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DIOPHANTINE APPROXIMATION ON VEECH SURFACES

by Pascal Hubert & Thomas A. Schmidt

Abstract. — We show that Y. Cheung's general Z-continued fractions can be adapted to give approximation by saddle connection vectors for any compact translation surface. That is, we show the finiteness of his Minkowski constant for any compact translation surface. Furthermore, we show that for a Veech surface in standard form, each component of any saddle connection vector dominates its conjugates in an appropriate sense. The saddle connection continued fractions then allow one to recognize certain transcendental directions by their developments.

Résumé (*Approximation diophantienne sur les surfaces de Veech*)

Nous montrons que les fractions continues generalisées Z de Y. Cheung s'adaptent pour exprimer l'approximation par vecteurs de connexion de selles sur n'importe quelle surface de translation compacte. C'est-à-dire, nous démontrons la finitude de la constant de Minkowski pour chaque surface de translation compacte. De plus, pour une surface de Veech en forme standard, nous montrons que chaque composant de n'importe quel vecteur de connexion de selle domine, dans un sens approprié, ses conjugués. Les fractions continues de connexions de selle permettent de reconnaître certaines directions transcendantales par leur développement.

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1. Introduction and Main Results

We show that Yitwah Cheung's generalization of the geometric interpretation of regular continued fractions gives a successful method for approximation of flow directions on translation surfaces by saddle connection vectors. Cheung [7], [8] generalizes the work of Poincaré and Klein by replacing approximation by the integer lattice in $\mathbb{R}^2$ with approximation by any infinite discrete set Z of nonzero vectors with finite "Minkowski constant", equal to one-fourth times the supremum taken over the areas of centro-symmetric bounded convex bodies disjoint from Z.

We prove, as Cheung certainly understood, that the set of saddle connection vectors of any translation surface has a finite Minkowski constant.

Theorem 1. — *Let S be a compact translation surface, and $Z = \mathrm{V}_{\mathrm{sc}}(S)$ the set of saddle connection vectors of S. Then*

$$\mu(Z) \leq \pi \operatorname{vol}(S)$$

where $\operatorname{vol}(S)$ *is the Lebesgue area of S and $\mu(Z)$ is as given in Definition 2.*

The following result is of independent interest; here, it allows us to reach transcendence results using approximation by saddle connection vectors. Recall that the group of linear parts (the so-called "derivatives") of the oriented affine diffeomorphisms of a compact finite genus translation surface, S, form a Fuchsian group, $\Gamma(S)$. The trace field of the surface is the algebraic number field generated over the rationals by the set of traces of the elements of $\Gamma(S)$, when this group is non-trivial. When $\Gamma(S)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, the surface is said to be a Veech surface.

Theorem 2. — *Suppose that S is a Veech surface normalized so that:* $\Gamma(S) \subset \mathrm{SL}_2(\mathbb{K})$*; the horizontal direction is periodic; and, both components of every saddle connection vector of S lie in $\mathbb{K}$, where $\mathbb{K}$ is the trace field of S. Then there exists a positive constant $c = c(S)$ such that for all holonomy vectors $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and $1 \leq i \leq 2$ one has*

$$|\, v_i \,| \geq c \,|\, \sigma(\, v_i\,)\,|\,,$$

where σ varies through the set of field embeddings of $\mathbb{K}$ into $\mathbb{R}$.

Note that in the above, each field embedding σ in fact takes values only in $\mathbb{R}$.

With S as above and $Z = \mathrm{V}_{\mathrm{sc}}(S)$ the set of saddle connection vectors of S, the Z-expansion of an inverse slope θ for a flow direction is defined in Section 2.1. Theorem 1 then implies that this gives a sequence of elements

$(p_n, q_n) \in \mathbb{K}^2$ such that $|\theta - p_n/q_n|$ goes to zero as n tends to infinity; see Lemma 2.

One criterion for a "good" continued fraction algorithm is that extremely rapid convergence to a real number implies that this number is transcendental. We show that the $Z = \mathrm{V}_{\mathrm{sc}}(S)$-fractions on Veech surfaces enjoy this property.

THEOREM 3. — *With S and $\mathbb{K}$ as above, let $D = [\mathbb{K} : \mathbb{Q}]$ be the field extension degree of $\mathbb{K}$ over the field of rational numbers. If a real number $\xi \in [0,1] \setminus \mathbb{K}$ has an infinite $\mathrm{V}_{\mathrm{sc}}(S)$-expansion, whose convergents p_n/q_n satisfy*

$$\limsup_{n\to\infty} \frac{\log \log q_n}{n} > \log(\,2D - 1\,)\,,$$

then ξ is transcendental.

1.1. Related work. — There exist algorithms that approximate flow directions on particular translation surfaces by so-called parabolic directions, see [1], [25], [24]. Roughly speaking, these algorithms can be viewed as continued fraction algorithms expressing real values in terms of the orbit of infinity under the action of a related Fuchsian group. Up to finite index and appropriate normalization, each underlying group in these examples is one of the infinite family of Hecke triangle Fuchsian groups, [28]. Some 60 years ago, for each Hecke group, D. Rosen [21] gave a continued fraction algorithm. Motivated in part by the use in [2] of the Rosen fractions to identify pseudo-Anosov directions with vanishing so-called SAF-invariant, with Y. Bugeaud, in [5] we recently gave the first transcendence results using Rosen continued fractions. Theorem 3 is the analog of a main result there.

Each Hecke group is contained in a particular $\mathrm{PSL}(2, K)$ with K a totally real number field. Key to the approach of [5] was the fact that any element in a Hecke group of sufficiently large trace is such that this trace is appropriately larger than each of its conjugates over $\mathbb{Q}$. This leads to a bound of the height of a convergent p_n/q_n in terms of q_n itself. The LeVeque form of Roth's Theorem, in combination with a bound on approximation in terms of $q_n q_{n+1}$, can then be used to show that transcendence is revealed by exceptionally high rates of growth of the q_n. We show here that all of this is possible for any Veech surface, replacing Rosen fractions by Z-expansions with $Z = \mathrm{V}_{\mathrm{sc}}(S)$. Key to this is our results that: (1) any nontrivial Veech group $\Gamma(S)$ has the property of the dominance of traces over their conjugates; and, (2) in the case of a Veech surface S, the dominance property for the group implies a (weaker) dominance of components of saddle vectors over their conjugates.

We mention that it would be interesting to compare the approximation in terms of saddle connection vectors with the known instances of approximation with parabolic directions.

1.2. Outline. — In the following section we sketch some of the disparate background necessary for our results; in Section 3 we prove the crucial result that the Minkowski constant is finite for any compact translation surface; in Section 4 we show that if S is a Veech surface then $\Gamma(S)$ has the property of dominating conjugates and from this that one can bound the heights in the Z-expansions, $Z = \mathrm{V}_{\mathrm{sc}}(S)$; finally, in Section 5 we very briefly show that the arguments of [5] are valid here: $Z = \mathrm{V}_{\mathrm{sc}}(S)$ -expansions with extremely rapidly growing denominators belong to transcendental numbers.

1.3. Thanks. — It is a pleasure to thank Curt McMullen for asking if the results of [5] could hold in the general Veech surface setting. We also thank Emmanuel Russ for pointing out the reference [3]. Finally, we thank the referee for a careful reading and for the suggestion of Corollary 2.

2. Background

2.1. Cheung's Z-expansions. — We briefly review Yitwah Cheung's definition of his Z-expansions — we follow Section 3 of [8], although we focus on approximation of a ray instead of a line. (This simplification is valid in our setting, as we can and do assume that the approximating set Z is symmetric about the origin.)

Fix a discrete set $Z \subset \mathbb{R}^2$, and assume that Z does not contain the zero vector. Given a positive real θ, consider the ray emitted from the origin with slope $1/\theta$. Our goal is to define a sequence of elements of Z that approximates this ray.

Remark 1. — Note that the number that is approximated here is the *inverse of the slope* of the ray. This choice accords well with the *projective* action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Let u be the unit vector in the direction of the ray. Denote the positive half plane of the ray by $H_+(\theta) = \{v \in \mathbb{R}^2 \mid u{\cdot}v > 0\}$, and let $Z_+(\theta) := Z \cap H_+(\theta)$. Let $v = (p, q) \in \mathbb{R}^2$; the difference vector between v and the vector whose endpoint is given by the intersection of $y = x/\theta$ and $y = q$ has length of absolute value $\mathrm{hor}_\theta(v) = |q\theta - p|$. The value q is the *height* of v and $\mathrm{hor}_\theta(v)$ is its *horizontal component*, see Figure 1.

Definition 1. — The *Z-convergents* of θ is the set of elements of Z in the half-plane of the ray such that each minimizes the horizontal component $\mathrm{hor}_\theta(v)$ amongst elements of equal or lesser height:

$$\mathrm{Conv}_Z(\theta) = \{v \in Z_+(\theta) \mid \forall w \in Z_+(\theta), |w_2| \le |v_2| \implies \mathrm{hor}_\theta(v) \le \mathrm{hor}_\theta(w)\}\,.$$

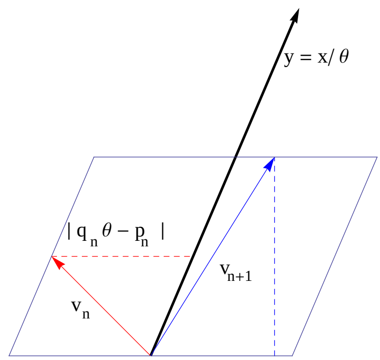

FIGURE 1. Vectors v_n approximate ray $y = x/\theta$; parallelogram related to Lemma 2.

The Z-*expansion* of θ is the sequence obtained by ordering the set $\mathrm{Conv}_Z(\theta)$ by height, where we choose as necessary between elements of the same height.

Recall from [8] that if Z contains some element of the x-axis, and there are infinitely many Z-convergents to θ, then certainly the heights of the sequence tend to infinity.

DEFINITION 2. — The *Minkowski constant* of Z is

$$\mu(Z) = \frac{1}{4} \sup \mathrm{area}(\mathscr{C})$$

where $\mathscr{C}$ varies through bounded, convex, $(0,0)$-symmetric sets that are disjoint from Z.

Finiteness of the Minkowski constant assures good approximation, see [8] for the proof of the following.

LEMMA 1 (Cheung *et al.*). — *Suppose both that $\mu(Z)$ is finite and that Z contains a non-zero vector on the x-axis. Then the Z-expansion of a direction with inverse slope θ is infinite if and only if no element of Z lies in this direction.*

Denote the n-th element of the Z-expansion of θ by (p_n, q_n). Then the following is also shown in [8], see our Figure 1.

LEMMA 2 (Cheung *et al.*). — *The Z-expansion of θ satisfies*

$$\frac{|p_n q_{n+1} - p_{n+1} q_n|}{2\, q_n\, q_{n+1}} < |\theta - p_n/q_n| \leq \mu(Z)/(q_n\, q_{n+1})\,.$$

2.2. Translation surfaces

2.2.1. *Translation surfaces, Veech surfaces.* — For all of this material, see the expository articles [18], [31]. A *translation surface* is a real surface with an atlas such that, off of a finite number of points, transition functions are translations. Here we consider only *compact* surfaces, and will continue to do so without further notice. From the Euclidean plane, this punctured surface inherits a flat metric, and this metric extends to the complete surface, with (possibly removable) conical singularities at the punctures. Due to the transition function being translations, directions of linear flow on a translation surface are well-defined, and Lebesgue measure is inherited from the plane. We define $\mathrm{vol}(S)$ to be the total Lebesgue measure of the surface.

Post-composing the coordinate function of a chart from the atlas of a translation surface with any element of $\mathrm{SL}_2(\mathbb{R})$ results in a new translation surface. This action preserves the underlying topology, the types of the conical singularities, and the area of the surface.

Related to this, an *affine diffeomorphism* of the translation surface is a homomorphism that restricts to be a diffeomorphism on the punctured flat surface whose derivative is a *constant* 2×2 real matrix. W. Veech [28] showed that for any compact translation surface S, the matrices that arise as such derivatives of (orientation- and area-preserving) affine diffeomorphisms form a Fuchsian group $\Gamma(S)$, now referred to as the *Veech group* of the surface.

A *Veech surface* is a translation surface such that the group $\Gamma(S)$ is a co-finite subgroup of $\mathrm{SL}(2,\mathbb{R})$; that is, such that the quotient of the Poincaré upper half-plane by $\Gamma(S)$ (using the standard fractional linear action) has finite hyperbolic area. Equivalently, $\Gamma(S)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$; indeed, some refer to a Veech surface as having the "lattice property".

2.2.2. *Saddle connections, ergodicity of action, parabolic directions.* — A *separatrix* of a translation surface S is a geodesic ray emanating from some singularity. A *saddle connection* is a separatrix connecting singularities (with no singularities on its interior). By using the local coordinates of the translation surface, each saddle connection defines a vector in $\mathbb{R}^2$. The collection of these (affine) *saddle connection vectors* is $\mathrm{V}_{\mathrm{sc}}(S)$. That $\mathrm{V}_{\mathrm{sc}}(S)$ contains an element of length at most $\sqrt{2\,\mathrm{vol}(S)}$ was shown by Vorobets [29](see the proof of Proposition 3.2 there).

Local coordinates on the *stratum* that is the space of translation surfaces of fixed genus and singularities type is provided by the integral relative to the set of singularities. In wording from [10], the saddle connections cut S into a collection of polygons which provide local coordinates. The stratum then inherits a Lebesgue measure, as [31] says, a key theorem is that of Masur and

Veech: the $\mathrm{SL}(2,\mathbb{R})$ action on translation surface preserves this measure and is ergodic on connected components of the strata.

Results of Veech [28] imply that if $\Gamma(S)$ is a lattice (and S has singularities), then the direction of any saddle connection vector is a parabolic direction — there is a parabolic element of $\Gamma(S)$ fixing a vector in this direction, and that there is some saddle connection vector in each parabolic direction. Since a lattice in $\mathrm{SL}_2(\mathbb{R})$ has only finitely many parabolic conjugacy classes, a Veech surface has only finitely many $\Gamma(S)$-orbits of parabolic directions.

2.2.3. *Trace field, standard form.* — Gutkin and Judge [11] defined the *trace field* of a translation surface to be the field extension of the rationals generated by the traces of derivatives of the affine diffeomorphisms of the surface; this is clearly independent of choice of a translation surface within an $\mathrm{SL}_2(\mathbb{R})$-orbit. A result of Gutkin and Judge (see Lemma 7.5 of [11]) implies that the ratio of the lengths of any two saddle connection vectors in a common parabolic direction gives an element of the trace field.

Möller [20], see Proposition 2.6 there, showed that the trace field of any Veech surface is totally real (that is, every field embedding into the complex numbers sends the field to a subfield of the real numbers). The result holds true under weaker hypotheses, see [12], [6].

Calta and Smillie in [6] introduced a notion of *standard form* of a translation surface; they expressed this notion in terms of the Kenyon-Smillie J-invariant of a translation surface, which in turn is related to the Sah-Arnoux-Fathi (SAF) invariant of interval exchanges. To simplify discussion, let us say that a translation surface is in *parabolic standard form* if the vertical, the horizontal and the diagonal are parabolic directions. It is immediate that a surface in parabolic standard form is in standard form (this as the SAF-invariant vanishes in any parabolic direction, and the Calta-Smillie definition merely requires that this invariant vanish in each of the three directions).

Calta and Smillie show that when a translation surface is in standard form, the parabolic directions all lie in the trace field. Combining this with the aforementioned result of Gutkin and Judge, one finds that, by scaling and choice within $\mathrm{SL}_2(\mathbb{R})$-orbit, the saddle connection vectors of a Veech surface can be assumed to have components in the trace field. Furthermore, Kenyon and Smillie [13], see the proof of Corollary 29 there, argue that the $\mathbb{Z}$-module generated by the saddle connection vectors has a submodule of finite index that is contained in $\mathcal{O}_K \oplus \mathcal{O}_K$, where $\mathcal{O}_K$ denotes the ring of integers of the trace field K. In particular, there is some $m \in \mathbb{N}$ such that for any $v \in \mathrm{V}_{sc}(S)$, the components of mv are in $\mathcal{O}_K$.

2.2.4. *Traces of hyperbolics dominate conjugates.* — A Fuchsian group Γ is said to have a *modular embedding* if there exists an arithmetic group Δ acting on $\mathbb{H}^n$ for an appropriate n, an inclusion $f : \Gamma \to \Delta$ and a holomorphic embedding $F = (F_1, ..., F_n) : \mathbb{H} \to \mathbb{H}^n$ such that $F_1 = \mathrm{id}$ and $F(\gamma \cdot z) = f(\gamma) \cdot F(z)$, see [9].

Schmutz Schaller and Wolfart [23] (see in particular their Corollary 5) show that if a Fuchsian group has a modular embedding, then the trace of any of its hyperbolic elements dominates its conjugates in absolute value. For ease of discussion, let us call this the *domination of conjugates* property.

M. Möller [20] shows that if S is a Veech surface, then $\Gamma(S)$ is commensurable to a Fuchsian group with a modular embedding (see his Corollary 2.11). Commensurability here means up to finite index and $\mathrm{SL}_2(\mathbb{R})$-conjugation; it directly follows that a Veech group of a Veech surface always has a finite index subgroup with the domination of conjugates property. Thus, any hyperbolic element of such a $\Gamma(S)$ has some power that whose trace dominates its conjugates. Since [13] (Theorem 28) shows that the trace of any hyperbolic in $\Gamma(S)$ generates the trace field of S over $\mathbb{Q}$, the following lemma completes our first argument that the Veech group of any Veech surface has the domination of conjugates property.

Lemma 3. — *Suppose that $M \in \mathrm{SL}_2(\mathbb{R})$ is a hyperbolic matrix and that there is some natural number n such that both $\mathbb{Q}(\mathrm{tr}(M^n)) = \mathbb{Q}(\mathrm{tr}(M))$ and the trace of M^n dominates its conjugates in norm. Then the trace of M dominates its conjugates in norm.*

Proof. — Recall that for any $M \in \mathrm{SL}_2(\mathbb{R})$, one has $M^2 = (\mathrm{tr}(M)) \cdot M - I$. Following [16], let

$$p_1(x) = 1,\ p_2(x) = x; \quad \text{and}\ \ p_n(x) = xp_{n-1}(x) - p_{n-2}(x),\ n \geq 3\,.$$

Then induction shows that $M^n = p_n(\mathrm{tr}(M)) \cdot M - p_{n-1}(\mathrm{tr}(M)) \cdot I$. From this, letting

$$s_n(x) = xp_n(x) - 2p_{n-1}(x)$$

for $n \geq 2$, by induction one finds

$$\mathrm{tr}(M^n) = s_n(\mathrm{tr}(M))\,.$$

We note that $s_n(x) \in \mathbb{Z}[x]$ for each n. In particular, when $\mathbb{Q}(\mathrm{tr}(M^n)) = \mathbb{Q}(\mathrm{tr}(M))$, we have

$$\sigma(\mathrm{tr}(M^n)) = \sigma(s_n(\mathrm{tr}(M))) = s_n(\sigma(\mathrm{tr}(M)))$$

for any field embedding of $\mathbb{Q}(\mathrm{tr}(M))$.

Elementary induction shows that $s_n(x)$ is an odd function if n is odd, and is an even function if n is even. Since M is hyperbolic, its trace certainly dominates in norm any conjugate that has absolute value less than or equal to two. It therefore suffices to show that each $s_n(x)$ is a strictly increasing function on the interval $(2, +\infty)$.

First note that elementary induction shows that $p_n(x) - p_{n-1}(x)$ is positive on $[2, +\infty)$; since $p'_n(x) - p'_{n-1}(x) = p_{n-1}(x) + (x-1)p'_{n-1}(x) - p'_{n-2}(x)$ induction also gives the positivity of this difference on the interval. Hence, still with $x \geq 2$, we have

$$\begin{aligned} s'_n(x) &= p_n(x) + xp'_n(x) - 2p'_{n-1}(x) \\ &= xp_{n-1}(x) - p_{n-2}(x) + xp'_n(x) - 2p'_{n-1}(x) \\ &\geq (\, p_{n-1}(x) - p_{n-2}(x)\,) + 2(\, p'_n(x) - p'_{n-1}(x)\,) \\ &\geq 0\,. \end{aligned}$$

Therefore, $s_n(x)$ is strictly increasing on this interval, and the result follows. □

In the proof of Lemma 6, we recall a more elementary derivation, showing that any Veech group containing a hyperbolic element has the domination of conjugates property.

2.2.5. *Zippered rectangles, pseudo-Anosov homeomorphisms, Rauzy-Veech induction.* — We give a terse summary of results of Veech [27]; see [14], [17] for more technical introductions.

In order to give a discrete version of geodesic flow over Teichmüller space, Veech [27] introduced his decomposition of a translation surface into *zippered rectangles* — rectangles have their vertical sides identified only up to a certain height (dependent upon the placement of singularities of the translation surface). The vertical flow on a zippered rectangle decomposition defines an interval exchange transformation; in other words, each zippered rectangle decomposition is the suspension of some interval exchange transformation. Veech gave an induction, now known as *Rauzy-Veech induction*, on the set of zippered rectangles that comports exactly with an induction on interval exchange maps. The interval exchanges that occur during a sequence of induction steps have various associated permutations; gathering these permutations into the *Rauzy graph*, Veech (using earlier work of Rauzy) related paths in this graph with the Teichmüller flow.

An affine diffeomorphism ϕ is a *pseudo-Anosov homeomorphism* exactly when its linear part is hyperbolic. The two real fixed points of the hyperbolic linear part correspond to directions that are expanding and contracting for ϕ; the larger eigenvalue of the linear part is then the expansion constant, known as the *dilatation* of the pseudo-Anosov ϕ. The pseudo-Anosov ϕ maps any geodesic ray in the expanding direction to some ray in this direction. (Thurston, see [26], first defined *pseudo-Anosov* homeomorphisms.)

Veech showed that appropriate loops in the graph of permutations arising from the Rauzy-Veech induction correspond to pseudo-Anosov homeomorphisms that fix a separatrix. Now, any affine diffeomorphism is bijective on the finite set of singularities; thus, some power acts as the identity on the singularities. Similarly, an affine pseudo-Anosov permutes the (finite) set of separatrices in its expanding direction and thus has a power that fixes some such separatrix. In particular, each affine pseudo-Anosov has a power that is identified by a closed loop in a Rauzy-Veech graph.

Furthermore, Veech showed that each edge in the loop defines a matrix related to the "suspension data", the product over the loop of these gives the integral *renormalization matrix* V that is *primitive*: a power has positive integer entries. The Perron-Frobenius Theorem thus applies; Veech points out that the Perron-Frobenius (that is, dominant) eigenvalue of V is the dilatation, λ^n, of ϕ^n. And, in fact that V gives the action of ϕ^n on integral homology relative to the singularities.

2.3. Approximation by algebraic numbers. — In the following, we repeat some lines of background from [5].

The following result was announced by Roth [22] and proven by LeVeque, see Chapter 4 of [15]. (The version below is Theorem 2.5 of [4].) Recall that given an algebraic number α, its *naive height*, denoted by $H(\alpha)$, is the largest absolute value of the coefficients of its minimal polynomial over $\mathbb{Z}$.

THEOREM 4. — *(LeVeque) Let K be a number field, and ξ a real algebraic number not in K. Then, for any $\epsilon > 0$, there exists a positive constant $c(\xi, K, \epsilon)$ such that*

$$|\,\xi - \alpha\,| > \frac{c(\xi, K, \epsilon)}{H(\alpha)^{2+\epsilon}}$$

holds for every α in K.

The *logarithmic Weil height* of α lying in a number field K of degree D over $\mathbb{Q}$ is $h(\alpha) = \frac{1}{D}\sum_{\nu} \log^+ ||\alpha||_\nu$, where $\log^+ t$ equals 0 if $t \leq 1$ and M_K denotes the places (finite and infinite "primes") of the field, and $||\cdot||_\nu$ is the ν-absolute value. This definition is independent of the field K containing α.

The *product formula* for the number field K is: $\prod_{\nu \in M_K} ||\alpha||_\nu = 1$. From this, $\forall\, \alpha, \beta \in \mathcal{O}_K$ with $\beta \neq 0$, such that the principal ideals $\langle\alpha\rangle, \langle\beta\rangle$ have no common prime divisors, one has $h(\alpha/\beta) = \frac{1}{D}\sum_{\sigma} \log^+ \max\{|\sigma(\alpha)|, |\sigma(\beta)|\,\}$, where σ runs through the infinite places of K, which we consider as field embeddings. Even upon dropping the relative primality condition, one finds

$$h(\alpha/\beta) \leq \frac{1}{D}\sum_{\sigma} \log^+ \max\{|\sigma(\alpha)|, |\sigma(\beta)|\,\}\,. \tag{1}$$

The two heights are related by

$$\log H(\alpha) \leq \deg(\alpha)\,(\,h(\alpha) + \log 2\,)\,, \tag{2}$$

for any non-zero algebraic number α, see Lemma 3.11 in [30].

Finally, recall that standard transcendence notation includes the use of $\ll$ and $\gg$ to denote inequality with implied constant.

3. Minkowski constants

3.1. Minkowski constants in strata. — The Minkowski constant of the nonzero holonomy of a translation surface defines a function that may be of true interest. The following shows that it has properties in common with the Siegel-Veech invariants (see [10]).

LEMMA 4. — *The function assigning to a translation surface S the Minkowski constant of the set of saddle connection vectors, $S \mapsto \mu(\,\mathrm{V}_{\mathrm{sc}}(S)\,)$, is constant on* $\mathrm{SL}_2(\mathbb{R})$*-orbits.*

Proof. — The action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces sends saddle connection vectors to saddle connection vectors. But, this standard action sends the collection of bounded, convex sets that are symmetric about the origin to itself. □

The following is now implied by the ergodicity of the $\mathrm{SL}(2,\mathbb{R})$ action.

COROLLARY 1. — *Any connected component of the moduli space of abelian differentials of a given signature has a subset of full measure on which the Minkowski constant is constant.*

We give an example where the Minkowski constant is small. See Figure 46 of [31] for a representation of the surface in question.

LEMMA 5. — *Consider the translation surface S given by the L-shaped square-tiled surface of three tiles. Then $\mu(\,\mathrm{V}_{\mathrm{sc}}(S)\,) = \mathrm{vol}(S)/3$.*

Proof. — We may assume that S has area 3. One easily finds that $\Gamma(S)$ is the Theta group, the subgroup of the modular group generated by $z \mapsto z+2$ and $z \mapsto -1/z$. The entries of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of this group satisfy $a \equiv d \equiv -b \equiv -c \pmod 2$. One immediately finds that S is in (parabolic) standard form: there are visibly connection vectors in the horizontal, vertical and diagonal directions. Indeed, these are primitive vectors in the full lattice $\mathbb{Z}^2$, and Theta acts so as to give that $\mathrm{V}_{\mathrm{sc}}(S)$ consists of all of the primitive vectors. Hence, $\mu(\,\mathrm{V}_{\mathrm{sc}}(S)\,) = \mu(\,\mathbb{Z}^2\,) = 1$. □

3.2. Finiteness of Minkowski constants. — Key to convergence of Cheung's Z-approximants is his hypothesis that the Minkowski constant $\mu(Z)$ is finite. Note that Theorem 1 justifies the statement in Corollary 3.9 of [8].

Proof. — (of Theorem 1) Fix any bounded convex region $\mathscr{C}$ that is symmetric about the origin in the plane. By a theorem of Fritz John, see say [3] for a discussion, the ellipse (symmetric about the origin) $\mathscr{E}$ of maximal area interior to $\mathscr{C}$ is such that the scaled ellipse $\sqrt{2}\,\mathscr{E}$ contains $\mathscr{C}$. It follows that $\operatorname{area}(\mathscr{E}) \geq \operatorname{area}(\mathscr{C})/2$.

Now, there is $A \in \mathrm{SL}(2,\mathbb{R})$ taking $\mathscr{E}$ to a circle. If $\operatorname{area}(\mathscr{C}) \geq 4\pi \operatorname{vol}(S)$, then $A \cdot \mathscr{E}$ contains any vector of length less than or equal to $\sqrt{2\operatorname{vol}(S)}$. But, $A \cdot \mathrm{V}_{\mathrm{sc}}(S) = \mathrm{V}_{\mathrm{sc}}(A \cdot S)$ has a saddle connection vector of length at most $\sqrt{2\operatorname{vol}(A \cdot S)} = \sqrt{2\operatorname{vol}(S)}$, where we have used the bound of Vorobets, mentioned in Section 2.2.2, for the length of the shortest saddle connection. Therefore, $A \cdot \mathscr{C}$ contains a saddle connection vector of $A \cdot S$ and hence $\mathscr{C}$ contains an element of $\mathrm{V}_{\mathrm{sc}}(S)$. We conclude that $\mu(\,\mathrm{V}_{\mathrm{sc}}(S)\,) \leq \pi \operatorname{vol}(S)$. □

4. Bounding the height of convergents

In the background discussion of Section 2.2.4, we sketched a proof showing that when S is a Veech surface the traces of hyperbolic elements in $\Gamma(S)$ dominate their conjugates. Here we give a more straightforward proof, with weaker hypotheses. For an earlier version of the following, see Theorem 5.3 of [19] and the remark thereafter. McMullen's proof is ultimately based upon Thurston's ideas, see [26]. Ours is based upon Veech's [27] use of his zippered rectangles and what is now called Rauzy-Veech induction, see Subsection 2.2.5 for a brief introduction.

Lemma 6. — *Let M be a hyperbolic element in the Veech group of a translation surface. Then the trace of M is larger in norm than any of its images under the non-trivial field embeddings of the trace field of the translation surface.*

Proof. — To M corresponds some affine pseudo-Anosov map, say ϕ whose dilatation we denote by λ. Some power ϕ^n fixes a separatrix. Hence, the results of Veech [27] imply that ϕ^n corresponds to a closed loop in the Rauzy graph corresponding to the sequence of interval exchanges related to a certain sequence of induction on the zippered rectangle decomposition built above the fixed separatrix. The corresponding renormalization matrix V is primitive in the sense that a power has positive integer entries. The Perron-Frobenius Theorem thus applies; Veech points out that the dominant eigenvalue of V is the dilatation, λ^n, of ϕ^n. The minimal polynomial over $\mathbb{Q}$ of λ^n divides the characteristic polynomial of V and thus λ^n dominates its conjugates in norm.

By Theorem 28 in [13], $\mathbb{Q}(\lambda^n) = \mathbb{Q}(\lambda)$. Since the map $x \mapsto x^n$ is increasing we find that λ itself dominates in norm all of its conjugates.

Any field embedding σ of $\mathbb{Q}(\lambda + \lambda^{-1})$ extends to a field embedding $\tilde{\sigma}$ of $\mathbb{Q}(\lambda)$. Since $f : x \mapsto x + x^{-1}$ is increasing for $x > 1$ and $\lambda + \lambda^{-1} > 2$, we find $|\sigma(\lambda + \lambda^{-1})| = |\tilde{\sigma}(\lambda) + \tilde{\sigma}(\lambda^{-1})| \leq f(|\tilde{\sigma}(\lambda)|) < f(|\lambda|)$. We find that $\mathrm{tr}(M) = \lambda + \lambda^{-1}$ dominates all of its conjugates. □

We now show that if a Fuchsian subgroup Γ of the determinant one matrices over a number field is known to have traces that dominate their conjugates, and Γ contains a translation, then every entry of every element of Γ weakly dominates its conjugates. We do this by refining arguments of [5] (see Lemma 3.1 there).

Lemma 7. — *Fix a totally real number field $\mathbb{K}$. Suppose that $\Gamma \subset \mathrm{PSL}_2(\mathbb{K})$ contains a parabolic element*

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

and set $c_1 = \min\{\,|\,\sigma(\lambda)/\lambda\,|\,\}$, where σ varies through the set of field embeddings of $\mathbb{K}$ into $\mathbb{R}$. Suppose further that for all $M \in \Gamma$ whose trace is sufficiently large in absolute value, that for all σ one has

$$|\,\mathrm{tr}(M)\,| \geq |\,\sigma(\,\mathrm{tr}(M)\,)\,|\,.$$

Then for all

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \Gamma\,,$$

and for all σ and for all $1 \leq i, j \leq 2$, one has

$$|\,a_{ij}\,| \geq c_1\,|\,\sigma(\,a_{ij}\,)\,| \textit{ if } i \neq j$$
$$\textit{and } \;|\,a_{ii}\,| \geq c_1^2\,|\,\sigma(\,a_{ii}\,)\,|\,.$$

Proof. — For ease of notation, let $P = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. Let $A \in \Gamma$ be arbitrary, with elements labeled as above. The trace of P^nA is $a_{11} + a_{22} + n\lambda a_{21}$; thus, either $a_{21} = 0$, or upon letting n tend to infinity this trace is eventually large in absolute value. Thus, we find that a_{21} is at least c_1 times the absolute value of any of its conjugates. Considering AP^n similarly gives that a_{12} is at least c_1 times the absolute value of any of its conjugates.

Now, the $(1,2)$-entry of AP^n is $na_{11}\lambda + a_{12}$ and as we considering arbitrary elements in Γ above, this must be at least c_1 times the absolute value of any of its conjugates. We thus find that $|\,a_{11}\,| \geq c_1^2\,|\,\sigma(\,a_{11}\,)\,|$. Replacing A by A^{-1} shows that also a_{22} has this property. □

In an initial version of this work, we had purposely avoided admitting further hypotheses in the above lemma, even though one can find stronger results. However, the referee has kindly suggested the following corollary and even provided the pleasant argument that we reproduce.

Corollary 2. — *Under the hypotheses of Lemma 7, for all*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \Gamma ,$$

and for all σ, one has

(i.) $|\, a_{ii} \,| \geq |\, \sigma(\, a_{ii}\,)\,|$, *for* $1 \leq i, j \leq 2$;
(ii.) *if $\exists \mu \in K$ such that μ strictly dominates its conjugates, then*

$$|\, a_{12} \,| \geq |\, \sigma(\, a_{12}\,)\,| \quad ;$$

(iii.) *if furthermore Γ contains an element of the form $Q = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$ then every entry of any element of Γ dominates its conjugates.*

Proof. — Since $P = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma$, we have that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. This conjugated group gives c_1 in the hypotheses of Lemma 7 the value 1; therefore, the diagonal entries of any element of this conjugated group dominate their conjugates. However, conjugation by $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ preserves diagonal entries.

Second, if μ strictly dominates its conjugates, then there is some positive k such that $\mu^k \lambda^{-1}$ strictly dominates its conjugates. Conjugating Γ by $\begin{pmatrix} \mu^k & 0 \\ 0 & 1 \end{pmatrix}$ again leads to a replacement of the value of c_1 by 1. For each $(1, 2)$-entry, a_{12}, of an element of Γ gives rise to a $(1, 2)$-entry $a_{12}\mu^{-k}$ of this conjugate group; the domination of this value of its conjugates and the fact that μ^{-k} is the smallest of its own conjugates in absolute value shows that a_{12} dominates its conjugates.

Finally, if Γ contains some Q as above, then the previous arguments show that any a_{21} dominates its conjugates. □

Remark 2. — We note that Corollary 2 allows one to confirm the conjecture made in Remark 5 of [5]. The denominator q_n of a Rosen approximant to a real number is an entry in a Hecke group G_m that fulfills all of the hypotheses of the corollary. Therefore, q_n dominates its conjugates.

REMARK 3. — We also note that every totally real number field K (other than $\mathbb{Q}$) does admit an element μ that strictly dominates its conjugates. By the Primitive Element Theorem, $K = \mathbb{Q}(\theta)$ for some θ; any non-trivial embedding σ is such that $\sigma(\theta) \neq \theta$ and thus we can find a rational p/q that is closer to θ than to any of its conjugates; hence, $\mu = 1/(\theta - p/q)$ belongs to K and is greater in absolute value than any of its conjugates.

We now prove the announced main result stating that each component of any saddle connection vector of a Veech surface is appropriately larger than any of its conjugates.

Proof. — (of Theorem 2) A translation surface has only finitely many singularities, and hence only finitely many saddle connection vectors in any given direction. Since S is a Veech surface we have both that each non-zero holonomy vector lies in some parabolic direction and that there are only finitely many Γ-orbits of parabolic directions. We choose a representative direction from each of these orbits, and let $\mathcal{V}$ be the set of all saddle connection vectors in these chosen directions.

Since S is in parabolic standard form, the (positively oriented) horizontal is certainly a parabolic direction for S. In particular, Γ has an element of the form $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$; but since S is a Veech surface, also the traces of hyperbolic elements in Γ dominate their conjugates, and thus Lemma 7 holds. We also can and do assume that in the above construction of $\mathcal{V}$, that the horizontal direction is chosen to represent its Γ-orbit.

Let $c' = \min\{|\,\sigma(\,v_1)\,|/|\,v_1\,|\,\}$, with the minimum taken over all horizontal $v = (v_1, 0) \in \mathcal{V}$, and embeddings σ. For $v \in \mathcal{V}$, there is $P \in \Gamma$ such that $Pv = v$. Since P is upper diagonalizable, we can find some vector w such that $Pw = v + w$. We can then express $e_1 = \alpha v + \beta w$ for some real number α, β, with $\beta \neq 0$ when v is non-horizontal. Note that since $Pe_1 = e_1 + \beta v$ and $P \in \Gamma$, we must have that $\beta \in \mathbb{K}$. Choosing such a w for each $v \in \mathcal{V}$, let $c'' = \min\{|\,\sigma(\,\beta)\,|/|\,\beta\,|\,\}$, over all non-horizontal $v \in \mathcal{V}$, and embeddings σ. Finally, set $c = c'''\,c_1^2$ with $c''' = \min\{c', c''\}$.

Now, if h is an arbitrary saddle connection vector of S, then there exists some $A \in \Gamma$ and $v \in \mathcal{V}$ such that $h = Av$. If $v = \alpha e_1$ is horizontal, then h is the multiple by α of the first column of A. Our result clearly holds in this case. Otherwise, with notation as above, induction gives $P^n e_1 = e_1 + n\beta v$, and thus $AP^n e_1 = Ae_1 + n\beta h$. The left hand side is the first column of an element of Γ, thus our standard argument allows us to conclude that each of βh_i with $i = 1, 2$ is greater in absolute value than c_1^2 times any of its conjugates. Here also we find that each of h_i is greater in absolute value than c times any of its conjugates.

Finally, by the finiteness of $\mathscr{V}$ one easily verifies that c may be taken to depend only on S. □

We now can bound the heights of the saddle connection approximants.

LEMMA 8. — *Fix a Veech surface S in parabolic standard form, with trace field $\mathbb{K}$. Let D denote the field extension degree $[\mathbb{K} : \mathbb{Q}]$. There exists a constant $c_2 = c_2(S)$ such that for all $x \in [0,1]$ of infinite $\mathrm{V}_{\mathrm{sc}}(S)$-expansion $(p_n/q_n)_{n\geq 1}$,*

$$H(p_n/q_n) \leq c_2 q_n^D .$$

Proof. — There is a positive integer m depending only on S such that $mv \in \mathscr{O}_K^2$ for all $v \in \mathrm{V}_{\mathrm{sc}}(S)$. Writing $(p_n, q_n) = (\alpha/m, \beta/m)$ with $\alpha, \beta \in \mathscr{O}_K$, Equation (1) gives $h(p_n/q_n) \leq \log m + \frac{1}{D}\sum_\sigma \log^+ \max\{|\sigma(p_n)|, |\sigma(q_n)|\}$. With c as in Theorem 2 we find $h(p_n/q_n) \leq \log cm + \log\max\{|p_n|, q_n\} \leq \log c'cm + \log q_n$ for c' depending only on S. Now, Equation (2) implies the result. □

5. Transcendence with Z-fractions

We prove our transcendance result in the traditional manner: by showing that the sequence of denominators of convergents to an algebraic number cannot grow too quickly. The ingredients are the result of Roth–LeVeque and the convergence bound with a denominator of $q_n q_{n+1}$.

Proof. — (of Theorem 3) Let ε be a positive real number. Let ζ be an algebraic number having an infinite $Z = \mathrm{V}_{\mathrm{sc}}(S)$-expansion with convergents r_n/s_n.

By the Roth–LeVeque Theorem 4, we have

$$|\zeta - r_n/s_n| \gg H(r_n/s_n)^{-2-\varepsilon}, \quad \text{for } n \geq 1.$$

And, hence by Lemma 8, for $n \geq 1$, we have $|\zeta - r_n/s_n| \gg s_n^{-2D-D\varepsilon}$.

The key to the proof is provided by applying Lemma 2 and thus finding that there exists a constant c_3 (independent of n) such that

$$s_{n+1} < c_3 s_n^{2D-1+D\varepsilon}.$$

Thereafter, standard manipulations, as in the proof of Theorem 1.1 of [5] give the result. □

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