RANDOM WALKS IN $(\mathbb{Z}_+)^2$ WITH NON-ZERO DRIFT ABSORBED AT THE AXES

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Abstract. — Spatially homogeneous random walks in $(\mathbb{Z}_+)^2$ with non-zero jump probabilities at distance at most 1, with non-zero drift in the interior of the quadrant and absorbed when reaching the axes are studied. Absorption probabilities generating functions are obtained and the asymptotic of absorption probabilities along the axes is made explicit. The asymptotic of the Green functions is computed along all different infinite paths of states, in particular along those approaching the axes.

Résumé (*Marches aléatoires dans $\mathbb{Z}_+^2$ avec un drift non nul, absorbées au bord*)

Dans cet article, nous étudions les marches aléatoires du quart de plan ayant des sauts à distance au plus un, avec un drift non nul à l'intérieur et absorbées au bord. Nous obtenons de façon explicite les séries génératrices des probabilités d'absorption au bord, puis leur asymptotique lorsque le site d'absorption tend vers l'infini. Nous calculons également l'asymptotique des fonctions de Green le long de toutes les trajectoires, en particulier selon celles tangentes aux axes.

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1. Introduction

Random walks in angles of $\mathbb{Z}^d$ conditioned in the sense of Doob's h-transform never to reach the boundary nowadays arouse enough interest in the mathematical community as they appear in several distinct domains.

An important class of such walks is the so-called "non-colliding" random walks. These walks are the processes $(Z_1(n), \ldots, Z_k(n))_{n \geq 0}$ composed of k independent and identically distributed random walks that never leave the Weyl chamber $W = \{z \in \mathbb{R}^k : z_1 < \cdots < z_k\}$. The distances between these random walks $U(n) = (Z_2(n) - Z_1(n), \ldots, Z_k(n) - Z_{k-1}(n))$ give a $k-1$ dimensional random process whose components are positive. These processes appear in the eigenvalue description of important matrix-valued stochastic processes: see [7] for an old well-known result on the eigenvalues of the process version of the Gaussian Unitary Ensemble and e.g. [5], [20], [19], [11], [13]. They are found in the analysis of corner-growth model, see [17] and [18]. Moreover, interesting connections between non-colliding walks, random matrices and queues in tandem are the subject of [28]. Paper [8] reveals a rather general mechanism of the construction of the suitable h-transform for such processes. But processes whose components are distances between independent random walks are not the only class of interest. In [21], random walks with exchangeable increments and conditioned never to exit the Weyl chamber are considered. In [29], the authors study a certain class of random walks, namely $(X_i(n))_{1 \leq i \leq k} = (|\{1 \leq m \leq n : \xi_m = i\}|)_{1 \leq i \leq k}$, where $(\xi_m, m \geq 1)$ is a sequence of i.i.d. random variables with common distribution on $\{1, 2, \ldots, k\}$. The authors identify in law their conditional version with a certain path-transformation. In [26] and [27], O'Connell relates these objects to the Robinson-Schensted algorithm.

Another important area where random processes in angles of $\mathbb{Z}^d$ conditioned never to reach the boundary appear is "quantum random walks". In [2], Biane constructs a quantum Markov chain on the von Neumann algebra of $\mathrm{SU}(n)$ and interprets the restriction of this quantum Markov chain to the algebra of a maximal torus of $\mathrm{SU}(n)$ as a random walk on the lattice of integral forms on $\mathrm{SU}(n)$ with respect to this maximal torus. He proves that the restriction of the quantum Markov chain to the center of the von Neumann algebra is a Markov chain on the same lattice obtained from the preceding by conditioning it in Doob's sense to exit a Weyl chamber at infinity. In [3], Biane extends these results to the case of general semi-simple connected and simply connected compact Lie groups, the basic notion being that of the minuscule weight. The corresponding random walk on the weight lattice in the interior of the Weyl chamber can be obtained as follows: if $2l$ is the order of the associated Weyl group, one draws the vector corresponding to the minuscule weight and its $l-1$

conjugates under the Weyl group ; then one translates these vectors to each point of the weight lattice in the interior of the Weyl chamber and assigns to them equal probabilities of jumps $1/l$.

For example, in the case U(3), the Weyl chamber of the corresponding Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is the "angle $\pi/3$", that is to say the domain of $(\mathbb{R}_+)^2$ delimited on the one hand by the x-axis and on the other by the axis making an angle equal to $\pi/3$ with the x-axis. One gets a spatially homogeneous random walk in the interior of the weights lattice, as in the left-hand side of Picture 1, the arrows designing transition probabilities equal to 1/3. In the cases of the Lie algebras $\mathfrak{sp}_4(\mathbb{C})$ or $\mathfrak{so}_5(\mathbb{C})$, the Weyl chamber is the angle $\pi/4$, see the second picture of Figure 1 for the transition probabilities. Both of these random walks can be of course thought as walks in $(\mathbb{Z}_+)^2$ with transition probabilities drawn in the third and fourth pictures of Figure 1. Biane shows that the

FIGURE 1. The walks on weights lattice of classical algebras—above, $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$-–can be viewed as random walks on $\mathbb{Z}_+{}^d$

suitable Doob's h-transform $h(x, y)$ for these random walks is the dimension of the representation with highest weight $(x - 1, y - 1)$. In [1], again thanks to algebraic methods, he computes the asymptotic of the Green functions $G_{x,y}$ for the random walk with jump probabilities 1/3 in the angle $\pi/3$ on the Picture 1, absorbed at the boundary, $x, y \to \infty$ and $y/x \to \tan(\gamma)$, γ lying in $\in [\epsilon, \pi/2 - \epsilon]$, $\epsilon > 0$. The asymptotic of the Green functions as $y/x \to 0$ or $y/x \to \infty$ could not be found by these technics.

In [3] Biane also studies some extensions to random walks with drift: these are spatially homogeneous random walks in the same Weyl chambers, with the same non-zero jump probabilities as previously, but now these jump probabilities are admitted not to be all equal to $1/l$, so that the mean drift vector may have positive coordinates. Due to Choquet-Denis theory, in [3], he finds all minimal non-negative harmonic functions for these random walks. Nevertheless this approach seems not allow to find the Martin compactification of these random walks, nor to compute the asymptotic of the Green functions along different paths.

In [14], Ignatiouk-Robert obtains, under general assumptions and for all $d \geq 2$, the Martin boundary of some random walks in the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ killed on the boundary. In this paper and in [15], Ignatiouk-Robert proposes a new large deviation approach to the analysis of the Martin boundary combined with the ratio-limit theorem for Markov-additive processes. Ignatiouk-Robert and Loree develop this original approach in a recent paper [16] and apply it with success to the analysis of spatially homogeneous random walks in $(\mathbb{Z}_+)^2$ killed at the axes, under hypotheses of unbounded jump probabilities (more precisely, having exponential decay) and non-zero drift. They compute the Martin compactification for these random walks and therefore obtain the full Martin boundary. These methods seem not to be powerful for a more detailed study, as for the computation of the asymptotic of the Green functions, or for the computation of the absorption probabilities at different points on the axes, or for the enumeration of lattice walks (see [4] and references therein for the study of this last problem for lattice walks on $(\mathbb{Z}_+)^2$ by analytic methods).

They also seem to be difficult to generalize to the random walks with zero drift.

In this paper we would like to study in detail the spatially homogeneous random walks $(X(n), Y(n))_{n \geq 0}$ in $(\mathbb{Z}_+)^2$ with jumps at distance at most 1. We denote by $\mathbb{P}(X(n+1) = i_0 + i, Y(n+1) = j_0 + j \mid X(n) = i_0, Y(n) = j_0) = p_{(i_0, j_0),(i+i_0, j+j_0)}$ the transition probabilities and do the hypothesis:

(H1) *For all (i_0, j_0) such that $i_0 > 0, j_0 > 0$, $p_{(i_0, j_0),(i_0, j_0)+(i,j)}$ does not depend on (i_0, j_0) and can thus be denoted by p_{ij}.*

(H2) *$p_{ij} = 0$ if $|i| > 1$ or $|j| > 1$.*

(H3) *The boundary $\{(0,0)\} \cup \{(i,0) : i \geq 1\} \cup \{(0,j) : j \geq 1\}$ is absorbing.*

(H4) *In the list $p_{11}, p_{10}, p_{1-1}, p_{0-1}, p_{-1-1}, p_{-1,0}, p_{-11}, p_{0-1}$ there are no three consecutive zeros.*

The last hypothesis (H4) is purely technical and avoids studying degenerated random walks.

In a companion paper we gave a rather complete analysis of such random walks under a simplifying hypothesis that also

(H2′) *$p_{-11} = p_{11} = p_{1-1} = p_{-1-1} = 0$.*

This hypothesis made the analysis more transparent for several reasons. First, the problem to find the generating functions of absorption probabilities, defined in (1) and (2) below, could be reduced to the resolution of Riemann boundary value problems on contours inside unit discs, where these functions are holomorphic, being generating functions of probabilities. These contours under general hypothesis (H2) may lie outside the unit disc, so that we are obliged first to continue these functions as holomorphic, then to exploit this continuation. Secondly, the conformal gluing function responsible for the conversion

between Riemann boundary value problem and Riemann-Hilbert problem we are faced with, has a particularly nice form. Thirdly, this hypothesis corresponds to an easy group of Galois automorphisms in terms of the book [9], as it is of order four. Fourthly, under (H2′), in the case of non zero drift, the asymptotic of the Green functions $G_{i,j}^{n_0,m_0}$ (recall that these are the mean numbers of visits to (i,j) starting from (n_0, m_0)) could be found very easily by means of Paper [22].

Let $M_x = \sum_{i,j} i p_{i,j}$, $M_y = \sum_{i,j} j p_{i,j}$ be the coordinates of the mean drift vector. If $M_x > 0$, $M_y > 0$, the time τ of absorption by the axes for this random walk is infinite with positive probability for any initial state (n_0, m_0), $n_0, m_0 > 0$. It is immediate that the suitable Doob's h-transform conditioning the process never to reach the axes is of course this probability and also that this h-transformed process is equal in distributional limit to the conditional process given $\{\tau > k\}$ as $k \to \infty$. What is the suitable h-transform under the hypothesis $M_x = M_y = 0$ and does this last statement stay true in this case ? The answer to this question in [30] under (H2′) came from the study of the exact tail's asymptotic of τ as $k \to \infty$. Namely it was shown that $P_{(n_0,m_0)}(\tau > k) \sim C n_0 m_0 k^{-1}$, where C does not depend on n_0, m_0 and $n_0 m_0$ is the unique non-negative harmonic function. Since extensions of these results on τ when (H2′) is relaxed are rather voluminous, we restrict ourselves in this paper to the case of positive drift:

(M) $M_x = \sum_{i,j} i p_{i,j} > 0$, $M_y = \sum_{i,j} j p_{i,j} > 0$,

and postpone to a future work the study of the random walks with zero drifts in the Weyl chambers of $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$.

It is the book [9] that gave us the main tool of analysis and has therefore inspired this paper. This book studies the random walks in $(\mathbb{Z}_+)^2$ under assumptions (H1) and (H2) but not (H3): the jump probabilities from the boundaries to the interior of $(\mathbb{Z}_+)^2$ are there not zero and the x-axis, the y-axis and $(0,0)$ are three other domains of spatial homogeneity. Moreover, the jumps from the boundaries are supposed such that the Markov chain is ergodic. The authors G. Fayolle, R. Iasnogorodski and V. Malyshev elaborate a profound and ingenious analytic approach to compute the generating functions of stationary probabilities of these random walks. This approach serves as a starting point for our investigation and by this reason plays an absolutely crucial role: preparatory Subsections 2.1, 2.2 and 2.3 proceed along the book [9] applied for the random walks killed at the boundary.

In Subsection 2.4, using this analytic approach, we analyze the absorption probability.Let

(1)
$$\begin{aligned}
h_i^{(n_0,m_0)} &= \mathbb{P}_{(n_0,m_0)} \left(\text{to be absorbed at } (i,0) \right), \\
\widetilde{h}_j^{(n_0,m_0)} &= \mathbb{P}_{(n_0,m_0)} \left(\text{to be absorbed at } (0,j) \right), \\
h_{00}^{(n_0,m_0)} &= \mathbb{P}_{(n_0,m_0)} \left(\text{to be absorbed at } (0,0) \right).
\end{aligned}$$

be the probabilities of being absorbed at points $(i,0)$, $(0,j)$ and $(0,0)$ starting from (n_0, m_0). Let $h^{n_0,m_0}(x)$ and $\tilde{h}^{n_0,m_0}(y)$ be their generating functions, initially defined for $|x| \leq 1$ and $|y| \leq 1$:

(2) $\qquad h^{n_0,m_0}(x) = \sum_{i \geq 1} h_i^{n_0,m_0} x^i, \qquad \widetilde{h}^{n_0,m_0}(y) = \sum_{j \geq 1} \widetilde{h}_j^{n_0,m_0} y^j.$

When no ambiguity on the initial state can arise, we drop the index (n_0, m_0) and write h_i, $\tilde{h}_j$, h_{00}, $h(x)$, $\tilde{h}(y)$ respectively.

In Section 3 the generating functions $h(x)$, $h(y)$ and h_{00} are computed. Subsection 3.1 gives the first integral representation of these functions on a smooth curve, which is almost directly deduced from [9]. In Subsection 3.2 we look closer at the conformal gluing function and transform this representation into one on a real segment, that suits better for further analysis, see Theorem 13.

In Section 4 we deduce the asymptotic of the absorption probabilities h_i and $\tilde{h}_j$ as $i \to \infty$ and $j \to \infty$. We show that $h_i \sim C(n_0, m_0) p^{-i} i^{-3/2}$, with some (made explicit) $p > 1$ and a constant $C(n_0, m_0)$. This constant $C(n_0, m_0)$ is also made explicit and turns out to depend quite interestingly on the "group of Galois automorphisms" of the random walk (in the sense of Definition 3), see Theorem 17.

In Section 5 we compute the asymptotic of the Green functions $G_{i,j}^{n_0,m_0}$ that is of the mean number of visits to (i,j) starting from (n_0, m_0) as $i, j \to \infty$, $j/i \to \tan(\gamma)$ where $\gamma \in [0, \pi/2]$.

In the case of $\gamma \in]0, \pi/2[$, thanks to [22] and [25], it is not a difficult task: the procedure used in [22] for the Green functions asymptotic (and in fact developed much earlier in [25] for the stationary probabilities' asymptotic) of the random walks in the quadrant under the simplifying hypothesis (H2′) in the interior and with some non-zero jump probabilities from the axes can be rather easily generalized to our random walks under (H2). To state the result, let $(u(\gamma), v(\gamma))$ be the unique solution of $\mathrm{grad}(\phi(u,v))/|\mathrm{grad}(\phi(u,v))| = (\cos(\gamma), \sin(\gamma))$ on $\phi(u,v) = \sum_{i,j} p_{ij} e^{iu} e^{jv} = 1$. Let $s_x(\tan(\gamma)) = \exp(u(\gamma))$ and $s_y(\tan(\gamma)) = \exp(v(\gamma))$. Then $G_{i,j}^{n_0,m_0} \sim Ci^{-1/2}[s_x(\tan(\gamma))^{n_0} s_y(\tan(\gamma))^{m_0} - h(s_x(\tan(\gamma))) - \tilde{h}(s_y(\tan(\gamma))) - h_{00}] s_x(j/i)^{-i} s_y(j/i)^{-j}$, the constant C does not depend on n_0, m_0, i, j and is made explicit, see Theorem 24.

It is a more delicate task to study the asymptotic of the Green functions $G_{i,j}^{n_0,m_0}$ in the case of $j/i \to 0$ (or $j/i \to \infty$) that has not been completed in previous works. This is the subject of Subsection 5.2. In Theorem 26 we prove that $G_{i,j}^{n_0,m_0} \sim C_0 i^{-1/2} j/i \left[m_0 s_x(0)^{n_0} s_y(0)^{m_0-1} - \tilde{h}'(s_y(0)) \right] s_x(j/i)^{-i} s_y(j/i)^{-j}$ where C_0 is independent of n_0 and m_0 and is made explicit. The explicit expression of $m_0 s_x(0)^{n_0} s_y(0)^{m_0-1} - \tilde{h}'(s_y(0))$ in terms of the parameters p_{ij} depends interestingly on the order of the "group of Galois automorphisms", see Remark 27. The result for $j/i \to \infty$ follows after exchanging i and j. The limit of $G_{i,j}^{n_0,m_0}/G_{i,j}^{n_1,m_1}$, as $i,j > 0$, $j/i \to 0$ is the same as the limit $h_i^{n_0,m_0}/h_i^{n_1,m_1}$ when $i \to \infty$ from Section 4. Furthermore the limits of $G_{i,j}^{n_0,m_0}/G_{i,j}^{n_1,m_1}$ when $j/i \to \tan(\gamma)$, $\gamma \in [0,\pi/2]$, provide explicitly all harmonic functions of the Martin compactification. This leads in particular to the result recently obtained in [16] that the Martin boundary is homeomorphic to $[0,\pi/2]$.

2. Analytic approach

2.1. A functional equation. — Define

(3)

$$G(x,y) = \sum_{i,j \geq 1} G_{i,j}^{n_0,m_0} x^{i-1} y^{j-1} = \sum_{i,j \geq 1} \mathbb{E}_{(n_0,m_0)} \left[\sum_{n \geq 0} 1_{\{(X(n),Y(n))=(i,j)\}} \right] x^{i-1} y^{j-1},$$

the generating function of the Green functions. With the notations of Section 1, we can state the following functional equation:

(4)
$$Q(x,y) G(x,y) = h(x) + \tilde{h}(y) + h_{00} - x^{n_0} y^{m_0},$$

where Q is the following polynomial, depending only on the walk's transition probabilities:

(5)
$$Q(x,y) = xy \left(\sum_{i,j} p_{ij} x^i y^j - 1 \right).$$

Prima facie, Equation (4) has a meaning in $\{x, y \in \mathbb{C} : |x| < 1, |y| < 1\}$. The proof of (4) comes from writing that for $k, l, n \in (\mathbb{Z}_+)^2$,

$$\mathbb{P}((X(n+1), Y(n+1)) = (k, l)) = \sum_{i,j \geq 1} \mathbb{P}((X(n), Y(n)) = (i, j)) \, p_{(i,j),(k,l)} +$$

$$+ \sum_{i \geq 1} \mathbb{P}((X(n), Y(n)) = (i, 0)) \, \delta_{(k,l)}^{(i,0)} + \sum_{j \geq 1} \mathbb{P}((X(n), Y(n)) = (0, j)) \, \delta_{(k,l)}^{(0,j)} +$$

$$+ \mathbb{P}((X(n), Y(n)) = (0, 0)) \, \delta_{(k,l)}^{(0,0)},$$

where $\delta_{(k,l)}^{(i,j)} = 1$ if $i = k$ and $j = l$, otherwise 0. It remains to multiply by $x^k y^l$ and then to sum with respect to k, l, n.

2.2. The algebraic curve $Q(x, y) = 0$. — The polynomial (5) can be written alternatively:

$$(6) \qquad Q(x, y) = a(x) y^2 + b(x) y + c(x) = \widetilde{a}(y) x^2 + \widetilde{b}(y) x + \widetilde{c}(y),$$

where

$$
\begin{aligned}
a(x) &= p_{11} x^2 + p_{01} x + p_{-11}, & \widetilde{a}(y) &= p_{11} y^2 + p_{10} y + p_{1-1}, \\
b(x) &= p_{10} x^2 - x + p_{-10}, & \widetilde{b}(y) &= p_{01} y^2 - y + p_{0-1}, \\
c(x) &= p_{1-1} x^2 + p_{0-1} x + p_{-1-1}, & \widetilde{c}(y) &= p_{-11} y^2 + p_{-10} y + p_{-1-1}, \\
d(x) &= b(x)^2 - 4a(x)c(x), & \widetilde{d}(y) &= \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y).
\end{aligned}
$$

We will now build the algebraic function $Y(x)$ defined by $Q(x, y) = 0$. Note first that $Q(x, y) = 0$ is equivalent to $(b(x) + 2a(x)y)^2 = d(x)$, so that the construction of the function Y is equivalent to that of the square root of the polynomial d. We need the following precisions on the roots of d:

LEMMA 1. — *(1) d is a third or fourth degree polynomial, whose all roots are real and mutually distinct. (2) We call its roots the x_i, $i \in \{1, \ldots, 4\}$, with eventually $x_4 = \infty$ if $\deg(d) = 3$. It turns out that there are two possibilities: either the modulus of the roots are mutually distinct and in this case we enumerate the roots in such a way that $|x_1| < |x_2| < |x_3| < |x_4|$, or there are two pairs of roots and inside of each pair the roots are opposed one from the other, in this case we enumerate them $0 < x_2 = -x_1 < x_3 = -x_4$. This last case corresponds to the walks having transition probabilities such that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$. (3) Moreover, $|x_1| < 1$, $|x_2| < 1$ and $|x_3| > 1$, $|x_4| > 1$. (4) x_2 and x_3 are positive. (5) $x_1 = 0$ (resp. $x_4 = \infty$) if and only if $p_{-10}^2 - 4p_{-11}p_{-1-1} = 0$ (resp. $p_{10}^2 - 4p_{11}p_{1-1} = 0$). (6) If $p_{-10}^2 - 4p_{-11}p_{-1-1} \neq 0$ (resp. $p_{10}^2 - 4p_{11}p_{1-1} \neq 0$) then $\mathrm{sign}(x_1) = \mathrm{sign}(p_{-10}^2 - 4p_{-11}p_{-1-1})$ (resp. $\mathrm{sign}(x_4) = \mathrm{sign}(p_{10}^2 - 4p_{11}p_{1-1})$).* □

Proof. — All these properties are proved in [9]. Note here that it is thanks to the hypothesis (H4), made in Section 1, that the polynomial d is of degree three or four.   □

There are two branches of the square root of d. Each determination leads to a well defined (i.e. single valued) and meromorphic function on the complex plane $\mathbb{C}$ appropriately cut, that is, in our case, on $\mathbb{C}\backslash[x_1, x_2]\cup[x_3, x_4]$. If $x_4 < 0$, then $[x_3, x_4]$ means $[x_3, +\infty[\cup] - \infty, x_4]$. We can write the analytic expression of these two branches Y_0 and Y_1 of Y: $Y_0(x) = Y_-(x)$ and $Y_1(x) = Y_+(x)$ where:

$$Y_\pm(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)}.$$

Just above, and in fact throughout the whole paper, we chose the principal determination of the logarithm as soon as we use the complex logarithm ; in this case to define the square root.

We now extend the domain of determination of Y from $\mathbb{C}$ to its Riemann surface S, so that Y becomes single-valued on S. Since there are two determinations of the square root of d (opposed one from the other), the Riemann surface S is formed by S_0 and S_1, two copies of the Riemann sphere $\mathbb{C}\cup\{\infty\}$ cut along $[x_1, x_2]$ and $[x_3, x_4]$ and joined across lines lying above these cuts. This gives a two-sheeted covering surface of $\mathbb{C}\cup\{\infty\}$, branched over $x_1, \ldots, x_4$. By opening out the cuts in the two sheets we see that the Riemann surface associated to Y is homeomorphic to a sphere with one handle attached, that is a Riemann surface of genus one, a torus. For more details about the construction of Riemann surfaces see for instance Book [31]. In a similar way, the functional equation (4) defines also an algebraic function $X(y)$. All the results concerning $X(y)$ can be deduced from those for $Y(x)$ after a proper change of the parameters, namely $p_{ij} \mapsto p_{ji}$.

To conclude this part, we give a lemma that clarifies some properties of the functions X and Y, useful in the sequel. It is proved in [9].

LEMMA 2. — *(1) $Y_0(1) = c(1)/a(1)$ and $Y_1(1) = 1$. (2) $Y_0(\{x \in \mathbb{C} : |x| = 1\}) \subset \{y \in \mathbb{C} : |y| < 1\}$ and $Y_1(\{x \in \mathbb{C} : |x| = 1\} \backslash \{1\}) \subset \{y \in \mathbb{C} : |y| > 1\}$. (3) For all x in this cut plane, $|Y_0(x)| \leq |Y_1(x)|$, with equality only on the cuts. (4) Suppose that $x_4 > 0$ and that the walk is non degenerated, see the hypothesis (H4). If $p_{1-1} = 0$, then $\lim_{x\to\infty} xY_0(x) \in] - \infty, 0[$ and if $p_{1-1} > 0$ then $\lim_{x\to\infty} Y_0(x) \in] - \infty, 0[$.*

2.3. Galois automorphisms and meromorphic continuation

Notation. — In this part and throughout the whole paper, $\imath$ denotes the complex number: $\imath^2 = -1$.

The Riemann surface S associated to the algebraic function Y is naturally endowed with a covering map $\pi : S \to \mathbb{C} \cup \{\infty\}$, such that for all $x \in \mathbb{C} \setminus [x_1, x_2] \cup [x_3, x_4]$, $\pi^{-1}(x)$ is composed of two elements, say s_0 and s_1, such that $s_i \in S_i$, $i = 0, 1$ and $\{Y(s_0), Y(s_1)\} = \{Y_0(x), Y_1(x)\}$. In the same way the Riemann surface $\tilde{S}$ associated to X is endowed with a map $\tilde{\pi} : \tilde{S} \to \mathbb{C} \cup \{\infty\}$, such that for all $y \in \mathbb{C} \setminus [y_1, y_2] \cup [y_3, y_4]$, $\tilde{\pi}^{-1}(y)$ is composed of two elements, say $\tilde{s}_0$ and $\tilde{s}_1$, such that $\tilde{s}_i \in \tilde{S}_i$, $i = 0, 1$ and $\{X(\tilde{s}_0), X(\tilde{s}_1)\} = \{X_0(y), X_1(y)\}$.

The surfaces S and $\tilde{S}$ having the same genus, we consider from now on *only one* surface T, conformally equivalent to S and $\tilde{S}$, with two coverings π and $\tilde{\pi}$. One can say that each $s \in S$ has two (not independent) "coordinates" $(x(s), y(s))$ such that $x(s) = \pi(s)$ and $y(s) = \tilde{\pi}(s)$ and of course $Q(x(s), y(s)) = 0$ for all $s \in T$.

We construct on T the following covering automorphisms ξ and η defined in the previous notations by $\xi(s_0) = s_1$ and $\eta(\tilde{s}_0) = \tilde{s}_1$. Thanks to (6), for any $s = (x, y) \in T$, ξ and η take the following explicit expressions:

$$(7) \qquad \xi(x, y) = \left(x, \frac{c(x)}{a(x)} \frac{1}{y} \right), \qquad \eta(x, y) = \left(\frac{\tilde{c}(y)}{\tilde{a}(y)} \frac{1}{x}, y \right).$$

ξ and η are of order two: $\xi^2 = \mathrm{id}$, $\eta^2 = \mathrm{id}$. In [24] and [9], for reasons explained there, they are also called Galois automorphisms.

DEFINITION 3. — *The group of the random walk $\mathcal{H}$ is the group generated by ξ and η.*

Being generated by a finite number of elements of order two, $\mathcal{H}$ is a Coxeter group. In fact, $\mathcal{H}$ is simply a dihedral group, since it is generated by two elements. Define $\delta = \eta\xi$. Then the order of $\mathcal{H}$ is equal to $2\inf\{n \in \mathbb{N}^* : \delta^n = \mathrm{id}\}$, it can be eventually infinite. The finiteness of this group, and in that event its order, will turn out to be decisive in the sequel, notably in Subsection 2.4 and Section 4.

As implied in [9], it is quite difficult to characterize geometrically the walks having an associated group $\mathcal{H}$ of order $2n$, except for little orders. That is how in [9] is proved that $\mathcal{H}$ is of order four if and only if

$$(8) \qquad \Delta = \begin{vmatrix} p_{11} & p_{10} & p_{1-1} \\ p_{01} & -1 & p_{0-1} \\ p_{-11} & p_{-10} & p_{-1-1} \end{vmatrix}$$

is equal to zero. In particular this is the case of the walks having transition probabilities verifying $p_{10} + p_{-10} + p_{01} + p_{0-1} = 1$, that we have studied in [30]. It is also proved in [9] that the the walks with transition probabilities $p_{-11} + p_{10} + p_{0-1} = 1$, in the Weyl chamber of $\mathfrak{sl}_3(\mathbb{C})$, see Figure 2, have a group of order six for any values of the parameters. As for the walks in the Weyl chamber of $\mathfrak{sp}_4(\mathbb{C})$, see Figure 1, they have, except for exceptional values of the parameters, a group of order infinite. We add here that the walks with $p_{11} = p_{10}$, $p_{-1-1} = p_{-10}$, $p_{11} + p_{-1-1} = 1/2$, drawn in Figure 2, have a group of order eight.
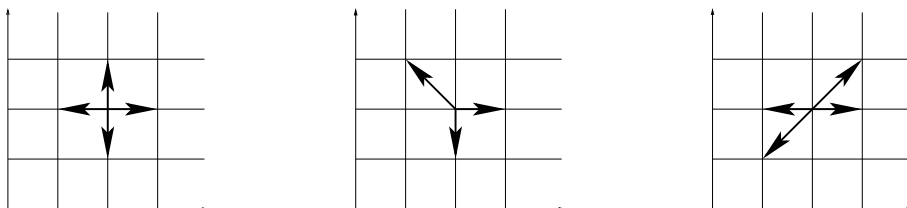

FIGURE 2. Random walks having groups of order four, six and eight respectively

We will now continue the functions h and $\tilde{h}$, initially defined on the unit disc, to $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$ as holomorphic functions. This continuation will have a twofold interest. First, in Section 3, we will have, in order to find explicit expressions of h and $\tilde{h}$, to solve a boundary value problem, with boundary condition on closed curves that lie in the exterior of the unit disc. Secondly in Section 5, when we will calculate the asymptotic of the Green functions, the quantity $x^{n_0} y^{m_0} - h(x) - \tilde{h}(y) - h_{00}$ will naturally appear, evaluated at some (x, y)--in fact, the saddle-point—that is not in $\mathscr{D}(0, 1)^2$.

To do this continuation, we will use a uniformization of the curve $\{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$. Being a Riemann surface of genus one, we already know that $Q = 0$ is homeomorphic to some quotient $\mathbb{C}/\Gamma$, where Γ is a two-dimensional lattice, that is to say to a parallelogram whose the opposed edges are identified. In [9], such a lattice Γ and also a bijection between $\mathbb{C}/\Gamma$ and $Q = 0$ are made explicit. Indeed, the authors find there $\omega_1 \in \imath\mathbb{R}$ and $\omega_2 \in \mathbb{R}$, two functions ϕ and ψ, such that $\{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\} = \{(x(\omega), y(\omega)), \omega \in \mathbb{C}/\Gamma\}$, where $x(\omega) = \phi(\wp_{1,2}(\omega), \wp'_{1,2}(\omega))$, $y(\omega) = \psi(\wp_{1,2}(\omega), \wp'_{1,2}(\omega))$, $\Gamma = \{n_1\omega_1 + n_2\omega_2, n_1, n_2 \in \mathbb{Z}\}$ and $\wp_{1,2}$ is the classical Weierstrass elliptic function associated to the periods ω_1 and ω_2, that are equal to:

$$(9) \quad \omega_1 = \imath \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-d(x)}}, \quad \omega_2 = \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{d(x)}}, \quad \omega_3 = \int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{d(x)}},$$

$\omega_3 \in]0, \omega_2[$ being a period that will turn out to be quite important in the sequel. The functions ϕ and ψ are also made explicit: for instance, if $x_4 \neq \infty$, then it is possible to take $\phi(p, p') = x_4 + d'(x_4)/(p - d''(x_4)/6)$ and if $x_4 = \infty$, $\phi(p, p') = (6p - d''(0))/d'''(0)$.
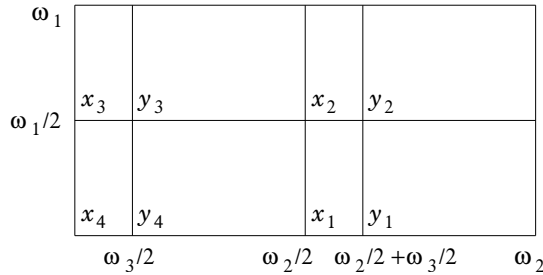

FIGURE 3. Location of the cuts on the covering surface

Moreover, on $\mathbb{C}/\Gamma$, the automorphisms ξ and η take the following particularly nice form:

$$(10) \quad \xi(\omega) = -\omega, \qquad \eta(\omega) = -\omega + \omega_3, \qquad \delta(\omega) = \eta(\xi(\omega)) = \omega + \omega_3.$$

In particular, the group $\mathscr{H}$ has a finite order if and only if $\omega_3/\omega_2 \in \mathbb{Q}$ and in this case the order is given by $2\inf\{n \in \mathbb{N}^* : n\omega_3/\omega_2 \in \mathbb{N}\}$.

Any function f of the variable x (resp. y) defined on some domain $\mathscr{D} \subset \mathbb{C}$ can be lifted on $\{\omega \in \mathbb{C}/\Gamma : x(\omega) \in \mathscr{D}\}$ (resp. $\{\omega \in \mathbb{C}/\Gamma : y(\omega) \in \mathscr{D}\}$) by setting $F(\omega) = f(x(\omega))$ (resp. $F(\omega) = f(y(\omega))$). In particular we can lift the generating functions h and $\tilde{h}$ and we set $H(\omega) = h(x(\omega))$ and $\tilde{H}(\omega) = \tilde{h}(y(\omega))$, well defined on $\{\omega \in \mathbb{C}/\Gamma : |x(\omega)| \leq 1\}$ and $\{\omega \in \mathbb{C}/\Gamma : |y(\omega)| \leq 1\}$ respectively. In particular, on $\{\omega \in \mathbb{C}/\Gamma : |x(\omega)| \leq 1, |y(\omega)| \leq 1\}$, using (4), we have $H(\omega) + \tilde{H}(\omega) + h_{00} - x(\omega)^{n_0}y(\omega)^{m_0} = 0$. Applying several times the Galois automorphisms ξ and η to any point of this domain and laying down $H(\omega) = H(\xi(\omega))$, $\tilde{H}(\omega) = \tilde{H}(\eta(\omega))$ the authors of [9] prove the following fundamental proposition.

PROPOSITION 4. — *The functions H and $\tilde{H}$ can be continued as meromorphic functions on respectively* $(\mathbb{C}/\Gamma) \setminus [0, \omega_1]$ *and* $(\mathbb{C}/\Gamma) \setminus [\omega_3/2, \omega_3/2 + \omega_1]$.

Furthermore,

$$H(\omega) = H(\xi(\omega)), \qquad \widetilde{H}(\omega) = \widetilde{H}(\eta(\omega)), \qquad \forall \omega \in \mathbb{C}/\Gamma,$$

$$H(\omega) + \widetilde{H}(\omega) + h_{00} - x(\omega)^{n_0} y(\omega)^{m_0} = 0, \qquad \forall \omega \in [\omega_3/2, \omega_2] \times [0, \omega_1/\imath].$$

COROLLARY 5. — *The function h and $\tilde{h}$ can be continued into meromorphic functions on $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$ respectively.*

2.4. Absorption probability. — In the next section we will find explicitly $h(x)$, $\tilde{h}(y)$ and h_{00} that will provide of course the absorption probability $h(1)+\tilde{h}(1)+h_{00}$. However, this expression is usable difficultly. In this subsection we prove that in a special case of finite groups of the random walk (see Definition 3) the probability of absorption takes a particularly nice form, see Corollary 8. Furthermore, in the case of the group of any order, Proposition 9 gives the precise exponential asymptotic of the absorption probability as $n_0, m_0 \to \infty$.

We first note that the quantity $H(\omega) + \tilde{H}(\omega) + h_{00} - x(\omega)^{n_0} y(\omega)^{m_0}$, for ω in $[0, \omega_3/2] \times [0, \omega_1/\imath]$, can be considerably simplified in some cases, namely when the group is finite (i.e. $\omega_2/\omega_3 \in \mathbb{Q}_+$) and when in addition $\omega_2/\omega_3 \in \mathbb{N}$. This is for example the case of the walks such that $\Delta = 0$, for which $\omega_2/\omega_3 = 2$ -- indeed, we have already seen that both assertions $\Delta = 0$ and $\omega_2/\omega_3 = 2$ are equivalent to the fact that $\mathcal{H}$ is of order four; this is also the case of the walk in the Weyl chamber of $\mathfrak{sl}_3(\mathbb{C})$, see Figure 2, since in this case the group is of order six, hence ω_2/ω_3 is equal to $3/2$ or 3, and by a direct calculation we show that $\omega_2/\omega_3 = 3$. On the other hand, this is not the case of the walk whose transition probabilities are represented on the right part of Figure 2: although the group is of order eight, we don't have $\omega_3 = \omega_2/4$ but $\omega_3 = 3\omega_2/4$.

PROPOSITION 6. — *Suppose that $\omega_2/\omega_3 \in \mathbb{N}$; in particular this implies that $\mathcal{H}$ is of order $2\omega_2/\omega_3$. Then if $\omega \in [0, \omega_3/2] \times [0, \omega_1/\imath]$,*
(11)
$$H(\omega) + \widetilde{H}(\omega) + h_{00} - x(\omega)^{n_0} y(\omega)^{m_0} = - \sum_{w \in \mathcal{H}} (-1)^{l(w)} x(w(\omega))^{n_0} y(w(\omega))^{m_0},$$

where $l(w)$ is the length of the word w, that is the smallest r for which we can write $w = s_1 \cdots s_r$, with s_i equal to ξ or η.

Proof. — The key point of the proof of Proposition 6, that also explains why we have done the hypothesis $\omega_2/\omega_3 \in \mathbb{N}$, is that in this only case, the fundamental domain $\chi_0 = [0, \omega_2/(2n)[\times [0, \omega_1/\imath[$ and the domain $[0, \omega_3/2[\times [0, \omega_1/\imath[$ of Proposition 4 coincide (by χ_0 is a fundamental domain we mean that each $\omega \in \mathbb{C}/\Gamma$ is conjugate under $\mathcal{H}$ to one and only one point of χ_0).

Let us first give a proof in the case of the groups of order four. Note that $H + \tilde{H} + h_{00} = H(\xi) + \tilde{H}(\eta) + h_{00}$, since H (resp. $\tilde{H}$) is invariant w.r.t. ξ (resp. η), thanks to Proposition 4. So $H + \tilde{H} + h_{00} = H(\xi) + \tilde{H}(\xi) + h_{00} + H(\eta) + \tilde{H}(\eta) + h_{00} - (H(\eta) + \tilde{H}(\xi) + h_{00})$. Using once again the invariance properties of H and $\tilde{H}$, we can write $H + \tilde{H} + h_{00} = H(\xi) + \tilde{H}(\xi) + h_{00} + H(\eta) + \tilde{H}(\eta) + h_{00} - (H(\xi\eta) + \tilde{H}(\eta\xi) + h_{00})$. Since the order of $\mathcal{H}$ is four, $\xi\eta = \eta\xi$ and the previous equation becomes: $H + \tilde{H} + h_{00} = H(\xi) + \tilde{H}(\xi) + h_{00} + H(\eta) + \tilde{H}(\eta) + h_{00} - (H(\xi\eta) + \tilde{H}(\xi\eta) + h_{00})$. If $\omega \in \chi_0$, then $w(\omega) \in (\mathbb{C}/\Gamma) \backslash \chi_0$ for all $w \in \mathcal{H} \backslash \{\mathrm{id}\}$. Indeed, we will prove in Lemma 7 that χ_0 is a fundamental domain. In addition, thanks to

Proposition 4, the functional equation $H(\omega) + \tilde{H}(\omega) + h_{00} - x(\omega)^{n_0} y(\omega)^{m_0} = 0$ is verified in $[\omega_3/2, \omega_2[\times[0, \omega_1/\imath[$ which coincides with $(\mathbb{C}/\Gamma) \setminus \chi_0$. In other words, we can replace $H(w(\omega)) + \tilde{H}(w(\omega)) + h_{00}$ by $x(w(\omega))^{n_0} y(w(\omega))^{m_0}$ for any of three elements $w \in \mathscr{H} \setminus \{\text{id}\}$. Proposition 6 is thus proved in the case of the groups $\mathscr{H}$ of order four.

In the general case $\omega_2/\omega_3 = n$, for $k \in \{1, \ldots, n-1\}$, denote by $w_{1,k}$ and $w_{2,k}$ the two reduced words of length k, i.e. the words $s_1 \cdots s_k$ and $s_2 \cdots s_k s_1$, where for $r \geq 1$, $s_{2r} = \xi$ and $s_{2r-1} = \eta$, and denote by w_n the only word of length n. The fact that there is only one word of length n follows from the equality $\inf\{n \in \mathbb{N}^* : \delta^n = \text{id}\} = \inf\{n \in \mathbb{N}^* : s_1 s_2 \cdots s_n = s_2 \cdots s_n s_1\}$. Then, by induction, we prove that

$$H(\omega) + \widetilde{H}(\omega) = \sum_{k=1}^{n-1} (-1)^{k+1} \left(H(w_{1,k}(\omega)) + \widetilde{H}(w_{1,k}(\omega)) \right.$$
$$\left. + H(w_{2,k}(\omega)) + \widetilde{H}(w_{2,k}(\omega)) \right)$$
$$- (-1)^n \left(H(w_n(\omega)) + \widetilde{H}(w_n(\omega)) \right).$$

Since $\mathscr{H} = \{\text{id}, w_{1,1}, w_{2,1}, \ldots, w_{1,n-1}, w_{2,n-1}, w_n\}$ and since $[0, \omega_3/2[\times[0, \omega_1/\imath[$ is a fundamental domain, if $\omega \in [0, \omega_3/2[\times[0, \omega_1/\imath[= [0, \omega_2/(2n)[\times[0, \omega_1/\imath[$ then thanks to Proposition 4, for any $w \in \mathscr{H} \setminus \{\text{id}\}$, $H(w(\omega)) + \tilde{H}(w(\omega)) + h_{00} = x(w(\omega))^{n_0} y(w(\omega))^{m_0}$. Moreover, $l(w_n) = n$ and for $k \in \{1, \ldots, n-1\}$ and $i \in \{1, 2\}$, $l(w_{i,k}) = k$, so (11) is proved. □

LEMMA 7. — *Suppose that the group $\mathscr{H}$ is finite of order $2n$. Then for any $k \in \{0, \ldots, 2n-1\}$, the domain $\chi_k = [k\omega_2/(2n), (k+1)\omega_2/(2n)[\times[0, \omega_1/\imath[$ is a fundamental domain, i.e. each $\omega \in [0, \omega_2[\times[0, \omega_1/\imath[$ is conjugate under $\mathscr{H}$ to one and only one point of χ_k.*

Proof. — Denote by $\Lambda_\mu = \mu + [0, \omega_1]$ the vertical segment with abscissa μ. Then, with (10), we can describe the actions of ξ and η on these segments. So, for any μ in $[0, \omega_2]$, $\xi(\Lambda_\mu) = \Lambda_{\omega_2-\mu}$. Also, if $\mu \in [0, \omega_3]$, then $\eta(\Lambda_\mu) = \Lambda_{\omega_3-\mu}$ and if $\mu \in]\omega_3, \omega_2]$ then $\eta(\Lambda_\mu) = \Lambda_{\omega_3+\omega_2-\mu}$. Of course, we also know the action of the elements of $\mathscr{H}$ on the domains $\chi_k = [k\omega_2/(2n), (k+1)\omega_2/(2n)[\times[0, \omega_1/\imath[$, since we know how these automorphisms act on the boundaries of these sets.

Suppose first that k is even. Then the $\cup_{p=0}^{n-1} \delta^p(\chi_k) = \cup_{q=0}^{n-1} \chi_{2q}$. In particular, there exists $m \in \{0, \ldots, n-1\}$ such that $\delta^m(\chi_k) = \chi_0$. Thanks to (10), we have $\xi(\chi_0) = \chi_{2n-1}$, so $\xi(\delta^m(\chi_k)) = \chi_{2n-1}$. Also, $\cup_{p=0}^{n-1} \delta^p(\chi_{2n-1}) = \cup_{q=0}^{n-1} \chi_{2q+1}$. Of course, $(\cup_{q=0}^{n-1} \chi_{2q+1}) \cup (\cup_{q=0}^{n-1} \chi_{2q}) = [0, \omega_2[\times[0, \omega_1/\imath[$ and $\mathscr{H} = \{\text{id}, \delta, \ldots, \delta^{n-1}, \xi\delta^m, \delta\xi\delta^m, \ldots, \delta^{n-1}\xi\delta^m\}$, for any $m \in \{0, \ldots, n-1\}$. For example, in Figure 4 are represented the domain χ_0 and its images under $\mathscr{H}$ in the particular case $\omega_3 = \omega_2/n$. Lemma 7 is thus proved if k is even. The proof
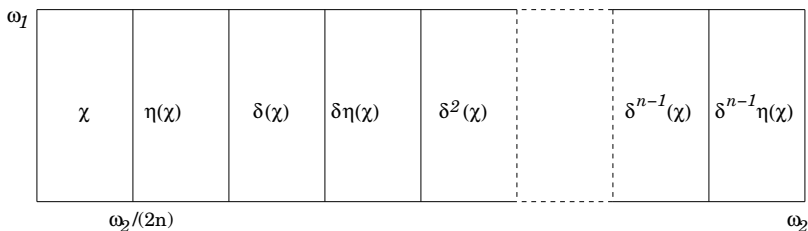
FIGURE 4. $\chi_0 = [0, \omega_2/(2n)[\times[0, \omega_1/\imath[$ is a fundamental domain

is quite similar in case of odd k, so we omit it.   □

COROLLARY 8. — *Suppose that $\omega_2/\omega_3 \in \mathbb{N}$. Then the probability of being absorbed is equal to:*

$$\mathbb{P}_{n_0,m_0}\left((X,Y) \text{ is absorbed}\right) = h(1) + \widetilde{h}(1) + h_{00} = 1 - \sum_{w \in \mathscr{H}} (-1)^{l(w)} f_{n_0,m_0}(w(1,1)),$$

where $f_{n_0,m_0}(x,y) = x^{n_0} y^{m_0}$ and the automorphisms of $\mathscr{H}$ are here defined by using (7).

Proof. — The proof is simply based on the fact that the point lying over $(1,1)$ belongs to $[0, \omega_3/2] \times [0, \omega_1/\imath]$, so that Corollary 8 is an immediate consequence of Proposition 6.   □

We can therefore easily calculate the probability of being absorbed for the walks verifying $\Delta = 0$, since in this case the group is of order four and $\omega_2/\omega_3 = 2$. In particular, we find again Proposition 28 of [30]. Corollary 8 applies also to the walk in the Weyl chamber of $\mathfrak{sl}_3(\mathbb{C})$, see the walk whose transition probabilities are drawn in the middle of Figure 2, since in this case $\omega_2/\omega_3 = 3$.

In the general case, the probability of being absorbed, $h_{00} + h(1) + \widetilde{h}(1)$, verifies the following inequality.

PROPOSITION 9. — *The probability of being absorbed can be bounded from above and below as follows:*

(12) $A/2 \le h_{00} + h(1) + \widetilde{h}(1) \le A,$

$$A = \left(\frac{p_{1-1} + p_{0-1} + p_{-1-1}}{p_{11} + p_{01} + p_{-11}}\right)^{n_0} + \left(\frac{p_{-1-1} + p_{-10} + p_{-11}}{p_{11} + p_{10} + p_{1-1}}\right)^{m_0}.$$

Proof. — We begin by writing the following equality:

$$2\big(h_{00} + H(\omega) + \widetilde{H}(\omega)\big)$$
$$= h_{00} + H(\xi(\omega)) + \widetilde{H}(\xi(\omega)) + h_{00} + H(\eta(\omega)) + \widetilde{H}(\eta(\omega))$$
$$+ H(\omega) - H(\eta(\omega)) + \widetilde{H}(\omega) - \widetilde{H}(\xi(\omega)),$$

obtained by using the invariance properties of H and $\tilde{H}$ claimed in Proposition 4. In particular, if $\omega \in [0, \omega_3/2] \times [0, \omega_1/\imath]$, then $\xi(\omega)$ and $\eta(\omega)$ belong to $\omega \in [\omega_3/2, \omega_2] \times [0, \omega_1/\imath]$, so that using once again Proposition 4, we obtain that $2(H(\omega) + \tilde{H}(\omega) + h_{00})$ is equal to:

(13)
$$x(\xi(\omega))^{n_0} y(\xi(\omega))^{m_0} + x(\eta(\omega))^{n_0} y(\eta(\omega))^{m_0} + H(\omega) - H(\eta(\omega)) + \widetilde{H}(\omega) - \widetilde{H}(\xi(\omega)).$$

In particular, if we take ω lying over $(1,1)$, that belongs to $[0, \omega_3/2] \times [0, \omega_1/\imath]$, as said in the proof of Corollary 8, and if we use that for this ω, $x(\eta(\omega)) = c(1)/a(1)$, $y(\xi(\omega)) = \tilde{c}(1)/\tilde{a}(1)$, we obtain:

$$2\left(h_{00} + h(1) + \widetilde{h}(1)\right) = A + h(1) - h\left(\widetilde{c}(1)/\widetilde{a}(1)\right) + \widetilde{h}(1) - \widetilde{h}\left(c(1)/a(1)\right).$$

Then, using that $c(1)/a(1) > 0$ and $\tilde{c}(1)/\tilde{a}(1) > 0$ (what implies that $h(\tilde{c}(1)/\tilde{a}(1)) > 0$ and $\tilde{h}(c(1)/a(1)) > 0$) allows to get the lower bound, and using that $c(1)/a(1) < 1$ and $\tilde{c}(1)/\tilde{a}(1) < 1$ (what is equivalent to the positivity of the two drifts, in accordance with our assumption (M)), added to the fact that h and $\tilde{h}$ are increasing, allows to get the upper bound.   □

3. Explicit form of the absorption probabilities generating functions

3.1. Riemann boundary value problem with shift. — Using the notations of Subsection 2.2, we define the two following curves:

$$(14) \qquad \mathcal{L} = Y_0\left(\left[\overleftarrow{x_1, x_2}\right]\right), \qquad \mathcal{M} = X_0\left(\left[\overleftarrow{y_1, y_2}\right]\right).$$

Just above, we use the notation of [9]: $\left[\overleftarrow{u, v}\right]$ stands for the contour $[u, v]$ traversed from u to v along the upper edge of the slit $[u, v]$ and then back to u along the lower edge of the slit.

In [9] is proved that the curves $\mathcal{L}$ and $\mathcal{M}$ are quartics, symmetrical w.r.t. the real axis, closed and simple, included in $\mathbb{C} \setminus [y_1, y_2] \cup [y_3, y_4]$ and $\mathbb{C} \setminus [x_1, x_2] \cup [x_3, x_4]$ respectively.

The reason why we have introduced these curves appears now: the functions h and $\tilde{h}$, defined in (2), verify the following boundary conditions on $\mathcal{M}$ and $\mathcal{L}$:

$$(15) \qquad \forall t \in \mathcal{M}: \quad h(t) - h(\bar{t}) = t^{n_0} Y_0(t)^{m_0} - \bar{t}^{n_0} Y_0(\bar{t})^{m_0},$$
$$\forall t \in \mathcal{L}: \quad \widetilde{h}(t) - \widetilde{h}(\bar{t}) = X_0(t)^{n_0} t^{m_0} - X_0(\bar{t})^{n_0} \bar{t}^{m_0}.$$

The way to obtain these boundary conditions is described in [9], so we refer to this book for the details. The function h, as a generating function of probabilities, is well defined on the closed unit disc and with Corollary 5 is continuable into a meromorphic function on $\mathbb{C} \setminus [x_3, x_4]$, domain that contains the bounded domain delimited by $\mathcal{M}$. Now we have the problem *to find h holomorphic inside $\mathcal{M}$, continuous up to the boundary $\mathcal{M}$ and verifying the boundary condition* (15). *In addition* $h(0) = 0$.

Problems with boundary conditions like (15) are called Riemann boundary value problems with shift. The classical way to study this kind of problems is to reduce them to Riemann-Hilbert problems, for which there exists a suitable and complete theory. The conversion between Riemann problems with shift and Riemann-Hilbert problems is done thanks to the use of conformal gluing functions, notion defined just above. For details about boundary value problems, we refer to [23].

DEFINITION 10. — *Let $\mathcal{C}$ be a simple closed curve, symmetrical w.r.t. the real axis. Denote by $\mathcal{G}_{\mathcal{C}}$ the interior of the bounded domain delimited by $\mathcal{C}$. w is called a conformal gluing function (CGF) for the curve $\mathcal{C}$ if* (i) *w is meromorphic in $\mathcal{G}_{\mathcal{C}}$, continuous up to its boundary* (ii) *w establishes a conformal mapping of $\mathcal{G}_{\mathcal{C}}$ onto the complex plane cut along a smooth arc U* (iii) *for all $t \in \mathcal{C}$, $w(t) = w(\bar{t})$.*

For the walks such that $p_{10} + p_{-10} + p_{01} + p_{0-1} = 1$, that we have studied in [30], we easily see that $\mathcal{L}$ and $\mathcal{M}$ are simply the circles $\mathcal{C}(0, (p_{0-1}/p_{01})^{1/2})$ and $\mathcal{C}(0, (p_{-10}/p_{10})^{1/2})$ and the functions $p_{01}t + p_{0-1}/t$ and $p_{10}t + p_{-10}/t$ are proper CGF.

In the general case, it is very pleasant to notice that we still have the existence and even the explicit expression of suitable CGF for the curves $\mathcal{L}$ and $\mathcal{M}$. The following result is due to [9]. Define for $t \in \mathbb{C}$

$$(16) \qquad w(t) = \wp_{1,3}\left(-\frac{\omega_1 + \omega_2}{2} + x^{-1}(t)\right),$$

where the ω_i, $i = 1, 2, 3$ are defined in (9), $\wp_{1,3}$ is the classical Weierstrass function associated to the periods ω_1 and ω_3, x^{-1} is the reciprocal function of the uniformization built in Subsection 2.3, we recall that it was $x(\omega) = x_4 + d'(x_4)/(\wp_{1,2}(\omega) - d''(x_4)/6)$ if $x_4 \neq \infty$ and $x(\omega) = (6\wp_{1,2}(\omega) - d''(0))/d'''(0)$ if $x_4 = \infty$, $\wp_{1,2}$ being the Weierstrass function with periods ω_1 and ω_2.

Then w is single-valued and meromorphic on $\mathcal{G}_{\mathcal{M}}$, continuous up to its boundary and establishes a conformal mapping of the domain $\mathcal{G}_{\mathcal{M}}$ onto the domain $\mathbb{C} \setminus U$, where $U = [w(X(y_1)), w(X(y_2))]$. Moreover, on $\mathcal{G}_{\mathcal{M}}$, w has one pole of order one, it is at x_2.

PROPOSITION 11. — *Let $\mathcal{M}$ be the curve defined in* (14). w, *defined in* (16), *is a CGF for $\mathcal{M}$.*

Proposition 11 and the different properties mentioned above it are proved in [9]. Then, following Subsection 5.4 of this book—though making use of the index lightly different—we obtain the following integral representation of the function h.

PROPOSITION 12. — *Let $\mathcal{M}$ be the curve defined in* (14) *and $\mathcal{M}_u = \mathcal{M} \cap \{t \in \mathbb{C} : \text{Im}(t) \geq 0\}$. The function h admits in $\mathscr{G}_{\mathcal{M}}$ the following integral representation, the function w being defined in* (16):

$$h(x) = \frac{1}{2\pi\imath} \int_{\mathcal{M}_u} \left(t^{n_0} Y_0(t)^{m_0} - \bar{t}^{n_0} Y_0(\bar{t})^{m_0} \right) \left(\frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right) \mathrm{d}t.$$

3.2. Integral representation of the generating functions. — In this subsection, we simplify the integral representation of h obtained in Proposition 12 for several reasons: indeed, this formulation does not highlight the singularities of h and hardly allows to obtain the asymptotic of the absorption probabilities, furthermore it makes appear h asymmetrically as an integral on $\mathcal{M}_u$. Before stating, in Theorem 13, the final result, we give the definition:

$$(17) \qquad \mu_{m_0}(t) = \frac{1}{(2a(t))^{m_0}} \sum_{k=0}^{\lfloor (m_0-1)/2 \rfloor} \binom{m_0}{2k+1} d(t)^k (-b(t))^{m_0-(2k+1)}.$$

The function μ_{m_0} is such that for all $t \downarrow [x_1, x_2]$ (resp. $t \uparrow [x_1, x_2]$), $Y_0(t)^{m_0} - \overline{Y_0(t)^{m_0}}$ is equal to $-2\imath(-d(t))^{1/2}\mu_{m_0}(t)$ (resp. $2\imath(-d(t))^{1/2}\mu_{m_0}(t)$).

THEOREM 13. — *The function h admits, on $\mathscr{G}_{\mathcal{M}}$, the following integral representation:*

(18)
$$h(x) = x^{n_0} Y_0(x)^{m_0} + \frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \mu_{m_0}(t) \left(\frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right) \sqrt{-d(t)} \mathrm{d}t.$$

where the function w is defined in (16), μ_{m_0} *in* (17).

The function h appears in (18) at the first sight as the sum of a function holomorphic on $\mathbb{C} \setminus [x_1, x_2] \cup [x_3, x_4]$ and an other function holomorphic on $\mathbb{C} \setminus \{w^{-1}(w([x_1, x_2])) \cup [x_3, x_4]\}$, set that is included in $\mathbb{C} \setminus [x_1, x_2] \cup [x_3, x_4]$. We will later split this representation into two terms (25) and (26), where the first one is holomorphic in $[x_1, x_2]$ by Lemma 18 and the second one is also holomorphic in $[x_1, x_2]$ by (30) and (32), so that this representation gives in fact that h is holomorphic in the neighborhood of $[x_1, x_2]$, what we already knew, since Lemma 1 implies that x_1 and x_2 lye in the unit disc.

Proof. — We start by expressing the integral obtained in Proposition 12 as an integral on a closed contour, namely $\mathcal{M}$. Making the change of variable $t \mapsto \bar{t}$ and using that on $\mathcal{M}$, $w(t) = w(\bar{t})$, we obtain

$$(19) \qquad h(x) = \frac{1}{2\pi i} \int_{\mathcal{M}} t^{n_0} Y_0(t)^{m_0} \left(\frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right) dt.$$

Then we transform (19) into an integral on the cut $[x_1, x_2]$. To do this, start by remarking that the function of two variables $(t, x) \mapsto w'(t)/(w(t) - w(x)) - (x_2 - x)/((x_2 - t)(t - x))$ is continuable into a holomorphic function in $\mathscr{G}_{\mathcal{M}}^2$. This property comes from the fact that w is one to one in $\mathscr{G}_{\mathcal{M}}$ and has a pole of order one at x_2. In particular, the function ϕ, initially well defined on $\mathscr{G}_{\mathcal{M}}^2 \setminus \{(y, y) : y \in \mathscr{G}_{\mathcal{M}}\}$ by

$$(20) \qquad \phi(t, x) = \frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} - \frac{x}{t(t - x)},$$

is continuable into a holomorphic function in $\mathscr{G}_{\mathcal{M}}^2$, again denoted by ϕ.

Consider now the contour $\mathcal{H}_\epsilon = \mathcal{M}_\epsilon \cup \mathscr{S}_\epsilon^1 \cup \mathscr{S}_\epsilon^2 \cup \mathscr{C}_\epsilon^1 \cup \mathscr{C}_\epsilon^2 \cup \mathscr{D}_\epsilon^1 \cup \mathscr{D}_\epsilon^2$, represented in Figure 5.
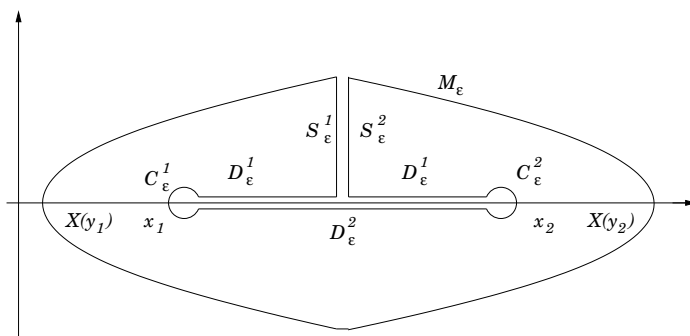
FIGURE 5. Contour of integration

A consequence of the holomorphy of ϕ in $\mathscr{G}_{\mathcal{M}}^2$ is that for all $x \in \mathscr{G}_{\mathcal{M}}$:

$$\int_{\mathcal{H}_\epsilon} t^{n_0} Y_0(t)^{m_0} \phi(t, x) \, dt = 0.$$

In particular, letting $\epsilon \to 0$ and using the definition of μ_{m_0} given in (17), we obtain that:

$$(21) \qquad \frac{1}{2\pi i} \int_{\mathcal{M}} t^{n_0} Y_0(t)^{m_0} \phi(t, x) \, dt = \frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \mu_{m_0}(t) \phi(t, x) \, dt.$$

Furthermore, the residue theorem implies that for all x in the bounded domain delimited by $\mathcal{H}_\epsilon$:

$$\frac{1}{2\pi\imath}\int_{\mathcal{H}_\epsilon}\frac{t^{n_0-1}Y_0\left(t\right)^{m_0}}{t-x}\mathrm{d}t = x^{n_0-1}Y_0\left(x\right)^{m_0}.$$

So, letting $\epsilon\to 0$ and using the definition of μ_{m_0} yield:

$$(22)\qquad \frac{1}{2\pi\imath}\int_{\mathcal{M}}\frac{t^{n_0-1}Y_0\left(t\right)^{m_0}}{t-x}\mathrm{d}t = x^{n_0-1}Y_0\left(x\right)^{m_0} + \frac{1}{\pi}\int_{x_1}^{x_2}\frac{t^{n_0-1}\mu_{m_0}\left(t\right)}{t-x}\mathrm{d}t.$$

Note that to obtain (21) and (22) we have used that the integral on $\mathcal{S}_\epsilon^1\cup\mathcal{S}_\epsilon^2$ of a function holomorphic in the neighborhood of $\mathcal{S}_\epsilon^1\cup\mathcal{S}_\epsilon^2$ goes to zero with ϵ, since the two contours $\mathcal{S}_\epsilon^1$ and $\mathcal{S}_\epsilon^2$ get closer of the same contour but covered in the two opposite directions. For $\mathcal{C}_\epsilon^1$ and $\mathcal{C}_\epsilon^2$, we have used that the integral of a function integrable goes to zero as the length of the contour goes to zero. Finally, Theorem 13 follows from (20), (21) and (22). □

In the particular case $\Delta = 0$ and $x_4 > 0$, we can quite simplify the integral representation found in Theorem 13. We have the following result, already mentioned but not proved in [30]:

PROPOSITION 14. — *Suppose that $\Delta = 0$ and that $x_4 > 0$. Then we have the equality:*

(23)
$$h\left(x\right) = \frac{x}{\pi}\int_{x_3}^{x_4}\left(t^{n_0}-\sigma\left(t\right)^{n_0}\right)\frac{\mu_{m_0}\left(t\right)}{t\left(t-x\right)}\sqrt{-d\left(t\right)}\mathrm{d}t + xP_\infty\left(x\mapsto x^{n_0-1}Y_0\left(x\right)^{m_0}\right)\left(x\right),$$

where if f is a function meromorphic at infinity, $P_\infty(f)$ denotes the principal part at infinity of f, that is to say the polynomial part of the Laurent expansion at infinity of f, $\sigma(t) = (l_1 + l_2)/2 + ((l_2 - l_1)/2)^2/(t - (l_1 + l_2)/2)$, $l_1 < 0$ and $l_2 > 0$ being the roots of the second degree polynomial $b(x)a'(x) - b'(x)a(x)$. In particular, the probabilities of absorption are in this case easily made explicit, by expanding in (23) $1/(t - x)$ according to the powers of x.

The proof of Proposition 14 is postponed to the end of Section 4, since necessary notations introduced there and some facts shown in the proofs of Lemmas 18 and 19 will shorten it.

Of course, by a similar analysis, we obtain integral representations of the function $\tilde{h}$. Also, to get the quantity h_{00}--as yet unknown—we evaluate Equation (4) at some (x,y) in $\{(x,y)\in\mathbb{C}^2 : |x| < 1, |y| < 1, Q(x,y) = 0\}$, for example $(1-\epsilon, Y_0(1-\epsilon))$, where ϵ is small enough, see Lemma 2.

4. Asymptotic of the absorption probabilities

In this part, we will study the asymptotic of the absorption probabilities, defined in (1). For that purpose we need to study the properties of the function w defined in (16), notably in relation with the finiteness of the group $\mathcal{H}$: we will have to distinguish the cases for which $\omega_2/\omega_3 \in \mathbb{N}$ from the other cases.

PROPOSITION 15. — *The function w, defined on $\mathscr{G}_{\mathscr{M}}$ by (16), can be continued ; this continuation is meromorphic on $\mathbb{C}$ if $\omega_2/\omega_3 \in \mathbb{N}$, meromorphic on $\mathbb{C} \setminus [x_3, x_4]$ and algebraic in the neighborhood of $[x_3, x_4]$ if $\omega_2/\omega_3 \notin \mathbb{N}$. In every case, w has a simple pole at x_2 and $\lfloor \omega_2/(2\omega_3) \rfloor$ double poles at points lying in $]x_2, x_3[\cap(\mathbb{C} \setminus \mathscr{G}_{\mathscr{M}})$. The behavior of w at x_3 depends strongly on the group $\mathcal{H}$:*

  (i) *Suppose first that $\omega_2/\omega_3 \in 2\mathbb{N}$. Then w has a simple pole at x_3.*

  (ii) *Suppose now that $\omega_2/\omega_3 \in 2\mathbb{N} + 1$. Then w is holomorphic at x_3.*

  (iii) *Suppose at last that $\omega_2/\omega_3 \notin \mathbb{N}$. Then w has an algebraic singularity at x_3, and more precisely in the neighborhood of x_3, w can be written as $w_1(t) + w_2(t)(x_3 - t)^{1/2}$, where w_1 and w_2 are holomorphic functions in the neighborhood of x_3 and $w_2(x_3) \neq 0$.*

Proof. — The explicit formula (16) of w shows that we have to study the reciprocal function of the uniformization $x(\omega)$. With Subsection 2.3 we get $x(\omega) = t$ if and only if $\wp_{1,2}(\omega) = f(t)$, where if $x_4 \neq \infty$ then $f(t) = d''(x_4)/6 + d'(x_4)/(t - x_4)$ and if $x_4 = \infty$ then $f(t) = (d''(0) + d'''(0)t)/6$. Moreover, by construction, in both cases, $f(X(y_1)) = \wp_{1,2}((\omega_2 + \omega_3)/2)$, $f(X(y_2)) = \wp_{1,2}((\omega_1 + \omega_2 + \omega_3)/2)$, $f(x_1) = \wp_{1,2}(\omega_2/2)$, $f(x_2) = \wp_{1,2}((\omega_1 + \omega_2)/2)$ and $f(\mathcal{M}) = \wp_{1,2}([(\omega_2 + \omega_3)/2, (\omega_2 + \omega_3)/2 + \omega_1])$. In particular, f is an automorphism of $\mathbb{C}$ and maps $\mathscr{G}_{\mathscr{M}}$ onto $]\omega_2/2, \omega_2/2 + \omega_3/2[\times]0, \omega_1/\imath[$, see Figure 3.

Recall that on a fundamental parallelogram $[0, \omega_2[\times[0, \omega_1/\imath[$, $\wp_{1,2}$ takes each value twice. Also, in $[0, \omega_2/2[\times[0, \omega_1/\imath[$ or in $[\omega_2/2, \omega_2[\times[0, \omega_1/\imath[$ $\wp_{1,2}$ is one to one. For these reasons and since $\omega_3 < \omega_2$, we obtain the existence of a function $t \mapsto \omega(t)$ defined on $\mathbb{C}$, two-valued for $t \in \mathbb{R} \setminus [x_2, x_3]$, one-valued everywhere else, that verifies, for all $t \in \mathbb{C}$, $\wp_{1,2}(\omega(t)) = f(t)$; moreover $\omega(\mathbb{C}) = [\omega_2/2, \omega_2] \times [0, \omega_1/\imath]$.

We show now that though ω is two-valued on $\mathbb{R} \setminus [x_2, x_3]$, w is single-valued on $\mathbb{C} \setminus [x_3, x_4]$. We do this by studying precisely the equation $\wp_{1,2}(\omega) = f(t)$ at points t where it has more than one solution:

  – For $t \in [x_4, x_1]$, the two values of $\omega(t)$, say $\omega_1(t)$ and $\omega_2(t)$, verify: $\omega_1(t) \in [\omega_2/2, \omega_2]$, $\omega_2(t) \in [\omega_1 + \omega_2/2, \omega_1 + \omega_2]$ and $\omega_2(t) - \omega_1(t) = \omega_1$.

  – For $t \in [x_1, x_2]$, the two values of $\omega(t)$ are such that: $\omega_1(t), \omega_2(t) \in [\omega_2/2, \omega_2/2 + \omega_1]$ and $\omega_1(t) - (\omega_1/2 + \omega_2/2) = (\omega_1/2 + \omega_2/2) - \omega_2(t)$.

  – For $t \in [x_3, x_4]$, the two values of $\omega(t)$ verify: $\omega_1(t), \omega_2(t) \in [\omega_2, \omega_2 + \omega_1]$ and $\omega_1(t) - \omega_2/2 = \omega_2/2 - \omega_2(t)$.

Using respectively the facts that $\wp_{1,2}$ is even and ω_1 periodic, w is in fact single-valued on $[x_1, x_2]$ and $[x_4, x_1]$ respectively, hence on $\mathbb{C} \setminus [x_3, x_4]$.

To show that w has a simple pole at x_2, we can for instance use an explicit expression of $\omega(t)$. For example, for all $t \in [x_2, X(y_2)]$,

$$\omega(t) = \frac{\omega_1 + \omega_2}{2} + \int_{f(x_2)}^{f(t)} \frac{dz}{\sqrt{4\,(z - f(x_1))\,(z - f(x_2))\,(z - f(x_3))}}.$$

Making an expansion of this quantity in the neighborhood of zero, using that $\wp_{1,3}(\omega) = 1/\omega^2 + \mathcal{O}(\omega^2)$ and that $\wp_{1,3}$ is even on $\mathbb{C}$, as well as (16), we obtain that

$$w(t) = \frac{(f(x_1) - f(x_2))\,(f(x_2) - f(x_3))}{f'(x_2)\,(x_2 - t)} + \widehat{w}(t),$$

where $\widehat{w}$ is holomorphic in the neighborhood of x_2.

We will now study the behavior of the function w in the neighborhood of x_3. Define $R = [0, \omega_3] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$ and note that $\omega(\mathbb{C}) - (\omega_1 + \omega_2)/2 = [0, \omega_2/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$.

We will consider separately the three cases $\Delta = 0$, $\Delta < 0$ and $\Delta > 0$. First of all let us show that $\Delta < 0$ (resp. $\Delta = 0$, $\Delta > 0$) implies $\omega_3 > \omega_2/2$ (resp. $\omega_3 = \omega_2/2$, $\omega_3 < \omega_2/2$). We already know from Subsections 2.3 and 2.4 that $\Delta = 0$ is equivalent to $\omega_2/\omega_3 = 2$. As a consequence, we obtain that to prove that $\Delta < 0$ (resp. $\Delta > 0$) is equivalent to $\omega_3 > \omega_2/2$ (resp. $\omega_3 < \omega_2/2$), it suffices to prove that there exists *one* walk verifying simultaneously $\Delta < 0$ and $\omega_3 > \omega_2/2$ (resp. $\Delta > 0$ and $\omega_3 < \omega_2/2$). Indeed, using the continuity of ω_2, ω_3 and Δ w.r.t. the parameters p_{ij} and the intermediate value theorem, we obtain the results for *all* walks. We have already seen that the second walk of Figure 2 (i.e. $p_{-11} + p_{10} + p_{0-1} = 1$) is such that $\omega_2/\omega_3 = 3$. Moreover, in this case, $\Delta = p_{-11}p_{10}p_{0-1} > 0$. Also, we can verify by a direct calculation starting from (9) that the walk $p_{-1-1} + p_{10} + p_{01} = 1$ verifies $\omega_2/\omega_3 = 3/2$; moreover, for this walk $\Delta = -p_{-1-1}p_{10}p_{01} < 0$.

(i) Suppose that $\Delta = 0$. Then $\omega_2/\omega_3 = 2$ and $R = \omega(\mathbb{C}) - (\omega_1 + \omega_2)/2$, so that by the same analysis as the one done just above for x_2, we find that w has a pole of order one at x_3.

(ii) Suppose now that $\Delta < 0$. Then $\omega_3 > \omega_2/2$ and $\omega(\mathbb{C}) - (\omega_1 + \omega_2)/2$ is strictly included in R, in such a way that w has no poles except at x_2 and is two-valued on $[x_3, x_4]$, for any (finite or infinite) order of the group $\mathcal{H}$.

(iii) Suppose at last that $\Delta > 0$. This implies $\omega_3 < \omega_2/2$, thus $\omega(\mathbb{C}) - (\omega_1 + \omega_2)/2$ contains strictly R. Moreover, we can write, with $n = \lfloor \omega_2/(2\omega_3) \rfloor$:

$$\omega(\mathbb{C}) - \frac{\omega_1 + \omega_2}{2} = \bigcup_{k=0}^{n-1} (k\omega_3 + R) \cup \widehat{R}, \qquad \widehat{R} = [n\omega_3/2, \omega_2/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)].$$

This equality, added to the fact that $\wp_{1,3}$ has, on the fundamental parallelogram $[0, \omega_3[\times[0, \omega_1/\imath[$, only one pole, at zero and of order two, shows that w has $\lfloor \omega_2/(2\omega_3) \rfloor$ double poles at points lying in $]x_2, x_3[\cap(\mathbb{C} \setminus \mathscr{G}_{\mathscr{M}})$.

Consider the particular case $\omega_2/\omega_3 \in 2\mathbb{N}$. Then the rectangle $\hat{R}$ is reduced to one point and $\omega([x_3, x_4]) - (\omega_1 + \omega_2)/2$ is congruent to $[-\omega_1/(2\imath), \omega_1/(2\imath)]$, so that for the same reasons as in the case $\Delta = 0$, we see that w has a simple pole at x_3.

Consider next the other particular case $\omega_2/\omega_3 \in 2\mathbb{N} + 1$. Then $\omega([x_3, x_4]) - (\omega_1 + \omega_2)/2$ is congruent to $\omega_3/2 + [-\omega_1/(2\imath), \omega_1/(2\imath)]$, in the neighborhood of which $\wp_{1,3}$ is holomorphic, so w has no pole at x_3 in this case.

If $\omega_2/\omega_3 \notin \mathbb{N}$, then $\omega([x_3, x_4]) - (\omega_1 + \omega_2)/2$ is congruent neither to $\omega_3/2 + [-\omega_1/(2\imath), \omega_1/(2\imath)]$ nor to $[-\omega_1/(2\imath), \omega_1/(2\imath)]$, in particular w has no pole at x_3.

Consider now more global aspects and show that w is meromorphic on $\mathbb{C}$ if and only if $\omega_2/\omega_3 \in \mathbb{N}$.

Recall from the beginning of the proof that for $t \in [x_3, x_4]$, the two values of $\omega(t)$ are such that $\omega_1(t), \omega_2(t) \in [\omega_2, \omega_2 + \omega_1]$ and $\omega_1(t) - \omega_2/2 = \omega_2/2 - \omega_2(t)$. In addition, we have shown just above that if $\omega_2/\omega_3 \in 2\mathbb{N}$ (resp. $\omega_2/\omega_3 \in 2\mathbb{N} + 1$), then $\omega([x_3, x_4]) - (\omega_1 + \omega_2)/2$ is congruent to $[-\omega_1/(2\imath), \omega_1/(2\imath)]$ (resp. $\omega_3/2 + [-\omega_1/(2\imath), \omega_1/(2\imath)]$). But $\omega \mapsto \wp_{1,3}(\omega)$ and $\omega \mapsto \wp_{1,3}(\omega_3/2 + \omega)$ are even functions, so that in both cases w is single-valued and meromorphic in the neighborhood of $[x_3, x_4]$.

Suppose now that $\omega_2/\omega_3 \notin \mathbb{N}$. Since $\omega(x_3) = \omega_1/2 + \omega_2$, then $w(t) = \wp_{1,3}(\omega_2/2 + (\omega(t) - \omega(x_3)))$. On the other hand, if ω is close to zero, we have:

$$\wp_{1,3}\left(\omega_2/2 + \omega\right) = \wp_{1,3}\left(\omega_2/2\right) + \sum_{k=1}^{+\infty} \frac{\wp_{1,3}^{(2k)}\left(\omega_2/2\right)}{(2k)!} \omega^{2k} + \omega \sum_{k=0}^{+\infty} \frac{\wp_{1,3}^{(2k+1)}\left(\omega_2/2\right)}{(2k+1)!} \omega^{2k}.$$

Also, by a similar calculation as the one done when we have studied the behavior of w in the neighborhood of x_2, we obtain:

$$\omega\left(t\right) - \omega\left(x_3\right) = \int_{f(t)}^{f(x_3)} \frac{\mathrm{d}z}{\sqrt{(z - f\left(x_1\right))(z - f\left(x_2\right))(z - f\left(x_3\right))}}$$

$$= -\left(\frac{-f'\left(x_3\right)\left(x_3 - t\right)}{4\left(f\left(x_3\right) - f\left(x_2\right)\right)\left(f\left(x_3\right) - f\left(x_1\right)\right)}\right)^{1/2} \left(1 + (t - x_3)\,\breve{w}\left(t\right)\right),$$

where $\breve{w}$ is holomorphic in a neighborhood of x_3. Thus, in a neighborhood of x_3, we can write $w(t)$ as the sum $w(t) = w_1(t) + w_2(t)(x_3 - t)^{1/2}$, where w_1 and w_2 are holomorphic in a neighborhood of x_3 and

$$(24) \qquad w_1\left(x_3\right) = \wp_{1,3}\left(\omega_2/2\right),$$

$$w_2\left(x_3\right) = -\left(\frac{-f'\left(x_3\right)}{4\left(f\left(x_3\right) - f\left(x_2\right)\right)\left(f\left(x_3\right) - f\left(x_1\right)\right)}\right)^{1/2} \wp_{1,3}'\left(\omega_2/2\right).$$

This closes the proof of Proposition 15. □

REMARK 16. — *As a consequence of Proposition 15, the CGF w is a rational function in the particular cases $\omega_2/\omega_3 \in \mathbb{N}$. The theory of transformation of elliptic functions gives a constructive way to write the expression of w on $\mathbb{C}$. Indeed, if we note $\wp_{1,2,n}$ the Weierstrass function associated to the periods ω_1 and ω_2/n, then the following formula, that can be found in [31],*

$$\wp_{1,2,n}(\omega) = \sum_{k=1}^{n-1} \left(\wp_{1,2}(\omega + k\omega_2/n) - \wp_{1,2}(k\omega_2/n) \right) + \wp_{1,2}(\omega),$$

allows to express $\wp_{1,3}(\omega)$ in terms of $\wp_{1,2}(\omega)$ and $\wp'_{1,2}(\omega)$, using also addition formulas for the Weierstrass function $\wp_{1,2}$. In practice, that is what we use. For instance, after some calculations, this method gives a suitable CGF for the second walk of Figure 2, with transition probabilities verifying $p_{-11} + p_{10} + p_{0-1} = 1$:

$$w(t) = \frac{t}{(t - x_2)\left(t - (p_{-11}p_{0-1}/(p_{10}^2 x_2))^{1/2}\right)^2}.$$

We are now ready to state and prove the main result of Section 4 on the asymptotic of the absorption probabilities.

Notation. — Throughout the whole paper, for two sequences $(a_k)_k$ and $(b_k)_k$ we will write $a_k \sim_{k\to+\infty} b_k$ or $a_k \sim b_k$ if $\lim_{k\to+\infty} a_k/b_k = 1$.

THEOREM 17. — *We recall that for $k \in \mathbb{N}^*$, h_k denotes $\mathbb{P}_{(n_0,m_0)}$ (to be absorbed at $(k,0)$).*

Suppose first that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$.
• *If $\omega_2/\omega_3 \in 2\mathbb{N}$, then $h_k \sim_{k\to+\infty} h_{1,k} + h_{2,k}$, $h_{1,k}$ and $h_{2,k}$ being defined in (27) and (28).*
• *If $\omega_2/\omega_3 \in 2\mathbb{N} + 1$, then $h_k \sim_{k\to+\infty} h_{1,k}$, $h_{1,k}$ being defined in (27).*
• *If $\omega_2/\omega_3 \notin \mathbb{N}$, then $h_k \sim_{k\to+\infty} h_{1,k} + h_{2,k}$, $h_{1,k}$ and $h_{2,k}$ being defined in (27) and (29).*

Suppose now that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$.
• *If k and $n_0 + m_0$ don't have the same parity, then $h_k = 0$, since $(k,0)$ is not reachable.*
• *If they have the same parity, then we obtain the asymptotic of h_k by multiplying by two the one of h_k in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$, and this in the three cases $\omega_2/\omega_3 \in 2\mathbb{N}$, $\omega_2/\omega_3 \in 2\mathbb{N} + 1$ and $\omega_2/\omega_3 \notin \mathbb{N}$.*

By Theorem 13 and thanks to (20), h can be split as $h = h_1 + h_2$, where

$$(25) \qquad h_1(x) = x^{n_0} Y_0(x)^{m_0} + \frac{x}{\pi} \int_{x_1}^{x_2} \frac{t^{n_0-1} \mu_{m_0}(t)}{t - x} \sqrt{-d(t)} \mathrm{d}t,$$

$$(26) \qquad h_2(x) = \frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \phi(t, x) \mu_{m_0}(t) \sqrt{-d(t)} \mathrm{d}t.$$

Theorem 17 is an immediate consequence of Lemmas 18 and 19 below on the asymptotic behavior of Taylor coefficients of h_1 and h_2 respectively.

LEMMA 18. — *The function h_1, initially defined in $\mathbb{C} \backslash [x_1, x_2] \cup [x_3, x_4]$ by (25), admits a holomorphic continuation in $\mathbb{C} \setminus [x_3, x_4]$. We still note h_1 the continuation of this function and we set $h_1(x) = \sum_{k=0}^{+\infty} h_{1,k} x^k$. Suppose first that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$; then*

$$(27) \quad h_{1,k} \sim \frac{m_0 x_3^{n_0+1/2}}{4\sqrt{\pi}} \left(\frac{-d'(x_3)}{a(x_3) c(x_3)} \right)^{1/2} \left(\frac{c(x_3)}{a(x_3)} \right)^{m_0/2} \frac{1}{k^{3/2} x_3^k}, \qquad k \to \infty.$$

Suppose now that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$; then the process can hit $(k, 0)$ if and only if k and $n_0 + m_0$ have the same parity. Therefore $h_k = 0$ if k and $n_0 + m_0$ don't have the same parity. If they have the same parity, then $h_{1,k}$ is equivalent to two times the right member of (27).

LEMMA 19. — *The function h_2, defined in (26), is holomorphic in $\mathbb{C} \setminus (w^{-1}([x_1, x_2]) \setminus [x_1, x_2])$ and we set $h_2(x) = \sum_{k=0}^{+\infty} h_{2,k} x^k$. Suppose first that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$; in addition,*

(i) *If $\omega_2/\omega_3 \in 2\mathbb{N}$, then:*
$$(28)$$
$$h_{2,k} \sim -\frac{m_0 x_2^{n_0} x_3^{1/2}}{4\sqrt{\pi}} \left(\frac{d'(x_2)}{a(x_2) c(x_2)} \right)^{1/2} \left(-\frac{\mathrm{Res}(w, x_2)}{\mathrm{Res}(w, x_3)} \right)^{1/2} \left(\frac{c(x_2)}{a(x_2)} \right)^{m_0/2} \frac{1}{k^{3/2} x_3^k},$$

   $k \to \infty$, *where for $i = 2, 3$, $\mathrm{Res}(w, x_i)$ denotes the residue of the function w at x_i, where from Proposition 15 it has a pole (of order one).*

(ii) *If $\omega_2/\omega_3 \in 2\mathbb{N} + 1$, then $h_{2,k} = o(h_{1,k})$.*

(iii) *If $\omega_2/\omega_3 \notin \mathbb{N}$, then the following asymptotic holds as k goes to infinity:*
$$(29)$$
$$h_{2,k} \sim \frac{\sqrt{x_3} \wp_{1,3}'(\omega_2/2) \sqrt{-f'(x_3)}}{\sqrt{\pi (f(x_3) - f(x_2)) (f(x_3) - f(x_1))}} \left(\int_{x_1}^{x_2} \frac{w'(t) t^{n_0} \mu_{m_0}(t) \sqrt{-d(t)}}{(w(t) - \wp_{1,3}(\omega_2/2))^2} \mathrm{d}t \right) \frac{1}{k^{3/2} x_3^k},$$

   where if $x_4 \neq \infty$ then $f(t) = d''(x_4)/6 + d'(x_4)/(t - x_4)$ and if $x_4 = \infty$ then $f(t) = (d''(0) + d'''(0)t)/6$.

Suppose now that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$; if k and $n_0 + m_0$ don't have the same parity, then $h_{2,k} = 0$. If they have the same parity and if $\omega_2/\omega_3 \in 2\mathbb{N}$ (resp. $\omega_2/\omega_3 \in 2\mathbb{N}+1$, $\omega_2/\omega_3 \notin \mathbb{N}$), then the asymptotic of $h_{2,k}$ is given by two

times the right member of (28) *(resp. is negligible w.r.t.* (27)*, is given by two times the right member of* (29)*).*

Proof of Lemma 18. — Suppose first that $x_4 > 0$. Apply the residue theorem at infinity on the contour $\mathscr{C}_\epsilon$, represented at the left of Figure 6. So we obtain that for all x inside the infinite domain delimited by $\mathscr{C}_\epsilon$,

$$\frac{1}{2\pi\imath} \int_{\mathscr{C}_\epsilon} \frac{t^{n_0-1} Y_0(t)^{m_0}}{t-x} \mathrm{d}t = x^{n_0-1} Y_0(x)^{m_0} - P_\infty\left(x \mapsto x^{n_0-1} Y_0(x)^{m_0}\right)(x),$$

where $P_\infty(x \mapsto x^{n_0-1} Y_0(x)^{m_0})$ is the principal part at infinity of the meromorphic function at infinity $x \mapsto x^{n_0-1} Y_0(x)^{m_0}$, see Proposition 14 for more details about the principal part at infinity of a function. Furthermore, by definition of μ_{m_0},

$$\lim_{\epsilon\to 0} \frac{1}{2\pi\imath} \int_{\mathscr{C}_\epsilon} \frac{t^{n_0-1} Y_0(t)^{m_0}}{t-x} \mathrm{d}t$$
$$= -\frac{1}{\pi} \int_{x_1}^{x_2} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} \mathrm{d}t + \frac{1}{\pi} \int_{x_3}^{x_4} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} \mathrm{d}t.$$

Therefore the function h_1 is just equal to:

(30)
$$h_1(x) = \frac{x}{\pi} \int_{x_3}^{x_4} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} \mathrm{d}t + x P_\infty\left(x \mapsto x^{n_0-1} Y_0(x)^{m_0}\right)(x).$$

Moreover, thanks to Lemma 2, we obtain that the degree of the polynomial defined by the principal part above is equal to n_0 if $p_{1-1} \neq 0$, $-\infty$ if $p_{1-1} = 0$ and $n_0 \leq m_0$, $n_0 - m_0$ if $p_{1-1} = 0$ and $n_0 > m_0$. In any case, if k is larger than this degree, then the following equality holds:

(31)
$$h_{1,k} = \frac{1}{\pi} \int_{x_3}^{x_4} \frac{\mu_{m_0}(t) \sqrt{-d(t)}}{t^{k+1-n_0}} \mathrm{d}t.$$

We can then easily obtain the asymptotic of this integral as k goes to infinity, using Laplace's method, see e.g. [6] page 275. We make an expansion of the numerator of the integrand in (31) in the neighborhood of x_3, we obtain $\mu_{m_0}(t)(-d(t))^{1/2} = \mu_{m_0}(x_3)(-d'(x_3))^{1/2}(t-x_3)^{1/2} + (t-x_3)^{3/2}f(t)$, where f is holomorphic at x_3. Classically, the second term in the previous sum will lead to a negligible contribution, so to get (27), it suffices therefore to use one hand that

$$\int_{x_3}^{x_4} \frac{\sqrt{t-x_3}}{t^k} \mathrm{d}t = \frac{\sqrt{\pi}}{2} \frac{1}{x_3^{k-3/2}} \frac{1}{k^{3/2}} + \mathcal{O}\left(\frac{1}{k^{5/2}}\right),$$

and on an other hand to simplify $\mu_{m_0}(x_3)$, using for this (17) and the fact that $d(x_3) = 0$.
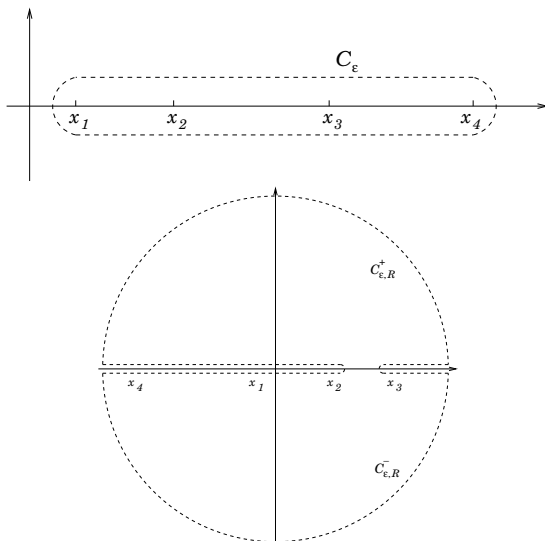
FIGURE 6. Contours of integration in the cases $x_4 > 0$ and $x_4 < 0$ respectively

Suppose now that $x_4 < 0$ and suppose in addition that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$. In this case, Y_0 is no more meromorphic at infinity and has no principal part at infinity, the previous argument does not run anymore. However, we will show that the asymptotic (27) is still correct. To that purpose, fix $R > -x_4$ and apply the classical residue theorem on the contour $\mathscr{C}_{\epsilon,R}$ described in the right side of Figure 6. After that ϵ has gone to zero, we obtain

$$(32) \qquad x^{n_0-1} Y_0(x)^{m_0} = -\frac{1}{\pi} \int_{x_1}^{x_2} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} dt$$

$$+ \frac{1}{\pi} \int_{x_3}^{R} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} dt + f_R(x),$$

where f_R is defined by:

$$f_R(x) = \frac{1}{2\pi i} \int_{\mathscr{C}(0,R)} \frac{t^{n_0-1} Y_0(t)^{m_0}}{t-x} dt + \frac{1}{\pi} \int_{-R}^{x_4} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} dt,$$

The first (resp. the second) Cauchy-type integral in the sum defining f_R is holomorphic in the disc $\mathscr{D}(0, R)$ (resp. $\mathscr{D}(0, -x_4)$) so that f_R is holomorphic in $\mathscr{D}(0, -x_4)$. Moreover, since we have supposed $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$, Lemma 1 yields $x_3 < -x_4$. In particular, this implies that, as k goes to infinity, the kth coefficient of the Taylor series at zero of f_R is $o(1/r^k)$, where

$x_3 < r < -x_4$. These coefficients will be therefore negligible w.r.t. those of

$$\frac{1}{\pi} \int_{x_3}^{R} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} dt.$$

We calculate the asymptotic of the coefficients of the function above using Laplace's method, as in the case $x_4 > 0$, the asymptotic (27) is thus still valid.

If $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$ (which implies, see Lemma 1, that $x_4 = -x_3 < 0$), then the process can reach $(k, 0)$ if and only if k and $n_0 + m_0$ have the same parity. In particular, if k and $n_0 + m_0$ don't have the same parity then $h_k = 0$. Besides, we can show that in this case $h_{1,k} = 0$, using that $[x_4, x_1] = [-x_3, -x_2]$ and the fact that Y_0, Y_1 and w are odd functions. If they have the same parity, then the asymptotic of the coefficients of f_R is no more negligible: indeed, thanks to Lemma 1, $x_3 = -x_4$ and after the use of $\mu_{m_0}(-t) = (-1)^{m_0-1}\mu_{m_0}(t)$, see (17), we obtain:
(33)
$$\frac{1}{\pi} \int_{-R}^{x_4} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t-x} dt = \frac{(-1)^{n_0+m_0}}{\pi} \int_{x_3}^{R} \frac{t^{n_0-1} \mu_{m_0}(t) \sqrt{-d(t)}}{t+x} dt.$$

For this reason, if k and n_0+m_0 have the same parity, we obtain that the asymptotic of $h_{1,k}$ is given by two times (27), using once again Laplace's method. □

Proof of Lemma 19. — This proof is based on the following principle, known as Pringsheim Theorem: the asymptotic of the coefficients of a Taylor series at 0 can be found starting from the precise knowledge of the first singularity of this Taylor series ; here "the first singularity" means the singularity the nearest in modulus from zero. So we have to find the first singularity of the function h_2, defined in (26). To do this, we have to find the singularities of the function ϕ, defined in (20), that appears in the integral defining h_2. *Grosso modo*, ϕ, as a function of two variables, can have singularities of two different kinds: either they are fixed in the sense that for all t (resp. all x) the function ϕ of the variable x (resp. t) has the same singularities, and in this case the set of all the singularities can be written as a direct product, or they are movable in the sense that the set of the singularities can not be written as a direct product. In our case, according to w is meromorphic or not, these two possibilities can arise, eventually simultaneously. Indeed, if w is not meromorphic, x_3 will be a fixed singularity of ϕ, hence also a singularity of h_2, and $\{(t, x) \in \mathbb{C}^2 : w(t) = w(x)\} \setminus \{(y, y) : y \in \mathbb{C}\}$ is the set of movable singularities of ϕ ; in particular if we define $\hat{x} = \inf\{x \in]x_2, +\infty[: w(x) \in w([x_1, x_2])\}$, then $\inf\{x_3, \hat{x}\}$ is the first singularity of h_2.

In our case, we will show that if $\omega_2/\omega_3 \in 2\mathbb{N}$, then $\hat{x} = x_3$, and if $\omega_2/\omega_3 \in 2\mathbb{N} + 1$ or $\omega_2/\omega_3 \notin \mathbb{N}$, then $\hat{x} > x_3$.

We first suppose that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$ and we will say at the end of the proof how to adapt our arguments to the walks having transition probabilities such that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$.

Suppose first that $\omega_2/\omega_3 \in 2\mathbb{N}$. In this case, we obtain $\lim_{x \xrightarrow{>} x_2} w(x) = \lim_{x \xrightarrow{<} x_3} w(x) = -\infty$, thanks to Proposition 15. More generally, from Proposition 15 and its proof we deduce that there exists a holomorphic function σ defined at least in a neighborhood of $[x_2, x_3]$ such that $\sigma \circ \sigma = \mathrm{id}$, $\sigma(x_3) = x_2$ and $w \circ \sigma = w$.

In addition, we have already seen that the function $(t, x) \mapsto w'(t)/(w(t) - w(x)) - (x_2 - x)/((x_2 - t)(t - x))$ is holomorphic in $\mathcal{G}_{\mathcal{M}}^2$, in particular in the neighborhood of (x_2, x_2) and more generally in the neighborhood of $\{(y, y) : y \in \mathcal{G}_{\mathcal{M}}\}$, but it is not holomorphic at (x_2, x_3) and in fact is not holomorphic at every point of $\{(y, \sigma(y)) : y \in \mathcal{G}_{\mathcal{M}}\}$. On the other hand, the function $(t, x) \mapsto w'(t)/(w(t) - w(x)) - (x_2 - x)/((x_2 - t)(t - x)) - \sigma'(t)(x_3 - x)/((x_3 - \sigma(t))(\sigma(t) - x))$ is holomorphic in the neighborhood of $\{(y, \sigma(y)) : y \in \mathcal{G}_{\mathcal{M}}\}$. This is why the function

$$(34) \qquad \widehat{\phi}(t, x) = \phi(t, x) - \frac{\sigma'(t) x}{\sigma(t)(\sigma(t) - x)}$$

is holomorphic in a neighborhood of $[x_1, x_2] \times \{x \in \mathbb{C} : |x| < R\}$ where $R > x_3$ and in particular for all $t \in [x_1, x_2]$, $x \mapsto \widehat{\phi}(t, x)$ is holomorphic on $\{x \in \mathbb{C} : |x| < R\}$. Therefore, the kth coefficient of the Taylor series at zero of the function

$$\frac{1}{\pi} \int_{x_1}^{x_2} \widehat{\phi}(t, x) \mu_{m_0}(t) t^{n_0} \sqrt{-d(t)} \mathrm{d}t$$

is $o(1/R^k)$ with $R > x_3$; in particular it is exponentially negligible w.r.t. to (27) and (28). In other words, it remains to evaluate the contribution of the coefficients of g_2, defined by

$$(35) \qquad g_2(x) = \frac{x}{\pi} \int_{x_1}^{x_2} \frac{\sigma'(t) \mu_{m_0}(t) t^{n_0}}{\sigma(t)(\sigma(t) - x)} \sqrt{-d(t)} \mathrm{d}t$$
$$= -\frac{x}{\pi} \int_{x_3}^{\sigma(x_1)} \frac{\sigma(t)^{n_0} \mu_{m_0}(\sigma(t))}{t(t - x)} \sqrt{-d(\sigma(t))} \mathrm{d}t,$$

where the second equality above comes from the change of variable $u = \sigma(t)$. Then, we apply Laplace's method, and we will obtain (28) as soon as we will have proved that $\sigma'(x_3) = \mathrm{Res}(w, x_2)/\mathrm{Res}(w, x_3)$. To do this, start by differentiating the equality $w(t) = w(\sigma(t))$, we obtain that $\sigma'(x_3) = \lim_{x \to x_3} w'(t)/w'(\sigma(t))$, what implies $\sigma'(x_3) = \lim_{x \to x_3} \mathrm{Res}(w, x_3)(\sigma(t) - x_2)^2/(\mathrm{Res}(w, x_2)(t - x_3)^2)$. Since $\sigma'(x_3) \neq 0$ it follows that $\sigma'(x_3) = \mathrm{Res}(w, x_2)/\mathrm{Res}(w, x_3)$.

Suppose now that $\omega_2/\omega_3 \in 2\mathbb{N}+1$. In this case, since w has a pole at x_2 but is holomorphic at x_3 by Proposition 15, the function ϕ is continuous on $[x_1, x_2] \times \{x \in \mathbb{C} : |x| < R\}$ where $R > x_3$ and for all $t \in [x_1, x_2]$, $x \mapsto \phi(t, x)$ is holomorphic on $\{x \in \mathbb{C} : |x| < R\}$. This is why the kth coefficient of the Taylor series at zero of h_2 is in this case $o(1/R^k)$, with $R > x_3$, that is, in particular $h_{2,k} = o(h_{1,k})$

Consider now the general case, namely $\omega_2/\omega_3 \notin \mathbb{N}$. In Proposition 15, we have shown that in the neighborhood of x_3, w can be written as $w_1(x) + w_2(x)(x_3 - x)^{1/2}$, where w_1 and w_2 are functions holomorphic at x_3, and their values at x_3 are made explicit in (24). In particular, in the neighborhood of x_3, we can write $h_2(x) = f(x) + g(x)(x_3 - x)^{1/2}$, with f and g holomorphic in the neighborhood of x_3, and with (26),

$$(36) \qquad g(x_3) = \frac{w_2(x_3)}{\pi} \int_{x_1}^{x_2} \frac{w'(t)}{(w(t) - w(x_3))^2} t^{n_0} \mu_{m_0}(t) \sqrt{-d(t)} dt.$$

We can now easily find the asymptotic of the coefficients of the Taylor series at 0 of the function h_2, following Pringsheim Theorem, mentioned at the beginning of the proof and summarized below: if $F(z) = \sum_k c_k z^k$ is a function (i) holomorphic in the open disc of radius r (ii) having a holomorphic continuation at every point of the circle of radius r except r (iii) having at r an algebraic singularity in the sense that in the neighborhood of r, F can be written as $F(z) = F_0(z) + \sum_{i=1}^{d} F_i(z)(1 - z/r)^{\theta_i}$ where the F_i, $i \geq 0$, are holomorphic functions in the neighborhood of r, not vanishing at r for $i \geq 1$, the $\theta_1 < \cdots < \theta_d$ are rational but not integer, then the asymptotic of the coefficients of the Taylor series at 0 can easily be calculated: $c_k \sim F_1(r) r^k / (\Gamma(-\theta_1) k^{\theta_1 + 1})$ as $k \to +\infty$, Γ being the classical Gamma function.

So, using Pringsheim Theorem with $F = h_2$––thanks to Corollary 5, or from the current proof, h_2 is continuable holomorphically through every point of the circle $\mathscr{C}(0, x_3)$ except x_3 ; this is here that we use the hypothesis $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$: indeed, under this assumption, h_2 is holomorphic at $-x_3$, $F_1 = g$, $r = x_3$, $F_1(r) = g(x_3)$ written in (36), $\theta_1 = 1/2$ and using the fact that $\Gamma(-1/2) = -2\sqrt{\pi}$, we get immediately the announced asymptotic.

Suppose now that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$, and note that for these walks, we can equally have $\omega_2/\omega_3 \in 2\mathbb{N}$, $\omega_2/\omega_3 \in 2\mathbb{N}+1$ or $\omega_2/\omega_3 \notin \mathbb{N}$. In fact, as in the proof of Lemma 18, we have, as well as the contribution of the point x_3, to take under account the one of the point x_4, equal in this case to $-x_3$. So we do the same analysis, but then we have in addition a term like (35) or (36) with x_4 instead x_3. By doing the change of variable $t \mapsto -t$, as in (33), we obtain that if k and $n_0 + m_0$ don't have the same parity, then the contribution is zero, and if they have, then the asymptotic of $h_{2,k}$ is given by two times (29),

using once again Laplace's method and an adaptation of Pringsheim Theorem for odd and even functions. □

REMARK 20. — *If $\omega_2/\omega_3 \in 2\mathbb{N}+1$, then $\omega_2/2$ is congruent to $\omega_3/2$ and in particular $\wp'_{1,3}(\omega_2/2)$ vanishes, what proves differently that there is in this case a negligible contribution of the integral (26) to the asymptotic of the coefficients of h.*

Theorem 17 can be summarized as follows.

PROPOSITION 21. — *The absorption probabilities h_i admit the following asymptotic as i goes to ∞:*

$$h_i \sim \frac{(-x_3 d'(x_3))^{1/2}}{4\pi^{1/2} a(x_3)} \left[m_0 \left(\frac{c(x_3)}{a(x_3)} \right)^{(m_0-1)/2} x_3^{n_0} - \widetilde{h}' \left(\left(\frac{c(x_3)}{a(x_3)} \right)^{1/2} \right) \right] \frac{1}{i^{3/2} x_3^i}.$$

Proof. — We chose to do the proof in the general case, namely $\omega_2/\omega_3 \notin \mathbb{N}$, since the ideas of the proof in the cases $\omega_2/\omega_3 \in 2\mathbb{N}$ and $\omega_2/\omega_3 \in 2\mathbb{N}+1$ are rather similar.

We will need the following consequence of Proposition 4 and Corollary 5: for all $x \in \mathbb{C} \setminus [x_3, x_4]$,

$$(37) \qquad h(x) = x^{n_0} Y_0(x)^{m_0} - \widetilde{h}(Y_0(x)) - h_{00}.$$

Then, use one hand the explicit expression of h, given in (18) of Theorem 13, and on the other hand (37) just above. We obtain:

$$\widetilde{h}(Y_0(x)) + h_{00} = -\frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \mu_{m_0}(t) \left(\frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right) \sqrt{-d(t)} \mathrm{d}t.$$

Differentiating this equality w.r.t. x, we get:

$$\widetilde{h}'(Y_0(x)) = -\frac{w'(x)}{Y_0'(x)} \frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \mu_{m_0}(t) \frac{w'(t)}{(w(t) - w(x))^2} \sqrt{-d(t)} \mathrm{d}t.$$

Since we have supposed $\omega_2/\omega_3 \notin \mathbb{N}$, we know from Proposition 15 that w has at x_3 an algebraic singularity, so that we can write w as the sum $w(x) = w_1(x) + w_2(x)(x_3 - x)^{1/2}$, where w_1 and w_2 are holomorphic at x_3, moreover the values of $w_1(x_3)$ and $w_2(x_3)$ are given in (24). In addition, using the explicit expression of Y_0, we obtain that $d(x)^{1/2} Y_0'(x)$ goes to $-d'(x_3)/(4a(x_3))$ as x goes to x_3. Also $(x_3 - x)^{1/2} w'(x)$ goes to $-w_2(x_3)/2$ as x goes to x_3. So we obtain:

$$\widetilde{h}'(Y_0(x_3)) = \widetilde{h}' \left(\left(\frac{c(x_3)}{a(x_3)} \right)^{1/2} \right)$$

$$= \frac{2a(x_3) w_2(x_3)}{(-d'(x_3))^{1/2}} \frac{1}{\pi} \int_{x_1}^{x_2} t^{n_0} \mu_{m_0}(t) \frac{w'(t)}{(w(t) - w(x_3))^2} \sqrt{-d(t)} \mathrm{d}t.$$

With this last equality, (27) and (29), we have proved Proposition 21.   □

We are now ready to prove Proposition 14, where we have given an explicit expression of h in the particular case $\Delta = 0$ and $x_4 > 0$.

For this we need two preliminary results, stated in Lemmas 22 and 23. Before, recall that in [9], the authors find the explicit expression of the curve $\mathcal{M}$, defined in (14) ; more precisely they make explicit q, q_1 and q_2, three polynomials of degree two, such that $\mathcal{M}$ is equal to $\{u + iv \in \mathbb{C} : q(u,v)^2 - q_1(u,v)q_2(u,v) = 0\}$. In [30] is observed that these polynomials can be written as:

(38)
$$\begin{vmatrix} p_{11} & 1 & p_{1-1} \\ p_{01} & -2u & p_{0-1} \\ p_{-11} & u^2+v^2 & p_{-1-1} \end{vmatrix}, \quad \begin{vmatrix} 1 & p_{10} & p_{1-1} \\ -2u & -1 & p_{0-1} \\ u^2+v^2 & p_{-10} & p_{-1-1} \end{vmatrix}, \quad \begin{vmatrix} p_{11} & p_{10} & 1 \\ p_{01} & -1 & -2u \\ p_{-11} & p_{-10} & u^2+v^2 \end{vmatrix}.$$

LEMMA 22. — *If $\Delta = 0$, then there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$ such that $q = \alpha q_1$ and $q_2 = \beta q_1$.*

Proof. — Take the following notations: $q(u,v) = \eta_{q,2}(u^2 + v^2) + \eta_{q,1}u + \eta_{q,0}$, $q_1(u,v) = -\eta_{q_1,2}(u^2+v^2)+\eta_{q_1,1}u+\eta_{q_1,0}$, $q_2(u,v) = \eta_{q_2,2}(u^2+v^2)+\eta_{q_2,1}u-\eta_{q_2,0}$. Of course, we get the explicit expression of these coefficients by expanding the determinants (38). In particular, we immediately notice that $\eta_{q_1,2}$, $\eta_{q_1,0}$, $\eta_{q_2,2}$ and $\eta_{q_2,0}$ are positive. Also, from straightforward calculations we obtain

$$\frac{\eta_{q_1,0}}{\eta_{q_1,2}} - \frac{\eta_{q_2,0}}{\eta_{q_2,2}} = \frac{\Delta}{\eta_{q_1,2}\eta_{q_2,2}}, \qquad \frac{\eta_{q_1,1}}{\eta_{q_1,2}} - \frac{\eta_{q_2,1}}{\eta_{q_2,2}} = \frac{2p_{10}\Delta}{\eta_{q_1,2}\eta_{q_2,2}}.$$

This immediately yields that if $\Delta = 0$ then there exists $\beta \neq 0$ such that $q_2 = \beta q_1$.

In turn, this fact and the equality $q(u,v) = p_{01}q_2(u,v) + p_{0-1}q_1(u,v) + \Delta u$, which is a Cramer relationship, consequence of (38), entail that there exists α, eventually zero, such that $q = \alpha q_1$.   □

Let $l_1 < 0$ and $l_2 > 0$ be the two roots of $q_1(x,0)$. Then, thanks to Lemma 22, $\mathcal{M}$ is simply the circle of center $\gamma = (l_2 + l_1)/2$ and radius $\rho = (l_2 - l_1)/2$.

The following result, that is a generalization of a straightforward result concerning the walks verifying $p_{10} + p_{-10} + p_{01} + p_{0-1} = 1$, will also be useful in the proof of Proposition 14.

LEMMA 23. — *Suppose here that $\Delta = 0$ and denote by $l_1 < 0$ and $l_2 > 0$ the two roots of $q_1(x,0)$, $\gamma = (l_2+l_1)/2$ and $\rho = (l_2-l_1)/2$. Then $(x_2-\gamma)(x_3-\gamma) = (x_1 - \gamma)(x_4 - \gamma) = \rho^2$.*

Proof. — Define $\tilde{l}_i = Y(x_i) = \epsilon_i(|c(x_i)/a(x_i)|)^{1/2}$, $i \in \{1, \ldots, 4\}$, with $\epsilon_2 = \epsilon_3 = -\epsilon_1 = -\epsilon_4 = 1$. From (47) of Subsection 5.1, it can be deduced that they are the four roots of $\tilde{P}_l(y) = \tilde{q}(y,0)^2 - \tilde{q}_1(y,0)\tilde{q}_2(y,0)$, the polynomial $\tilde{q}$ (resp. $\tilde{q}_1$, $\tilde{q}_2$) being obtained by making in q (resp. q_1, q_2), defined in (38), the change of parameters $p_{ij} \mapsto p_{ji}$. Of course, we could prove an analogous of Lemma 22 that would be: if $\Delta = 0$ then $\tilde{P}_l$ has two double roots ; in particular $\tilde{l}_2 = \tilde{l}_3$, so that, a and c being positive for positive values of the argument, $c(x_2)/a(x_2) = c(x_3)/a(x_3)$. In addition, by a direct calculation, we notice that $c(x)/a(x) = c(y)/a(y)$ if and only if x and y are joined together by:

$$(p_{1-1}p_{01} - p_{0-1}p_{11})\,xy + (p_{1-1}p_{-11} - p_{-1-1}p_{11})\,(x+y)$$
$$+ (p_{0-1}p_{-11} - p_{-1-1}p_{01}) = 0.$$

• If $p_{1-1}p_{01} - p_{0-1}p_{11} \neq 0$, then the polynomial $q(x,0)$ is non zero and thanks to Lemma 22, l_1 and l_2 are the two roots of $q(x,0)$; in other words they are the roots of $(p_{1-1}p_{01} - p_{0-1}p_{11})x^2 + 2(p_{1-1}p_{-11} - p_{-1-1}p_{11})x + (p_{0-1}p_{-11} - p_{-1-1}p_{01})$. Therefore, using the root-coefficient relationships, we have $x_2x_3 - (x_2 + x_3)(l_1 + l_2)/2 + l_1l_2 = 0$, what exactly means that $(x_2 - \gamma)(x_3 - \gamma) = \rho^2$.

• If $p_{1-1}p_{01} - p_{0-1}p_{11} = 0$ then $q(x,0)$ is the null polynomial. Indeed, in this case, $\deg(q) \leq 1$; in addition, thanks to Lemma 22, there exists $\beta \in \mathbb{R}$ such that $q = \beta q_1$. Since $\deg(q_1) = 2$, $\beta = 0$ and q is the null polynomial. In particular, we have, for all x and y, $c(x)/a(x) = c(y)/a(y)$ which leads to the equality $a = \delta c$, where δ is some positive constant. In such cases, x_2 and x_3 are easily calculated since they are the two roots of $-b(x) - 2(a(x)c(x))^{1/2}$, which is a polynomial of degree two. So, it suffices to calculate explicitly x_2, x_3, also l_1, l_2, and to notice that $(x_2 - \gamma)(x_3 - \gamma) = \rho^2$.

With the same arguments, we show that $(x_1 - \gamma)(x_4 - \gamma) = \rho^2$. □

Proof of Proposition 14. — Proposition 14 will result from the three following facts.

(1) First, since $\mathcal{M}$ is a circle of center γ and radius ρ and thanks to Lemma 23, we easily verify that the function $w(t) = (x_2 - \gamma)/(t - x_2) - (x_3 - \gamma)/(t - x_3)$ is a suitable CGF for the curve $\mathcal{M}$. In particular, the function $\sigma(t) = \gamma + \rho^2/(t - \gamma)$, defined in Proposition 14, lets w invariant ; also, $\sigma'(x_3) = \text{Res}(w, x_2)/\text{Res}(w, x_3) = -(x_2 - \gamma)/(x_3 - \gamma)$.

(2) Then, the key point is that since $\Delta = 0$, the function $\hat{\phi}$ defined in (34) is equal to zero. So, $h(x)$ is equal to the sum $h_1(x) + g_2(x)$, h_1 being defined in (30) and g_2 in (35).

(3) Moreover, once again thanks to Lemma 23, $\sigma(x_1) = x_4$, so that the integral (35) is an integral between x_3 and x_4. Moreover, by a direct calculation, we can show that if P stands for a, b or c, then $P(\sigma(t)) = (\rho/(t-\gamma))^2 P(t)$. In particular, $d(\sigma(t)) = (\rho/(t - \gamma))^4 d(t)$ and $\mu_{m_0}(\sigma(t)) = ((t - \gamma)\rho)^2 P(t)$.

Bringing together all these facts, we obtain (23).   □

5. Asymptotic of the Green functions

In this section we find the asymptotic of the Green functions $G_{i,j}^{n_0,m_0} = \mathbb{E}_{n_0,m_0}\left[\sum_{n\geq 0} 1_{\{(X(n),Y(n))=(i,j)\}}\right]$ if $i,j > 0$ and $j/i \to \tan(\gamma)$ where $\gamma \in [0,\pi/2]$.

5.1. Case $\gamma \in]0,\pi/2[$. — As it has been said in the introduction, in the case $\gamma \in]0,\pi/2[$ the procedure is essentially the same as in [22] and [25], we just outline some details that are different.

It follows from (4) that by Cauchy formula, for ϵ small enough,

$$(39) \qquad G_{i,j}^{n_0,m_0} = \frac{1}{(2\pi\imath)^2}\int_{|x|=1-\epsilon}\frac{h(x)}{x^i}\left(\int_{|y|=1-\epsilon}\frac{dy}{Q(x,y)\,y^j}\right)dx$$

$$(40) \qquad + \frac{1}{(2\pi\imath)^2}\int_{|y|=1-\epsilon}\frac{\widetilde{h}(y)+h_{00}}{y^j}\left(\int_{|x|=1-\epsilon}\frac{dx}{Q(x,y)\,x^i}\right)dy$$

$$(41) \qquad - \frac{1}{(2\pi\imath)^2}\int_{|y|=1-\epsilon}\frac{1}{y^{j-m_0}}\left(\int_{|x|=1-\epsilon}\frac{dx}{Q(x,y)\,x^{i-n_0}}\right)dy.$$

Then we apply the residue theorem at infinity to each inner integral above. Since $Q(x,y) = a(x)(y - Y_0(x))(y - Y_1(x)) = \tilde{a}(y)(x - X_0(y))(x - X_1(y))$, we have to know the positions of $Y_i(x)$ and $X_i(y)$ w.r.t. the circle $\mathscr{C}(0,1-\epsilon)$ when $|x| = |y| = 1 - \epsilon$. In fact, we will prove that for any x, y such that $|x| = |y| = 1 - \epsilon$ and $\epsilon > 0$ small enough:

$$(42)\quad |Y_0(x)| < 1-\epsilon, \quad |Y_1(x)| > 1-\epsilon, \quad |X_0(y)| < 1-\epsilon, \quad |X_1(y)| < 1-\epsilon.$$

Thanks to a proper change of parameters, it suffices of course to prove the first two inequalities. We already know from Lemma 2 that $Y_1(\{x \in \mathbb{C} : |x| = 1\} \setminus \{1\}) \subset \{y \in \mathbb{C} : |y| > 1\}$ and $Y_0(\{x \in \mathbb{C} : |x| = 1\}) \subset \{y \in \mathbb{C} : |y| < 1\}$. In particular, by continuity, this immediately leads to the first inequality in (42), for sufficiently small values of ϵ. This also entails that there exists $\theta_0(\epsilon)$, going to 0 as ϵ goes to 0, such that for all $x = (1-\epsilon)\exp(\imath\theta)$ with $\theta \in]\theta_0(\epsilon), 2\pi - \theta_0(\epsilon)[$, $|Y_1(x)| > 1 - \epsilon$. To conclude, it suffices to show that for all $x = (1-\epsilon)\exp(\imath\theta)$ with $\theta \in]-\theta_0(\epsilon), \theta_0(\epsilon)[$, $|Y_1(x)| > 1 - \epsilon$. For this we will prove that there exists a neighborhood of 1, independent of ϵ, where the curves $Y_1(\{x \in \mathbb{C} : |x| = 1 - \epsilon\})$ and $Y_1(\{x \in \mathbb{C} : |x| = 1\})$ don't intersect ; then we will also show that $Y_1(1-\epsilon) > Y_1(1) = 1$. In order to show that the two above curves don't intersect, remark that if they do, this means that $Y_1(x) = Y_1(\hat{x})$, with some $x, \hat{x}$ such that $|x| = 1$, $|\hat{x}| = 1 - \epsilon$. This last equality is equivalent to $\hat{x}x = \tilde{c}(Y_1(x))/\tilde{a}(Y_1(x))$. Since $Y_1(1) = 1$ and $\tilde{c}(1)/\tilde{a}(1) \in]0,1[$, the previous

equality is not possible in a neighborhood of 1 for x and $\hat{x}$. To prove that $Y_1(1 - \epsilon) > 1$, we remark that an explicit calculation shows that $Y_1(x) > 1$ if and only if $a(x) + b(x) + c(x) < 0$. But the polynomial $a + b + c$ goes to ∞ when $x \to \pm\infty$ and has two real roots, 1 and $\tilde{c}(1)/\tilde{a}(1) < 1$, so that $Y_1(1 - \epsilon) > 1$.

Hence the inner integral of (39) (resp. of (40), (41)) equals the residue at $Y_1(x)$ (resp. at $X_1(y)$) with the constant $-2\pi\imath$, the residue at infinity being zero. Then, letting $\epsilon \to 0$, $G_{i,j}^{n_0,m_0}$ is represented as the sum of the simple integrals

(43)

$$G_{i,j}^{n_0,m_0} = -\frac{1}{2\pi\imath}\int_{|x|=1}\frac{h(x)}{d(x)^{1/2}\,x^i Y_1(x)^j}\mathrm{d}x - \frac{1}{2\pi\imath}\int_{|y|=1}\frac{\tilde{h}(y) + h_{00} - X_1(y)^{n_0}\,y^{m_0}}{\tilde{d}(y)^{1/2}\,X_1(y)^i\,y^j}\mathrm{d}y.$$

These integrals are typical to apply the saddle-point method, see [10].

To find the suitable saddle-point for $\ln(xY_1(x)^{\tan(\gamma)})$ or equivalently for $\ln(X_1(y)y^{\tan(\gamma)})$, let us first have a closer look on the critical points of $\chi_{\gamma,0}$ and $\chi_{\gamma,1}$, defined by

(44) $\qquad \chi_{\gamma,0}(x) = xY_0(x)^{\tan(\gamma)}, \qquad \chi_{\gamma,1}(x) = xY_1(x)^{\tan(\gamma)}, \qquad \gamma \in\,]0, \pi/2[.$

The equations $\chi'_{\gamma,0}(x) = 0$ and $\chi'_{\gamma,1}(x) = 0$ are equivalent to

(45) $\quad \pm d(x)^{1/2}\,(a(x)\,c(x) - x\,(a'(x)\,c(x) - a(x)\,c'(x))\tan(\gamma)/2)$
$\qquad = x\tan(\gamma)\,(a(x)\,c(x)\,b'(x) - b(x)\,(a'(x)\,c(x) + a(x)\,c'(x))/2).$

Taking the square of both sides, we obtain that $P(\gamma, x) = 0$, where $P(\gamma, x)$ is the eight degree polynomial:

(46) $\qquad P(\gamma, x) = -(a(x)\,c(x) + x\tan(\gamma)\,r(x))\,d(x) + (x\tan(\gamma))^2\,P_l(x)$

where we note

(47) $\qquad\qquad P_l(x) = r(x)^2 - r_1(x)\,r_2(x) = \lim_{\gamma\to\pi/2}\frac{P(\gamma, x)}{(x\tan(\gamma))^2}$

and

(48) $\qquad\qquad r = ac' - a'c, \qquad r_1 = ba' - b'a, \qquad r_2 = cb' - c'b.$

Note that with the notations (38), $r(x) = q(x, 0)$ and for $i = 1, 2$, $r_i(x) = q_i(x, 0)$.

The eight roots of the polynomial $P(\gamma, x)$ are the four critical points of $\chi_{\gamma,0}(x)$ and those four of $\chi_{\gamma,1}(x)$, $\gamma \in]0, \pi/2[$. It is immediate that in the limiting case $\gamma = 0$ its roots are the branch points x_i, $i \in \{1, \ldots, 4\}$ and the roots of a and c. If $\gamma = \pi/2$, two of its roots are 0, two equal ∞ and four of them are the $X(y_i)$, $i \in \{1, \ldots, 4\}$, that are roots of $P_l(x)$.

Note that under the restricted hypothesis (H2′) the critical points can be made explicit. The polynomial (46) equals $(\tan(\gamma)^2 - 1)x^2 P_{1,4}(\gamma, x)P_{2,3}(\gamma, x)$, where $P_{1,4}(\gamma, x)$ and $P_{2,3}(\gamma, x)$ are polynomials of second degree, namely:

$$P_{1,4}(\gamma, x) = p_{10}x^2 - \frac{1 + \left(1 - (1-\tan(\gamma)^2)(1-4p_{0-1}p_{01} + 4p_{-10}p_{10}\tan(\gamma)^2)\right)^{1/2}}{1-\tan(\gamma)^2}x + p_{-10},$$

$$P_{2,3}(\gamma, x) = p_{10}x^2 - \frac{1 - \left(1 - (1-\tan(\gamma)^2)(1-4p_{0-1}p_{01} + 4p_{-10}p_{10}\tan(\gamma)^2)\right)^{1/2}}{1-\tan(\gamma)^2}x + p_{-10}.$$

The saddle-point for the first integral in (43) is the biggest root of $P_{2,3}(\gamma, x)$. This is the (unique) critical point of $\chi_{\gamma,1}(x)$ such that $x > 0$ and $Y_1(x) > 0$. In [22] it has been characterized as the solution of (49) below. Let us do it under (H2). We need to introduce the function $\phi(u, v) = \sum_{i,j} p_{ij}e^{iu}e^{jv}$ for $(u, v) \in \mathbb{R}^2$. The equation $Q(x, y) = 0$ with $x, y > 0$ is equivalent to $\phi(u, v) = 1$ with $u = \ln(x)$, $v = \ln(y)$. If $x > 0$ is the critical point of $\chi_{0,\gamma}(x)$ such that $Y_0(x) > 0$ (resp. the one of $\chi_{1,\gamma}(x)$ such that $Y_1(x) > 0$), then after some algebraic manipulations with $u = \ln(x)$ and $v = \ln(Y_0(x))$ (resp. $v = \ln(Y_1(x))$) we see that the equation (45) is equivalent to

$$(49) \qquad \frac{\partial \phi(u, v)/\partial u}{\partial \phi(u, v)/\partial v} = \tan(\gamma).$$

Then either

$$(50) \qquad \frac{\mathrm{grad}(\phi(u, v))}{|\mathrm{grad}(\phi(u, v))|} = (\cos(\gamma), \sin(\gamma))$$

or

$$(51) \qquad \frac{\mathrm{grad}(\phi(u, v))}{|\mathrm{grad}(\phi(u, v))|} = (\cos(\gamma + \pi), \sin(\gamma + \pi)).$$

The mapping $(u, v) \mapsto \mathrm{grad}(\phi(u, v))/|\mathrm{grad}(\phi(u, v))|$ is a homeomorphism from $D = \{(u, v) \in \mathbb{R}^2 : \phi(u, v) = 1\}$ to the unit two-dimensional sphere, see [12]. Hence, for any $\gamma \in [0, \pi/2]$ there is one solution of (50) on D, called $(u(\gamma), v(\gamma))$ and one solution of (51) on D, called $(u(\gamma + \pi), v(\gamma + \pi))$. Thus the positive critical point of $\chi_{\gamma,0}(x)$ with $Y_0(x) > 0$ and the one of $\chi_{\gamma,1}(x)$ with $Y_1(x) > 0$ are among $e^{u(\gamma)}$ and $e^{u(\gamma+\pi)}$. In addition $e^{u(\gamma)}$ (resp. $e^{u(\gamma+\pi)}$) is critical for $\chi_{i,\gamma}(x)$ if and only if $e^{v(\gamma)}$ (resp. $e^{v(\gamma+\pi)}$) equals $Y_i(e^{u(\gamma)})$ (resp. $Y_i(e^{u(\gamma+\pi)})$), $i = 0, 1$. We verify that $e^{v(\gamma)} = Y_1(e^{u(\gamma)})$ and $e^{v(\gamma+\pi)} = Y_0(e^{u(\gamma+\pi)})$, so that $e^{u(\gamma)}$ is the critical point of $\chi_{\gamma,1}(x)$ and $e^{u(\gamma+\pi)}$ is the one of $\chi_{\gamma,0}(x)$. Indeed, for $i = 0, 1$,

$$(52) \qquad \frac{\partial \phi}{\partial v}\left(u(\gamma), Y_i\left(e^{u(\gamma)}\right)\right) = \left[2Y_i\left(e^{u(\gamma)}\right)a\left(e^{u(\gamma)}\right) + b\left(e^{u(\gamma)}\right)\right]Y_i\left(e^{u(\gamma)}\right)^{-1}.$$

Moreover, on $[x_2, x_3]$, $2a(x)Y_1(x) + b(x) = d(x)^{1/2}$ and $2a(x)Y_0(x) + b(x) = -d(x)^{1/2}$, so that (52) is negative for $Y = Y_0$, positive for $Y = Y_1$, what answers to the problem.

Notation. — We put $s_x(\tan(\gamma)) = e^{u(\gamma)}$ and $s_y(\tan(\gamma)) = Y_1(e^{u(\gamma)})$.
The mapping $\gamma \mapsto (s_x(\tan(\gamma)), s_y(\tan(\gamma)))$ is a homeomorphism between $[0, \pi/2]$ and $\{(x, y) \in \mathbb{C}^2 : x > 0, y > 0, Q(x, y) = 0\}$.

We note that $s_x(0) = x_3$, $s_y(0) = Y(x_3)$ and $s_x(\infty) = X(y_3)$, $s_y(\infty) = y_3$. When γ runs $[0, \pi/2]$, $s_x(\tan(\gamma))$ monotonously decreases from x_3 to $Y(x_3)$ and $s_y(\tan(\gamma))$ monotonously increases from $Y(x_3)$ to y_3.

Obviously the unique critical point of $\tilde{\chi}_{\gamma,1}(y) = X_1(y)y^{\tan(\gamma)}$ with $y > 0$ and $X_1(y) > 0$ is $s_y(\tan(\gamma))$ defined above and $X_1(s_y(\tan(\gamma))) = s_x(\tan(\gamma))$.

THEOREM 24. — *Let $\gamma \in]0, \pi/2[$. If $j/i \to \tan(\gamma)$, then*

$$G_{i,j}^{n_0,m_0} = \frac{s_x(\tan(\gamma))^{n_0} s_y(\tan(\gamma))^{m_0} - h(s_x(\tan(\gamma))) - \tilde{h}(s_y(\tan(\gamma))) - h_{00}}{s_x(j/i)^i s_y(j/i)^j} \left(\frac{C(\gamma)}{\sqrt{i}} + \mathcal{O}\left(\frac{1}{i\sqrt{i}} \right) \right),$$

where the constant $C(\gamma)$, that does not depend on (n_0, m_0), is equal to:

$$C(\gamma) = (2\pi)^{-1/2} \tilde{d}(s_y(\tan(\gamma)))^{-1} s_y(\tan(\gamma)) \left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} \left\{ \frac{X_1(s_y(\tan(\gamma))y)}{s_x(\tan(\gamma))} y^{\tan(\gamma)} \right\} \bigg|_{y=1} \right)^{-1/2}.$$

Proof. — One appropriately shifts the contour of integration $|x| = 1$ (resp. $|y| = 1$) in the first term of (43) (resp. the second) to a contour Γ_γ (resp. $\tilde{\Gamma}_\gamma$) passing through $s_x(\tan(\gamma))$ (resp. $s_y(\tan(\gamma))$), which is the saddle-point of order one. Γ_γ is the contour of steepest descent (i.e. $\mathrm{Im}(xY_1(x)^{\tan(\gamma)}) = 0$ on it) in a neighborhood of $s_x(\tan(\gamma))$ and outside this neighborhood it remains "higher" than $s_x(\tan(\gamma))$ in the sense of the level curves of the function $\chi_{\gamma,1}$. The construction of Γ_γ is done as in [22] and [25], therefore we omit the details. Likewise, we construct the contour $\tilde{\Gamma}_\gamma$.

Then by Cauchy theorem, the first (resp. the second) term in (43) equals the integral over Γ_γ (resp. over $\tilde{\Gamma}_\gamma$), whose asymptotic is computed by the saddle-point method. $\square$

5.2. Asymptotic of the Green functions in the cases $\gamma = 0, \pi/2$. — For that purpose, we first need to know the behavior of $s_x(j/i) - s_x(0)$ and $s_y(j/i) - s_y(0)$ when j/i is in a neighborhood of 0.

LEMMA 25. — *Let P_l be the polynomial defined in (47). As $j/i \to 0$, the following expansions hold:*

$$(53) \qquad s_x(0) - s_x(j/i) = \frac{x_3^2 P_l(x_3)}{-d'(x_3) a(x_3) c(x_3)} (j/i)^2 + \mathcal{O}((j/i)^3),$$

$$(54) \qquad s_y(j/i) - s_y(0) = \frac{x_3 P_l(x_3)^{1/2}}{2a(x_3)^{3/2} c(x_3)^{1/2}} j/i + \mathcal{O}((j/i)^2).$$

Proof. — Start by proving (53). One hand, using (46), we obtain that $P(\arctan(j/i), x_3) = s_x(j/i)^2 P_l(x_3)$; on the other hand, by definition of $P(\arctan(j/i), x)$ and also with (46), we have $P(\arctan(j/i), x_3) = (x_3 - s_x(j/i))R(j/i)$, with $R(0) = -d'(x_3)a(x_3)c(x_3) \neq 0$. Equation (53) follows immediately. Then, to prove (54), start by remarking that $Y_1(x) - Y_1(x_3) = b(x_3)/(2a(x_3)) - b(x)/(2a(x)) + d(x)^{1/2}/(2a(x))$, so that in the neighborhood of x_3, $Y_1(x) - Y_1(x_3) = f(x) + (-d'(x_3)^{1/2}/(2a(x_3)))(x_3 - x)^{1/2}(1 + g(x))$, with f and g holomorphic in the neighborhood of x_3 where they take the value 0. Moreover, for all $j/i \in [0, \infty]$, $s_y(j/i) = Y_1(s_x(j/i))$, see Subsection 5.1. This yields $s_y(\gamma) - s_y(0) = (-d'(x_3))^{1/2}/(2a(x_3))(x_3 - s_x(\gamma))^{1/2} + \mathcal{O}(x_3 - s_x(\gamma))$. Finally, using (53), we obtain (54). □

THEOREM 26. — *Suppose first that* $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$. *Then the Green functions admit the following asymptotic as* $i \to \infty$, $j > 0$ *and* $j/i \to 0$:

$$(55) \qquad G_{i,j}^{n_0,m_0} = \frac{C_0 \left(m_0 s_x(0)^{n_0} s_y(0)^{m_0-1} - \tilde{h}'(s_y(0)) \right)}{s_x(j/i)^i s_y(j/i)^j} \frac{j}{i\sqrt{i}} (1 + o(1)),$$

where the constant C_0 *is equal to:*

$$(56) \qquad C_0 = \left(\frac{2}{\pi} \right)^{1/2} \frac{s_y'(0) s_x(0)^{1/2}}{\left(-\tilde{d}(s_y(0)) X_1''(s_y(0)) \right)^{1/2}},$$

$s_x(0) = x_3$, $s_y(0) = Y_1(x_3)$ *and* $s_y'(0)$ *is obtained from Lemma* 25.

   The analogous result as $j \to \infty$, $i > 0$ *and* $j/i \to \infty$ *holds after the proper change of parameters.*

   Suppose now that $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$. *If* $n_0 + m_0$ *and* $i + j$ *don't have the same parity then* $G_{i,j}^{n_0,m_0} = 0$; *if they have the same parity, the asymptotic of* $G_{i,j}^{n_0,m_0}$ *is given by two times the right member of* (55).

REMARK 27. — *Theorem* 17 *and Proposition* 21 *give immediately an explicit expression for* $m_0 s_x(0)^{n_0} s_y(0)^{m_0-1} - \tilde{h}'(s_y(0))$ *in the three cases* $\omega_2/\omega_3 \in 2\mathbb{N}$, $\omega_2/\omega_3 \in 2\mathbb{N} + 1$ *and* $\omega_2/\omega_3 \notin \mathbb{N}$ *that we do not list here.*

Proof. — We detail the proof in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$ and explain at the end what changes if $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$. $G_{i,j}^{n_0,m_0}$ appears in (43) as the sum of two integrals, one on the contour $|x| = 1$, the other on $|y| = 1$. Using Cauchy Theorem we will move these contours up to $s_x(j/i)$ and $s_y(j/i)$ respectively in a such way that the asymptotic of the integrals on the new contours will be calculated by the saddle-point method.

In order to define these new contours of integration, we need to introduce the following—eventually multivalued—functions:

$$(57) \qquad \kappa_{j/i}(x) = \ln(x) + \frac{j}{i} \ln\left(\frac{Y_1(s_x(j/i)x)}{s_y(j/i)}\right),$$

$$\widetilde{\kappa}_{j/i}(y) = \ln\left(\frac{X_1(s_y(j/i)y)}{s_x(j/i)}\right) + \frac{j}{i}\ln(y).$$

According to Subsection 5.1, the function $\kappa_{j/i}$ (resp. $\widetilde{\kappa}_{j/i}$) has, for all $j/i > 0$, a critical point at 1 where it equals 0. Consider now the functions $x_{j/i}(t)$ and $y_{j/i}(t)$ defined in neighborhoods $V_{x,j/i}(0)$ and $V_{y,j/i}(0)$ of 0 by

$$(58) \qquad \kappa_{j/i}\left(x_{j/i}(t)\right) = t^2, \qquad \widetilde{\kappa}_{j/i}\left(y_{j/i}(t)\right) = t^2,$$

and $\mathrm{sign}(\mathrm{Im}(x_{j/i}(t))) = \mathrm{sign}(t)$, $\mathrm{sign}(\mathrm{Im}(y_{j/i}(t))) = \mathrm{sign}(t)$. These last relationships are fixed in order to define $x_{j/i}$ and $y_{j/i}$ not ambiguously. By inverting the relationships (58), we obtain the explicit expression of $x_{j/i}$ and $y_{j/i}$. Here, inverting means using the so-called Bürman-Lagrange formula, see e.g. [6], that allows to write the coefficients of the Taylor series of a reciprocal function as integrals in terms of the direct function. As $j/i \to 0$, then $s_y(j/i) \to s_y(0) \in]y_2, y_3[$. We may define

$$\widetilde{\rho} = \inf_{j/i \in [0,1]} \inf\{y_3/s_y(j/i) - 1, 1 - y_2/s_y(j/i)\},$$

which verifies $\widetilde{\rho} > 0$ and $\widetilde{\kappa}_{j/i}$ is holomorphic in the disc $\mathcal{D}(1,\widetilde{\rho})$ for all $j/i \in [0,1]$. Using the Bürman-Lagrange formula we see that the radius of $V_{y,j/i}(0)$ does not vanish as $j/i \to 0$: $y_{j/i}(t)$ is in fact defined and holomorphic in $D(0,\rho)$, ρ being positive and independent of $j/i \in [0,1]$. Moreover, the functions $x_{j/i}(t)$ and $y_{j/i}(t)$ are joined together by:

$$(59) \quad x_{j/i}(-t) = \frac{X_1\left(s_y(j/i)y_{j/i}(t)\right)}{s_x(j/i)}, \qquad y_{j/i}(-t) = \frac{Y_1\left(s_x(j/i)x_{j/i}(t)\right)}{s_y(j/i)}.$$

This is a consequence of the automorphy relationships proved in [9]: for x (resp. y) exterior to some curve (what is the case here), $X_1(Y_1(x)) = x$ (resp. $Y_1(X_1(y)) = y$). For this reason and the fact that 1 is a critical point of order one of $\kappa_{j/i}$, we obtain that

$$x_{j/i}(t) \in \{X_1(s_y(j/i)y_{j/i}(-t))/s_x(j/i), X_1(s_y(j/i)y_{j/i}(t))/s_x(j/i)\}.$$

Then it suffices to calculate the sign of the imaginary part in order to identify which of the two possibilities happens: we have $x_{j/i}(t) = X_1(s_y(j/i)y_{j/i}(-t))/s_x(j/i)$. The first equality of (59) shows that $x_{j/i}$ is holomorphic as well in $V_{x,j/i}(0) = D(0,\rho)$ for all $j/i \in [0,1]$.

The functions $\hat{x}_{j/i}(t) = s_x(j/i)x_{j/i}(t)$ and $\hat{y}_{j/i}(t) = s_y(j/i)y_{j/i}(t)$ determine of course the paths of steepest descent for $\ln(xY_1(x)^{j/i})$ and $\ln(X_1(y)y^{j/i})$ respectively. Note that the limiting curve $\hat{x}_0(t)$ runs the real line decreasing from $\hat{x}_0(\rho)$ to $\hat{x}_0(0) = x_3$ and then increasing from x_3 to $\hat{x}_0(\rho)$ when t runs $[-\rho, \rho]$; indeed $x_0(t) = \exp(t^2)$. The function $\hat{y}_0(t) = Y_1(\hat{x}_0(-t))$ runs the values $Y_1^-(\hat{x}_0(t)) = \lim_{x \uparrow \hat{x}_0(t)} Y_1(x)$ from $Y_1^-(\hat{x}_0(\rho))$ up to $Y_1(x_3)$ and then the values $Y_1^+(\hat{x}_0(t)) = \lim_{x \downarrow \hat{x}_0(t)} Y_1(x)$ to $Y_1^+(\hat{x}_0(\rho))$.

For any ρ small enough we will now define two closed contours $\mathscr{C}_{\rho,j/i,x} = x_{j/i}([-\rho, \rho]) \cup \mathscr{A}_{\rho,j/i,x}$ and $\mathscr{C}_{\rho,j/i,y} = y_{j/i}([-\rho, \rho]) \cup \mathscr{A}_{\rho,j/i,y}$, where $\mathscr{A}_{\rho,j/i,x}$, $\mathscr{A}_{\rho,j/i,y}$ verify the following three properties.

(i) There exists a constant $c(\rho) > 0$ such that $|\kappa_{j/i}(x)| > c(\rho)$ for any $x \in \mathscr{A}_{\rho,j/i,x}$ and any j/i small enough, and such that $|\tilde{\kappa}_{j/i}(x)| > c(\rho)$ for any $y \in \mathscr{A}_{\rho,j/i,y}$ and any j/i small enough.

(ii) The first integrand in (43) does not have any singularities in the domain bounded by $|x| = 1$ and the contour $s_x(j/i)\mathscr{C}_{\rho,j/i,x}$, the second integrand in (43) does not have any singularities in the domain bounded by $|y| = 1$ and the contour $s_y(j/i)\mathscr{C}_{\rho,j/i,y}$.

(iii) There exists a constant $L(\rho)$ such that the lengths of the contours $\mathscr{C}_{\rho,j/i,x}$ and $\mathscr{C}_{\rho,j/i,y}$ are bounded by $L(\rho)$ for all j/i small enough.

Let us construct such $\mathscr{A}_{\rho,j/i,x}$. We may take for $\mathscr{A}_{\rho,0,x}$ the circle of radius $x_0(\rho) > 1$. Then $|\kappa_0(x)| = |\ln(x)| = \ln(x_0(\rho)) > (\ln(x_0(\rho)))/2$ for any $x \in \mathscr{A}_{\rho,0,x}$. Let us then take for $\mathscr{A}_{\rho,j/i,x}$ the arc $\{|x_{j/i}(\rho)| \exp(\imath\theta) : \theta \in]\arg(x_{j/i}(\rho)), 2\pi - \arg(x_{j/i}(\rho))[\}$. Since $\hat{x}_{j/i}(\rho) \to \hat{x}_0(\rho) = x_3 \exp(\rho^2)$ as $j/i \to 0$ and since Y_1 has no zero on $\mathbb{C}$ (it can be easily shown that under (H4), Y_1 does not vanish on $\mathbb{C}$), the property (i) remains valid for $\kappa_{j/i}(x)$ with $c(\rho) = (\ln(x_0(\rho)))/4$ for all j/i small enough. Furthermore, the singularities of the first integrand in (43) are the zeros of d, i.e. the branch points x_i, $i \in \{1, \ldots, 4\}$. But with Lemma 1, x_1 and x_2 are inside the unit circle $|x| = 1$; as for x_3 and x_4, they are outside $s_x(j/i)\mathscr{C}_{\rho,j/i,x}$: x_3 is outside this contour by construction and x_4 also lies outside $s_x(j/i)\mathscr{C}_{\rho,j/i,x}$ for ρ small enough, because $|x_4| > x_3$, since we have supposed $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$, see once again Lemma 1. So (ii) is verified and (iii) is also and obviously verified. We can also construct $\mathscr{A}_{\rho,j/i,y}$ starting by $\mathscr{A}_{\rho,0,y}$. Since $s_y(0)$ is a critical point of X_1, the level line $\{y \in \mathbb{C} : |X_1(y)| = x_3\}$ has a double point at $s_y(0)$ and $\{y \in \mathbb{C} : |X_1(y)| = x_3\} \setminus \{s_y(0)\}$ has two connected components. Moreover, thanks to Lemma 28, the circle $\mathscr{C}(0, s_y(0)) \setminus \{s_y(0)\}$ lies in the domain $\{y \in \mathbb{C} : |X_1(y)| > x_3\}$. For this reason and since $-s_y(0) \in]-y_3, -y_2[\subset]-|y_4|, -|y_1|[$, one can clearly construct a contour $\mathscr{A}_{\rho,0,y}$ that verifies (i), (ii) and (iii). Then, by continuity of the different quantities w.r.t. j/i, one can build contours $\mathscr{A}_{\rho,j/i,y}$ verifying (i), (ii) and (iii) for all j/i small enough.

Let us go back to (43). Before passing from double integrals in (39), (40), (41) to the simple ones in (43), it will be convenient to subtract the constant $h(x_3)$ from the numerator of (39) and add it to the numerator of (40). Next, we move the contours $|x| = 1$ and $|y| = 1$ to $s_x(j/i)\mathscr{C}_{\rho,j/i,x}$ and $s_y(j/i)\mathscr{C}_{\rho,j/i,y}$; thanks to Cauchy theorem and since by construction the contours avoid the singularities of the integrands, the value of the integrals is not changed. After the change of variables $x \mapsto x s_x(j/i)$ and $y \mapsto y s_y(j/i)$ in (43), we get $G_{i,j}^{n_0,m_0} = -K_{i,j}^{n_0,m_0}/(s_x(j/i)^i s_y(j/i)^j)$, where $K_{i,j}^{n_0,m_0}$ is defined by:

(60)
$$
K_{i,j}^{n_0,m_0} = \frac{s_x(j/i)}{2\pi\imath} \int_{\mathscr{C}_{\rho,j/i,x}} \frac{h(s_x(j/i)x) - h(x_3)}{d(s_x(j/i)x)^{1/2}} e^{-i\kappa_{j/i}(x)} \mathrm{d}x
$$
$$
+ \frac{s_y(j/i)}{2\pi\imath} \int_{\mathscr{C}_{\rho,j/i,y}} \frac{\widetilde{h}(s_y(j/i)y) + h_{00} + h(x_3) - X_1(s_y(j/i)y)^{n_0}(s_y(j/i)y)^{m_0}}{\widetilde{d}(s_y(j/i)y)^{1/2}} e^{-i\widetilde{\kappa}_{j/i}(y)} \mathrm{d}y.
$$

We split $K_{i,j}^{n_0,m_0} = K_{i,j,1}^{n_0,m_0} + K_{i,j,2}^{n_0,m_0}$, where $K_{i,j,1}^{n_0,m_0}$ (resp. $K_{i,j,2}^{n_0,m_0}$) is obtained from $K_{i,j}^{n_0,m_0}$ by integrating only on the contours $x_{j/i}([-\rho,\rho])$ and $y_{j/i}([-\rho,\rho])$ (resp. $\mathscr{C}_{\rho,j/i,x}$ and $\mathscr{C}_{\rho,j/i,y}$). We will prove that the asymptotic $K_{i,j,1}^{n_0,m_0}$ will lead to the result announced in Theorem 26 and that $K_{i,j,2}^{n_0,m_0}$ will be exponentially negligible, see (61).

We start by studying $K_{i,j,2}^{n_0,m_0}$. Consider the following two quantities $S_{1,\delta}$ and $S_{2,\delta}$ and prove that for δ sufficiently small they are finite.

$$
S_{1,\delta} = \sup_{j/i\in[0,\delta]} \sup_{x\in\mathscr{C}_{\rho,j/i,x}} \left| \frac{h(s_x(j/i)x) - h(x_3)}{d(s_x(j/i)x)^{1/2}} \right|,
$$

$$
S_{2,\delta} = \sup_{j/i\in[0,\delta]} \sup_{y\in\mathscr{C}_{\rho,j/i,y}} \left| \frac{\widetilde{h}(s_y(j/i)y) + h_{00} + h(x_3) - X_1(s_y(j/i)y)^{n_0}(s_y(j/i)y)^{m_0}}{\widetilde{d}(s_y(j/i)y)^{1/2}} \right|.
$$

The fact that for δ small enough $S_{1,\delta}$ is finite comes from the three following properties. First, we recall (see Corollary 5 and the proof of Lemma 19) that h is meromorphic on $\mathbb{C} \setminus [x_3,x_4]$, holomorphic in $\mathscr{D}(0,x_3)$ and continuable holomorphically through every point of $\mathscr{C}(0,x_3)$ except x_3. This is why $h(x)$, and therefore also $(h(x)-h(x_3))/d(x)^{1/2}$, are bounded in a neighborhood of every point of the circle $\mathscr{C}(0,x_3)$ except at x_3. Secondly, it can be easily deduced from the proofs of Lemmas 18 and 19 that in the neighborhood of x_3, $h(x) = h(x_3) + c(x_3-x)^{1/2} + \mathcal{O}(x_3-x)$, where c is some non zero constant. So the quantity $(h(x)-h(x_3))/d(x)^{1/2}$ is bounded in the neighborhood of x_3. Thirdly, the contours $s_x(j/i)\mathscr{C}_{\rho,j/i,x}$ avoid, by construction, the branch points x_1, x_2 and x_4. The fact that $S_{2,\delta}$ is finite follows similarly: first, by construction, the contours $s_y(j/i)\mathscr{C}_{\rho,j/i,y}$ avoid the branch points y_i, $i \in \{1,\ldots,4\}$, also, the poles of X_1 and $\widetilde{h}$ being isolated, the contours $s_y(j/i)\mathscr{C}_{\rho,j/i,y}$ can be chosen

such that they remain away from these poles. Recalling the properties (i) and (iii) of $A_{\rho,j/i,x}$ and $A_{\rho,j/i,y}$ we deduce that for any $i > 0$ and any j/i small enough,

$$(61) \qquad \left| K_{i,j,2}^{n_0,m_0} \right| \leq (L(\rho)/(2\pi))(x_3 S_{1,\delta} + y_3 S_{2,\delta}) \exp(-ic(\rho)).$$

Let us now turn to $K_{i,j,1}^{n_0,m_0}$. Making in (60) the change of variable $x = x_{j/i}(t)$ and $y = y_{j/i}(t)$, we represent $K_{i,j,1}^{n_0,m_0}$ as an integral on the segment $[-\rho, \rho]$. Moreover, using (43) and the following equality—that comes from (4) and (59)

$$(62)$$
$$s_x(j/i) x'_{j/i}(-t) \widetilde{d}\left(s_y(j/i) y_{j/i}(-t)\right)^{1/2} = -s_y(j/i) y'_{j/i}(t) d\left(s_x(j/i) x_{j/i}(-t)\right)^{1/2},$$

we obtain that $K_{i,j,1}^{n_0,m_0} = \int_{-\rho}^{\rho} f_{j/i}(t) \exp(-it^2) \mathrm{d}t$, where

$$f_{j/i}(t) = \left[h\left(X_1\left(\widehat{y}_{j/i}(t)\right)\right) + \widetilde{h}\left(\widehat{y}_{j/i}(t)\right) \right.$$
$$\left. + h_{00} - X_1\left(\widehat{y}_{j/i}(t)\right)^{n_0} \widehat{y}_{j/i}(t)^{m_0} \right] \widehat{y}'_{j/i}(t) \widetilde{d}\left(\widehat{y}_{j/i}(t)\right)^{-1/2},$$

and $\widehat{y}_{j/i}(t) = s_y(j/i) y_{j/i}(t)$. In addition, the formula (37) yields that $f_{j/i}$ is equal to:

$$(63)$$
$$f_{j/i}(t) = \left[\widetilde{h}\left(\widehat{y}_{j/i}(t)\right) - \widetilde{h}\left(\frac{c\left(X_1\left(\widehat{y}_{j/i}(t)\right)\right)}{a\left(X_1\left(\widehat{y}_{j/i}(t)\right)\right) \widehat{y}_{j/i}(t)} \right) \right.$$
$$\left. - X_1\left(\widehat{y}_{j/i}(t)\right)^{n_0} \left(\widehat{y}_{j/i}(t)^{m_0} - \left(\frac{c\left(X_1\left(\widehat{y}_{j/i}(t)\right)\right)}{a\left(X_1\left(\widehat{y}_{j/i}(t)\right)\right) \widehat{y}_{j/i}(t)} \right)^{m_0} \right) \right] \widehat{y}'_{j/i}(t) \widetilde{d}\left(\widehat{y}_{j/i}(t)\right)^{-1/2}.$$

In particular, this representation (63), added to the—already noticed—holomorphy of $y_{j/i}$ in $\mathscr{D}(0,\rho)$, ρ being independent of $j/i \in [0,1]$, implies that $f_{j/i}$ is holomorphic in a disc of center 0 and of radius positive and independent of j/i for j/i small enough. Therefore $f_{j/i}$ can be expanded in its Taylor series in the neighborhood of 0: $f_{j/i}(t) = f_{j/i}(0) + t f'_{j/i}(0) + t^2 f''_{j/i}(0)/2 + t^3 f'''_{j/i}(0)/6 + t^4 g_{j/i}(t)$, where $g_{j/i}$ is also holomorphic in some centered disc of radius positive and independent of j/i for j/i sufficiently small. Reducing eventually ρ and δ, we have that $G = \sup_{j/i \in [0,\delta]} \sup_{t \in [-\rho,\rho]} |g_{j/i}(t)|$ is finite. Then, applying Laplace's method, we obtain the bound

$$(64)$$
$$\left| K_{i,j,1}^{n_0,m_0} - \frac{1}{2\pi i} \left(\frac{\pi^{1/2} f_{j/i}(0)}{i^{1/2}} + \frac{\pi^{1/2} f''_{j/i}(0)}{4i^{3/2}} \right) \right| \leq \frac{3\pi^{1/2} G}{4i^{5/2}} + C \exp\left(-i\rho^2\right).$$

with some constant $C > 0$ for any $i, j > 0$, $j/i \in [0,\delta]$.

To conclude the analysis of $K_{i,j,1}^{n_0,m_0}$, it remains to evaluate the asymptotic expansions of $f_{j/i}(0)$ and $f''_{j/i}(0)$ as $j/i \to 0$. Taking $t = 0$ in (63) we derive

that $f_{j/i}(0)$ is equal to:

(65)
$$\frac{\tilde{h}\left(s_y\left(j/i\right)\right) - \tilde{h}\left(\frac{c(s_x(j/i))}{a(s_x(j/i))s_y(j/i)}\right) - s_x\left(j/i\right)^{n_0}\left(s_y\left(j/i\right)^{m_0} - \left(\frac{c(s_x(j/i))}{a(s_x(j/i))s_y(j/i)}\right)^{m_0}\right)}{\tilde{d}\left(s_y\left(j/i\right)\right)^{1/2}} s_y\left(j/i\right) y'_{j/i}\left(0\right).$$

Note that since $s_y(0)^2 = c(s_x(0))/a(s_x(0))$ (indeed, $s_y(0) = Y_1(s_x(0))$ and $s_x(0) = x_3$), then for any suitable function F,

$$F(s_y(j/i)) - F(c(s_x(j/i))/(a(s_x(j/i)))s_y(j/i)) = 2F'(s_y(0))s'_y(0)j/i + o(j/i),$$

$s'_y(0)$ being obtained from Lemma 25. We use this fact successively with $F(y) = \tilde{h}(y)$ and $F(y) = y^{m_0}$ to expand the numerator of (65) as $2\left(m_0 s_x(0)^{n_0} s_y(0)^{m_0-1} - \tilde{h}'(s_y(0))\right)s'_y(0)j/i + o(j/i)$, $j/i \to 0$. The functions $s_x(j/i)$ and $s_y(j/i)$ being continuous on $[0, +\infty]$, the Taylor coefficients of $y_{j/i}(t)$ depend continuously on j/i, so that $y'_{j/i}(0) \to y'_0(0)$ as $j/i \to 0$. Let us compute the value of $y'_0(0)$. To get it, we differentiate twice (58) ; this yields

(66)
$$y'_{j/i}\left(0\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{\frac{X_1\left(s_y\left(j/i\right)y\right)}{s_x\left(j/i\right)}y^{j/i}\right\}\bigg|_{y=1} = 2.$$

In addition, an explicit calculation gives:

(67)
$$\lim_{j/i \to 0} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{\frac{X_1\left(s_y\left(j/i\right)y\right)}{s_x\left(j/i\right)}y^{j/i}\right\}\bigg|_{y=1} = \frac{s_y\left(0\right)^2}{s_x\left(0\right)}X_1''\left(s_y\left(0\right)\right).$$

(66) and (67) imply $y'_0(0) = \imath\sqrt{2}s_x(0)^{1/2}(-X_1''(s_y(0)))^{-1/2}s_y(0)^{-1}$, the $\imath$ coming from the fact that $X_1''(s_y(0))$ is negative. Hence, we obtain that $f_{j/i}(0) = l_1 j/i(1 + o(1))$, where

(68)
$$l_1 = -\imath\frac{2\sqrt{2}s'_y\left(0\right)s_x\left(0\right)^{1/2}\left(m_0 s_x\left(0\right)^{n_0} s_y\left(0\right)^{m_0-1} - \tilde{h}'\left(s_y\left(0\right)\right)\right)}{\left(-\tilde{d}\left(s_y\left(0\right)\right)X_1''\left(s_y\left(0\right)\right)\right)^{1/2}}.$$

The Taylor coefficients of $y_{j/i}(t)$ depending continuously on j/i, so do those of $f_{j/i}(t)$. Then $f''_{j/i}(0) \to f''_0(0)$ as $j/i \to 0$. But $f(t)$ is an odd function on $[-\rho, \rho]$. To see this, first, remark that (6) yields $c(\hat{x}_0(t))/(a(\hat{x}_0(t))Y_1(\hat{x}_0(t))) = Y_0(\hat{x}_0(t))$. Moreover, $Y_1(\hat{x}_0(-t)) = Y_0(\hat{x}_0(t))$, so that $\hat{x}_0$ being even ($\hat{x}_0(t)$ is equal to $x_3 \exp(t^2)$), the function within the brackets in (63) is odd. In addition, using (62) we obtain that $\hat{y}'_0(t)\tilde{d}(\hat{y}_0(t))^{-1/2} = -\hat{x}'_0(-t)d(\hat{x}_0(-t))^{-1/2}$. Being the product of two odd functions, $\hat{x}'_0 d(\hat{x}_0)^{-1/2}$ is even, so that $\hat{y}'_0 \tilde{d}(\hat{y}_0)^{-1/2}$ is also even. This implies that f_0 is odd and as an immediate consequence $f''_0(0) = 0$ and $f''_{j/i}(0) = o(1)$ as $j/i \to 0$. Bringing together (61), (64), the expansions $f_{j/i}(0) = l_1 j/i(1 + o(1))$ and $f''_{j/i}(0) = o(1)$ with l_1 defined in (68)

we obtain:

(69)
$$G_{i,j}^{n_0,m_0} = \frac{-1}{s_x\,(j/i)^i\,s_y\,(j/i)^j}\left(\frac{1}{2\pi\imath}\frac{\pi^{1/2}l_1 j/i\,(1+o\,(1))}{i^{1/2}} + o\left(i^{-3/2}\right)\right) + \mathcal{O}\big(\exp\left(-ic\left(\rho\right)\right)\big),\ \ j/i \to 0.$$

This concludes the proof of the theorem in the case $p_{11}+p_{-1-1}+p_{1-1}+p_{-11} < 1$.

We now briefly explain the notable differences in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$. In this case, all the functions considered are odd or even: for instance Y_i and X_i, $i = 0, 1$ are odd, d and $\tilde{d}$ are even (see Lemma 28), h and $\tilde{h}$ have the parity of $n_0 + m_0$. In particular, it is immediate from (43) that if $i+j$ and $n_0 + m_0$ don't have the same parity, then $G_{i,j}^{n_0,m_0} = 0$. If they have the same parity, then we can obtain the asymptotic of the Green functions with essentially the same analysis as in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$, the only significant change being that we have now to take under account the contribution of two critical points: $(s_x(j/i), s_y(j/i))$, as before, but now also $(-s_x(j/i), -s_y(j/i))$. In particular, the new contour of integration $\mathcal{C}_{\rho,j/i,x}$ (resp. $\mathcal{C}_{\rho,j/i,x}$) have to go at once through $s_x(j/i)$ and through $-s_x(j/i)$ (resp. $s_y(j/i)$ and through $-s_y(j/i))$, for instance they can be taken symmetrical w.r.t. the imaginary axis. This fact underlies that the asymptotic of the $G_{i,j}^{n_0,m_0}$ is, in this case, twice, in accordance with the conclusions of Theorem 26. $\square$

The following result has been used in the proof of Theorem 26.

LEMMA 28. — *If $p_{11}+p_{-1-1}+p_{-11}+p_{1-1} < 1$, for all $y \in \mathcal{C}(0, s_y(0))\backslash\{s_y(0)\}$, $|X_1(y)| > x_3$. If $p_{11}+p_{-1-1}+p_{-11}+p_{1-1} = 1$, for all $y \in \mathcal{C}(0, s_y(0))\backslash\{\pm s_y(0)\}$, $|X_1(y)| > x_3$.*

Proof. — As a direct consequence of Lemma 1, we obtain that X_1 is meromorphic in the neighborhood of every point of $] - y_3, -y_2[$, since $] - y_3, -y_2[\subset$ $] - |y_4|, -|y_1|[$. Let us now show that

(i) if $p_{11} + p_{-1-1} + p_{-11} + p_{1-1} < 1$, then for all $y \in [y_2, y_3]$, $|X_1(-y)| > |X_1(y)|$,

(ii) if $p_{11} + p_{-1-1} + p_{-11} + p_{1-1} = 1$, then for all $y \in [y_2, y_3] = [-y_1, -y_4]$, $X_1(-y) = -X_1(y)$.

Start by remarking that $X_1(y) \leq 0$ (resp. $X_1(y) \geq 0$) on $[y_4, y_1]$ (resp. on $[y_2, y_3]$), this is proved in [9]. Thus, on $[y_2, y_3]$, the unique possibility to have $|X_1(-y)| = |X_1(y)|$ is that $X_1(y) = -X_1(-y)$. After calculation, we obtain that this is equivalent to $(p_{01}y^2 + p_{0-1})^2(p_{11}y^2 + p_{1-1})(p_{-11}y^2 + p_{-1-1}) + \{(p_{-10}p_{1-1}-p_{10}p_{-1-1})^2 + p_{0-1}(p_{-10}p_{1-1}+p_{10}p_{-1-1})\}y^2 + \{p_{-10}p_{10}(1 - 2(p_{1-1}p_{-11}+p_{-1-1}p_{11})) + p_{-10}p_{1-1}(p_{01}+p_{-10}p_{11}) + p_{-10}p_{11}(p_{0-1}+p_{-10}p_{1-1}) + p_{10}p_{-1-1}(p_{01}+p_{10}p_{-11}) + p_{10}p_{-11}(p_{0-1}+p_{10}p_{-1-1})\}y^4 + \{(p_{10}p_{-11}-p_{-10}p_{11})^2 + p_{01}(p_{10}p_{-11} + p_{-10}p_{11})\}y^6 = 0$. If $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$ (resp. $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$), then the previous equality holds for none

(resp. any) $y \in [y_2, y_3]$. Therefore (ii) is proved. To prove (i), we remark that an explicit calculation leads to $|X_1(-1)| > 1 = X_1(1)$ so that by continuity, for all $y \in [y_2, y_3]$, $|X_1(-y)| > |X_1(y)|$.

We prove now Lemma 28 in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} < 1$. We will show that $\mathscr{C}(0, s_y(0)) \cap \{y \in \mathbb{C} : |X_1(y)| = x_3\} = \{s_y(0)\}$. This suffices since one hand, this implies that either for all $y \in \mathscr{C}(0, s_y(0)) \setminus \{s_y(0)\}$, $|X_1(y)| > x_3$ or for all $y \in \mathscr{C}(0, s_y(0)) \setminus \{s_y(0)\}$, $|X_1(y)| < x_3$; but on the other hand, thanks to (i), $|X_1(-s_y(0))| > X_1(s_y(0)) = x_3$, so that by continuity we will conclude. Let $y^* \in \mathscr{C}(0, s_y(0))$ be such that $|X_1(y^*)| = x_3$. Setting $\hat{x} = X_1(y^*)/x_3$, $\hat{y} = y^*/s_y(0)$ and using $Q(x_3, s_y(0)) = 0$, $Q(X_1(y^*), y^*) = 0$, we obtain $\hat{Q}(\hat{x}, \hat{y}) = 0$, where $\hat{Q}(\hat{x}, \hat{y}) = (\sum_{i,j} \hat{p}_{ij} \hat{x}^i \hat{y}^j - 1)\hat{x}\hat{y}$ and $\hat{p}_{ij} = p_{ij} x_3^i s_y(0)^j$. In particular, for all i and j, $\hat{p}_{ij} > 0$ and $\sum_{i,j} \hat{p}_{ij} = 1$, since $Q(x_3, s_y(0)) = 0$. But from elementary considerations about sums of complex numbers, having simultaneously $\sum_{i,j} \hat{p}_{ij} = 1$, $\sum_{i,j} \hat{p}_{ij} \hat{x}^i \hat{y}^j = 1$ and $|\hat{x}| = |\hat{y}| = 1$ leads necessarily to $\hat{x} = \hat{y} = 1$, so that $y^* = s_y(0)$.

We conclude the proof in the case $p_{11} + p_{-1-1} + p_{1-1} + p_{-11} = 1$ by using similar arguments and the fact that $x_4 = -x_3$. $\qquad \square$

REMARK 29. — *Thanks to Proposition 21, it is immediate now that for any* $n_0, m_0, n_1, m_1 > 0$:
(70)
$$\lim_{i,j>0, j/i \to 0} \frac{G_{i,j}^{n_0, m_0}}{G_{i,j}^{n_1, m_1}} = \lim_{i \to \infty} \frac{h_i^{n_0, m_0}}{h_i^{n_1, m_1}} = \frac{m_0 s_x(0)^{n_0} s_y(0)^{m_0 - 1} - \widetilde{h}^{n_0, m_0 \,'}(s_y(0))}{m_1 s_x(0)^{n_1} s_y(0)^{m_1 - 1} - \widetilde{h}^{n_1, m_1 \,'}(s_y(0))},$$

the probabilities of absorption $h_i^{n_0, m_0}$ *being defined in* (1). *In addition, using* (37) *and Lemma 25, we obtain that as* γ *goes to zero, the limit* $\lim_{i,j>0, j/i \to \tan(\gamma)} G_{i,j}^{n_0, m_0}/G_{i,j}^{n_1, m_1}$ *converges also to the right member of* (70). *In other words, the Martin kernel is continuous at* 0 *and likewise, we verify that it is also continuous at* $\pi/2$.

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