the Jacobian map $\mathscr{J} : \Gamma_{2[\frac{n}{2}]} \to E'_{n,2[\frac{n}{2}]}$, $\tau \mapsto \mathscr{J}(\tau)$, is a group *homomorphism* ($\mathscr{J}(\sigma\tau) = \mathscr{J}(\sigma)\sigma(\mathscr{J}(\tau)) = \mathscr{J}(\sigma)\mathscr{J}(\tau)$) with the kernel Σ, hence Σ is a normal subgroup of $\Gamma_{2[\frac{n}{2}]}$ such that the factor group $\Gamma_{2[\frac{n}{2}]}/\Sigma$ is abelian since the group $E'_{n,2[\frac{n}{2}]}$ is abelian. □

The elements of the group Γ_{2m} are solutions to the system of polynomial equations over K,

$$\Gamma_{2m} = \{\gamma_b \in \Gamma \mid \sum_{i=1}^{n} \mathrm{tr}_i(B) \in \mathfrak{m}^{2m}\}. \tag{60}$$

So, Γ_{2m} is an algebraic unipotent group over the ring K. By Theorem 2.3.(1), the system of polynomial equations in (60) can be made explicit

(61)

$$\Gamma_{2m} = \{\gamma_b \in \Gamma \mid \phi(\partial^\alpha(\sum_{i=1}^{n} \mathrm{tr}_i(B))) = 0, \quad \text{for all even } 0 \neq \alpha \in \mathscr{B}_n,\ 1 \leq |\alpha| < 2m\}.$$

Lemma 4.4. — *Let K be a commutative ring. Then, $\Gamma^{2m+1} \subseteq \Gamma_{2m}$, for each $m = 1, 2, \dots, [\frac{n}{2}] + 1$.*

Proof. — Let $\sigma \in \Gamma^{2m+1}$. Then $\sigma(x_1) = x_1 + b_1, \dots, \sigma(x_n) = x_n + b_n$, where all $b_i \in \Lambda^{\mathrm{od}}_{n,\geq 2m+1}$. Now, the result follows from

$$\mathscr{J}(\sigma) \equiv 1 + \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} \equiv 1 \mod \mathfrak{m}^{2m}, \tag{62}$$

i.e. $\mathscr{J}(\sigma) \in E'_{2m}$ since all $\frac{\partial b_i}{\partial x_i} \in \mathfrak{m}^{2m}$. □

Now, we introduce two important subgroups of Σ, namely Σ' and Σ'', and prove that $\Gamma = \Phi\Sigma''$ (Theorem 4.9.(1)) and $\Sigma = \Sigma'\Sigma''$ (Corollary 4.11.(1)).

The group Σ'. — Consider the following subgroup of Σ,

$$\begin{aligned}\Sigma' := \Sigma \cap \Phi &= \{\sigma \in \Sigma \mid \sigma(x_1) \in (x_1), \dots, \sigma(x_n) \in (x_n)\} \\ &= \{\sigma \in \Sigma \mid \sigma(x_1) = x_1(1 + a_1), \dots, \sigma(x_n) = x_n(1 + a_n)\}\end{aligned}$$

where each element $a_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{ev}}_{\geq 2}$. The group Σ' is a closed subgroup of Σ as the intersection of two closed subgroups Σ and Φ of Γ. It contains the descending chain of normal subgroups

$$\Sigma' = \Sigma'^3 \supseteq \Sigma'^5 \supseteq \cdots \supseteq \Sigma'^{2m+1} := \Sigma' \cap \Gamma^{2m+1} \supseteq \cdots \supseteq \Sigma'^{2[\frac{n}{2}]+1} \supseteq \Sigma'^{2[\frac{n}{2}]+3} = \{e\}$$

with the abelian factors $\{\Sigma'^{2m+1}/\Sigma'^{2m+3}\}$.

Example. Let $a_1 \in K\lfloor x_2, \dots, x_n\rfloor_{2m}$ and $a_2 \in K\lfloor x_1, x_3, \dots, x_n\rfloor_{2m}$ be homogeneous even elements of the same graded degree $2m \geq 2$, and $\sigma \in \Gamma$: $x_1 \mapsto x_1(1+a_1), x_2 \mapsto x_2(1+a_2)$, $x_j \mapsto x_j$, $j \geq 3$. Then $\sigma \in \Sigma$ iff

$$1 = \mathcal{J}(\sigma) = \det \begin{pmatrix} 1+a_1 & -x_1\frac{\partial a_1}{\partial x_2} \\ -x_2\frac{\partial a_2}{\partial x_1} & 1+a_2 \end{pmatrix} = 1 + a_1 + a_2 + a_1a_2 + x_1x_2\frac{\partial a_1}{\partial x_2}\frac{\partial a_2}{\partial x_1}$$

iff $a_1 = -a_2 \in K\lfloor x_3, \dots, x_n\rfloor_{2m}$ and $a_1^2 = 0$. So, *for each even homogeneous element $a \in K\lfloor x_3, \dots, x_n\rfloor_{2m}$ such that $a^2 = 0$ the automorphism*

(63) $$\sigma \in \Gamma : x_1 \mapsto x_1(1+a),\ x_2 \mapsto x_2(1-a),\ x_j \mapsto x_j,\ j \geq 3,$$

belongs to the group Σ'.

The group Σ''. — For each $i = 1, \dots, n$, and $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3}$, consider the element of Σ:

(64) $$\xi_{i,b_i} : x_i \mapsto x_i + b_i,\ x_j \mapsto x_j,\ \forall\, j \neq i.$$

Let Σ'' be the subgroup of Σ generated by all the elements ξ_{i,b_i}, $1 \leq i \leq n$. For each $i = 1, \dots, n$, let

$$\Sigma''_i := \{\xi_{i,b_i} \mid b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3}\}.$$

Since $\xi_{i,b_i}\xi_{i,b'_i} = \xi_{i,b_i+b'_i}$ and $\xi^{-1}_{i,b_i} = \xi_{i,-b_i}$, the set Σ''_i is an *abelian* group canonically isomorphic to the abelian additive group $K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3}$ via $\xi_{i,b_i} \mapsto b_i$. Therefore, the group Σ''_i is the direct product of its one-dimensional abelian subgroups isomorphic to $(K, +)$,

(65) $$\Sigma''_i = \prod_{\alpha} \{\xi_{i,\lambda x^\alpha}\}_{\lambda \in K}$$

where α runs through all the odd subsets of the set $\{1, \dots, \widehat{i}, \dots, n\}$ with $|\alpha| \geq 3$; the map $(K, +) \to \{\xi_{i,\lambda x^\alpha}\}_{\lambda\in K}$, $\lambda \mapsto \xi_{i,\lambda x^\alpha}$, is a group isomorphism.

So, *the group Σ'' is generated by its abelian subgroups $\Sigma''_1, \dots, \Sigma''_n$.*

The commutator of the elements $\xi_{i,b_i} \in \Sigma''_i$ and $\xi_{j,b_j} \in \Sigma''_j$ where $i \neq j$ is given by the rule

(66) $$[\xi_{i,b_i}, \xi_{j,b_j}] :\ x_i \mapsto (1 - bb')x_i - b^2b'x_j - b(c' + b'c),$$

(67) $$x_j \mapsto bb'^2x_i + (1 + bb' + (bb')^2)x_j + b'(c + bc' + bb'c),$$

(68) $$x_k \mapsto x_k,\ k \neq i, j,$$

where $b_i = bx_j + c$ and $b_j = b'x_i + c'$ for unique elements $b, b' \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\rfloor^{\mathrm{ev}}_{\geq 2}$ and $c, c' \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\rfloor^{\mathrm{od}}_{\geq 3}$.

Given automorphisms $\xi_{1,b_1}, \dots \xi_{n,b_n}$, and $\xi := \prod_{i=1}^n \xi_{i,b_i}$ is their product in an arbitrary (fixed) order, then

(69) $$\xi(x_i) = x_i + b_i + \cdots,\ \ i = 1, \dots, n.$$

The group Σ'' contains the descending chain of normal subgroups

$$\Sigma'' = \Sigma''^3 \supseteq \Sigma''^5 \supseteq \cdots \supseteq \Sigma''^{2m+1} := \Sigma'' \cap \Gamma^{2m+1} \supseteq \cdots \supseteq \Sigma''^{2[\frac{n}{2}]+1} \supseteq \Sigma''^{2[\frac{n}{2}]+3} = \{e\}$$

with the abelian factors $\{\Sigma''^{2m+1}/\Sigma''^{2m+3}\}$.

A direct calculation gives

$$[\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x^\beta}] = \xi_{i,-\lambda\mu x^\alpha x^\beta} \tag{70}$$

where $i \neq j$; α is an even set and β is an odd set such that the sets $\{i,j\}$ and $\alpha \cup \beta$ are disjoint. Similarly,

$$[\xi_{i,\lambda x^\alpha}, \xi_{j,\mu x^\beta}] = e \tag{71}$$

for all $i \neq j$; $\lambda, \mu \in K$; α and β are odd sets such that $|\alpha| \geq 3$ and $|\beta| \geq 3$, and the sets $\{i,j\}$ and $\alpha \cup \beta$ are disjoint.

One can verify that for each $i \neq j$, and even sets α and β such that $\{i,j\}$ and $\alpha \cup \beta$ are disjoint, $\lambda, \mu \in K$, the commutator (which is an element of Σ'')

(72)
$$[\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x_i x^\beta}] : x_i \mapsto x_i(1-\lambda\mu x^\alpha x^\beta),\ x_j \mapsto x_j(1+\lambda\mu x^\alpha x^\beta),\ x_k \mapsto x_k,\ k \neq i,j,$$

belongs to the group Σ', and

$$[\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x_i x^\beta}]^{-1} : x_i \mapsto x_i(1+\lambda\mu x^\alpha x^\beta),\ x_j \mapsto x_j(1-\lambda\mu x^\alpha x^\beta),\ x_k \mapsto x_k,\ k \neq i,j.$$

Now, the next corollary is obvious since $[\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x_i x^\beta}] \in \Sigma' \cap \Sigma'' \subseteq \Phi \cap \Sigma''$.

COROLLARY 4.5. — *If K is a commutative ring and $n \geq 6$ then $\Sigma' \cap \Sigma'' \neq \{e\}$ and $\Phi \cap \Sigma'' \neq \{e\}$.*

A straightforward calculation gives

$$[\xi_{i,\nu x_j x^\gamma}, [\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x_i x^\beta}]] = \xi_{i,-2\lambda\mu\nu x_j x^\alpha x^\beta x^\gamma}, \tag{73}$$

for all $\lambda, \mu, \nu \in K$; the sets α, β, and γ are even and non-empty; the sets $\{i,j\}$ and $\alpha \cup \beta \cup \gamma$ are disjoint. Similarly,

$$[\xi_{i,\nu x^\gamma}, [\xi_{i,\lambda x_j x^\alpha}, \xi_{j,\mu x_i x^\beta}]] = \xi_{i,-\lambda\mu\nu x^\alpha x^\beta x^\gamma}, \tag{74}$$

for all $\lambda, \mu, \nu \in K$; the sets α and β, are even and non-empty; the set γ is odd and $|\gamma| \geq 3$; the sets $\{i,j\}$ and $\alpha \cup \beta \cup \gamma$ are disjoint.

The group Σ'' for $n = 4, 5, 6$. — Let A be a group and $A_1, \ldots, A_s$ be its subgroups. We say that the group A is an *exact product* of the groups A_i,

$$A := A_1 \cdots A_s[\text{exact}] := {}^{\text{ex}}\prod_{i=1}^{n} A_i$$

if each element $a \in A$ is a *unique* product $a = a_1 \cdots a_s$ of elements $a_i \in A_i$.

The next lemma describes the structure of the group Σ'' for small values of $n = 4, 5, 6$. These values are rather peculiar as Theorem 4.7 shows.

Lemma 4.6. — *Let K be a commutative ring.*

1. *If $n = 4$ then $\Sigma'' = \Sigma_1'' \times \cdots \times \Sigma_4''$ is the abelian group.*
2. *If $n = 5$ then $\Sigma'' = \Sigma_1'' \times \cdots \times \Sigma_5''$ is the abelian group.*
3. *If $n = 6$ then*
   (a) *$\Sigma'' = Z(\Sigma'') \times \prod_{i,j,k,l} \Sigma''_{i;j,k,l}$ is the exact product of the centre $Z(\Sigma'')$ of Σ'' and the one dimensional abelian subgroups $\Sigma''_{i;j,k,l} := \{\xi_{i,\lambda x_j x_k x_l}\}_{\lambda \in K} \simeq K$ where $i = 1, \ldots, 6$; $j < k < l$; $i \notin \{j, k, l, \}$.*
   (b) *$[\Sigma'', \Sigma''] = \Sigma' \cap \Sigma''$ and the group $\Sigma' \cap \Sigma''$ is the direct product $\prod_{i<j} C_{ij}$ of its subgroups $C_{ij} := \{c_{ij,\lambda} : x_i \mapsto x_i(1 - \lambda x^\alpha), x_j \mapsto x_j(1 + \lambda x^\alpha), x_k \mapsto x_k, k \neq i, j\}$ where $\alpha := \{1, \ldots, 6\} \backslash \{i, j\}$ and $C_{ij} \simeq (K, +)$ via $c_{ij,\lambda} \mapsto \lambda$.*
   (c) *$Z(\Sigma'') = \Sigma''^5 = (\Sigma' \cap \Sigma'') \times \prod_{i=1}^6 \Sigma''_{i,5}$ where $\Sigma''_{i,5} := \{\xi_{i,b_i} \,|\, b_i \in K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_6 \rfloor_5\}$.*

Proof. — 1 and 2. If $n = 4, 5$ then the elements of Σ_i'' commute with the elements of Σ_j'', hence statements 1 and 2 are obvious.

3. Let $n = 6$. The group Σ'' is generated by its abelian subgroups $\Sigma_1'', \ldots, \Sigma_6''$, and so, by (65), the group Σ'' is generated by the 1-dimensional abelian subgroups $\langle \xi_{i,\lambda x^\alpha} \rangle_{\lambda \in K} \simeq (K, +)$, $\xi_{i,\lambda x^\alpha} \mapsto \lambda$, where $|\alpha| = 3, 5$. If $|\alpha| = 5$ then all the $\xi_{i,\lambda x^\alpha} \in Z := Z(\Sigma'')$, the centre of the group Σ''. Since $n = 6$, $\Sigma''^5 \subseteq Z$ and the RHS of (70) is equal to the identity. By Theorem 2.10.(2), $[\Sigma'', \Sigma''] \subseteq \Sigma''^5$, and so $[\Sigma'', \Sigma''] \subseteq Z$. By (70), (71), and (72), the only nontrivial commutators come from (72) and only in the case when $i < j$ and $\{i, j\} \cup \alpha \cup \beta = \{1, \ldots, 6\}$ there. In this case, the commutators (72) is the automorphism $c_{ij,\lambda\mu} \in Z$. It is obvious that the product of the groups C_{ij} is the direct product $\prod_{i<j} C_{ij}$, and $[\Sigma'', \Sigma''] = \prod_{i<j} C_{ij} \subseteq \Sigma' \cap \Sigma''^5$. It follows that $\Sigma''^5 = [\Sigma'', \Sigma''] \times \prod_{i=1}^6 \Sigma''_{i,5}$ is the direct product of groups since the abelian group Σ''^5 is generated by the subgroups $[\Sigma'', \Sigma'']$ and $\prod_{i=1}^6 \Sigma''_{i,5}$, and their intersection is trivial. Since $n = 6$, $\Sigma'' = \Sigma''^5 \times \prod_{i,j,k,l} \Sigma''_{i;j,k,l}$ is the

exact product of groups where i,j,k,l are as in (a), then $\Sigma'\cap\Sigma''=\Sigma'\cap\Sigma''^5$. Since $[\Sigma'',\Sigma'']\subseteq\Sigma'$ and $\Sigma''^5=[\Sigma'',\Sigma'']\times\prod_{i=1}^6\Sigma''_{i,5}$, we have

$$\Sigma'\cap\Sigma''=[\Sigma'',\Sigma'']\times(\Sigma'\cap\prod_{i=1}^6\Sigma''_{i,5})=[\Sigma'',\Sigma'']=\prod_{i<j}C_{ij}$$

since $\Sigma'\cap\prod_{i=1}^6\Sigma''_{i,5}=\{e\}$. This proves statement (b).

It follows from the decomposition $\Sigma''=\Sigma''^5\times\prod_{i,j,k,l}\Sigma''_{i;j,k,l}$ and (72) that $Z\subseteq\Sigma''^5$. Since $\Sigma''^5\subseteq Z$, we have $Z=\Sigma''^5$. Since $\Sigma''=\Sigma''^5\times\prod_{i,j,k,l}\Sigma''_{i;j,k,l}$, (a) follows. Since $\Sigma''^5=[\Sigma'',\Sigma'']\times\prod_{i=1}^6\Sigma''_{i,5}$, (c) follows. □

A minimal set of generators for the group Σ''. — The next result provides a (minimal) set of generators for the group Σ''.

THEOREM 4.7. — *Let K be a commutative ring and $n\geq 4$. Then*

1. *if either n is odd; or $n=4$; or n is even and $n\geq 8$ and $\frac{1}{2}\in K$, then*
$$\Sigma''=\langle\sigma_{i,\lambda x_jx_kx_l}\,|\,\lambda\in K; i=1,\dots,n; j<k<l; i\not\in\{j,k,l\}\rangle,$$
and the subgroups $\{\sigma_{i,\lambda x_jx_kx_l}\}_{\lambda\in K}\simeq K$ of Σ'' form a minimal set of generators for Σ''.
2. *If n is even, then*
$$\Sigma''=\langle\sigma_{i,\lambda x_jx_kx_l},\ \sigma_{i,\lambda x_1\cdots\widehat{x_i}\cdots x_n}\,|\,\lambda\in K; i=1,\dots,n; j<k<l; i\not\in\{j,k,l\}\rangle.$$

Proof. — 1. Recall that the group Σ'' is generated by its abelian subgroups $\Sigma''_1,\dots\Sigma''_n$. In order to prove the claims that the elements above generate the group Σ'' it suffices to show that each automorphism $\xi_{i,\lambda x^\alpha}$ (where α is odd, $|\alpha|\geq 3$, and $i\not\in\alpha$) is a product of some of them. By (70), for $\lambda\in K$ and distinct indices $i,j,k_1,l_1,k_2,l_2,\dots,k_{2m},l_{2m},p,q,r$ there is the equality

$$(75)\qquad \xi_{i,\lambda(x_{k_1}x_{l_1})(x_{k_2}x_{l_2})\cdots(x_{k_{2m}}x_{l_{2m}})x_px_qx_r}=[\xi_{i,x_jx_{k_1}x_{l_1}},[\xi_{j,x_ix_{k_2}x_{l_2}},[\xi_{i,x_jx_{k_3}x_{l_3}},[\xi_{j,x_ix_{k_4}x_{l_4}},\dots[\xi_{j,x_ix_{k_{2m}}x_{l_{2m}}},\xi_{i,\lambda x_px_qx_r}]\dots].$$

Similarly, for $\lambda\in K$ and distinct indices $i,j,k_1,l_1,k_2,l_2,\dots,k_{2m+1},l_{2m+1},p,q,r$ there is the equality

(76)
$$\xi_{i,\lambda(x_{k_1}x_{l_1})(x_{k_2}x_{l_2})\cdots(x_{k_{2m+1}}x_{l_{2m+1}})x_px_qx_r}=[\xi_{i,x_jx_{k_1}x_{l_1}},[\xi_{j,x_ix_{k_2}x_{l_2}},[\xi_{i,x_jx_{k_3}x_{l_3}},[\xi_{j,x_ix_{k_4}x_{l_4}},\dots[\xi_{i,x_jx_{k_{2m+1}}x_{l_{2m+1}}},\xi_{j,-\lambda x_px_qx_r}]\dots].$$

Suppose that n is odd. The set $\{i\}\cup\alpha$ (for $\xi_{i,\lambda x^\alpha}$) is an even set hence it is not equal to the set $\{1,\dots,n\}$, and so one can find element j such that $j\not\in\{i\}\cup\alpha$.

It follows from (75) and (76) that the 1-dimensional subgroups $\{\sigma_{i,\lambda x_j x_k x_l}\}$ are generators for the group Σ''. They are a minimal set of generators due to the isomorphism in Theorem 2.10.(2) in the case $m = 1$ there.

Let $n \geq 4$ be an even number. If $n = 4$ the result is obvious. If $n = 6$ the result is *not true* by Lemma 4.6.(3). So, let $n \geq 8$ and $\frac{1}{2} \in K$. If $\{i\} \cup \alpha \neq \{1, \dots, n\}$ then we have already proven that the automorphism $\xi_{i,\lambda x^\alpha}$ is a product of the elements $\sigma_{i',\lambda' x_j x_k x_l}$. If $\{i\} \cup \alpha = \{1, \dots, n\}$ then the automorphism $\xi_{i,\lambda x^\alpha} = \xi_{i,\lambda x_1 \cdots \widehat{x_i} \cdots x_n}$ is a product of the elements $\sigma_{i',\lambda' x_j x_k x_l}$, by (73).

2. This statement has been proven already in the proof of statement 1 (see the last two sentences above). □

The Jacobian group is a complicated group. To understand its structure first we intersect it with the small group $\mathscr{E}_{n,i}$.

Lemma 4.8. — *Let K be a commutative ring. Then*

$$\mathscr{E}_{n,i} \cap \Sigma = \mathscr{E}''_{n,i} := \{\xi_{i,b_i} \mid b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n \rfloor^{\mathrm{od}}_{\geq 3}\}.$$

Proof. — A typical element of the group $\mathscr{E}_{n,i}$ is as follows $\gamma_{1+a,b} : x_i \mapsto x_i(1+a)+b$, $x_j \mapsto x_j$, $j \neq i$ (see (39)). Then $\mathscr{J}(\gamma_{1+a,b}) = 1+a$. So, $\gamma_{1+a,b} \in \mathscr{E}_{n,i} \cap \Sigma$ iff $a = 0$, i.e. $\mathscr{E}_{n,i} \cap \Sigma = \mathscr{E}''_{n,i}$. □

The equality $\Gamma = \Phi\Sigma''$. — For a natural number n, let $\mathrm{od}(n)$ be the largest odd number such that $\leq n$, and let $\mathrm{Od}(n)$ be the set of all odd natural numbers j such that $3 \leq j \leq n$, i.e. $\mathrm{Od}(n) = \{3, 5, \dots, \mathrm{od}(n)\}$.

For each $t = 3, 5, \dots, \mathrm{od}(n)$, let

$$\mathscr{T}''_{n,t} := \{\sigma \in \Sigma'' \mid \sigma(x_i) = x_i + b_i,\ b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor_t,\ i = 1, \dots, n\}.$$

The set $\mathscr{T}''_{n,t}$ is an affine variety over K of dimension $n\binom{n-1}{t}$, i.e. the algebra of regular functions on $\mathscr{T}''_{n,t}$ is a polynomial algebra in $n\binom{n-1}{t}$ variables where the coordinate functions are the coefficients of the polynomials $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor_t$. Let

$$\mathscr{T}''_n := \mathscr{T}''_{n,\mathrm{od}(n)} \times \cdots \times \mathscr{T}''_{n,5} \times \mathscr{T}''_{n,3} \tag{77}$$

be the product of algebraic varieties. Then

$$\dim(\mathscr{T}''_n) = n \sum_{s=1}^{\mathrm{od}(n)} \binom{n-1}{2s+1} = n\Big(\sum_{s=0}^{\mathrm{od}(n)} \binom{n-1}{2s+1} - n + 1\Big) = n(2^{n-2} - n + 1). \tag{78}$$

In general, $\Phi \cap \Sigma'' \neq \{e\}$ (Corollary 4.5). The next theorem shows that $\Gamma = \Phi\Sigma''$ and that any element of Γ is a unique product of elements from Φ and Σ'' when one puts certain conditions on the choice of the multiples.

THEOREM 4.9. — *Let K be a commutative ring. Then*

1. $\Gamma = \Phi\Sigma''$.
2. $\Gamma^j = \Phi^j\Sigma''^j$, $j = 3, 5, \dots, \mathrm{od}(n)$.
3. *Let* $j = 3, 5, \dots, \mathrm{od}(n)$. *Each* $\sigma \in \Gamma^j$ *is a unique product* $\sigma = \phi_j\xi_{\mathrm{od}(n)}\cdots\xi_{j+2}\xi_j$ *where* $\phi_j \in \Phi^j$ *and each* ξ_k *is as in* (69) *with* $\xi_k(x_i) - x_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor_k$, $i = 1, \dots, n$. *Moreover, for all* $i = 1, \dots, n$ *the following conditions hold:*
   (a) $\sigma^{-1}(x_i) \equiv -\xi_j(x_i) \mod ((x_i) + \Lambda^{\mathrm{od}}_{n,\geq j+2})$,
   (b) $\xi_k\cdots\xi_{j+2}\xi_j\sigma^{-1}(x_i) \equiv -\xi_{k+2}(x_i) \mod ((x_i)+\Lambda^{\mathrm{od}}_{n,\geq k+4})$, $k = j, j+2, \dots, \mathrm{od}(n)-2$.
4. *For each odd natural number j such that $3 \leq j \leq n$, let* $\mathscr{F}''^j_n := \mathscr{F}''_{\mathrm{od}(n)} \times \cdots \times \mathscr{F}''_{j+2} \times \mathscr{F}''_j$. *The map*

$$\Phi^j \times \mathscr{F}''^j_n \to \Gamma^j, \quad (\phi, \xi_{\mathrm{od}(n)}, \dots, \xi_{j+2}, \xi_j) \mapsto \phi\xi_{\mathrm{od}(n)}\cdots\xi_{j+2}\xi_j,$$

   is an isomorphism of the algebraic varieties with the inverse map $\sigma \mapsto \phi\xi_{\mathrm{od}(n)}\cdots\xi_{j+2}\xi_j$ *given by the decomposition of statement 3.*
5. *The map*

$$\Phi \times \mathscr{F}''_n \to \Gamma, \quad (\phi, \xi_{\mathrm{od}(n)}, \dots, \xi_5, \xi_3) \mapsto \phi\xi_{\mathrm{od}(n)}\cdots\xi_5\xi_3,$$

   is an isomorphism of the algebraic varieties with the inverse map $\sigma \mapsto \phi\xi_{\mathrm{od}(n)}\cdots\xi_5\xi_3$ *given by the decomposition of statement 3.*

Proof. — 1. Statement 1 is a particular case of statement 2 when $j = 3$ since $\Gamma = \Gamma^3$, $\Phi = \Phi^3$, and $\Sigma'' = \Sigma''^3$.

2. Statement 2 follows from statement 3.

3. First, we prove that there exists a decomposition for σ that satisfies the conditions (a) and (b), then we prove the uniqueness.

By the inversion formula (Theorem 3.1), the map $\Gamma \to \Gamma$, $\sigma \mapsto \sigma^{-1}$, is an automorphism of the algebraic variety. Let $\sigma \in \Gamma^j$. Then $\sigma^{-1} \in \Gamma^j$, and, for each $i = 1, \dots, n$, $\sigma^{-1}(x_i) = x_i(1 + a_i) + b_i$ for some elements $a_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor^{\mathrm{ev}}_{\geq j-1}$ and $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor^{\mathrm{od}}_{\geq j}$. Then $b_i = c_i + \cdots$ for some element $c_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor_j$. Clearly,

$$\sigma^{-1}(x_i) \equiv c_i \mod ((x_i) + \Lambda^{\mathrm{od}}_{n,\geq j+2}). \tag{79}$$

Define the element $\xi_j \in \Sigma''^j$ by the rule

$$\xi_j(x_i) = x_i - c_i, \quad \text{for all } i. \tag{80}$$

Then the automorphism ξ_j satisfies the condition (a). Consider the automorphism

$$\sigma' := \xi_j\sigma^{-1} \in \Gamma^j : x_i \mapsto x_i(1 + a'_i) + b'_i$$

where $a_i' \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor^{\mathrm{ev}}_{\geq j-1}$ and $b_i' \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor^{\mathrm{od}}_{\geq j+2}$. Then $b_i' = c_i' + \cdots$ for some $c_i' \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n\rfloor_{j+2}$. Clearly,

$$\xi_j\sigma^{-1}(x_i) \equiv c_i' \mod ((x_i) + \Lambda^{\mathrm{od}}_{n,\geq j+4}). \tag{81}$$

Define the element $\xi_{j+2} \in \Sigma''^{j+2}$ by the rule $\xi_{j+2}(x_i) = x_i - c_i'$ for all i. Then ξ_{j+2} satisfies the condition (b) for $k = j$. Now, we can repeat the same argument for the automorphism $\sigma'' := \xi_{j+2}\xi_j\sigma^{-1}$. Continue in this fashion we finally come to the inclusion

$$\xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j\sigma^{-1} \in \Phi^j,$$

and obtain a decomposition for σ that satisfies the properties (a) and (b) exists.

Uniqueness: Let $\sigma = \phi'\xi'_{\mathrm{od}(n)} \cdots \xi'_{j+2}\xi'_j$ be another decomposition with $\xi_j'(x_i) = x_i - \lambda_i$ for some $\lambda_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor_j$, $1 \leq i \leq n$. Then $\xi_j'^{-1}(x_i) = x_i + \lambda_i$. Since $\sigma^{-1}(x_i) \equiv \xi_j'^{-1}(x_i) \equiv \lambda_i \mod ((x_i) + \Lambda^{\mathrm{od}}_{n,\geq j+2})$, we must have $\xi_j' = \xi_j$, by (79). Similarly, (81) yields the equality $\xi'_{j+2} = \xi_{j+2}$. Using the same argument again and again (or by induction) we see that $\xi_k' = \xi_k$ for all $k = j, j+2, \dots, \mathrm{od}(n)$. These equalities imply that $\phi' = \phi$.

4. This statement follows from statement 3 since the map $\sigma \mapsto \phi\xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j$ is a polynomial map as the proof of statement 3 shows.

5. This statement is a particular case of statement 4 for $j = 3$ since $\Phi = \Phi^3$, $\mathscr{F}'' = \mathscr{F}''^3$, and $\Gamma = \Gamma^3$. □

In the proof of Theorem 4.9, the algorithm is given for finding the automorphisms ϕ and ξ_i in the presentation $\sigma = \phi\xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j$.

Corollary 4.10. — *Let K be a commutative ring. Then $\mathscr{J}(\Gamma) = \mathscr{J}(\Phi)$.*

Proof. — Let $\sigma \in \Gamma$. Then $\sigma = \tau\sigma''$ for some $\tau \in \Phi$ and $\sigma'' \in \Sigma''$ (Theorem 4.9.(1)). Then

$$\mathscr{J}(\sigma) = \mathscr{J}(\tau\sigma'') = \tau(\mathscr{J}(\sigma''))\mathscr{J}(\tau) = \tau(1)\mathscr{J}(\tau) = \mathscr{J}(\tau).$$

Therefore, $\mathscr{J}(\Gamma) = \mathscr{J}(\Phi)$. □

The equality $\Sigma = \Sigma'\Sigma''$. — The next result shows that the Jacobian group Σ is the product of its subgroups Σ' and Σ'', i.e. each element $\sigma \in \Sigma$ is a product $\sigma = \sigma'\sigma''$ of some elements $\sigma' \in \Sigma'$ and $\sigma'' \in \Sigma''$. This product is not unique as, in general, $\Sigma' \cap \Sigma'' \neq \{e\}$ (Corollary 4.5). Though, by putting extra conditions on the choice of the elements σ' and σ'' the uniqueness can be preserved.

Corollary 4.11. — *Let K be a commutative ring. Then*

1. $\Sigma = \Sigma'\Sigma''$.
2. $\Sigma^j = \Sigma'^j\Sigma''^j$, $j = 3, 5, \dots, \mathrm{od}(n)$.

3. *Let* $j = 3, 5, \dots, \mathrm{od}(n)$. *Each* $\sigma \in \Sigma^j$ *is a unique product* $\sigma = \sigma' \xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j$ *where* $\sigma' \in \Sigma'^j$ *and each* ξ_k *is as in* (69) *with* $\xi_k(x_i) - x_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n\rfloor_k$, $i = 1, \dots, n$. *Moreover, for all* $i = 1, \dots, n$ *the following conditions hold:*
   (a) $\sigma^{-1}(x_i) \equiv -\xi_j(x_i) \mod ((x_i) + \Lambda^{\mathrm{od}}_{n, \geq j+2})$,
   (b) $\xi_k \cdots \xi_{j+2}\xi_j\sigma^{-1}(x_i) \equiv -\xi_{k+2}(x_i) \mod ((x_i) + \Lambda^{\mathrm{od}}_{n, \geq k+4})$, $k = j, j+2, \dots, \mathrm{od}(n) - 2$.
4. *For each odd natural number* j *such that* $3 \leq j \leq n$, *the map*
$$\Sigma'^j \times \mathscr{F}''^j_n \to \Sigma^j, \;\; (\sigma', \xi_{\mathrm{od}(n)}, \dots, \xi_{j+2}, \xi_j) \mapsto \sigma'\xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j,$$
is an isomorphism of the algebraic varieties with the inverse map $\sigma \mapsto \sigma'\xi_{\mathrm{od}(n)} \cdots \xi_{j+2}\xi_j$ *given by the decomposition of statement 3.*
5. *The map*
$$\Sigma' \times \mathscr{F}''_n \to \Sigma, \;\; (\sigma', \xi_{\mathrm{od}(n)}, \dots, \xi_5, \xi_3) \mapsto \sigma'\xi_{\mathrm{od}(n)} \cdots \xi_5\xi_3,$$
is an isomorphism of the algebraic varieties with the inverse map $\sigma \mapsto \sigma'\xi_{\mathrm{od}(n)} \cdots \xi_5\xi_3$ *given by the decomposition of statement 3.*

Proof. — 1. By Theorem 4.9.(1), $\Gamma = \Phi\Sigma''$. Note that $\Sigma \subseteq \Gamma$, $\Sigma'' \subseteq \Sigma$, and $\Sigma' = \Sigma \cap \Phi$. Now,
$$\Sigma = \Sigma \cap \Gamma = \Sigma \cap \Phi\Sigma'' = (\Sigma \cap \Phi)\Sigma'' = \Sigma'\Sigma''.$$
This proves statement 1. The rest follows at once from Theorem 4.9. □

By Corollary 4.11 and (78), in order to find the dimension (and generators) for the Jacobian group Σ it suffices to find the dimension (and generators) for Σ'.

The largest normal subgroup of Γ in Σ. — Let $A \subseteq B$ be groups. Then

(82) $$\mathscr{N}(A, B) := \{a \in A \mid bab^{-1} \in A, \;\; \text{for all} \;\; b \in B\},$$

is a normal subgroup of B contained in A. If N is a normal subgroup of B that is contained in A then $N \subseteq \mathscr{N}(A, B)$. Therefore, $\mathscr{N}(A, B)$ is the *largest normal* subgroup of B that is contained in A. *The group A is a normal subgroup of B iff $A = \mathscr{N}(A, B)$.*

Theorem 4.12. — *Let K be a commutative ring. Then*
$$\mathscr{N}(\Sigma, \Gamma) = \{\tau \in \Sigma \mid \tau(\lambda) = \lambda \;\; \text{for all} \;\; \lambda \in \mathrm{im}(\mathscr{J})\}.$$

Proof. — $\tau \in \mathscr{N}(\Sigma, \Gamma)$ iff $\sigma\tau\sigma^{-1} \in \Sigma$ for all $\sigma \in \Gamma$ iff
$$1 = \mathscr{J}(\sigma\tau\sigma^{-1}) = \mathscr{J}(\sigma)\sigma(\mathscr{J}(\tau))\sigma\tau(\mathscr{J}(\sigma^{-1})) = \mathscr{J}(\sigma)\sigma\tau(\mathscr{J}(\sigma^{-1}))$$
iff $\tau(\mathscr{J}(\sigma^{-1})) = \sigma^{-1}(\mathscr{J}(\sigma)^{-1})$ iff $\tau(\mathscr{J}(\sigma^{-1})) = \mathscr{J}(\sigma^{-1})$ for all $\sigma \in \Sigma$ (since $\sigma^{-1}(\mathscr{J}(\sigma)^{-1}) = \mathscr{J}(\sigma^{-1})$) iff $\tau(\lambda) = \lambda$ for all $\lambda \in \mathrm{im}(\mathscr{J})$. □

COROLLARY 4.13. — *Let K be a commutative ring and $n \geq 4$. Then the group Σ is not a normal subgroup of Γ iff $n \geq 5$.*

Proof. — For $n = 4$, the group Γ is abelian, and so Σ is a normal subgroup of Γ. Let $n \geq 5$. By Theorem 4.12, Σ is a normal subgroup of Γ iff $\Sigma = \mathscr{N}(\Sigma, \Gamma)$ iff

$$\mathrm{im}(\mathscr{J}) \subseteq E'^{\Sigma}_n := \{e \in E'_n \,|\, \sigma(e) = e,\ \forall \sigma \in \Sigma\}.$$

For the automorphism $\Gamma \ni \sigma : x_1 \mapsto x_1(1 + x_2x_3), x_i \mapsto x_i, i \neq 1$, the Jacobian $\mathscr{J}(\sigma) = 1 + x_2x_3$ does not belong to E'^{Σ}_n since $\tau(\mathscr{J}(\sigma)) \neq \mathscr{J}(\sigma)$ where $\Sigma \ni \tau : x_2 \mapsto x_2 + x_1x_4x_5, x_j \mapsto x_j, j \neq 2$. Therefore, Σ is not a normal subgroup of Γ. □

5. The algebraic group Σ' and its dimension

In this section, the group Σ' is studied in detail over a commutative ring K. It is proved that the group Σ' is a unipotent affine group over K of dimension $(n - 2)2^{n-2} - n + \pi_n$ (Corollary 5.6). Important subgroups $\{\Phi'^{2s+1}\}$ are introduced and results are proved for these groups (Lemma 5.3 and Theorem 5.4) that play a crucial role in finding the dimension and coordinates of the Jacobian group Σ.

LEMMA 5.1. — *Let K be an arbitrary ring, $n \geq 4$, and $s = 1, 2, \dots, [\frac{n-1}{2}]$. Each element $a \in \Lambda_{n,2s}$ is a unique sum $a = a_{n-2s} + a_{n-2s+1} + \cdots + a_n$ where*

$$a_{n-2s} := \lambda x_{n-2s+1}x_{n-2s+2} \cdots x_n,$$

$$a_{n-2s+p} := c_p x_{n-2s+p+1}x_{n-2s+p+2} \cdots x_n,\ c_p := \sum_{1 \leq i_1 < \cdots < i_p \leq n-2s+p-1} \lambda_{i_1,\dots,i_p} x_{i_1} \cdots x_{i_p},$$

$$a_n := c_{2s} := \sum_{1 \leq i_1 < \cdots < i_{2s} \leq n-1} \lambda_{i_1,\dots,i_{2s}} x_{i_1} \cdots x_{i_{2s}},$$

$1 \leq p \leq 2s - 1$, and the lambdas are from K.

Proof. — This follows directly from Theorem 8.1.(1). □

Let K be a commutative ring and $n \geq 4$. For each *fixed* natural number s such that $1 \leq s \leq [\frac{n-1}{2}]$, we define the K-module

$$V := V_{n,2s} := \Lambda_{n,2s}(1) \oplus \cdots \oplus \Lambda_{n,2s}(n),\ \ \Lambda_{n,2s}(i) := K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor_{2s},$$

the direct sum of free K-modules of finite rank over K. Each element $v = (v_1, \dots, v_n) \in V$ is a unique sum $v = \sum_{i=1}^n v_i e_i$ where $v_i \in \Lambda_{n,2s}(i)$ and $e_1 := (1, 0, \dots, 0), \dots, e_n := (0, \dots, 0, 1)$. By Lemma 5.1, the K-module homomorphism

$$\overline{\mathscr{J}} := \overline{\mathscr{J}}_{n,2s} : V \to \Lambda_{n,2s},\ (v_1, \dots, v_n) \mapsto v_1 + \cdots + v_n, \tag{83}$$

is a surjection, and the K-homomorphism (where $a = a_{n-2s} + \cdots + a_n$ as in Lemma 5.1)

$$f = f_{n,2s} : \Lambda_{n,2s} \to V, \quad a = a_{n-2s} + \cdots + a_n \mapsto \sum_{i=n-2s}^{n} a_i e_i = (0, \dots, 0, a_{n-2s}, \dots, a_n)$$

is a section of $\overline{\mathscr{J}}$, i.e. $\overline{\mathscr{J}} f = \mathrm{id}$. Hence,

$$V = \ker(\overline{\mathscr{J}}) \oplus f(\Lambda_{n,2s}), \quad f(\Lambda_{n,2s}) = A_{n-2s} \oplus \cdots \oplus A_n, \tag{84}$$

where $A_i := A_{n,2s,i} := f(\Lambda_{n,2s}) \cap \Lambda_{n,2s}(i)$. By Lemma 5.1,

$$\begin{aligned} A_{n-2s} &= K x_{n-2s+1} x_{n-2s+2} \cdots x_n, \\ A_{n-2s+p} &= \Big(\bigoplus_{1 \le i_1 < \cdots < i_p \le n-2s+p-1} K x_{i_1} \cdots x_{i_p}\Big) x_{n-2s+p+1} x_{n-2s+p+2} \cdots x_n, \\ A_n &= \bigoplus_{1 \le i_1 < \cdots < i_{2s} \le n-1} K x_{i_1} \cdots x_{i_{2s}}, \end{aligned}$$

where $1 \le p \le 2s - 1$. So, the K-modules $A_{n-2s}, \dots, A_n$ are free finitely generated K-modules generated by monomials (as above) of degree $2s$.

We will see later that the map $\overline{\mathscr{J}}$ in (83) is, up to isomorphism, the Jacobian map $\overline{\mathscr{J}}$ (88). Our goal is to find a special K-basis for the kernel $\ker(\overline{\mathscr{J}})$ that has connection with certain automorphisms of the group Σ'. For, we will define, so-called, avoidance functions which allows one to produce the required basis and then explicit automorphisms of the group Σ'.

Avoidance functions. — For each monomial $u = x_{i_1} \cdots x_{i_t}$ of Λ_n the set $\{i_1, \dots, i_t\}$ is called the *support* of the monomial u. Let us fix a natural number s such that $1 \le s \le [\frac{n-1}{2}]$. Next, for each $i = 1, \dots, n-1$, we are going to define a set S'_i and a function j_i on it. We do this in two steps: first, for $i = n-2s, \dots, n-1$; and then for $i = 1, \dots, n-2s-1$.

For each $i = n-2s, n-2s+1, \dots, n$, let $S_i := S_{i,s} := \mathrm{Supp}(A_i)$ be the set of supports of all the monomials from the module A_i (the K-module A_i is generated by monomials), and let $S'_i := S'_{i,s}$ be its complement in the set $\mathrm{Supp}(\Lambda_{n,2s}(i)) = \{\alpha \subseteq \{1, \dots, \widehat{i}, \dots, n\} \mid |\alpha| = 2s\}$ where $|\alpha|$ is the number of elements in the set α. In more detail,

$$\begin{aligned}
S_{n-2s,s} &= \{\{n-2s+1,\dots,n\}\},\\
S_{n-2s+p,s} &= \{\alpha\subseteq\{1,\dots,\widehat{n-2s+p},\dots,n\}\,|\,\alpha\supseteq\{n-2s+p+1,\dots,n\},|\alpha|=2s\},\\
S_{n,s} &= \{\alpha\subseteq\{1,\dots,n-1\}\,|\,|\alpha|=2s\},\ \text{and}\\
S'_{n-2s,s} &= \{\alpha\subseteq\{1,\dots,\widehat{n-2s},\dots,n\}\,|\,\alpha\neq\{n-2s+1,\dots,n\},|\alpha|=2s\},\\
S'_{n-2s+p,s} &= \{\alpha\subseteq\{1,\dots,\widehat{n-2s+p},\dots,n\}\,|\,\alpha\not\supseteq\{n-2s+p+1,\dots,n\},|\alpha|=2s\},\\
S'_{n,s} &= \varnothing,
\end{aligned}$$

where $1\le p\le 2s-1$. One can easily verify that the following sets are *the only empty sets* among the sets $\{S'_{n-2s+p',s}\,|\,s=1,\dots,[\frac{n-1}{2}];\ p'=0,1,\dots,2s-1\}$: if n is an *odd* number, $s=[\frac{n-1}{2}]$, and $p'=0,1,\dots,2s-1$, i.e.

$$S'_{1,[\frac{n-1}{2}]}=S'_{2,[\frac{n-1}{2}]}=\cdots=S'_{n-1,[\frac{n-1}{2}]}=\varnothing. \tag{85}$$

In particular, for $n=5$ we have

$$S'_{1,2}=S'_{2,2}=\cdots=S'_{4,2}=\varnothing. \tag{86}$$

Let us stress that *for each* $i=n-2s,\dots,n-1$, *the set* $S'_i:=S'_{i,s}$ *is equal to the set of all* $\alpha\in\mathrm{Supp}(\Lambda_{n,2s}(i))$ *such that* $\{i+1,i+2,\dots,n\}\backslash\alpha\neq\varnothing$. So, we can fix a function

$$j_i:=j_{i,s}:S'_i\to\{i+1,i+2,\dots,n\},\quad \alpha\mapsto j_i(\alpha)\in\{i+1,i+2,\dots,n\}\backslash\alpha.$$

If the set S'_i is an empty set then this definition is vacuous since we have an ‘empty function’ defined on the empty set. It is convenient to have these ‘empty functions’ in order to save on notation.

For each $i=1,\dots,n-2s-1$, let $S'_i:=S'_{i,s}:=\mathrm{Supp}(\Lambda_{n,2s}(i))$, and we can fix a function

$$j_i:=j_{i,s}:S'_i\to\{i+1,i+2,\dots,n\},\quad \alpha\mapsto j_i(\alpha)\in\{i+1,i+2,\dots,n\}\backslash\alpha,$$

(if not then $\{i+1,\dots,n\}\subseteq\alpha$ for some $\alpha\in S'_i$, and so $\alpha\cup\{1,\dots,i\}=\{1,\dots,n\}$ but then one has the contradiction: $n=|\alpha\cup\{1,\dots,i\}|\le|\alpha|+i\le 2s+n-2s-1=n-1<n$).

Definition. For the fixed number s (as above), the functions $\{j_i\,|\,1\le i\le n-1\}$ are called *avoidance functions*.

Note that there are many avoidance functions, in general. The importance of avoidance functions $\{j_i\}$ is the fact that, for any $\alpha\in S'_i$, we can attach the 1-dimensional abelian subgroup of Σ', $\{\rho_{i,j_i(\alpha);\lambda x^\alpha}\}\simeq K$, $\rho_{i,j_i(\alpha);\lambda x^\alpha}\mapsto\lambda$, by the rule

(87)

$$\rho_{i,j_i(\alpha);\lambda x^\alpha}:x_i\mapsto x_i(1+\lambda x^\alpha),\quad x_{j_i(\alpha)}\mapsto x_{j_i(\alpha)}(1-\lambda x^\alpha),\quad x_k\mapsto x_k,\quad k\neq i,j_i(\alpha).$$

Lemma 5.2. — *Let K be a commutative ring, $n \geq 4$, and $\{j_i = j_{i,s}\}$ be avoidance functions for a fixed number s such that $1 \leq s \leq [\frac{n-1}{2}]$. Then the set $\cup_{i=1}^{n-1}\{x^\alpha(e_i - e_{j_i(\alpha)}) \,|\, \alpha \in S'_i\}$ is a K-basis for the kernel $\ker(\overline{\mathscr{J}})$ of the map $\overline{\mathscr{J}} = \overline{\mathscr{J}}_{n,2s}$, (83); and the rank of the free K-module $\ker(\overline{\mathscr{J}}_{n,2s})$ is equal to*

$$\mathrm{rk}_K(\ker(\overline{\mathscr{J}}_{n,2s})) = \sum_{i=1}^{n-1} |S'_{i,s}| = n\binom{n-1}{2s} - \binom{n}{2s}.$$

Proof. — By the very definition, the elements from the union are from the kernel $\ker(\overline{\mathscr{J}})$ and K-linear independent (use the fact that $i < j_i(\alpha)$ for all i and $\alpha \in S'_i$; and the fact that, for each $i = 1, \ldots, n-1$, the monomials x^α, $\alpha \in S'_{i,s}$ are K-linear independent). Let U be the K-submodule of V that these elements generate. It follows from the definition of the sets S'_i that $U + f(\Lambda_{n,2s}) = V$, hence $U = \ker(\overline{\mathscr{J}})$, by (84) and the inclusion $U \subseteq \ker(\overline{\mathscr{J}})$.

By (84), the rank of the free K-module $\ker(\overline{\mathscr{J}}_{n,2s})$ is equal to

$$\begin{aligned}\mathrm{rk}_K(\ker(\overline{\mathscr{J}}_{n,2s})) &= \sum_{i=1}^{n-1} |S'_{i,s}| = \mathrm{rk}_K(V_{n,2s}) - \mathrm{rk}_K(f(\Lambda_{n,2s}))\\ &= n\,\mathrm{rk}_K(\Lambda_{n-1,2s}) - \mathrm{rk}_K(\Lambda_{n,2s}) = n\binom{n-1}{2s} - \binom{n}{2s}. \quad \square\end{aligned}$$

The groups Φ'^{2s+1}. — Let K be a commutative ring, and $n \geq 4$. For each number $s = 1, 2, \ldots, [\frac{n-1}{2}]$, let Φ'^{2s+1} be the subset of Φ that contains all the elements of the following type:

$$\begin{aligned}
\sigma(x_i) &= x_i(1 + \cdots), \quad 1 \leq i \leq n - 2s - 1,\\
\sigma(x_{n-2s}) &= x_{n-2s}(1 + \lambda x_{n-2s+1}x_{n-2s+2}\cdots x_n + \cdots),\\
\sigma(x_{n-2s+1}) &= x_{n-2s+1}(1 + (\sum_{1 \leq i_1 \leq n-2s} \lambda_{i_1}x_{i_1})x_{n-2s+2}\cdots x_n + \cdots),\\
\cdots &= \cdots,\\
\sigma(x_{n-2s+p}) &= x_{n-2s+p}(1 + (\sum_{1 \leq i_1 < \cdots < i_p \leq n-2s+p-1} \lambda_{i_1,\ldots,i_p}x_{i_1}\cdots x_{i_p})x_{n-2s+p+1}\\
&\qquad \cdots x_n + \cdots),\\
\cdots &= \cdots,\\
\sigma(x_{n-1}) &= x_{n-1}(1 + (\sum_{1 \leq i_1 < \cdots < i_{2s-1} \leq n-2} \lambda_{i_1,\ldots,i_{2s-1}}x_{i_1}\cdots x_{i_{2s-1}})x_n + \cdots),\\
\sigma(x_n) &= x_n(1 + \sum_{1 \leq i_1 < \cdots < i_{2s} \leq n-1} \lambda_{i_1,\ldots,i_{2s}}x_{i_1}\cdots x_{i_{2s}} + \cdots),
\end{aligned}$$

where the lambdas are elements of K and the three dots mean higher terms. By Theorem 2.10.(2), Φ'^{2s+1} is a subgroup of Φ. In the notation of Lemma 5.1, the automorphism $\sigma = \sigma_a$ above (where $a = a_{n-2s} + \cdots + a_n$ as in Lemma 5.1) can be written as

$$\sigma(x_i) = x_i(1 + \cdots), \quad 1 \leq i \leq n - 2s - 1,$$
$$\sigma(x_i) = x_i(1 + a_i + \cdots), \quad n - 2s \leq i \leq n.$$

Clearly,

$$\Phi^{2s+3} \subseteq \Phi'^{2s+1} \subseteq \Phi^{2s+1},$$

and Φ^{2s+3} is a normal subgroup of Φ'^{2s+1} since Φ^{2s+3} is a normal subgroup of Φ. For each $s = 1, \ldots, [\frac{n-1}{2}]$, we have the Jacobian map

$$\mathcal{J} : \Phi^{2s+1} \to E'_{n,2s} := 1 + \sum_{i \geq s} \Lambda_{n,2s}, \quad \sigma \mapsto \mathcal{J}(\sigma).$$

The factor group $E'_{n,2s}/E'_{n,2s+2} = \{(1 + a)E'_{n,2s+2} \,|\, a \in \Lambda_{n,2s}\}$ is canonically isomorphic to the additive group $\Lambda_{n,2s}$ via the isomorphism

$$E'_{n,2s}/E'_{n,2s+2} \to \Lambda_{n,2s}, \quad (1 + a)E'_{n,2s+2} \mapsto a.$$

The Jacobian map $\mathcal{J} : \Phi^{2s+1} \to E'_{n,2s}$ yields the Jacobian maps

$$\overline{\mathcal{J}} : \Phi^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2}, \quad \sigma\Phi^{2s+3} \mapsto \mathcal{J}(\sigma)E'_{n,2s+2}, \tag{88}$$

and

$$\overline{\mathcal{J}} : \Phi'^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2}, \quad \sigma\Phi^{2s+3} \mapsto \mathcal{J}(\sigma)E'_{n,2s+2}.$$

There is the natural isomorphism of the abelian groups:

$$\Phi^{2s+1}/\Phi^{2s+3} \to \bigoplus_{i=1}^{n} \Lambda_{n,2s}(i) = V = V_{n,2s},$$
$$\{\sigma : x_i \mapsto x_i(1 + a_i)\Phi^{2s+3}\} \mapsto (a_1, \ldots, a_n),$$

where $a_i \in \Lambda_{n,2s}(i)$ for all $i = 1, \ldots, n$. When we identify the groups Φ^{2s+1}/Φ^{2s+3} and V on the one hand, and the groups $E'_{n,2s}/E'_{n,2s+2}$ and $\Lambda_{n,2s}$ on the other via the isomorphisms above, then *the Jacobian map* $\overline{\mathcal{J}} : \Phi^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2}$, $\sigma\Phi^{2s+3} \mapsto \mathcal{J}(\sigma)E'_{n,2s+2}$, *coincides with the map* (83), $\overline{\mathcal{J}} : V \to \Lambda_{n,2s}$, $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$. Then Lemma 5.3 follows, which is one of the key results in finding generators for the Jacobian group Σ and its subgroup Σ'.

LEMMA 5.3. — *Let K be a commutative ring, $n \geq 4$, and $s = 1, \ldots, [\frac{n-1}{2}]$. The Jacobian map* $\overline{\mathcal{J}} : \Phi'^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2}$, $\sigma\Phi^{2s+3} \mapsto \mathcal{J}(\sigma)E'_{n,2s+2}$, *is an isomorphism of the abelian groups which is given by the rule*

$$\overline{\mathcal{J}}(\sigma\Phi^{2s+3}) = (1+\lambda x_{n-2s+1}\cdots x_n + \sum_{p=1}^{2s-1}(\sum_{1\leq i_1<\cdots<i_s\leq n-2s+p-1}\lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_p})x_{n-2s+p+1}\cdots x_n + \sum_{1\leq i_1<\cdots<i_{2s}\leq n-1}\lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_{2s}})E'_{n,2s+2}$$

for the element $\sigma\in\Phi'^{2s+1}$ as above (i.e. in the definition of Φ'^{2s+1}).

Proof. — When one writes down the determinant $\mathcal{J}(\sigma)$ for the element $\sigma\in\Phi'^{2s+1}$ as above it is easy to see that $\mathcal{J}(\sigma)E'_{n,2s+2}$ is the product of the diagonal elements in the determinant $\mathcal{J}(\sigma)$ modulo $E'_{n,2s+2}$:

$$\begin{aligned}\overline{\mathcal{J}}(\sigma\Phi^{2s+3}) &= (1+\lambda x_{n-2s+1}\cdots x_n)\cdots \\ &\cdots((1+\sum_{1\leq i_1<\cdots<i_s\leq n-2s+p-1}\lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_p})x_{n-2s+p+1}\cdots x_n)\cdots \\ &\cdots(1+\sum_{1\leq i_1<\cdots<i_{2s}\leq n-1}\lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_{2s}})E'_{n,2s+2} \\ &= (1+\lambda x_{n-2s+1}\cdots x_n + \sum_{p=1}^{2s-1}(\sum_{1\leq i_1<\cdots<i_s\leq n-2s+p-1} \\ &\qquad \lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_p})x_{n-2s+p+1}\cdots x_n \\ &+ \sum_{1\leq i_1<\cdots<i_{2s}\leq n-1}\lambda_{i_1,\dots,i_p}x_{i_1}\cdots x_{i_{2s}})E'_{n,2s+2},\end{aligned}$$

and so we obtain the formula for $\overline{\mathcal{J}}(\sigma\Phi^{2s+3})$ in Lemma 5.3. Now, it is obvious that the map $\overline{\mathcal{J}}$ is the isomorphism of the abelian groups since each element of $\Lambda_{n,2s}$ can be uniquely written as a sum s in the formula for $\overline{\mathcal{J}}(\sigma\Phi^{2s+3}) = (1+s)E'_{n,2s+2}$ above (see Lemma 5.1). □

THEOREM 5.4. — *Let K be a commutative ring, $n\geq 4$, and $s=1,\dots,[\frac{n-1}{2}]$. Then*

1. $\Phi^{2s+1}=\Phi'^{2s+1}\Sigma'^{2s+1}=\Sigma'^{2s+1}\Phi'^{2s+1}$.
2. $\Phi^{2s+1}/\Phi^{2s+3}\simeq\Sigma'^{2s+1}/\Sigma'^{2s+3}\times\Phi'^{2s+1}/\Phi^{2s+3}$, *the direct product of abelian groups.*
3. *Each automorphism $\sigma\in\Phi^{2s+1}$: $x_i\mapsto x_i(1+b_i+\cdots)$, $b_i\in\Lambda_{n,2s}(i)$, $i=1,\dots,n$, is the unique product modulo Φ^{2s+3} as follows,*

$$\sigma\equiv\phi'\prod_{i=1}^{n-1}\prod_{\alpha\in S'_i}\rho_{i,j_i(\alpha);\lambda_\alpha x^\alpha}\mod\Phi^{2s+3}, \tag{89}$$

where

$$\phi'(x_k) := x_k,\ \ 1 \le k \le n-2s-1,$$
$$\phi'(x_i) := x_i(1+a_i),\ \ n-2s \le i \le n,$$

and

$$(b_1, \dots, b_n) = \sum_{i=1}^{n-1} \sum_{\alpha \in S'_i} \lambda_\alpha x^\alpha (e_i - e_{j_i(\alpha)}) + \sum_{i=n-2s}^{n} a_i e_i$$

in $V = \ker(\overline{\mathscr{J}}) \oplus f(\Lambda_{n,2s})$ *for unique* $\lambda_\alpha \in K$ *and unique elements* a_i *as in Lemma 5.1.*

Proof. — Recall that the Jacobian map $\overline{\mathscr{J}} : \Phi^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2}$, $\sigma\Phi^{2s+3} \mapsto \mathscr{J}(\sigma)E'_{n,2s+2}$, is naturally identified with the map (83),

$$\overline{\mathscr{J}} : V \to \Lambda_{n,2s},\ \ (a_1, \dots, a_n) \mapsto a_1 + \dots + a_n,$$

under the identifications $\Phi^{2s+1}/\Phi^{2s+3} \equiv V$ and $E'_{n,2s}/E'_{n,2s+2} \equiv \Lambda_{n,2s}$. Consider the direct sum (84): $V = \ker(\overline{\mathscr{J}}) \oplus f(\Lambda_{n,2s})$. By Lemma 5.2, the free K-module $\ker(\overline{\mathscr{J}})$ has the K-basis $\cup_{i=1}^{n-1}\{x^\alpha(e_i - e_{j_i(\alpha)}) \mid \alpha \in S'_i\}$, and each element of $f(\Lambda_{n,2s})$ is a unique sum $\sum_{i=n-2s}^{n} a_i e_i$ where a_i are as in Lemma 5.1. To each basis element $x^\alpha(e_i - e_{j_i(\alpha)})$ we attach the 1-dimensional abelian subgroup of Σ'^{2s+1}: $\{\rho_{i,j_i(\alpha);\lambda x^\alpha}\}_{\lambda \in K} \simeq K$, by (87). To each element $a = \sum_{i=n-2s}^{n} a_i e_i$ it corresponds (under the identification $\Phi^{2s+1}/\Phi^{2s+3} \equiv V$) the automorphism $\phi_a \in \Phi'^{2s+1}$:

$$\phi_a(x_k) := x_k,\ \ 1 \le k \le n-2s-1,$$
$$\phi_a(x_i) := x_i(1+a_i),\ \ n-2s \le i \le n.$$

Each element v of V is a unique sum

$$v = \sum_{i=1}^{n-1} \sum_{\alpha \in S'_i} \lambda_\alpha x^\alpha (e_i - e_{j_i(\alpha)}) + \sum_{k=n-2s}^{n} a_k e_k$$

for unique $\lambda_\alpha \in K$ and unique a_k as in Lemma 5.1. Under the identification $\Phi^{2s+1}/\Phi^{2s+3} \equiv V$, the element v can be identified with the automorphism σ_v modulo Φ^{2s+3} (i.e. $v \equiv \sigma_v \Phi^{2s+3}$) where

$$\sigma_v = \phi_a \sigma',\ \ \sigma' := \prod_{i=1}^{n-1} \prod_{\alpha \in S'_i} \rho_{i,j_i(\alpha);\lambda_\alpha x^\alpha}. \tag{90}$$

Conversely, any coset $\sigma\Phi^{2s+3}$ where $\sigma \in \Phi^{2s+1}$ can be identified with the element $v \in V$ (i.e. $\sigma\Phi^{2s+3} \equiv v$) by the rule: let $\sigma(x_i) = x_i(1 + b_i + \cdots)$ for some $b_i \in \Lambda_{n,2s}(i)$, $i = 1, \dots, n$, then $(b_1, \dots, b_n) \in V = \oplus_{i=1}^{n} \Lambda_{n,2s}(i)$ and $\sigma\Phi^{2s+3} \equiv (b_1, \dots, b_n)$. Now, statement 1 follows immediately from (90). Under the identification $\Phi^{2s+1}/\Phi^{2s+3}\equiv V$, the decomposition $V=\ker(\overline{\mathscr{J}})\oplus f(\Lambda_{n,2s})$ corresponds to the decomposition (the direct product of groups)

$$\Phi^{2s+1}/\Phi^{2s+3}\simeq \Sigma'^{2s+1}\Phi^{2s+3}/\Phi^{2s+3}\times \Phi'^{2s+1}/\Phi^{2s+3}.$$

Since $\Sigma'^{2s+1}\Phi^{2s+3}/\Phi^{2s+3}\simeq \Sigma'^{2s+1}/\Sigma'^{2s+3}\cap\Phi^{2s+3}\simeq \Sigma'^{2s+1}/\Sigma'^{2s+3}$, the statement 2 follows. Statement 3 is just (90). □

THEOREM 5.5. — *Let K be a commutative ring, and $n\geq 4$.*

1. *Then each automorphism $\sigma\in\Phi$ is a unique product*

$$\sigma=\prod_{i=1}^{[\frac{n-1}{2}]}\phi_{2s+1}\sigma_{2s+1}=\phi_3\sigma_3\phi_5\sigma_5\cdots\phi_{2[\frac{n-1}{2}]+1}\sigma_{2[\frac{n-1}{2}]+1} \tag{91}$$

for unique elements $\phi_{2s+1}\in\Phi'^{2s+1}$ and $\sigma_{2s+1}\in\Sigma'^{2s+1}$ from (89). Moreover,

$$\sigma\equiv\phi_3\sigma_3 \mod \Phi^5,$$
$$(\phi_3\sigma_3\cdots\phi_{2s-1}\sigma_{2s-1})^{-1}\sigma\equiv\phi_{2s+1}\sigma_{2s+1} \mod \Phi^{2s+3},\ \ 2\leq s\leq[\frac{n-1}{2}].$$

2. *Each automorphism $\sigma\in\Sigma'$ is a unique product*

$$\sigma=\prod_{i=1}^{[\frac{n-1}{2}]}\sigma_{2s+1}=\sigma_3\sigma_5\cdots\sigma_{2[\frac{n-1}{2}]+1} \tag{92}$$

for unique elements $\sigma_{2s+1}\in\Sigma'^{2s+1}$ from (89). Moreover,

$$\sigma\equiv\sigma_3 \mod \Phi^5,$$
$$(\sigma_3\cdots\sigma_{2s-1})^{-1}\sigma\equiv\sigma_{2s+1} \mod \Phi^{2s+3},\ \ 2\leq s\leq[\frac{n-1}{2}].$$

Proof. — 1. This statement follows from Theorem 5.4.

2. We need only to show that, for $\sigma\in\Sigma'$, $\phi_3=\cdots=\phi_{2[\frac{n-1}{2}]+1}=e$ in (91). Suppose that $\phi_{2s+1}\neq e$ for some s and the s is the least possible with this property. We seek a contradiction. Without loss of generality we may assume that $\sigma_3=\cdots=\sigma_{2s-1}=e$, i.e. $\sigma=\phi_{2s+1}\sigma_{2s+1}\cdots$. Since $\mathscr{J}(\sigma)=1$, we must have $\overline{\mathscr{J}}(\sigma)=1$ in $E'_{n,2s}/E'_{n,2s+2}$. On the other hand, $\overline{\mathscr{J}}(\sigma)=\overline{\mathscr{J}}(\phi_{2s+1})\neq 1$ in $E'_{n,2s}/E'_{n,2s+2}$ (Lemma 5.3), by the choice of the ϕ_{2s+1}, a contradiction. □

The dimension of the algebraic group Σ'. — Recall that $n \geq 4$ and for each number $s = 1, \dots, [\frac{n-1}{2}]$, we defined the sets $S'_i := S'_{i,s}$, $1 \leq i \leq n-1$, and the avoidance functions $\{j_i := j_{i,s}\}$. By Theorem 5.5.(2) and Theorem 5.4.(3), each element σ of Σ' is a unique *ordered* product

$$\sigma = \prod_{s=1}^{[\frac{n-1}{2}]} \prod_{i=1}^{n-1} \prod_{\alpha \in S'_{i,s}} \rho_{i, j_{i,s}(\alpha); \lambda_\alpha x^\alpha} \tag{93}$$

where $\alpha = \alpha_{i,s}$ (they depend on i and s) and $\lambda_\alpha = \lambda_{\alpha,i,s} \in K$. Therefore, $\{\lambda_\alpha = \lambda_{\alpha,i,s}\}$ are affine coordinates for the algebraic group Σ' over the ring K, and the algebra of (regular) functions $\mathcal{O}(\Sigma')$ on the algebraic group Σ' is a *polynomial algebra* in

$$\dim(\Sigma') = \sum_{s=1}^{[\frac{n-1}{2}]} \sum_{i=1}^{n-1} |S'_{i,s}| \tag{94}$$

variables. Consider the function

$$\pi_n := \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Corollary 5.6. — *Let K be a commutative ring and $n \geq 4$. The group Σ' is a unipotent affine group over K of dimension*

$$\dim(\Sigma') = \sum_{s=1}^{[\frac{n-1}{2}]} (n\binom{n-1}{2s} - \binom{n}{2s}) = \sum_{s=1}^{[\frac{n-1}{2}]} (n-2s-1)\binom{n}{2s}$$
$$= \begin{cases} (n-2)2^{n-2} - n + 2 & \textit{if } n \textit{ is even,} \\ (n-2)2^{n-2} - n + 1 & \textit{if } n \textit{ is odd.} \end{cases}$$

over K, i.e. the algebra of regular functions on Σ' is a polynomial algebra over the ring K in $\dim(\Sigma')$ *variables* $\{\lambda_\alpha\}$.

Proof. — The only statement which is needed to be proven is the formula for the dimension. For each s, by Lemma 5.2,

$$\sum_{i=1}^{n-1} |S'_{i,s}| = n\binom{n-1}{2s} - \binom{n}{2s} = (n-2s-1)\binom{n}{2s}.$$

The first part of the formula for $\dim(\Sigma')$ then follows from (94). Note that

$$n\sum_{s=1}^{[\frac{n-1}{2}]} \binom{n-1}{2s} = n(\sum_{s=0}^{[\frac{n-1}{2}]} \binom{n-1}{2s} - 1) = n(2^{n-2} - 1).$$

If n is even then

$$\sum_{s=1}^{[\frac{n-1}{2}]}\binom{n}{2s}=\sum_{s=0}^{[\frac{n-1}{2}]}\binom{n}{2s}-1=\sum_{s=0}^{[\frac{n}{2}]}\binom{n}{2s}-\binom{n}{2[\frac{n}{2}]}-1=2^{n-1}-2.$$

If n is odd then

$$\sum_{s=1}^{[\frac{n-1}{2}]}\binom{n}{2s}=\sum_{s=1}^{[\frac{n}{2}]}\binom{n}{2s}=\sum_{s=0}^{[\frac{n}{2}]}\binom{n}{2s}-1=2^{n-1}-1.$$

By the first part of the formula for $\dim(\Sigma')$ and the calculations above, we have

$$\dim(\Sigma')=n(2^{n-2}-1)-(2^{n-1}-\pi_n)=(n-2)2^{n-2}-n+\pi_n. \qquad \square$$

The subgroup Σ' of Γ is 'twice smaller' than Γ in the following sense (see (34))

$$\lim_{n\to\infty}\frac{\dim(\Sigma')}{\dim(\Gamma)}=\lim_{n\to\infty}\frac{(n-2)2^{n-2}-n+\pi_n}{n(2^{n-1}-n)}=\frac{1}{2}. \tag{95}$$

The group Σ' is not a normal subgroup of Σ if $n\geq 6$ and $2\neq 0$ in K. — Clearly, $\Sigma\ni\sigma: x_1\mapsto x_1+x_2x_3x_4,\ x_i\mapsto x_i,\ i\neq 1$; and $\Sigma'\ni\tau: x_1\mapsto x_1(1+x_5x_6), x_2\mapsto x_2(1-x_5x_6),\ x_i\mapsto x_i,\ i\neq 1,2$. Then $\sigma\tau\sigma^{-1}\tau^{-1}(x_1)=x_1+2x_2x_3x_4x_5x_6$, hence $\sigma\tau\sigma^{-1}\tau^{-1}\notin\Sigma'$. This means that the subgroup Σ' of Σ is not normal if $n\geq 6$ and $2\neq 0$ in K.

We will see later that the group Σ'' is a *closed normal* subgroup of the Jacobian group Σ (Theorem 6.4.(2)).

6. A (minimal) set of generators for the Jacobian group Σ and its dimension

Let K be a *commutative* ring. In this section, a minimal set of generators for the Jacobian group Σ is found explicitly (Theorem 6.1). The dimensions and coordinates of the following algebraic groups are found explicitly: Σ (Theorem 6.3), $\Sigma'\cap\Sigma''$ (Lemma 6.2), Σ'' (Theorem 6.4). It is proved that the sets of cosets $\Sigma'\diagdown\Sigma$ and $\Sigma'\cap\Sigma''\diagdown\Sigma''$ have natural structure of an affine variety of dimension $n(2^{n-2}-n+1)$ over K (Corollary 6.5).

(Minimal) set of generators for the Jacobian group Σ. — We keep the notations of Section 5. For $s=1$, $1 \le p \le 2s-1=1$, i.e. $p=1$. Consider the sets $S'_{i,1}$, $1 \le i \le n-1$, defined in Section 5:

$$S'_{i,1} = \mathrm{Supp}(\Lambda_{n,2}(i)), \ \ 1 \le i \le n-3,$$
$$S'_{n-2,1} = \{\alpha \subseteq \{1, \dots, \widehat{n-2}, n-1, n \,|\, \alpha \neq \{n-1, n\}, |\alpha| = 2\},$$
$$S'_{n-1,1} = \{\alpha \subseteq \{1, \dots, \widehat{n-1}, n \,|\, \alpha \not\ni n, |\alpha| = 2\}.$$

Fix avoidance functions

$$j_i : S'_{i,1} \mapsto \{i+1, \dots, n\}, \ \ 1 \le i \le n-1.$$

The next theorem provides a (minimal) set of generators for the Jacobian group Σ for $n \ge 7$.

Theorem 6.1. — *Let K be a commutative ring, $n \ge 7$, and for $s=1$ let $\{j_i := j_{i,1}\}$ be avoidance functions. If either n is odd; or n is even and $\frac{1}{2} \in K$; then*

$$\Sigma = \langle \rho_{i,j_i(\alpha);\lambda x^\alpha}, \sigma_{i',\lambda x_j x_k x_l} \,|\, \lambda \in K; 1 \le i \le n-1; \alpha \in S'_{i,1}; 1 \le i' \le n; j<k<l; i' \notin \{j,k,l\}\rangle$$

and the 1-dimensional abelian subgroups $\{\rho_{i,j_i(\alpha);\lambda x^\alpha}\} \simeq K$ and $\{\sigma_{i',\lambda x_j x_k x_l}\} \simeq K$ of Σ form a minimal set of generators for Σ in the sense that no subgroup can be dropped.

Proof. — Recall that $\Sigma = \Sigma'\Sigma''$ (Corollary 4.11.(1)); and, by (93),

$$\Sigma' = \langle \rho_{i,j_{i,s}(\alpha_s);\lambda x^{\alpha_s}} \,|\, \lambda \in K; 1 \le i \le n-1; 1 \le s \le [\frac{n-1}{2}]; \alpha_s \in S'_{i,s}\rangle$$

where $j_{i,s}$ are avoidance functions;

$$\Sigma'' = \langle \sigma_{i',\lambda x_j x_k x_l} \,|\, \lambda \in K; 1 \le i' \le n; j<k<l; i' \notin \{j,k,l\}\rangle$$

(Theorem 4.7.(1)), and, by the definition, the group Σ'' is generated by all the automorphisms

$$\xi_{i,b_i} : x_i \mapsto x_i + b_i, \ \ x_j \mapsto x_j, \ \ j \neq i,$$

where $b_i \in K\lfloor x_1, \dots, \widehat{x_i}, \dots, x_n \rfloor^{\mathrm{od}}_{\ge 3}$.

If $s=1$ then the elements $\{\rho_{i,j_{i,1}(\alpha_1);\lambda x^{\alpha_1}}\}$ are precisely the ρ-part of the generators in the theorem. If $s \ge 2$ then, by (72), each element $\rho_{i,j_{i,s}(\alpha_s);\lambda x^{\alpha_s}}$ belongs to the group Σ''. Therefore,

$$\Sigma = \langle \rho_{i,j_i(\alpha);\lambda x^\alpha}, \sigma_{i',\lambda x_j x_k x_l} \,|\, \lambda \in K; 1 \le i \le n-1; \alpha \in S'_{i,1}; 1 \le i' \le n; j<k<l; i' \notin \{j,k,l\}\rangle,$$

i.e. the first part of the theorem is proved. To prove the second part of the theorem (about minimality), note that, by Theorem 2.10.(2), the map

$$(\Lambda_{n,3})^n \to U^3/U^5, \ \ a = (a_1, \dots, a_n) \mapsto \sigma_a U^5,$$

is a group isomorphism where $\sigma_a \in U^3 : x_i \mapsto x_i + a_i$. We identify these two groups via the isomorphism above, then the elements $\rho_{i,j_i(\alpha);\lambda x^\alpha} U^5$ and $\sigma_{i';\lambda x_j x_k x_l} U^5$ are identified correspondingly with the elements $\lambda x^\alpha (x_i e_i - x_{j_i(\alpha)} e_{j_i(\alpha)})$ and $\lambda x_j x_k x_l e_{i'}$ of the K-module $W := (\Lambda_{n,3})^n = \oplus_{i=1}^n \Lambda_{n,3} e_i$ where $e_1 := (1,0,\dots,0), \dots, e_n := (0,\dots,0,1)$. To prove the minimality it suffices to show that the elements $\{x^\alpha(x_i e_i - x_{j_i(\alpha)} e_{j_i(\alpha)}),\ x_j x_k x_l e_{i'}\}$ are K-linearly independent. To prove this let us consider the descending filtration $\{W_i := \oplus_{j=i}^n \Lambda_{n,3} e_j\}$ on W. Clearly, $W_i / W_{i+1} = (\Lambda_{n,3} e_i \oplus W_{i+1})/W_{i+1} \simeq \Lambda_{n,3} e_i \simeq \Lambda_{n,3}$, $i \geq 1$. Suppose that $r := \sum \lambda_{i\alpha} x^\alpha (x_i e_i - x_{j_i(\alpha)} e_{j_i(\alpha)}) + \sum \mu_{i',j,k,l} x_j x_k x_l e_{i'} = 0$ is a nontrivial relation. Let i be minimal index such that either some $\lambda_{i\alpha} \neq 0$ or some $\mu_{i,j,k,l} \neq 0$. Then $r \in W_i$. Taking the relation r modulo W_{i+1} we have

$$r \equiv \Big(\sum \lambda_{i\alpha} x^\alpha x_i + \sum \mu_{i,j,k,l} x_j x_k x_l\Big) e_i \equiv 0 \mod W_{i+1},$$

(we used the fact that $j_i(\alpha) > i$), i.e. $\sum \lambda_{i\alpha} x^\alpha x_i + \sum \mu_{i,j,k,l} x_j x_k x_l = 0$ in $\Lambda_{n,3}$, hence all $\lambda_{i\alpha} = 0$ and all $\mu_{i,j,k,l} = 0$ since all the monomials are distinct, a contradiction. This finishes the proof of the theorem. $\square$

The dimension of $\Sigma' \cap \Sigma''$. — Recall that

$$\pi_n := \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Let $\Sigma/\Sigma'' := \{\sigma\Sigma'' \,|\, \sigma \in \Sigma\} = \{\sigma\Sigma'' \,|\, \sigma \in \Sigma'\}$ since $\Sigma = \Sigma'\Sigma''$ (Corollary 4.11.(1)).

The next result shows that the subgroup Σ'' of Σ is quite large and that the intersection $\Sigma' \cap \Sigma''$ is a large subgroup of Σ'.

Lemma 6.2. — *Let K be a commutative ring and $n \geq 4$. Then*

1. $\Sigma' \cap \Sigma'' = \Sigma'^5$ *where* $\Sigma'^5 := \{\sigma \in \Sigma' \,|\, (\sigma - 1)(\mathfrak{m}) \subseteq \mathfrak{m}^5\}$, *and so* $\Sigma' \cap \Sigma''$ *is a closed subgroup of* Σ.
2. *The group* $\Sigma' \cap \Sigma''$ *is a unipotent affine group over* K *of dimension*
$$\dim(\Sigma' \cap \Sigma'') = (n-2)2^{n-2} - n + \pi_n - (n-3)\binom{n}{2}$$
over K.
3. *There is the natural bijection* $\Sigma/\Sigma'' \to \Sigma'/\Sigma'^5 \simeq \prod_{i=1}^{n-1} \prod_{\alpha \in S'_{i,1}} \{\rho_{i,j_i(\alpha);\lambda x^\alpha}\}_{\lambda \in K}$, $\sigma\Sigma'' \mapsto \sigma\Sigma'$, *where* $\sigma \in \Sigma'$ *(see Corollary 4.11.(1)). The set* Σ'/Σ'^5 *is an affine variety of dimension*
$$\dim(\Sigma'/\Sigma'^5) = n\binom{n-1}{2} - \binom{n}{2} = (n-3)\binom{n}{2}.$$

Proof. — 1. For $n = 4$, the first statement is obvious as $\Sigma'^5 = \Sigma''^5 = \{e\}$ and $\Sigma' \cap \Sigma'' = \{e\}$, by the very definitions of the groups Σ' and Σ''. We assume that $n \geq 5$.

By (92), any element σ of Σ' is a product $\sigma = \sigma_3\sigma_5 \cdots \sigma_{2[\frac{n-1}{2}]+1}$ where σ_i is a product of elements of the type $\rho_{i,j_{i,s}(\alpha_s);\lambda x^{\alpha_s}}$, $\alpha_s \in S'_{i,s}$, $1 \leq s \leq [\frac{n-1}{2}]$, by (93). Any element σ of Σ'^5 is a product $\sigma = \sigma_5\sigma_7 \cdots \sigma_{2[\frac{n-1}{2}]+1}$ (with $\sigma_3 = e$) where each σ_i is product of elements of the type $\rho_{i,j_{i,s}(\alpha_s);\lambda x^{\alpha_s}}$, $\alpha_s \in S'_{i,s}$, $2 \leq s \leq [\frac{n-1}{2}]$. For $n \geq 6$, by (72), if $s \geq 2$ then all $\rho_{i,j_{i,s}(\alpha_s);\lambda x^{\alpha_s}} \in \Sigma''$. Therefore, an element $\sigma = \sigma_3\sigma_5 \cdots \sigma_{2[\frac{n-1}{2}]+1} \in \Sigma'$ belongs to the group Σ'' (i.e. $\sigma \in \Sigma' \cap \Sigma''$) iff $\sigma_3 \in \Sigma''$ (since $\sigma_5 \cdots \sigma_{2[\frac{n-1}{2}]+1} \in \Sigma''$) iff $\sigma_3 = e$.

An element $\sigma = \sigma_3\sigma_5 \cdots \sigma_{2[\frac{n-1}{2}]+1} \in \Sigma'$ belongs to the group Σ'^5 iff $\sigma_3 \in \Sigma'^5$ (since $\sigma_5 \cdots \sigma_{2[\frac{n-1}{2}]+1} \in \Sigma'^5$) iff $\sigma_3 = e$. Therefore, $\Sigma' \cap \Sigma'' = \Sigma'^5$, if $n \geq 6$.

If $n = 5$ then $\Sigma'^5 = \{e\}$ by (86), and so $\Sigma' \cap \Sigma'' = \{e\}$, by the very definitions of the groups Σ' and Σ''.

3. By Corollary 4.11.(1), $\Sigma = \Sigma'\Sigma''$. Using statement 1, we se that

$$\Sigma/\Sigma'' = \Sigma'\Sigma''/\Sigma'' \simeq \Sigma'/\Sigma' \cap \Sigma'' \simeq \Sigma'/\Sigma'^5 \simeq \Sigma'^3/\Sigma'^5 \simeq \prod_{i=1}^{n-1} \prod_{\alpha \in S'_{i,1}} \{\rho_{i,j_i(\alpha);\lambda x^\alpha}\}_{\lambda \in K}.$$

So, Σ'/Σ'^5 is an affine variety. Now, using the identifications as in the proof of Theorem 5.4 in the case $s = 1$ there it follows at once that

$$\dim(\Sigma'/\Sigma'^5) = \mathrm{rk}_K(V) - \mathrm{rk}_K(\Lambda_{n,2}) = n\binom{n-1}{2} - \binom{n}{2} = (n-3)\binom{n}{2}.$$

One can prove this fact directly. Note that $|S'_{i,1}| = \binom{n-1}{2}$, $1 \leq i \leq n-3$; $|S'_{n-2,1}| = \binom{n-1}{2} - 1$ and $|S'_{n-1,1}| = \binom{n-2}{2}$. Then

$$\dim(\Sigma'/\Sigma'^5) = \sum_{i=1}^{n-1} |S'_{i,1}| = (n-3)\binom{n-1}{2} + \binom{n-1}{2} - 1 + \binom{n-2}{2} = (n-3)\binom{n}{2}.$$

2. By statement 1, $\Sigma' \cap \Sigma'' = \Sigma'^5$. Hence,

$$\dim(\Sigma' \cap \Sigma'') = \dim(\Sigma') - \dim(\Sigma'/\Sigma'^5) = (n-2)2^{n-2} - n + \pi_n - (n-3)\binom{n}{2},$$

by Corollary 5.6. □

The dimension of Σ. — The next theorem gives the dimension of the Jacobian group Σ.

THEOREM 6.3. — *Let K be a commutative ring and $n \geq 4$. The Jacobian group Σ is a unipotent affine group over K of dimension*

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{if } n \text{ is odd,} \end{cases}$$

over K, i.e. the algebra of regular functions on Σ is a polynomial algebra in $\dim(\Sigma)$ *variables over K.*

Proof. — Recall that the algebraic group Σ' is affine and $\dim(\Sigma') = (n-2)2^{n-2} - n + \pi_n$ (Corollary 5.6); $\Sigma \simeq \Sigma' \times \mathcal{F}''_n$ (Corollary 4.11.(5)); $\dim(\mathcal{F}''_n) = n(2^{n-2} - n + 1)$, see (78). Therefore, the algebraic group Σ is affine and

$$\dim(\Sigma) = \dim(\Sigma') + \dim(\mathcal{F}''_n) = (n-2)2^{n-2} - n + \pi_n + n(2^{n-2} - n + 1) = (n-1)2^{n-1} - n^2 + \pi_n.$$

□

The Jacobian group Σ is a large subgroup of Γ since

$$\lim_{n\to\infty} \frac{\dim(\Sigma)}{\dim(\Gamma)} = \lim_{n\to\infty} \frac{(n-1)2^{n-1} - n^2 + \pi_n}{n(2^{n-1} - n)} = 1. \tag{96}$$

The coordinates of Σ. — The isomorphism (93) and the isomorphism in Theorem 4.9.(5) provide the explicit coordinates for the Jacobian group Σ if $n \geq 4$.

The dimension of Σ''. — The following theorem gives the dimension of the group Σ'' and proves that the group Σ'' is a *closed normal* subgroup of Σ (which is not obvious from the outset).

THEOREM 6.4. — *Let K be a commutative ring and $n \geq 4$. Then*

1. $\Sigma'' = (\Sigma' \cap \Sigma'')\mathcal{F}''_n = \Sigma'^5 \mathcal{F}''_n$.
2. *Σ'' is the closed normal algebraic subgroup of Σ. Moreover, Σ'' is an affine group of dimension*

$$\dim(\Sigma'') = \begin{cases} (n-1)2^{n-1} - n^2 + 2 - (n-3)\binom{n}{2} & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 - (n-3)\binom{n}{2} & \text{if } n \text{ is odd,} \end{cases}$$

   and the factor group $\Sigma/\Sigma'' \simeq \Sigma'/\Sigma'^5$ is an abelian affine group of dimension $\dim(\Sigma/\Sigma'') = n\binom{n-1}{2} - \binom{n}{2} = (n-3)\binom{n}{2}$.
3. *The map $\Sigma' \cap \Sigma'' \times \mathcal{F}''_n \to \Sigma''$, $(\sigma', \xi_{\mathrm{od}(n)}, \dots, \xi_5, \xi_3) \mapsto \sigma' \xi_{\mathrm{od}(n)} \cdots \xi_5 \xi_3$, is an isomorphism of algebraic varieties over K with the inverse $\sigma \mapsto \sigma' \xi_{\mathrm{od}(n)} \cdots \xi_5 \xi_3$ given in Corollary 4.11.(5).*

Proof. — 1. The first equality follows from statement 3, then the second equality follows from $\Sigma' \cap \Sigma'' = \Sigma'^5$ (Lemma 6.2.(1)).

3. Since $\mathscr{T}''_n \subseteq \Sigma'' \subseteq \Sigma$, statement 3 follows from Corollary 4.11.(5).

2. By Lemma 6.2.(2) and statement 3, the group Σ'' is affine and

$$\begin{aligned}\dim(\Sigma'') &= \dim(\Sigma' \cap \Sigma'') + \dim(\mathscr{T}''_n)\\ &= (n-2)2^{n-2} - n + \pi_n - (n-3)\binom{n}{2} + n(2^{n-2} - n + 1)\\ &= (n-1)2^{n-1} - n^2 + \pi_n - (n-3)\binom{n}{2},\end{aligned}$$

by Lemma 6.2.(2) and (78). Recall that Σ' is a closed subgroup of Σ and Σ'^5 is a closed subgroup of Σ', hence Σ'' is a closed subgroup of Σ since

$$\Sigma'' \simeq (\Sigma' \cap \Sigma'') \times \mathscr{T}''_n = \Sigma'^5 \times \mathscr{T}''_n \subseteq \Sigma' \times \mathscr{T}''_n \simeq \Sigma.$$

Let us prove that the group Σ'' is a normal subgroup of the Jacobian group Σ. First, note that

$$\Sigma^5 \subseteq \Sigma'', \tag{97}$$

since $\Sigma^5 = \Sigma'^5\Sigma''^5$ (Corollary 4.11.(2)) and $\Sigma'^5 = \Sigma' \cap \Sigma'' \subseteq \Sigma''$ (Lemma 6.2.(1)). The subgroup Σ'' is normal in Σ iff $\sigma\Sigma'' = \Sigma''\sigma$ for all $\sigma \in \Sigma$. Note that

$$[\Sigma, \Sigma] \subseteq [\Gamma, \Gamma] \cap \Sigma = [\Gamma^3, \Gamma^3] \cap \Sigma \subseteq \Gamma^5 \cap \Sigma = \Sigma^5.$$

For any $\tau \in \Sigma''$,

$$\sigma\tau = \sigma\tau\sigma^{-1}\tau^{-1}\tau\sigma = [\sigma, \tau]\tau\sigma \in [\Sigma, \Sigma]\Sigma''\sigma \subseteq \Sigma^5\Sigma''\sigma = \Sigma''\sigma,$$

by (97). This means that $\sigma\Sigma'' \subseteq \Sigma''\sigma$. Similarly,

$$\tau\sigma = \sigma\tau[\tau^{-1}, \sigma^{-1}] \in \sigma\Sigma''[\Sigma, \Sigma] \subseteq \sigma\Sigma''\Sigma^5 = \sigma\Sigma'',$$

by (97). This means that $\Sigma''\sigma \subseteq \sigma\Sigma''$. Therefore, $\sigma\Sigma'' = \Sigma''\sigma$ for all $\sigma \in \Sigma$, i.e. Σ'' is a normal subgroup of Σ. Finally, the factor group

$$\Sigma/\Sigma'' = \Sigma'\Sigma''/\Sigma'' \simeq \Sigma'/\Sigma' \cap \Sigma'' = \Sigma'/\Sigma'^5$$

is an abelian affine group of dimension $(n-3)\binom{n}{2}$ (Lemma 6.2.(3)), and its coordinates are given explicitly by Lemma 6.2.(3). □

The coordinates on Σ''. — By Theorem 6.4.(3), each automorphism $\sigma \in \Sigma''$ is a unique product $\sigma = \sigma'\xi_{\mathrm{od}(n)} \cdots \xi_5\xi_3$ where, by (93), the σ' is a unique product

$$\sigma' = \prod_{s=2}^{[\frac{n-1}{2}]} \prod_{i=1}^{n-1} \prod_{\alpha \in S'_{i,s}} \rho_{i,j_{i,s}(\alpha);\lambda_\alpha x^\alpha} \tag{98}$$

where $\alpha = \alpha_{i,s}$ (they depend on i and s) and $\lambda_\alpha = \lambda_{\alpha,i,s} \in K$. Therefore, $\{\lambda_\alpha = \lambda_{\alpha,i,s}\}$ and the coefficients of the elements that define the automorphisms ξ_i are affine coordinates for the algebraic group Σ'' over the ring K. The group Σ'' is a large subgroup of the Jacobian group Σ and the group Σ' is of 'half size' of Σ'' since

$$\lim_{n\to\infty} \frac{\dim(\Sigma'')}{\dim(\Sigma)} = 1, \qquad \lim_{n\to\infty} \frac{\dim(\Sigma')}{\dim(\Sigma'')} = \frac{1}{2}. \tag{99}$$

The dimension of $\Sigma'\backslash\Sigma$. — For groups $A \subseteq B$, let $A\backslash B := \{Ab \,|\, b \in B\}$ and $B/A := \{bA \,|\, b \in B\}$. If a group G acts on sets X and Y then a map $f : X \to Y$ that respects the actions of the group G on the sets X and Y is called a *G-map*, i.e. $f(ax) = af(x)$ for all $a \in A$ and $x \in X$. The isomorphism in Corollary 4.11.(5) is a Σ'-isomorphism where the group Σ' acts by left multiplication on Σ and Σ' (in $\Sigma' \times \mathcal{F}''_n$). Therefore, the set $\Sigma'\backslash\Sigma$ is naturally isomorphic to the set $\Sigma'\backslash\Sigma' \times \mathcal{F}''_n \simeq \mathcal{F}''_n$. The set $\mathcal{F}''_n$ is an affine variety over K of dimension $n(2^{n-2} - n + 1)$ (by (78)), hence so is $\Sigma'\backslash\Sigma$. Note that

$$\Sigma'\backslash\Sigma = \Sigma'\backslash\Sigma'\Sigma'' \simeq \Sigma' \cap \Sigma''\backslash\Sigma'' \simeq \Sigma'^5\backslash\Sigma''$$

since $\Sigma = \Sigma'\Sigma''$ (Corollary 4.11.(1)) and $\Sigma' \cap \Sigma'' = \Sigma'^5$ (Lemma 6.2.(1)). So, we have proved the next corollary.

COROLLARY 6.5. — *Let K be a commutative ring and $n \geq 4$. There are natural isomorphisms of affine varieties over K: $\mathcal{F}''_n \simeq \Sigma'\backslash\Sigma \simeq \Sigma' \cap \Sigma''\backslash\Sigma'' \simeq \Sigma'^5\backslash\Sigma''$, each of them has dimension $n(2^{n-2} - n + 1)$ over K.*

7. The image of the Jacobian map, the dimensions of the Jacobian ascents and of Γ/Σ

In this section, it is proved that all the Jacobian ascents Γ_{2s} are distinct groups with a single exception (Corollary 7.7) and that their structure is completely determined by the Jacobian group, $\Gamma_{2s} = \Gamma^{2s+1}\Sigma$ (Theorem 7.1); each quotient space Γ_{2s}/Γ_{2t} is an affine variety (Corollary 7.4) which is, via the Jacobian map, canonically isomorphic to the affine variety $E'_{n,2s}/E'_{n,2t}$ (Theorem 7.6). In particular, the quotient space Γ/Σ is an affine variety of dimension $2^{n-1} - \pi_n$ (Corollary 7.8). The Jacobian map is a surjective map if n is odd and is not if n is even. (Theorem 7.9).

The equalities $\Gamma_{2s} = \Gamma^{2s+1}\Sigma$. — It follows directly from (25) and (51) that $\Gamma^{2s+1}\Sigma \subseteq \Gamma_{2s}$ for all $s = 1, 2, \ldots, [\frac{n-1}{2}]$. The next theorem states that, in fact, the equalities hold, i.e. $\Gamma^{2s+1}\Sigma = \Gamma_{2s}$. The groups $\{\Gamma^{2s+1}\}$ have clear structure and are given explicitly, therefore studying the Jacobian ascents $\{\Gamma_{2s}\}$ is immediately reduced to studying the Jacobian group Σ.

THEOREM 7.1. — *Let K be a commutative ring and $n \geq 4$. Then*

1. $\Gamma_{2s} = \Gamma^{2s+1}\Sigma = \Phi^{2s+1}\Sigma = \Phi'^{2s+1}\Sigma$ *for each* $s = 1, 2, \ldots, [\frac{n-1}{2}]$.
2. *If n is an even number then $\Gamma_n = \Sigma$, i.e. $\Gamma_n = \Gamma_{n+2} = \Sigma$.*

Proof. — 1. The equalities $\Gamma^{2s+1}\Sigma = \Phi^{2s+1}\Sigma$ are obvious due to Theorem 4.9.(2). The equalities $\Phi^{2s+1}\Sigma = \Phi'^{2s+1}\Sigma$ are obvious due to Theorem 5.4.(1). The inclusions $\Gamma^{2s+1}\Sigma \subseteq \Gamma_{2s}$ are obvious for $s = 1, 2, \ldots, [\frac{n-1}{2}]$. To prove the reverse inclusions let $\sigma \in \Gamma_{2s}$ (i.e. $\mathcal{J}(\sigma) \in E'_{2s}$) where $1 \leq s \leq [\frac{n-1}{2}]$. We have to show that $\sigma \in \Gamma^{2s+1}\Sigma$. If $\mathcal{J}(\sigma) = 1$, i.e. $\sigma \in \Sigma$, there is nothing to prove. So, we assume that $\mathcal{J}(\sigma) \neq 1$, i.e. $\sigma \notin \Sigma$. By Theorem 4.9.(1), $\sigma = \phi\xi$ for some elements $\phi \in \Phi^{2m+1} \backslash \Phi^{2m+3}$ and $\xi \in \Sigma''$. Now, we fix a presentation, say $\sigma = \phi\xi$, with the largest m. Using Lemma 5.3 and Theorem 5.4.(2,3), we see that (by the minimality of m)

$$\mathcal{J}(\sigma) = \mathcal{J}(\phi\xi) = \mathcal{J}(\phi)\phi(\mathcal{J}(\xi)) = \mathcal{J}(\phi) \in E'_{2m} \backslash E'_{2m+2},$$

and so $\sigma \in \Gamma_{2m} \backslash \Gamma_{2m+2}$, hence $s \leq m$ by the choice of m and since $\sigma \in \Gamma_{2s}$. This proves that $\sigma \in \Phi^{2m+1}\Sigma \subseteq \Gamma^{2m+1}\Sigma \subseteq \Gamma^{2s+1}\Sigma$, as required.

2. Note that $\Gamma_n \subseteq \Gamma_{n-2} = \Phi'^{n-1}\Sigma$ (by statement 1) and $\Phi^{n+1} = \{e\}$, and so $\Phi'^{n-1}/\Phi^{n+1} = \Phi'^{n-1}$. If $\sigma \in \Gamma_n$ then $\sigma = \phi\tau$ for some automorphisms $\phi \in \Phi'^{n-1}$ and $\tau \in \Sigma$, and $E'_{n,n} = 1 + Kx_1x_2 \cdots x_n \ni \mathcal{J}(\sigma) = \mathcal{J}(\phi\tau) = \mathcal{J}(\phi)$, hence $\phi \in \Phi^{n+1} = \{e\}$, by Theorem 5.4.(3) and Lemma 5.3. Therefore, $\sigma = \tau \in \Sigma$. This proves that $\Gamma_n = \Sigma$. □

Note that the number $t := 2[\frac{n-1}{2}] + 2$ is equal to $n + 1$ if n is odd; and to n if n is even. Correspondingly,

$$\Gamma_{2[\frac{n-1}{2}]+2} = \begin{cases} \Gamma_{n+1} = \Sigma & \text{if } n \text{ is odd,} \\ \Gamma_n = \Sigma & \text{if } n \text{ is even (by Theorem 7.1.(2)).} \end{cases} \tag{100}$$

Combining two results together, namely Theorem 7.1.(1) and Theorem 5.4.(2), for each $s = 1, 2, \ldots, [\frac{n-1}{2}]$, there is a natural isomorphism of the abelian groups:

$$\Gamma_{2s}/\Gamma_{2s+2} \simeq \Phi'^{2s+1}/\Phi^{2s+3}. \tag{101}$$

In more detail,

$$\begin{aligned}\Gamma_{2s}/\Gamma_{2s+2} &= \Phi^{2s+1}\Sigma/\Phi^{2s+3}\Sigma \simeq \Phi^{2s+1}/\Phi^{2s+3}(\Sigma\cap\Phi^{2s+1}) = \Phi^{2s+1}/\Phi^{2s+3}\Sigma'^{2s+1}\\ &\simeq (\Phi^{2s+1}/\Phi^{2s+3})/(\Phi^{2s+3}\Sigma'^{2s+1}/\Phi^{2s+3})\\ &\simeq (\Sigma'^{2s+1}/\Sigma'^{2s+3}\times\Phi'^{2s+1}/\Phi^{2s+3})/(\Sigma'^{2s+1}/\Sigma'^{2s+3}) \simeq \Phi'^{2s+1}/\Phi^{2s+3}.\end{aligned}$$

By Lemma 5.3 and (101), for each $s = 1,2,\ldots,[\frac{n-1}{2}]$, there is the natural isomorphism of the abelian groups:

(102) $$\Gamma_{2s}/\Gamma_{2s+2} \simeq \Phi'^{2s+1}/\Phi^{2s+3} \to E'_{n,2s}/E'_{n,2s+2},\quad \sigma\Gamma_{2s+2}\mapsto \mathcal{J}(\sigma)E'_{n,2s+2}.$$

This isomorphism and its inverse, (103), are some of the key results in finding the image of the Jacobian map (Theorem 7.9). Recall that the map $\Lambda_{n,2s}\to E'_{n,2s}/E'_{n,2s+2}$, $a\mapsto (1+a)E'_{n,2s+2}$, is an isomorphism of the abelian groups. By Theorem 5.4.(3), (see also Lemma 5.3), the map

(103)
$$\Lambda_{n,2s}\simeq E'_{n,2s}/E'_{n,2s+2}\to\Phi'^{2s+1}/\Phi^{2s+3}\simeq\Gamma_{2s}/\Gamma_{2s+2},\ \ (1+a)E'_{n,2s+2}\mapsto\phi'_a\Phi^{2s+3}(\simeq\phi'_a\Gamma_{2s+2}),$$

is the *inverse* map to the isomorphism (102) where $\Lambda_{n,2s}\ni a = a_{n-2s}+\cdots+a_n$ is the unique sum as in Lemma 5.1 and the automorphism ϕ'_a is defined in (89), namely,

$$\begin{aligned}\phi'_a(x_k) &:= x_k,\quad 1\le k\le n-2s-1,\\ \phi'_a(x_i) &:= x_i(1+a_i),\quad n-2s\le i\le n.\end{aligned}$$

The fact that the map (103) is the inverse of the map (102) means that, for all $a\in\Lambda_{n,2s}$,

(104) $$\mathcal{J}(\phi'_a)\equiv 1+a \mod E'_{n,2s+2},$$

or, equivalently, for all $\sigma\in\Gamma_{2s}$,

(105) $$\sigma\equiv\phi'_a \mod \Gamma_{2s+2},$$

where $\mathcal{J}(\sigma)\equiv 1+a \mod E'_{n,2s+2}$ for a unique element $a\in\Lambda_{n,2s}$.

Recall that the group Γ is equipped with the Jacobian filtration $\{\Gamma_{2s}\}$, and the group E'_n is equipped with the filtration $\{E'_{n,2s}\}$. Both filtrations are descending. The Jacobian map $\mathcal{J}:\Gamma\to E'_n$, $\sigma\mapsto\mathcal{J}(\sigma)$, is a filtered map, i.e. $\mathcal{J}(\Gamma_{2s})\subseteq E'_{n,2s}$ for all s.

THEOREM 7.2. — *Let K be a commutative ring, $n\ge 4$, and $s=1,\ldots,[\frac{n-1}{2}]$. Then each automorphism $\sigma\in\Gamma$ is a unique product $\sigma=\phi'_{a(2)}\phi'_{a(4)}\cdots\phi'_{a(2[\frac{n-1}{2}])}\gamma$*

for unique elements $a(2s)\in\Lambda_{n,2s}$ *and* $\gamma\in\Gamma_{2[\frac{n-1}{2}]+2}=\Sigma$ *(by* (100)*). Moreover,*

$$\begin{aligned}
a(2) &\equiv \mathscr{J}(\sigma)-1 \mod E'_{n,4},\\
a(2t) &\equiv \mathscr{J}(\phi'^{-1}_{a(2t-2)}\cdots\phi'^{-1}_{a(2)}\sigma)-1 \mod E'_{n,2t+2},\ t=2,\dots,[\frac{n-1}{2}],\\
\gamma &= (\phi'_{a(2)}\phi'_{a(4)}\cdots\phi'_{a(2[\frac{n-1}{2}])})^{-1}\sigma.
\end{aligned}$$

Proof. — In brief, the theorem is a direct consequence of repeated application of (105). For $s=1$, by (105), $\sigma\equiv\phi'_{a(2)}\mod\Gamma_4$ for a unique element $a(2)\in\Lambda_{n,2}$ such that $\mathscr{J}(\sigma)\equiv 1+a(2)\mod E'_{n,4}$. Now, $\sigma=\phi'_{a(2)}\sigma_4$ where $\sigma_4:=\phi'^{-1}_{a(2)}\sigma\in\Gamma_4$. Repeating the same argument for the automorphism $\sigma_4\in\Gamma_4$ (i.e. for $s=2$), we have $\sigma_4\equiv\phi'_{a(4)}\mod\Gamma_6$ for a unique element $a(4)\in\Lambda_{n,4}$ such that $\mathscr{J}(\sigma_4)\equiv 1+a(4)\mod E'_{n,6}$. Then, $\sigma=\phi'_{a(2)}\phi'_{a(4)}\sigma_6$ where $\sigma_6:=\phi'^{-1}_{a(4)}\phi'^{-1}_{a(2)}\sigma\in\Gamma_6$. Continue in this way we prove the theorem. □

We know already that the group Γ is an affine variety over K where the coefficients $\{\lambda_{\sigma,i,\alpha}\}$ of the monomials x^α in the decomposition $\sigma(x_i)=x_i+\sum_{|\alpha|\geq 2}\lambda_{\sigma,i,\alpha}x^\alpha$ (where $\sigma\in\Gamma$) are the coordinate functions on Γ. Theorem 7.2 introduces the isomorphic affine structure on Γ where the coefficients of the monomials x^α in $a(2s)$ and the coordinate functions on the Jacobian group Σ are new coordinate functions on Γ. We will see that this affine structure on Γ is very useful in studying the spaces Γ_{2s}/Γ_{2t}.

The next corollary is a direct consequence of Theorem 7.2.

Corollary 7.3. — *Let* K *be a commutative ring,* $n\geq 4$*, and* $s=1,\dots,[\frac{n-1}{2}]$*. Then each automorphism* $\sigma\in\Gamma_{2s}$ *is a unique product* $\sigma=\phi'_{a(2s)}\phi'_{a(2s+2)}\cdots\phi'_{a(2[\frac{n-1}{2}])}\gamma$ *for unique elements* $a(2t)\in\Lambda_{n,2t}$*,* $s\leq t\leq[\frac{n-1}{2}]$*, and* $\gamma\in\Gamma_{2[\frac{n-1}{2}]+2}=\Sigma$ *(by* (100)*). Moreover,*

$$\begin{aligned}
a(2s) &\equiv \mathscr{J}(\sigma)-1 \mod E'_{n,2s+2},\\
a(2t) &\equiv \mathscr{J}(\phi'^{-1}_{a(2t-2)}\cdots\phi'^{-1}_{a(2s)}\sigma)-1 \mod E'_{n,2t+2},\ s<t\leq[\frac{n-1}{2}],\\
\gamma &= (\phi'_{a(2s)}\phi'_{a(4)}\cdots\phi'_{a(2[\frac{n-1}{2}])})^{-1}\sigma.
\end{aligned}$$

The dimension of Γ_{2s}/Γ_{2t}**.** — The next corollary shows that the sets Γ_{2s}/Γ_{2t} are affine varieties over K.

Corollary 7.4. — *Let* K *be a commutative ring,* $n\geq 4$*, and* $1\leq s<t\leq[\frac{n-1}{2}]+1$*. Then the set*

$$\Gamma_{2s}/\Gamma_{2t}=\{\phi'_{a(2s)}\cdots\phi'_{a(2t-2)}\Gamma_{2t}\,|\,a(2s)\in\Lambda_{n,2s},\dots,a(2t-2)\in\Lambda_{n,2t-2}\}$$

is an affine variety over K *of dimension* $\dim(\Gamma_{2s}/\Gamma_{2t}) = \sum_{k=s}^{t-1} \binom{n}{2k}$.

Proof. — The first part of the corollary follows from Corollary 7.3, where the coefficients of the elements $a(2s), \dots, a(2t-2)$ are coordinate functions of the affine variety Γ_{2s}/Γ_{2t}. Clearly, $\dim(\Gamma_{2s}/\Gamma_{2t}) = \sum_{k=s}^{t-1} \mathrm{rk}_K(\Lambda_{n,2k}) = \sum_{k=s}^{t-1} \binom{n}{2k}$. □

The dimension of the Jacobian ascents. — By Corollary 7.3, for each $n \geq 4$ and $s = 1, \dots, [\frac{n-1}{2}]$, the Jacobian group

$$\Gamma_{2s} = \{\phi'_{a(2s)} \cdots \phi'_{a(2[\frac{n-1}{2}])}\gamma \mid a(2i) \in \Lambda_{n,2i}, \gamma \in \Sigma\} \tag{106}$$

is an affine variety of dimension

$$\dim(\Gamma_{2s}) = \dim(\Sigma) + \sum_{i=s}^{[\frac{n-1}{2}]} \binom{n}{2i}. \tag{107}$$

The coordinate functions for the affine group Γ_{2s} are the coefficients of all the elements $a(2i)$ and the coordinate functions on the Jacobian group Σ. In the particular case when $s = 1$, one has

$$\Gamma = \Gamma_2 = \{\phi'_{a(2)} \cdots \phi'_{a(2[\frac{n-1}{2}])}\gamma \mid a(2i) \in \Lambda_{n,2i}, \gamma \in \Sigma\}. \tag{108}$$

It follows from (106) and (108) that each Jacobian ascent Γ_{2s}, $s = 1, \dots, [\frac{n-1}{2}]$, is a *closed* subgroup of Γ that satisfies *exactly* $\dim(\Gamma) - \dim(\Gamma_{2s}) = \sum_{i=1}^{s-1} \binom{n}{2i}$ defining equations, namely, all coefficients of the elements $a(2), a(4), \dots, a(2s-2)$ are equal to zero.

Note that, for an even number n, $\Gamma_{2[\frac{n}{2}]} = \Gamma_n = \Sigma$ (Theorem 7.1.(2)). This means that, for each $n \geq 4$ (not necessarily even), the groups Σ and Γ_{2s}, $s = 1, \dots, [\frac{n-1}{2}]$, are *all* the Jacobian ascents.

Corollary 7.5. — *Let* K *be a commutative ring,* $n \geq 4$*. Then all the Jacobian ascents are affine groups over* K *and closed subgroups of* Γ*, and* $\dim(\Gamma_{2s}) = \dim(\Sigma) + \sum_{i=s}^{[\frac{n-1}{2}]} \binom{n}{2i}$, $s = 1, \dots, [\frac{n-1}{2}]$.

The isomorphisms $\overline{\mathcal{J}}_{s,t}$. — For $1 \leq s < t \leq [\frac{n-1}{2}] + 1$, the abelian group $E'_{n,2s}/E'_{n,2t}$ is an affine variety over K of the same dimension as the affine variety Γ_{2s}/Γ_{2t}. The next result shows that the Jacobian map

$$\overline{\mathcal{J}}_{s,t} : \Gamma_{2s}/\Gamma_{2t} \to E'_{n,2s}/E'_{n,2t}, \quad \sigma\Gamma_{2t} \mapsto \mathcal{J}(\sigma)E'_{n,2t}, \tag{109}$$

is an isomorphism of affine varieties.

Theorem 7.6. — *Let* K *be a commutative ring,* $n \geq 4$*, and* $1 \leq s < t \leq [\frac{n-1}{2}] + 1$*. Then the Jacobian map* $\overline{\mathcal{J}}_{s,t}$*, (109), is an isomorphism of affine varieties.*

Proof. — By the definition, the map $\overline{\mathscr{J}}_{s,t}$ is a polynomial map. In order to finish the proof of the theorem it suffices to show that the map $\overline{\mathscr{J}}_{s,t}$ is a bijection and its inverse is also a polynomial map. For a given t, to prove these two statements, we will use downward induction on s starting at $s = t - 1$ where the result is known due to (102), (103), (104), and (105). If $t = 2$ then $s = 1$, and we are done. So, let $t \geq 3$ and $s < t - 1$, and, by the inductive hypothesis, we assume that the map $\overline{\mathscr{J}}_{s+1,t}$ is an isomorphism of affine varieties. We are going to present the inverse map for $\overline{\mathscr{J}}_{s,t}$ which is, by construction, a polynomial map.

By Corollary 7.4, each element of Γ_{2s}/Γ_{2t} can be written uniquely in the form $\phi'_a\tau\Gamma_{2t}$ where $a \in \Lambda_{n,2s}$ and $\tau\Gamma_{2t} \in \Gamma_{2s+2}/\Gamma_{2t}$. Similarly, each element of $E'_{n,2s}/E'_{n,2t}$ can be written uniquely in the form $(1+a)bE'_{n,2t}$ where $a \in \Lambda_{n,2s}$ and $bE'_{n,2t} \in E'_{n,2s+2}/E'_{n,2t}$. To finish the proof we have to show that, for a given element $(1+a)bE'_{n,2t} \in E'_{n,2s}/E'_{n,2t}$, and an unknown $\phi'_{a'}\tau\Gamma_{2t} \in \Gamma_{2s}/\Gamma_{2t}$, the equation

$$\overline{\mathscr{J}}_{s,t}(\phi'_{a'}\tau\Gamma_{2t}) = (1+a)bE'_{n,2t}$$

has a unique solution $\phi'_{a'}\tau\Gamma_{2t}$ that depends polynomially on the RHS. By taking the equation modulo $E'_{n,2s+2}$, we obtain the equality $a = a'$, by (104):

$$1 + a \equiv \mathscr{J}(\phi'_{a'}) \equiv 1 + a' \mod E'_{n,2s+2}.$$

Now, we can solve the equation explicitly which can be written as follows

$$\mathscr{J}(\phi'_a)\phi'_a(\overline{\mathscr{J}}_{s+1,t}(\tau\Gamma_{2t})) = \overline{\mathscr{J}}_{s,t}(\phi'_a\tau\Gamma_{2t}) = (1+a)bE'_{n,2t}.$$

Namely,

$$\tau\Gamma_{2t} = (\overline{\mathscr{J}}_{s+1,t})^{-1}\phi_a'^{-1}(\mathscr{J}(\phi'_a)^{-1}(1+a)bE'_{n,2t}) \tag{110}$$

is the unique solution that depends polynomially on the RHS (note that $\mathscr{J}(\phi'_a)^{-1}(1+a) \in E'_{n,2s+2}$, by (104)), as required. □

The Jacobian ascents are distinct groups except one case. — Now, we are ready to give an answer to the question of whether the Jacobian ascents are distinct groups or not.

Corollary 7.7. — *Let K be a commutative ring and $n \geq 4$.*

1. *If n is an odd number then the Jacobian ascents*

$$\Gamma = \Gamma_2 \supset \Gamma_4 \supset \cdots \supset \Gamma_{2s} \supset \cdots \supset \Gamma_{2[\frac{n}{2}]} \supset \Gamma_{2[\frac{n}{2}]+2} = \Sigma$$

are distinct groups.

2. *If n is an even number then the Jacobian ascents*

$$\Gamma = \Gamma_2 \supset \Gamma_4 \supset \cdots \supset \Gamma_{2s} \supset \cdots \supset \Gamma_{2[\frac{n}{2}]-2} \supset \Gamma_{2[\frac{n}{2}]} = \Gamma_{2[\frac{n}{2}]+2} = \Sigma$$

are distinct groups except the last two groups, i.e. $\Gamma_{2[\frac{n}{2}]} = \Gamma_{2[\frac{n}{2}]+2}$.

Proof. — 1. If n is odd then $[\frac{n-1}{2}] = [\frac{n}{2}]$ and the result follows from Theorem 7.6 since the groups $\{E'_{n,2s}\}$ are distinct for $s = 1, 2, \dots, [\frac{n-1}{2}]$.

2. If n is even then $[\frac{n-1}{2}] = [\frac{n}{2}] - 1$ and $2[\frac{n}{2}] = n$. By Theorem 7.6, the following groups are distinct: $\Gamma = \Gamma_2 \supset \Gamma_4 \supset \cdots \supset \Gamma_{2s} \supset \cdots \supset \Gamma_{2[\frac{n}{2}]-2} \supset \Sigma$. By Theorem 7.1.(2), $\Gamma_{2[\frac{n}{2}]} = \Gamma_{2[\frac{n}{2}]+2} = \Sigma$, and so the result. □

The dimension of Γ/Σ. — By taking the extreme values for s and t in Corollary 7.4, namely, $s = 1$ and $t = [\frac{n-1}{2}] + 1$, we see that Γ/Σ is an affine variety due to (100) and $\Gamma = \Gamma_2$. The next corollary gives the dimension of the variety Γ/Σ.

COROLLARY 7.8. — *Let K be a commutative ring and $n \geq 4$. Then Γ/Σ is an affine variety.*

1. *If n is odd then the Jacobian map $\Gamma/\Sigma \to E'_n$, $\sigma\Sigma \mapsto \mathscr{J}(\sigma)$, is an isomorphism of the affine varieties over K, and $\dim(\Gamma/\Sigma) = 2^{n-1} - 1$.*
2. *If n is even then the Jacobian map $\Gamma/\Sigma \to E'_n/E'_{n,n}$, $\sigma\Sigma \mapsto \mathscr{J}(\sigma)E'_{n,n}$, is an isomorphism of the affine varieties over K (where $E'_{n,n} = 1 + Kx_1 \cdots x_n$), and $\dim(\Gamma/\Sigma) = 2^{n-1} - 2$.*

Proof. — 1. Take $s = 1$ and $t = [\frac{n-1}{2}] + 1$ in Theorem 7.6. Since n is an odd number, $2t + 2 = n + 1$, and so $E'_{n,2t+2} = E'_{n,n+1} = \{1\}$ and $\Gamma_{2t+2} = \Gamma_{n+1} = \Sigma$ (Corollary 7.7.(1)). By Theorem 7.6, the Jacobian map $\Gamma/\Sigma \to E'_n$, $\sigma\Sigma \mapsto \mathscr{J}(\sigma)$, is an isomorphism of the affine varieties over K. Now,

$$\dim(\Gamma/\Sigma) = \dim(E'_n) = \sum_{s=1}^{[\frac{n}{2}]} \binom{n}{2s} = \sum_{s=0}^{[\frac{n}{2}]} \binom{n}{2s} - 1 = 2^{n-1} - 1.$$

2. Similarly, take $s = 1$ and $t = [\frac{n-1}{2}] + 1$ in Theorem 7.6. Since n is an even number, $2t + 2 = n$, and so $E'_{n,2t+2} = E'_{n,n} = 1 + Kx_1 \cdots x_n$ and $\Gamma_{2t+2} = \Gamma_n = \Sigma$ (Corollary 7.7.(2)). By Theorem 7.6, the Jacobian map $\Gamma/\Sigma \to E'_n/E'_{n,n}$, $\sigma\Sigma \mapsto \mathscr{J}(\sigma)E'_{n,n}$, is an isomorphism of the affine varieties over K, and

$$\dim(\Gamma/\Sigma) = \dim(E'_n/E'_{n,n}) = \sum_{s=1}^{[\frac{n}{2}]} \binom{n}{2s} - 1 = \sum_{s=0}^{[\frac{n}{2}]} \binom{n}{2s} - 2 = 2^{n-1} - 2. \quad \square$$

Theorem 7.9 gives an answer to the natural question of whether the Jacobian map $\mathscr{J} : \Gamma \to E'_n$, $\sigma \mapsto \mathscr{J}(\sigma)$, is surjective? The answer is 'yes' for odd numbers n, and, surprisingly, 'no' for even numbers n. In the second case, the image of the Jacobian map is large. More precisely, it is a *closed* subvariety of E'_n of codimension 1 which is defined by a single equation. Moreover, it is canonically isomorphic to the affine variety $E'_n/E'_{n,n}$.

THEOREM 7.9. — *Let K be a commutative ring, $n \geq 4$, $\mathscr{J} : \Gamma \to E'_n$, $\sigma \mapsto \mathscr{J}(\sigma)$, be the Jacobian map, and $s = 1, 2, \ldots, [\frac{n-1}{2}]$. Then,*

1. *for an odd number n, the Jacobian map $\mathscr{J}$ is surjective. Moreover, for each s, the map $\mathscr{J} : \Gamma_{2s} \to E'_{n,2s}$, $\sigma \to \mathscr{J}(\sigma)$, is surjective; and*
2. *for an even number n, the Jacobian map $\mathscr{J}$ is not surjective but very close to be a surjective map. In more detail,*
   (a) *the image $\mathrm{im}(\mathscr{J})$ is a closed algebraic variety of E'_n of codimension 1 (i.e. $\dim(\mathrm{im}(\mathscr{J})) = 2^{n-1} - 2$) which is defined by a single equation (see the proof),*
   (b) $\mathrm{im}(\mathscr{J}) \cap E'_{n,n} = \{1\}$ *where* $E'_{n,n} = 1 + Kx_1 \cdots x_n$,
   (c) *the image $\mathrm{im}(\mathscr{J})$ is canonically isomorphic to the algebraic group $E'_n / E'_{n,n}$ via the map $\mathrm{im}(\mathscr{J}) \to E'_n / E'_{n,n}$, $\alpha \mapsto \alpha E'_{n,n}$.*

Proof. — 1. The fact that the Jacobian map $\mathscr{J} : \Gamma \to E'_n$, $\sigma \mapsto \mathscr{J}(\sigma)$, is surjective follows from Corollary 7.8.(1) and (55).

2. By Corollary 7.8.(2) and (55), the Jacobian map $\mathscr{J}$ is not surjective, though there is a bijection between the image $\mathrm{im}(\mathscr{J})$ and $E'_n / E'_{n,n}$. The set $E'_n / E'_{n,n}$ may be identified with the closed affine subvariety of the affine variety E'_n that is given by a single equation: the coefficient of the element $x_1 \cdots x_n$ is equal to zero. In more detail, $E'_n = 1 + \oplus_{s=1}^{[\frac{n}{2}]} \Lambda_{n,2s}$ and $E'_n / E'_{n,n}$ is identified with $1 + \oplus_{s=1}^{[\frac{n}{2}]-1} \Lambda_{n,2s}$. Then the bijection between $\mathrm{im}(\mathscr{J})$ and $E'_n / E'_{n,n}$ means that the last coordinate, say $\lambda = \lambda(\sigma)$, of each element $\mathscr{J}(\sigma) = 1 + \cdots + \lambda x_1 \cdots x_n$ is a polynomial function of the previous coordinates (in the three dots expression). This is the defining equation of the image $\mathrm{im}(\mathscr{J})$ in E'_n. So, the statements (a) and (c) follow. The statement (b) follows at once from the equality $\Gamma_n = \Sigma$ (Theorem 7.1.(2)): $\sigma \in \mathrm{im}(\mathscr{J}) \cap E'_{n,n}$ iff $\sigma \in \Gamma_n = \Sigma$. □

8. Analogues of the Poincaré Lemma

In this section, two results (Theorems 8.2 and 8.3) are proved that have flavor of the Poincaré Lemma. Theorem 8.2 is used in the proof of Theorem 9.1.

THEOREM 8.1. — *Let K be an arbitrary (not necessarily commutative) ring. Then*

1. *the Grassmann ring $\Lambda_n(K)$ is a direct sum of right K-modules*

$$\Lambda_n(K) = x_1 \cdots x_n K \oplus x_1 \cdots x_{n-1} K \oplus x_1 \cdots x_{n-2} K \lfloor x_n \rfloor \oplus \cdots$$
$$\cdots \oplus x_1 \cdots x_i K \lfloor x_{i+2} \ldots, x_n \rfloor \oplus \cdots \oplus x_1 K \lfloor x_3 \ldots, x_n \rfloor \oplus K \lfloor x_2 \ldots, x_n \rfloor.$$

2. *So, each element* $a \in \Lambda_n(K)$ *is a unique sum*

$$a = x_1 \cdots x_n a_n + x_1 \cdots x_{n-1} b_n + \sum_{i=1}^{n-2} x_1 \cdots x_i b_{i+1} + b_1$$

where $a_n, b_n \in K$, $b_i \in K\lfloor x_{i+1} \ldots, x_n \rfloor$, $1 \leq i \leq n-1$. *Moreover,*

$$\begin{aligned} a_n &= \partial_n \partial_{n-1} \cdots \partial_1(a), \\ b_{i+1} &= \partial_i \partial_{i-1} \cdots \partial_1 (1 - x_{i+1}\partial_{i+1})(a), \ 1 \leq i \leq n-1, \\ b_1 &= (1 - x_1 \partial_1)(a). \end{aligned}$$

So,

$$a = x_1 \cdots x_n \partial_n \partial_{n-1} \cdots \partial_1(a) + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 (1 - x_{i+1}\partial_{i+1})(a) + (1 - x_1\partial_1)(a).$$

Proof. — For each $i = 1, \ldots, n$, let $K_i := K\lfloor x_i, \ldots, x_n \rfloor$ and $K_{n+1} := K$.

1. Existence of the decomposition is a consequence of a repeated use of the fact that $K_i = x_i K_{i+1} \oplus K_{i+1}$. Namely,

$$\begin{aligned} K_n &= x_1 K_2 \oplus K_2 = x_1(x_2 K_3 \oplus K_3) \oplus K_2 = x_1 x_2 K_3 \oplus x_1 K_3 \oplus K_2 \\ &= x_1 x_2 (x_3 K_4 \oplus K_4) \oplus x_1 K_3 \oplus K_2 \\ &= x_1 x_2 x_3 (x_4 K_5 \oplus K_5) \oplus x_1 x_2 K_4 \oplus x_1 K_3 \oplus K_2 = \cdots, \end{aligned}$$

when this process stops after n steps we get the required decomposition.

2. The crucial steps in finding the coefficients for the element a are (i) $\partial_i^2 = \cdots = \partial_n^2 = 0$, and (ii) for each $i = 1, \ldots, n$, the map $\phi_i := 1 - x_i \partial_i : \Lambda_n \to \Lambda_n$ is the projection onto the Grassmann subring $K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_n \rfloor$ in the decomposition $\Lambda_n = K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_n \rfloor \oplus x_i K\lfloor x_1, \ldots, \widehat{x_i}, \ldots, x_n \rfloor$ (Lemma 2.2). The tail t in the sum $a = x_1 \cdots x_n a_n + t$ has (total) degree in the variables $x_1, \ldots, x_n$ strictly less than n, hence t is killed by the map $\partial_n \cdots \partial_1$. Therefore,

$$\partial_n \cdots \partial_1(a) = \partial_n \cdots \partial_1(x_1 \cdots x_n a_n) = \partial_n \cdots \partial_2(x_2 \cdots x_n a_n) = \cdots = a_n.$$

To find the elements b_i we use induction on i. Since the map $\phi_1 = (1 - x_1\partial_1) : \Lambda_n \to \Lambda_n$ is a projection onto $K\lfloor x_2, \ldots, x_n \rfloor$ and all summands of a but the last belong to the ideal (x_1) (which is annihilated by ϕ_1), it follows at once that $\phi_1(a) = \phi_1(b_1) = b_1$. Similarly, applying ϕ_2 to a we see that $\phi_2(a) = x_1 b_2 + \phi_2(b_1)$. Since $\phi_2(b_1) \in K\lfloor x_3, \ldots, , x_n \rfloor$, we have $\partial_1 \phi_2(b_1) = 0$, and so $\partial_1 \phi_2(a) = \partial_1(x_1 b_2) = b_2$. Suppose that the formula for the b_k in the theorem is true for all $k = 1, \ldots, i$, we have to prove it for $i+1$. The cases $i = 1, 2$ have been established already. So, let $i \geq 3$. Now,

$$\phi_{i+1}(a) = x_1 \cdots x_i b_{i+1} + \phi_{i+1}(\sum_{k=1}^{i-1} x_1 \cdots x_k \partial_k \cdots \partial_1 \phi_{k+1}(a)) + \phi_{i+1}(b_1).$$

Note that the skew derivations $\partial_1, \dots, \partial_i$ *commute* with ϕ_{i+1}. $\phi_{i+1}(b_1) \in K\lfloor x_2, \dots, \widehat{x_{i+1}}, \dots, x_n \rfloor$ implies $\partial_1\phi_{i+1}(b_1) = 0$, and so $\partial_i \cdots \partial_1\phi_{i+1}(b_1) = 0$. For each $k = 1, \dots, i-1$, let $c_k = \phi_{i+1}(x_1 \cdots x_k \partial_k \cdots \partial_1 \phi_{k+1}(a))$. Using the commutation relations for the Grassmann K-algebra $\Lambda_k = \oplus_{\alpha,\beta \in \mathscr{B}_k} \partial^\alpha x^\beta K$ one can write (in Λ_k)

$$x_1 \cdots x_k \partial_k \cdots \partial_1 = 1 + d_k \quad \text{where} \quad d_k \in \oplus_{0 \neq \alpha \in \mathscr{B}_k, \beta \in \mathscr{B}_k} \partial^\alpha x^\beta K.$$

Since $\partial_k \cdots \partial_1 d_k = 0$ (as $\partial_1^2 = \cdots = \partial_k^2 = 0$) and $i > k$, we have

$$\begin{aligned}\partial_i \cdots \partial_1 c_k &= \phi_{i+1}\partial_i \cdots \partial_1(1 + d_k)\phi_{k+1}(a) = \phi_{i+1}\partial_i \cdots \partial_{k+1} \cdots \partial_1 \phi_{k+1}(a) \\ &= (-1)^k \phi_{i+1}\partial_i \cdots \widehat{\partial_{k+1}} \cdots \partial_1 \partial_{k+1}\phi_{k+1}(a) = 0\end{aligned}$$

since $\partial_{k+1}\phi_{k+1} = 0$. Now, we see that

$$\begin{aligned}\partial_i \cdots \partial_1 \phi_{i+1}(a) &= \partial_i \cdots \partial_1(x_1 \cdots x_i b_{i+1}) \\ &= \partial_i \cdots \partial_1(x_1 \cdots x_i) \cdot b_{i+1} \\ &\qquad (\text{as } \ b_{i+1} \in K\lfloor x_{i+2}, \dots, x_n \rfloor \subseteq \cap_{k=1}^i \ker(\partial_k)) \\ &= b_{i+1},\end{aligned}$$

as required. □

By Theorem 8.1, the identity map $\mathrm{id}_{\Lambda_n} : \Lambda_n \to \Lambda_n$ is equal to

(111)

$$\mathrm{id}_{\Lambda_n} = x_1 \cdots x_n \partial_n \partial_{n-1} \cdots \partial_1 + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 (1 - x_{i+1}\partial_{i+1}) + (1 - x_1\partial_1).$$

If $n' \geq n$ then the RHS of (111) is a map from $\Lambda_{n'}$ to itself. Therefore,

(112)

$$\mathrm{id}_{\Lambda_n'} = x_1 \cdots x_n \partial_n \partial_{n-1} \cdots \partial_1 + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 (1 - x_{i+1}\partial_{i+1}) + (1 - x_1\partial_1).$$

THEOREM 8.2. — *Let K be an arbitrary ring, $u_1, \dots, u_n \in \Lambda_n(K)$, and $a \in \Lambda_n(K)$ be an unknown. Then the system of equations*

$$\begin{cases} x_1 a = u_1 \\ x_2 a = u_2 \\ \quad \vdots \\ x_n a = u_n \end{cases}$$

has a solution in Λ_n iff the following two conditions hold

1. $u_1 \in (x_1), \dots, u_n \in (x_n)$, *and*
2. $x_i u_j = -x_j u_i$ *for all* $i \neq j$.

In this case,

$$a = x_1 \cdots x_n a_n + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}) + \partial_1(u_1), \quad a_n \in K, \tag{113}$$

are all the solutions.

Remark. An analogue of the Poincaré Lemma for Λ_n is given later (Theorem 8.3). Theorem 8.2 is a sort of Poincaré Lemma for the Grassmann algebra since the map $l_{x_i} : \Lambda_n \to \Lambda_n$, $u \mapsto x_i u$, the left multiplication by x_i, is a sort of skew partial derivatives on x_i as follows from the following two properties:

1. Each element $a \in \Lambda_n$ is a unique sum $a = x_i\alpha + \beta$ with $\alpha, \beta \in K\lfloor x_1, \dots, \widehat{x_i}, \dots x_n \rfloor$ and $l_{x_i}(a) = x_i\beta$; and
2. for any two elements $a_s \in \Lambda_{n,s}$ and $a_t \in \Lambda_{n,t}$ where $s, t \in \mathbb{Z}_2$:

$$x_i(a_s a_t) = \frac{1}{2} x_i a_s a_t + \frac{1}{2} x_i a_s a_t = (\frac{1}{2} x_i a_s) a_t + (-1)^s a_s (\frac{1}{2} x_i a_t),$$

provided $\frac{1}{2} \in K$.

Proof. — Suppose that $a \in \Lambda_n$ is a solution then $u_i = x_i a \in (x_i)$ for all i; and, for all $i \neq j$,

$$x_i u_j + x_j u_i = x_i x_j a + x_j x_i a = x_i x_j a - x_i x_j a = 0.$$

So, conditions 1 and 2 hold. Evaluating the skew derivation ∂_i at the equality $u_i = x_i a$ one sees that

$$\partial_i(u_i) = \partial_i(x_i a) = (1 - x_i\partial_i)(a). \tag{114}$$

Let us write the element a as the sum in Theorem 8.1. Note that if a is a solution to the system then $a + x_1 \cdots x_n a_n$ is also a solution for an arbitrary choice of $a_n \in K$, and vice versa. By (114) and Theorem 8.1.(2),

$$a = x_1 \cdots x_n a_n + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}) + \partial_1(u_1).$$

This proves (113).

It remains to show that if conditions 1 and 2 hold then (113) are solutions to the system. We prove directly that $x_j a = u_j$ for all j. An idea of the proof is to use the identity (112). For $j = 1$, note that $x_1\partial_1(u_1) = u_1$ since $u_1 \in (x_1)$, and so $x_1 a = x_1\partial_1(u_1) = u_1$. Suppose that $2 \le j \le n$. Then

$$x_j a = x_1 \cdots x_{j-1}\partial_{j-1} \cdots \partial_1 x_j \partial_j(u_j) + \sum_{i=1}^{j-2} x_1 \cdots x_i \partial_i \cdots \partial_1 x_j \partial_{i+1}(u_{i+1}) + x_j\partial_1(u_1).$$

Note that $x_j\partial_{i+1}(u_{i+1}) = -\partial_{i+1}(x_ju_{i+1}) = -\partial_{i+1}(-x_{i+1}u_j) = (1 - x_{i+1}\partial_{i+1})(u_j)$; $x_j\partial_j(u_j) = u_j$ since $u_j \in (x_j)$; and $x_j\partial_1(u_1) = -\partial_1(x_ju_1) = -\partial_1(-x_1u_j) = (1-x_1\partial_1)(u_j)$. Using these equalities, we see that

$$x_ja = (x_1\cdots x_{j-1}\partial_{j-1}\cdots\partial_1 + \sum_{i=1}^{j-2} x_1\cdots x_i\partial_i\cdots\partial_1(1-x_{i+1}\partial_{i+1}) + (1-x_1\partial_1))(u_j) = u_j,$$

by (112). □

THEOREM 8.3. — *Let K be an arbitrary ring, $u_1,\dots,u_n \in \Lambda_n(K)$, and $a \in \Lambda_n(K)$ be an unknown. Then the system of equations*

$$\begin{cases} \partial_1(a) = u_1 \\ \partial_2(a) = u_2 \\ \quad\vdots \\ \partial_n(a) = u_n \end{cases}$$

has a solution in Λ_n iff the following two conditions hold

1. *for each $i = 1,\dots,n$, $u_i \in K\lfloor x_1,\dots,\widehat{x_i},\dots,x_n\rfloor$, and*
2. *$\partial_i(u_j) = -\partial_j(u_i)$ for all $i \neq j$.*

In this case,

$$a = \lambda + \sum_{0\neq\alpha\in\mathscr{B}_n} \phi(u_\alpha)x^\alpha, \quad \lambda\in K, \tag{115}$$

are all the solutions where ϕ is defined in Lemma 2.2.(3) and, for $\alpha = \{i_1 < \cdots < i_k\}$, $u_\alpha := \partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_2}(u_{i_1})$.

Proof. — Suppose that $a \in \Lambda_n$ is a solution then $u_i = \partial_i(a) \in \mathrm{im}(\partial_i) = K\lfloor x_1,\dots,\widehat{x_i},\dots,x_n\rfloor$, and so the first condition holds. For all $i \neq j$,

$$\partial_i(u_j) = \partial_i\partial_j(a) = -\partial_j\partial_i(a) = -\partial_j(u_i),$$

and so the second condition holds. Note that if a is a solution then $a+\lambda$, $\lambda \in K$, are all the solutions since $K = \cap_{i=1}^n \ker(\partial_i)$. By Theorem 2.3.(1),

$$a = \sum_{\alpha\in\mathscr{B}_n} \phi(\partial^\alpha(a))x^\alpha = \lambda + \sum_{0\neq\alpha\in\mathscr{B}_n} \phi(\partial^\alpha(a))x^\alpha = \lambda + \sum_{0\neq\alpha\in\mathscr{B}_n} \phi(u_\alpha)x^\alpha,$$

so (115) holds.

It remains to show that if conditions 1 and 2 hold then (115) are solutions to the system. We prove directly that $\partial_i(a) = u_i$ for all i. An idea of the proof is to use the equality of Theorem 2.3.(1) together with conditions 1 and 2.

$$\partial_i(a) = \sum_{i\in\alpha\in\mathscr{B}_n} \phi(u_\alpha)(-1)^{\alpha_1+\cdots+\alpha_{i-1}}x^{\alpha\setminus\{i\}} = \sum_{i\in\alpha\in\mathscr{B}_n} \phi(\partial^{\alpha\setminus\{i\}}(u_i))x^{\alpha\setminus\{i\}} = u_i.$$

The second equality above is due to the fact that $(-1)^{\alpha_1+\cdots+\alpha_{i-1}}u_\alpha = \partial^{\alpha\backslash\{i\}}(u_i)$, by condition 2. The last equality follows from Theorem 2.3.(1) and condition 1. □

9. The unique presentation $\sigma = \omega_{1+a}\gamma_b\sigma_A$ for $\sigma \in \mathrm{Aut}_K(\Lambda_n)$

In this section, K is a *reduced commutative* ring with $\frac{1}{2} \in K$. By Theorem 2.14.(3), $G = \Omega\Gamma\mathrm{GL}_n(K)^{op}$. So, each element $\sigma \in G$ has the unique presentation as the product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ where $\omega_{1+a} \in \Omega$ ($a \in \Lambda_n'^{\mathrm{od}}$), $\gamma_b \in \Gamma$, $\sigma_A \in \mathrm{GL}_n(K)^{op}$ where $\Lambda_n'^{\mathrm{od}} := \oplus_i \Lambda_{n,i}$ and i runs through all *odd* natural numbers such that $1 \leq i \leq n-1$.

Theorem 9.1. — *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then each element $\sigma \in G$ is a unique product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ (Theorem 2.14.(3)) where $a \in \Lambda_n'^{\mathrm{od}}$ and*

1. $\sigma(x) = Ax + \cdots$ *(i.e. $\sigma(x) \equiv Ax \mod \mathfrak{m}$) for some $A \in \mathrm{GL}_n(K)$,*
2. $b = A^{-1}\sigma(x)^{\mathrm{od}} - x$, *and*
3. $a = \frac{1}{2}(-1 + x_1\cdots x_n\partial_n\cdots\partial_1)\gamma_b(\sum_{i=1}^{n-1} x_1\cdots x_i\partial_i\cdots\partial_1\partial_{i+1}(a'_{i+1}) + \partial_1(a'_1))$ *where $a'_i := (A^{-1}\gamma_b^{-1}(\sigma(x)^{\mathrm{ev}}))_i$, the i'th component of the column-vector $A^{-1}\gamma_b^{-1}(\sigma(x)^{\mathrm{ev}})$.*

Remark. Recall that $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $\sigma(x) = \begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{pmatrix}$, $\sigma(x)^{\mathrm{ev}} = \begin{pmatrix} \sigma(x_1)^{\mathrm{ev}} \\ \vdots \\ \sigma(x_n)^{\mathrm{ev}} \end{pmatrix}$, $\sigma(x)^{\mathrm{od}} = \begin{pmatrix} \sigma(x_1)^{\mathrm{od}} \\ \vdots \\ \sigma(x_n)^{\mathrm{od}} \end{pmatrix}$, and any element $u \in \Lambda_n$ is a unique sum $u = u^{\mathrm{ev}} + u^{\mathrm{od}}$ of its even and odd components.

Proof. — Statement 1 is obvious. Note that

$$\sigma(x) = \omega_{1+a}\gamma_b(Ax) = \omega_{1+a}(A(x+b)) = A(x+b) + 2aA(x+b). \tag{116}$$

Then $\sigma(x)^{\mathrm{od}} = A(x+b)$ and $\sigma(x)^{\mathrm{ev}} = 2aA(x+b)$. The first equality is equivalent to statement 2, and the second equality can be rewritten as follows,

$$-\frac{1}{2}A^{-1}\sigma(x)^{\mathrm{ev}} = (x+b)a = \gamma_b(x)a = \gamma_b(x\gamma_b^{-1}(a)),$$

or, equivalently, $x\gamma_b^{-1}(a) = -\frac{1}{2}A^{-1}\gamma_b^{-1}(\sigma(x)^{\mathrm{ev}})$. This is the system of equations

$$\begin{cases} x_1\gamma_b^{-1}(a) = -\frac{1}{2}a'_1, \\ x_2\gamma_b^{-1}(a) = -\frac{1}{2}a'_2, \\ \quad\vdots \\ x_n\gamma_b^{-1}(a) = -\frac{1}{2}a'_n. \end{cases}$$

Its solutions are given by Theorem 8.2,

$$a = -\frac{1}{2}\gamma_b(\sum_{i=1}^{n-1} x_1\cdots x_i\partial_i\cdots\partial_1\partial_{i+1}(a'_{i+1}) + \partial_1(a'_1)) + a_n x_1\cdots x_n, \quad a_n \in K,$$

where we have used the fact that $\gamma_b(a_n x_1\cdots x_n) = a_n x_1\cdots x_n$. The element a_n can be found by applying the skew differential operator $\partial_n\cdots\partial_1$ to the equation above and taking into account that it kills the element $a \in \Lambda_n'^{\mathrm{od}}$. Now, statement 3 follows. □

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