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Chris Woodward

Moment maps and geometric invariant theory

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Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

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Moment maps and geometric invariant theory

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Contents

1. Introduction

These are expanded notes from a set of lectures given at the school "Actions Hamiltoniennes: leurs invariants et classification" at Luminy in April 2009. The topics center around the theorem of Kempf and Ness [52], which describes the equivalence between the notion of quotient in geometric invariant theory introduced by Mumford in the 1960's [74], and the notion of symplectic quotient introduced by Meyer [73] and Marsden-Weinstein [71] in the 1970's. Infinite-dimensional generalizations of this equivalence have played an increasingly important role in geometry, starting with the theorem of Narasimhan and Seshadri [75] connecting unitary structures on a bundle with holomorphic stability, which by historical accident preceded the finite-dimensional theorem.

The proof of the Kempf-Ness theorem depends on the convexity of certain *Kempf-Ness functions* whose minima are zeros of the moment map. The convexity also plays an important role in the relation to geometric quantization discovered by Guillemin and Sternberg [35]: it corresponds to the fact that "invariant quantum states concentrate near zeros of the moment map". Roughly speaking these notes were written as an exercise in "just how far" one can carry the convexity of the Kempf-Ness function. For example, using convexity I give alternative proofs of some of the results in Kirwan's book [53] as well as finite-dimensional versions of Harder-Narasimhan and Jordan-Hölder filtrations; the former appears in the algebraic literature under the name of Hesselink one-parameter subgroups [46] but the latter seems to have been undeveloped.

The text is interspersed with applications to existence of invariants in representation theory, such as the problem of determining the existence of invariants in tensor products of irreducible representations, and various techniques for computing moment polytopes. For example, the last section describes Teleman's improved version of quantization commutes with reduction [95] which also covers the behavior of the higher cohomology groups, and the non-abelian localization formula which computes the difference between the sheaf cohomology of the quotient and the invariant

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cohomology of the action. Some of the topics not treated are notably: Duistermaat-Heckman theory, symplectic normal forms, localization theorems in equivariant cohomology, and connections to classical invariant theory, to name a few.

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2. Actions of Lie groups

To establish notation we review the basics of Lie group actions.

2.1. **Lie groups.** A *Lie group* is a smooth manifold K equipped with a group structure so that group multiplication $K \times K \to K$ is a smooth map. The *Lie algebra* $\mathfrak{k}$ is the space of left-invariant vector fields on K, and may be identified with the tangent space of K at the identity $e \in K$. The *exponential map* $\exp : \mathfrak{k} \to K$ is defined by evaluating the time-one flow at the identity.

Suppose that K is compact and connected. Let $T \subset K$ be a maximal torus. We denote by $\Lambda := \exp^{-1}(e) \cap \mathfrak{t}$ the integral lattice and by $\Lambda^\vee \subset \mathfrak{t}^\vee$ its dual, the weight lattice. Any element $\mu \in \Lambda^\vee$ defines a character $T \to U(1), t \mapsto t^\mu$ given for $\xi \in \mathfrak{t}$ by $\exp(\xi)^\mu := \exp(2\pi i \mu(\xi))$. The *Weyl group* of T is denoted $W = N(T)/T$. The Lie algebra $\mathfrak{k}$ splits under the action of T into the direct sum of the Lie algebra $\mathfrak{t}$ and a finite sum of *root spaces* $\mathfrak{k}_\alpha, \alpha \in \mathcal{R}(\mathfrak{k})$ where $\mathcal{R}(\mathfrak{k}) \subset \Lambda^\vee/\{\pm 1\}$ is the set of roots and each $\mathfrak{k}_\alpha$ is identified with a one complex-dimensional representation on which T acts by $\exp(\xi)^\alpha := \exp(2\pi i \alpha(\xi))$. The kernels $\ker(\alpha)$ of the roots $\alpha \in \mathcal{R}(\mathfrak{k})$ divide $\mathfrak{t}$ into a set of (open) *Weyl chambers*; given a generic linear function on $\mathfrak{t}$ there is a unique open *positive Weyl chamber* on which the function is positive; we denote by $\mathfrak{t}_+$ its closure.

2.2. **Smooth actions and quotients.** Let X be a smooth manifold. A *(left) action* of K on X is a smooth map $K \times X \to X,\ (k, x) \mapsto kx$ with the properties that $k_0(k_1 x) = (k_0 k_1)x$ and $ex = x$ for all $k_0, k_1 \in K$ and $x \in X$. A *K-manifold* is a smooth manifold equipped with a smooth K-action. Let X_0, X_1 be K-manifolds. A smooth map $\varphi : X_0 \to X_1$ is *K-equivariant* if $\varphi(kx) = k\varphi(x)$ for all $k \in K, x \in X_0$.

Both the Lie algebra and its dual are naturally K-manifolds: The *adjoint* action of an element $k \in K$ on the Lie algebra $\mathfrak{k}$ is denoted $\mathrm{Ad}(k) \in \mathrm{End}(\mathfrak{k})$. The coadjoint action of k on the dual $\mathfrak{k}^\vee$ is $\mathrm{Ad}^\vee(k) := (\mathrm{Ad}(k^{-1}))^\vee$. The group K itself is a K-manifold in three different ways: the left action, the (inverted) right action, and the *adjoint* action by conjugation $\mathrm{Ad}(k_0)k_1 := k_0 k_1 k_0^{-1}$. The exponential map $\exp : \mathfrak{k} \to K$ is equivariant with respect to the adjoint action on $\mathfrak{k}$ and K. If K is compact, then the dual $\mathfrak{t}^\vee$ of the Lie algebra $\mathfrak{t}$ of the maximal torus T admits a canonical embedding in $\mathfrak{k}^\vee$, whose image is the T-fixed point set for the coadjoint action of T on $\mathfrak{k}^\vee$, and so $\mathfrak{k}^\vee$ admits a canonical projection onto $\mathfrak{t}^\vee$.

Let X be K-manifold. Let $\mathrm{Diff}(X)$ denote the infinite-dimensional group of diffeomorphisms of X and $\mathrm{Vect}(X)$ the Lie algebra of vector fields on X. The K-action induces a canonical group homomorphism

$$K \to \mathrm{Diff}(X), \quad k \mapsto k_X,\ k_X(x) = kx$$

and a Lie algebra homomorphism

$$\mathfrak{k} \to \mathrm{Vect}(X), \quad \xi \mapsto \xi_X,\ \xi_X(x) = \frac{d}{dt}_{t=0} \exp(-t\xi)x.$$

The sign here arises because the Lie bracket is defined using left-invariant vector fields which are the generating vector fields for the *right* action of the group on itself, whereas our actions are by default from the *left*. The *orbit* of a point $x \in X$ is the set $Kx := \{kx | k \in K\} \subset X$. The *stabilizer* of a point $x \in X$ is $K_x := \{k \in K | kx = x\}$; its Lie algebra is the set $\mathfrak{k}_x := \{\xi \in \mathfrak{k} \,|\, \xi_X(x) = 0\}$. A (co)adjoint orbit is an orbit of the (co)adjoint action of K on $\mathfrak{k}$ resp. $\mathfrak{k}^\vee$.

Let $\psi : K_0 \to K_1$ be a homomorphism of Lie groups and let X be a K_1-manifold. The action of K_1 and the homomorphism ψ induce a K_0-action on X by $k_0 x := \psi(k_0)x$. The orbits of the K_0 action are those of the K_1-action, while the stabilizers $(K_0)_x = \psi^{-1}((K_1)_x)$ are inverse images under ψ.

Let X be a K-manifold. A *slice* at x is a K_x-invariant submanifold $V \subset X$ containing x such that KV is open in X and the natural smooth K-equivariant map $K \times_{K_x} V \to KV$ is a diffeomorphism onto its image. It follows from the existence of geodesic flows etc. that actions of *compact* groups have slices. A *quotient* of a K-space is a pair (Y, π) consisting of a space Y and

a K-invariant morphism $\pi : X \to Y$ such that any other K-invariant morphism factors through π. The existence of slices implies that any free action of a compact group K on a manifold X has a manifold quotient X/K; more generally if the action is not free then the quotient exists in the category of Hausdorff topological spaces. (Strictly speaking one should write the quotient on the left, since our actions are by convention left actions. However, I find this rather cumbersome since in English X/K reads "the quotient of X by K").

2.3. **Equivariant differential forms.** Recall that a *graded derivation* of a graded algebra A of degree d is an operator $D \in \mathrm{End}(A)_d$ such that $D(a_0a_1) = D(a_0)a_1 + (-1)^{d|a_0|}a_0D(a_1)$ for homogeneous elements $a_0, a_1 \in A$. The space of graded derivations $\mathrm{Der}(A)$ (direct sum over degrees) forms a *graded Lie algebra* with bracket given by the *graded commutator*: given graded derivations D_0, D_1 of degrees $|D_0|, |D_1|$, define $\{D_0, D_1\} = D_0D_1 - (-1)^{|D_0||D_1|}D_1D_0$.

Let X be a smooth manifold of dimension n. We denote by $\mathrm{Vect}(X)$ the Lie algebra of smooth vector fields on X, and by $\Omega(X) = \bigoplus_{j=0}^{n} \Omega^j(X)$ the graded algebra of smooth forms on X. For any $v \in \mathrm{Vect}(X)$ we have the derivations defined by *contraction* $\iota_v : \Omega^j(X) \to \Omega^{j-1}(X)$ and *Lie derivative* $L_v : \Omega^j(X) \to \Omega^{j+1}(X)$. Let d denote the de Rham operator, the graded derivation $\mathrm{d} : \Omega^j(X) \to \Omega^{j+1}(X)$ such that $\mathrm{d}f(v) = L_v f, \mathrm{d}\mathrm{d}f = 0$ for $f \in \Omega^0(X), v \in \mathrm{Vect}(X)$. The operators ι_v, L_v, d generate a finite dimensional graded Lie algebra of $\mathrm{Der}(\Omega(X))$ with graded commutation relations for $v, w \in \mathrm{Vect}(X)$ given by

$\{\ ,\ \}$	ι_v	L_v	d
ι_w	0	$\iota_{[v,w]}$	L_w
L_w	$\iota_{[w,v]}$	$L_{[w,v]}$	0
d	L_v	0	0

.

It suffices to check the commutation relations by verifying them on generators $f \in \Omega^0(X), \mathrm{d}g \in \Omega^1(X)$ of $\Omega(X)$. We denote by $Z^j(X)$ the space of *closed forms* $Z^j(X) = \{\alpha \in \Omega^j(X) | \mathrm{d}\alpha = 0\}$ by $B^j(X) = \{\alpha \in \Omega^j(X) | \exists \beta \in \Omega^{j-1}(X), \mathrm{d}\beta = \alpha\}$ the space of *exact forms* and by $H^j(X)$ the *de Rham cohomology* $H^j(X) = Z^j(X)/B^j(X)$.

Suppose that X admits a smooth action of a Lie group K. Cartan (see [37]) introduced a space $\Omega_K(X)$ of *K-equivariant forms*

$$\Omega_K^j(X) = \bigoplus_{2a+b=j} \mathrm{Hom}^a(\mathfrak{k}, \Omega^b(X))^K, \quad \Omega_K(X) = \bigoplus_{j=0}^{\infty} \Omega_K^j(X)$$

where $\mathrm{Hom}^a(\cdot)^K$ denotes equivariant polynomial maps of homogeneous degree a. The *equivariant de Rham operator* is defined by

$$\mathrm{d}_K : \Omega_K^j(X) \to \Omega_K^{j+1}(X), \quad (\mathrm{d}_K(\alpha))(\xi) = (\mathrm{d} + \iota_{\xi_X})(\alpha(\xi)).$$

Let $Z_K^j(X)$ resp. B_K^j denote the equivariant closed resp. exact forms. The *equivariant de Rham cohomology* is

$$H_K^j(X) = Z_K^j(X)/B_K^j(X), \quad H_K(X) = \bigoplus_{j=0}^{\infty} H_K^j(X).$$

If K action is free, $H_K(X)$ is isomorphic to the cohomology $H(X/K)$ of the quotient, see for example [37].

3. Hamiltonian group actions

This section contains a quick review of equivariant symplectic geometry. More detailed treatments can be found in Cannas [21], Guillemin-Sternberg [36], Abraham-Marsden [1], or Delzant's lectures in this volume.

3.1. **Symplectic manifolds.** Let X be a smooth manifold. A *symplectic form* on X is a closed non-degenerate two-form $\omega \in \Omega^2(X)$. A *symplectic manifold* is a manifold equipped with a symplectic two-form. A *symplectomorphism* of symplectic manifolds $(X_0, \omega_0), (X_1, \omega_1)$ is a diffeomorphism $\varphi : X_0 \to X_1$ with $\varphi^*\omega_1 = \omega_0$. The term *symplectic* is the Greek translation of the Latin word *complex*, and was used by Weyl to distinguish the classical groups of linear symplectomorphisms resp. complex linear transformations.

The simplest example of a symplectic manifold is $\mathbb{R}^{2n}$ equipped with the standard two-form $\sum_{j=1}^n \mathrm{d}q_j \wedge \mathrm{d}p_j$; Darboux's theorem says that any symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2n}$ equipped with the standard form. There are simple cohomological restrictions on which manifolds admit symplectic forms: Suppose that X has dimension $2n$. Non-degeneracy of a two-form $\omega \in \Omega^2(X)$ is equivalent to the non-vanishing of the highest wedge power $\omega^n \in \Omega^{2n}(X)$; if X is compact and ω is symplectic then the cohomology class $[\omega^n] = [\omega]^n$ must be non-zero, since its integral is non-vanishing, which implies that the classes $[\omega], [\omega]^2, \ldots, [\omega]^{n-1}$ are also non-vanishing. For example this argument rules out the existence of symplectic structures on spheres except for the two-sphere, where any area form gives a symplectic structure.

Symplectic manifolds provide a natural framework for Hamiltonian dynamics as follows. For any symplectic manifold (X, ω) let $\mathrm{Symp}(X, \omega) \subset \mathrm{Diff}(X)$ denote the group of symplectomorphisms and $\mathrm{Vect}^s(X) \subset \mathrm{Vect}(X)$ the Lie subalgebra of *symplectic vector fields* $v \in \mathrm{Vect}(X), L_v\omega = 0$. Any smooth function $H \in C^\infty(X)$ defines a symplectic vector field $H^\# \in \mathrm{Vect}^s(X)$ by $\iota_{H^\#}\omega = \mathrm{d}H$. In local Darboux coordinates, $H^\#$ is given by

$$H^\# = \sum_{j=1}^n \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}.$$

The image of $C^\infty(X)$ in $\mathrm{Vect}^s(X)$ is the space $\mathrm{Vect}^h(X)$ of *Hamiltonian* vector fields. Thus a vector field $v \in \mathrm{Vect}(X)$ is symplectic resp. Hamiltonian iff the associated closed one-form $\iota_v\omega$ is closed resp. exact. The *Poisson bracket* is the Lie bracket on $C^\infty(X)$ defined by the formula

$$\{H_0, H_1\} = \omega(H_0^\#, H_1^\#). \tag{3.1}$$

The map $H \mapsto -H^\#$ extends to an exact sequence of Lie algebras

$$0 \to H^0(X, \mathbb{R}) \to C^\infty(X) \to \mathrm{Vect}^s(X) \to H^1(X, \mathbb{R}) \to 0$$

where the Lie bracket on the de Rham cohomology groups $H^0, H^1(X, \mathbb{R})$ is taken to be trivial. A *Hamiltonian dynamical system* is a pair (X, H) consisting of a symplectic manifold X and an *energy function* $H \in C^\infty(X)$. Time evolution is given by the flow of $H^\# \in \mathrm{Vect}(X)$. If $K \in C^\infty(X)$ is another function, such as a component of angular momentum, then $\{K, H\} = -L_{K^\#}H = L_{H^\#}K$, so H is invariant under the flow generated by $K^\#$ iff K is conserved in time. This equivalence is often called *Noether's theorem*: for every symmetry of a Hamiltonian system there is a conserved quantity.

The cotangent bundle $T^\vee Q$ of a smooth manifold Q possesses a canonical symplectic structure: Let $\pi : T^\vee Q \to Q, (q, p) \to q$ be the canonical projection. The *canonical one-form* on $T^\vee Q$ is

$$\alpha \in \Omega^1(T^\vee Q), \quad \alpha_{(q,p)}(v) = p(D\pi_{q,p}(v)).$$

Local coordinates $q_1, \ldots, q_n$ on Q induce dual coordinates $p_1, \ldots, p_n$ in which $\alpha = \sum_{j=1}^n p_j \mathrm{d}q_j$. It follows that the *canonical two-form* ω on $T^\vee Q$ given by $\omega = -\mathrm{d}\alpha$ is symplectic. These forms are canonical in the sense that any diffeomorphism $Q_0 \to Q_1$ induces an isomorphism $T^\vee Q_0 \to T^\vee Q_1$ preserving the canonical one-forms, which is therefore a symplectomorphism. Physically $T^\vee Q$ represents the space of states of a classical particle moving on a manifold Q. However, many Hamiltonian dynamical systems have symplectic manifolds that are not cotangent bundles. For example, the two-sphere is the natural symplectic manifold for the study of the evolution of the angular momentum vector of a rigid body.

Proposition 3.1.1. *The following are natural operations on symplectic manifolds:*

(a) *(Sums) Let $(X_0, \omega_0), (X_1, \omega_1)$ be symplectic manifolds. Then the disjoint union $(X_0 \sqcup X_1, \omega_0 \sqcup \omega_1)$ is a symplectic manifold.*
(b) *(Products) Let (X_j, ω_j) be symplectic manifolds, $j = 0, 1$. Then the product $X_0 \times X_1$ equipped with two-form $\pi_0^*\omega_0 + \pi_1^*\omega_1$ is a symplectic manifold, where $\pi_j : X_0 \times X_1 \to X_j, j = 0, 1$ is the projection onto X_j.*
(c) *(Duals) Let (X, ω) be a symplectic manifold. Then the dual $(X, -\omega)$ (or more generally, $(X, \lambda\omega)$ for any non-zero $\lambda \in \mathbb{R}$) is a symplectic manifold.*

Symplectomorphism is a very restrictive notion of morphism, since in particular the symplectic manifolds must be the same dimension. A more flexible notion of morphism in the symplectic category is given by the notion of *Lagrangian correspondence* [99]. (The discussion of correspondences

is only used to formulate the universal property for symplectic quotients; readers not interested in this can skip all discussion of correspondences and the symplectic category.) Let (X, ω) be a symplectic manifold. A *Lagrangian submanifold* of X is a submanifold $i : L \to X$ with $i^*\omega = 0$ and $\dim(L) = \dim(X)/2$. Let $(X_j, \omega_j), j = 0, 1$ be symplectic manifolds. A *Lagrangian correspondence* from X_0 to X_1 is a Lagrangian submanifold of $X_0^- \times X_1$. Let $L_{01} \subset X_0^- \times X_1$ and $L_{12} \subset X_1^- \times X_2$ be Lagrangian correspondences. Let π_{02} denote the projection from $X_0^- \times X_1 \times X_1^- \times X_2$. Then

$$L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{X_1} L_{12})$$

is, if smooth and embedded, a Lagrangian correspondence in $X_0^- \times X_2$ called the *composition* of L_{01} and L_{12}. The graph $\mathrm{graph}(\psi_{01})$ of any symplectomorphism ψ_{01} from X_0 to X_1 is automatically a Lagrangian correspondence, and if ψ_{01}, ψ_{12} are two such symplectomorphisms then $\mathrm{graph}(\psi_{01} \circ \psi_{12}) = \mathrm{graph}(\psi_{01}) \circ \mathrm{graph}(\psi_{12})$. With this notion of composition, the pair (symplectic manifolds, Lagrangian correspondences) becomes a partially defined category, with identity given by the diagonal correspondence. The partially defined composition leads to an honest category, obtained by allowing sequences of morphisms and identifying sequences if they are related by geometric composition [98].

Symplectic geometry can be considered a special case of Poisson geometry: A *Poisson bracket* on a manifold X is a Lie bracket $\{\ ,\ \} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ that is a derivation with respect to multiplication of functions, that is, $\{f, gh\} = \{f, g\}h + g\{f, h\}$. A *Poisson manifold* is a manifold equipped with a Poisson bracket. A *morphism* of Poisson manifolds is a smooth map $\psi : X_0 \to X_1$ such that $\{\psi^* f, \psi^* g\} = \psi^*\{f, g\}$. Given any Poisson bracket on a manifold X, for each $H \in C^\infty(X)$ the derivation $\{H,\ \}$ is equal to $L_{H^\#}$ for some vector field $H^\#$. The span of the vector fields $H^\#$ defines a decomposition of X into *symplectic leaves*, each of which is equipped with a symplectic structure so that (3.1) holds. On the other hand, the notion of symplectic geometry as a special case of Poisson geometry is not particularly compatible with the idea that Lagrangian correspondences should serve as morphisms.

3.2. **Hamiltonian group actions.** Let K be a Lie group acting smoothly on a manifold X. The action is *symplectic* if it preserves the symplectic form, that is, $k_X \in \mathrm{Symp}(X, \omega)$ for all $k \in K$, *infinitesimally symplectic* if $\xi_X \in \mathrm{Vect}^s(X)$ for all $\xi \in \mathfrak{k}$, and *weakly Hamiltonian* if $\xi_X \in \mathrm{Vect}^h(X)$ for all $\xi \in \mathfrak{k}$. A *symplectic K-manifold* is a symplectic manifold equipped with a symplectic action of K.

Let (X, ω) be a symplectic K-manifold. The action is *Hamiltonian* if the map $\mathfrak{k} \to \mathrm{Vect}(X)$, $\xi \mapsto \xi_X$ lifts to an equivariant map of Lie algebras $\mathfrak{k} \to C^\infty(X)$. Such a map is called a *comoment map*. A *moment map* is an equivariant map $\Phi : X \to \mathfrak{k}^\vee$, satisfying

$$\iota_{\xi_X}\omega = -d\langle \Phi, \xi \rangle, \quad \forall \xi \in \mathfrak{k} \tag{3.2}$$

Any comoment map $\phi : \mathfrak{k} \to C^\infty(X)$ defines a moment map by $\langle \Phi(x), \xi \rangle = (\phi(\xi))(x)$.

Example 3.2.1. Let $K = V$ be a vector space acting on $X = T^\vee V$ by translation. After identifying $\mathfrak{k} \to V$ and so $\mathfrak{k}^\vee \to V^\vee$, a moment map for the action is given by the projection $X \cong V \times V^\vee \to V^\vee, (q, p) = p$, that is, by the ordinary momentum, hence the terminology *moment map*.

The notion of moment map was introduced in independent work of Kirillov, Kostant, and Souriau, in connection with geometric quantization and representation theory. See [14] for a discussion of the history of the moment map and the relationship of the work between these authors. Unfortunately there is no standard sign convention for (3.2); our convention agrees with that of Kirwan [53]. More generally, if X is a smooth manifold equipped with a closed two-form ω and an action of K leaving ω invariant, then we say that Φ is a moment map if (3.2) holds.

A *Hamiltonian resp. degenerate Hamiltonian K-manifold* is a datum (X, ω, Φ) consisting of a symplectic K-manifold (X, ω) resp. smooth K-manifold X equipped with an invariant closed two-form ω, and a moment map Φ for the action. Let (X_0, ω_0, Φ_0) and (X_1, ω_1, Φ_1) be Hamiltonian K-manifolds. An *isomorphism* of Hamiltonian K-manifolds is a K-equivariant symplectomorphism $\varphi : (X_0, \omega_0) \to (X_1, \omega_1)$ such that $\varphi^*\Phi_1 = \Phi_0$.

Archimedes' computation of the area of the two-sphere is essentially a moment map calculation. Let $S^2 = \{x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \in \mathrm{Vect}(\mathbb{R}^3)$. The two-form $\omega = \iota_v(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y$ restricts to a symplectic form on S^2, invariant under rotation on $\mathbb{R}^3$.

Proposition 3.2.2. *A moment map for the action of S^1 on S^2 by rotation clockwise around the z-axis is given by $(x, y, z) \mapsto z$, under the identification of the Lie algebra of S^1 and its dual with $\mathbb{R}$.*

Proof. The generating vector field for $\xi = 1$ is $\xi_X = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$. A computation shows that $\iota_{\xi_X}\omega = -dz$. □

To relate this to Archimedes' formula, note that if r, θ, z are cylindrical coordinates on $\mathbb{R}^3$, then $\iota_{\frac{\partial}{\partial\theta}}\omega = \mathrm{d}z$ and so $\omega = \mathrm{d}z \wedge \mathrm{d}\theta$. Thus

Corollary 3.2.3 (Archimedes)**.** *The area of the unit two-sphere between any two values $z_1, z_2 \in (-1, 1)$ of z is the same as the area of the cylinder $S^1 \times [-1, 1]$ between those two values, $2\pi(z_2 - z_1)$.*

In particular (and this is the result reported by Cicero to be inscribed on Archimedes' tombstone) the area of the unit two-sphere S^2 is equal to the area of the cylinder $S^1 \times [-1, 1]$, namely 4π. We

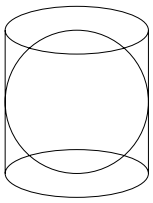

Figure 3.1: $S^1 \times [-1, 1]$ has the same area as S^2

can deduce from the moment map for the circle action the moment map for the full rotation group $SO(3)$ as follows. We identify $\mathfrak{so}(3) \to \mathbb{R}^3$ so that the infinitesimal rotation around the j-th basis vector e_j maps to e_j.

Corollary 3.2.4. *After identifying $\mathfrak{so}(3)^\vee \to \mathbb{R}^3$, the action of $SO(3)$ on S^2 has moment map the inclusion $S^2 \to \mathbb{R}^3$.*

Proof. By symmetry, moment maps for the rotation around the other two axes are given by $(x, y, z) \mapsto x$ resp. y. Hence the inclusion satisfies the equation (3.2). In addition Φ is equivariant and so defines a moment map. □

The following are natural operations on Hamiltonian K-manifolds:

Proposition 3.2.5. (a) *(Sums) Let $(X_0, \omega_0, \Phi_0), (X_1, \omega_1, \Phi_1)$ be Hamiltonian K-manifolds. Then the disjoint union $X_0 \sqcup X_1$ is a Hamiltonian K-manifold, equipped with moment map $\Phi_0 \sqcup \Phi_1$.*

(b) *(Exterior Products) Let (X_j, ω_j, Φ_j) be Hamiltonian K_j-manifolds, $j = 0, 1$. Then the product $X_0 \times X_1$ is a Hamiltonian $K_0 \times K_1$-manifold, equipped with moment map $\pi_0^*\Phi_0 \times \pi_1^*\Phi_1$, where $\pi_j : X_0 \times X_1 \to X_j, j = 0, 1$ is the projection onto X_j.*

(c) *(Duals) Let (X, ω, Φ) be a Hamiltonian K-manifold. Then the dual $(X, -\omega, -\Phi)$ (or more generally, any rescaling by a non-zero constant) is a Hamiltonian K-manifold.*

(d) *(Pull-backs) Let $\varphi : K_0 \to K_1$ be a homomorphism of Lie groups, and (X, ω, Φ) a Hamiltonian K_1-manifold. The Lie algebra homomorphism $D\varphi : \mathfrak{k}_0 \to \mathfrak{k}_1$ induces a dual map $D\varphi^\vee : \mathfrak{k}_1^\vee \to \mathfrak{k}_0^\vee$. The action of K_0 induced by ϕ has moment map $D\varphi^\vee \circ \Phi$.*

(e) *(Interior products) Let (X_j, ω_j, Φ_j) be Hamiltonian K-manifolds, $j = 0, 1$. Then the product $X_0 \times X_1$ is a Hamiltonian K-manifold, equipped with moment map $\pi_0^*\Phi_0 + \pi_1^*\Phi_1$. This is a combination of the previous two items, using the diagonal embedding $\mathfrak{k} \to \mathfrak{k} \times \mathfrak{k}$ whose adjoint is $\mathfrak{k}^\vee \times \mathfrak{k}^\vee \to \mathfrak{k}^\vee, (\xi_0, \xi_1) \mapsto \xi_0 + \xi_1$.*

More generally one can speak of Hamiltonian actions on Poisson manifolds. The dual $\mathfrak{k}^\vee$ of the Lie algebra $\mathfrak{k}$ has a canonical *Lie-Poisson* bracket, $C^\infty(\mathfrak{k}^\vee) \times C^\infty(\mathfrak{k}^\vee) \to C^\infty(\mathfrak{k}^\vee)$ with the property that $\{\xi, \eta\} = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{k}$. A *Poisson moment map* for a K-action on a Poisson manifold X is a Poisson map $\Phi : X \to \mathfrak{k}^\vee$. A *Hamiltonian-Poisson K-manifold* is a Poisson K-manifold equipped with a Poisson moment map.

Proposition 3.2.6. *Any Hamiltonian K-manifold (X, ω, Φ) is a Hamiltonian-Poisson K-manifold.*

Proof. For $\lambda, \xi \in \mathfrak{k}$ we have $\Phi^*\{\lambda, \xi\} = \Phi^*[\lambda, \xi] = L_{\lambda_X}\Phi^*\xi = \{\Phi^*\lambda, \Phi^*\xi\}$. The case of non-linear functions is similar. □

Conversely, any Poisson moment map induces an ordinary moment map on its symplectic leaves. In particular the coadjoint action is Poisson-Hamiltonian with moment map the identity, and the symplectic leaves are the coadjoint orbits. Thus as observed by Kirillov, Kostant, and Souriau,

Proposition 3.2.7. *Any coadjoint orbit $K\lambda, \lambda \in \mathfrak{k}^\vee$ of K has the canonical structure of a Hamiltonian K-manifolds with moment map given by the inclusion $K\lambda \to \mathfrak{k}^\vee$.*

Example 3.2.8. Identify $\mathbb{R}^3 \cong \mathfrak{so}(3) \cong \mathfrak{so}(3)^*$. The Proposition gives Hamiltonian $SO(3)$-structures on the orbits of $SO(3)$ on $\mathbb{R}^3$, which are either spheres (for non-zero radii λ) or a point (if $\lambda = 0$.) This reproduces the Corollary 3.2.4.

For any *transitive* Hamiltonian action, the moment map is a local diffeomorphism and so gives a covering of the coadjoint orbit that is its image, see Kostant [62].

The Darboux theorem has various equivariant generalizations that we will not discuss here; we only mention that as a consequence:

Proposition 3.2.9. *(see* [53]*) Let X be a Hamiltonian K-manifold, K compact. For any $\xi \in \mathfrak{k}$, the function $\langle \Phi, \xi \rangle$ is a Morse function with even index.*

In the remainder of the section we explain two other ways in which moment maps can be naturally interpreted. The first is closely related to the notion of *equivariantly closed differential form* introduced in Section 2.3, see Atiyah and Bott [7]:

Proposition 3.2.10. *Let (X, ω) be a symplectic K-manifold. There exists a one-to-one correspondence between moment maps for the action of K, and equivariantly closed extensions of $\omega \in \Omega^2(X)$ to $\Omega^2_K(X)$.*

Proof. Since $\Omega^2_K(X) \cong \Omega^2(X)^K \oplus \mathrm{Hom}(\mathfrak{k}, \Omega^0(X))^K$ any extension in $\Omega^2_K(X)$ is equal to $\omega + \Phi$ for some $\Phi \in \mathrm{Map}_K(X, \mathfrak{k}^\vee) \cong \mathrm{Hom}(\mathfrak{k}, \Omega^0(X))^K$. The extension if equivariantly closed iff $0 = \mathrm{d}_K(\omega + \Phi) = (\mathrm{d}\omega, \iota_{\xi_X}\omega + \mathrm{d}\langle \Phi, \xi \rangle)$. Since ω is by assumption closed, $\mathrm{d}_K(\omega + \Phi) = 0$ iff Φ is a moment map. □

The second interpretation of a moment map depends on the notion of *linearization* of an action, as we now explain. Suppose that $L \to X$ is a Hermitian line bundle with unit circle bundle L_1 with generating vector fields $\xi_L \in \mathrm{Vect}(L_1), \xi \in \mathbb{R}$. The circle group $U(1)$ acts on L_1 by scalar multiplication. Let $\alpha \in \Omega^1(L_1)^{U(1)}, \alpha(\xi_L) = \xi$ be a connection one-form with curvature $(2\pi/i)\omega \in \Omega^2(X)$. (That is, to fix conventions, $\mathrm{d}\alpha = \pi^*\omega$ where $\pi : L_1 \to X$ is the projection.) The group $\mathrm{Aut}(L_1, \alpha)$ of unitary automorphisms of L preserving α naturally maps to the symplectomorphism group $\mathrm{Symp}(X, \omega)$ of X, defining an exact sequence $1 \to U(1) \to \mathrm{Aut}(L_1, \alpha) \to \mathrm{Symp}(X, \omega)$. A *linearization* of the action of K on X is a lift $K \to \mathrm{Aut}(L_1, \alpha)$. An *infinitesimal linearization* is a lift $\mathfrak{k} \to \mathrm{Vect}(L_1)^{U(1)}$.

Proposition 3.2.11. *Let X be a K-manifold, $\omega \in \Omega^2(X)^K$ a closed invariant two-form, and $\pi : L \to X$ a Hermitian line-bundle with connection one-form $\alpha \in \Omega^1(L_1)^{K \times U(1)}$ whose curvature is equal to $(2\pi/i)\omega$. The set of moment maps Φ for the K-action is in one-to-one correspondence with the set of infinitesimal linearizations of the action of K.*

Proof. Let $\pi_1 : L_1 \to X$ denote the projection. Given a lift $\mathfrak{k} \to \mathrm{Vect}(L_1)^{U(1)}$, define a moment map $\Phi : X \to \mathfrak{k}^\vee$ by $\langle \Phi(x), \xi \rangle = (\alpha(\xi_L))(l)$, for any $l \in \pi^{-1}(x)$, independent of the choice of l. Then

$$\begin{aligned} \pi_1^*\mathrm{d}\langle \Phi, \xi \rangle &= \mathrm{d}(\alpha(\xi_L)) = \mathrm{d}\iota_{\xi_L}\alpha(l) = (L_{\xi_L} - \iota_{\xi_L}\mathrm{d})\alpha \\ &= L_{\xi_L}\alpha - \iota_{\xi_L}\pi_1^*\omega = -\pi_1^*\iota_{\xi_X}\omega. \end{aligned}$$

Since α is invariant, Φ is equivariant, and so defines a moment map. Conversely, given a moment map define $\xi_L \in \mathrm{Vect}(L_1)^{S^1}$ by $\langle \Phi(x), \xi \rangle = (\alpha(\xi_L))(l)$. Then the same computation shows that $L_{\xi_L}\alpha = 0$. To see that $\xi \mapsto \xi_L$ defines a lift of $\mathfrak{k} \to \mathrm{Vect}^s(X, \omega)$ to $\mathrm{Vect}(L_1)^{U(1)}$, note that given $\xi, \eta \in \mathfrak{k}$, the vectors $[\xi, \eta]_L$ and $[\xi_L, \eta_L]$ agree up to a vertical vector field. To see that they are equal, note $\alpha([\xi_L, \eta_L]) = [L_{\xi_L}, \iota_{\eta_L}]\alpha = \pi^* L_{\xi_L}\langle \Phi, \eta \rangle = \pi^*\langle \Phi, [\xi, \eta] \rangle = \alpha([\xi, \eta]_L)$. □

The following is immediate from the definitions:

Proposition 3.2.12. *Suppose that Φ is the moment map induced by a lift of the action to a Hermitian line bundle with connection L. Then $\exp(\xi), \xi \in \mathfrak{k}_x$ acts on the fiber L_x via $l \mapsto \exp(i\langle\Phi(x), \xi\rangle)l$.*

In other words, the value of the moment map at a fixed point determines the action of the identity component of the group on the fiber over that point.

The notion of Lagrangian correspondence generalizes to Hamiltonian actions as follows. (again, readers not interested in universal properties of quotients may skip this discussion):

Definition 3.2.13. Let X be a Hamiltonian K-manifold with moment map $\Phi : X \to \mathfrak{k}^\vee$. A *$K$-Lagrangian submanifold* is a K-invariant Lagrangian submanifold on which Φ vanishes. Let (X_j, ω_j, Φ_j) be Hamiltonian K-manifolds for $j = 0, 1$. A *K-Lagrangian correspondence* is a K-Lagrangian submanifold of $X_0^- \times X_1$.

Allowing sequences of K-Lagrangian correspondences and identifying sequences related by a geometric composition gives an honest category as in non-equivariant case.

3.3. **Symplectic quotients.** Naturally one would like a notion of quotient of a Hamiltonian K-manifold, which should be an object in the symplectic category and satisfy a universal property for morphisms in the equivariant symplectic category. It is easy to see that the most naive definition, of the actual quotient, is unsatisfactory for several reasons. For example, even if the action is free, then the quotient will not necessarily have even dimension, and so may not admit a symplectic structure. Also the action will not in general be free, and so the quotient will not even have the structure of a manifold.

The construction of Meyer [73] and Marsden-Weinstein [71] is free of these problems, at least under suitable hypotheses: Let (X, ω, Φ) be a Hamiltonian K-manifold with moment map $\Phi : X \to \mathfrak{k}^\vee$. Define the *symplectic quotient*

$$X/\!\!/K := \Phi^{-1}(0)/K.$$

Theorem 3.3.1 (Meyer [73], Marsden-Weinstein [71])**.** *Let X be a Hamiltonian K-manifold. If K acts freely and properly on $\Phi^{-1}(0)$, then $X/\!\!/K$ has the structure of a smooth manifold of dimension $\dim(X) - 2\dim(K)$ with a unique symplectic form ω_0 satisfying $i^*\omega = p^*\omega_0$, where $i : \Phi^{-1}(0) \to X$ and $p : \Phi^{-1}(0) \to X/\!\!/K$ are the inclusion and projection respectively.*

The double slash in the notation $X/\!\!/K$ is meant to reflect that the dimension drops by $2\dim(K)$, in contrast to the ordinary quotient X/K for which dimension drops by $\dim(K)$, if the action is free. The proof depends on the following. Let $\mathrm{ann}(\mathfrak{k}_x) \subset \mathfrak{k}^\vee$ be the annihilator of $\mathfrak{k}_x$.

Lemma 3.3.2. *Let X be a Hamiltonian K-manifold. For any $x \in X$,*

(a) $\mathrm{Im}\, D_x\Phi = \mathrm{ann}(\mathfrak{k}_x)$.
(b) $\mathrm{Ker}\, D_x\Phi = \{\xi_X(x), \xi \in \mathfrak{k}\}^{\omega_x}$.

Proof. (a) We have $\langle D_x\Phi(v), \eta\rangle = \omega_x(v, \eta_X(x))$ for $v \in T_xX$ which vanishes for all $v \in T_xX$ iff $\eta_X(x) = 0$. (b) The same identity shows $\omega_x(\xi_X(x), v) = 0$ for $v \in \mathrm{Ker}\, D_x\Phi$, so the left-hand-side of (b) is contained in the right. Equality now follows by a dimension count, using (a). □

Proof of Theorem. By part (a) of the Lemma, the pull-back i^*w vanishes on the orbits of K and is K-invariant, and so descends to a form ω_0 on $X/\!\!/K$. Part (b) shows that ω_0 is non-degenerate. Since $p^*\mathrm{d}\omega_0 = \mathrm{d}i^*\omega = i^*\mathrm{d}\omega = 0$, ω_0 is closed, hence symplectic. □

The following is a fundamental example:

Example 3.3.3. (Products of spheres) Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers and $X = S^2_{\lambda_1} \times \ldots \times S^2_{\lambda_n}$, where S^2_λ denotes the unit two-sphere with invariant area form re-scaled by λ.

Lemma 3.3.4. *The group $K = SO(3)$ acts diagonally on $X = (S^2)^n$ with moment map $\Phi : X \to \mathfrak{k}^\vee \cong \mathbb{R}^3$, $\quad (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$.*

Proof. By 3.2.4 and 3.2.5 (e). □

The symplectic quotient is the *moduli space of closed n-gons with lengths* $\lambda_1, \ldots, \lambda_n$

$$X/\!\!/SO(3) = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3)^n \mid \|x_j\| = \lambda_j,\ x_1 + \ldots + x_n = 0\}/SO(3).$$

Its topology depends on the choice of $\lambda_1, \ldots, \lambda_n$, see for example Hausmann-Knutson [42]. In general there are a finite number of "chambers" in which the topology of $X/\!\!/SO(3)$ is constant. The chambers in which $X/\!\!/SO(3)$ is non-empty are described by the following:

Proposition 3.3.5. *$X/\!\!/SO(3) \neq \emptyset$ iff $\lambda_j \leq \sum_{i \neq j} \lambda_i$ for all $j = 1, \ldots, n$.*

Proof. For $n = 3$, these are the triangle inequalities. For $n > 3$, we assume without loss of generality that $\lambda_1 \geq \ldots \geq \lambda_n$. Then the inequalities above are equivalent to the single inequality $\lambda_1 \leq \lambda_2 + \ldots + \lambda_n$. One checks that there exists j so that $|\lambda_2 + \ldots + \lambda_j - \lambda_{j+1} - \ldots - \lambda_n| < \lambda_1$. The general case follows from that for $n = 3$, which implies that there exists a triangle with side lengths $\lambda_1, \lambda_2 + \ldots + \lambda_j, \lambda_{j+1} + \ldots + \lambda_n$. □

This ends the example.

We end this section with two remarks on the definition of symplectic quotient. First, the symplectic quotient of a Hamiltonian action can be viewed as a symplectic leaf of the quotient of the corresponding Hamiltonian-Poisson action in the following sense. Suppose that X is a Hamiltonian-Poisson K-manifold such that K acts freely. The restriction of the Poisson bracket to $C^\infty(X)^K$ defines a canonical Poisson structure on X/K. Then $X/\!\!/K$ is a symplectic leaf on the smooth locus in X/K [4]; the other leaves are symplectic quotients at other coadjoint orbits, discussed in Section 8.

Second, the symplectic quotient satisfies the following universal property for quotients. Suppose that (X, ω, Φ) is a Hamiltonian K-manifold and K acts freely on $\Phi^{-1}(0)$. We denote by $L_\Phi \subset X^- \times (X/\!\!/K)$ the image of $\Phi^{-1}(0)$ under $i \times p$. Then L_Φ is a K-Lagrangian correspondence.

Theorem 3.3.6. *Suppose that X is a Hamiltonian K-manifold. If Y is a symplectic manifold with trivial K-action, then any K-Lagrangian correspondence from X to Y factors through L_Φ.*

Proof. Suppose for simplicity that the morphism consists of a single correspondence $L \subset X^- \times Y$. By definition of K-Lagrangian correspondence, $L \subset \Phi^{-1}(0) \times Y$. Since K acts freely on $\Phi^{-1}(0)$, L/K is a submanifold of $X^-/\!\!/K \times Y$ and is easily checked to be Lagrangian. Then $L = L/K \circ L_\Phi$. □

Unfortunately the generalization of this universal property to arbitrary morphisms in the symplectic category requires rather complicated freeness assumptions.

3.4. **Fubini-Study actions.** Kähler manifolds are complex manifolds with symplectic structures that are compatible, in a certain sense, with the complex structure. An *almost complex structure* on a manifold X is an endomorphism $J \in \mathrm{End}(TX)$ with $J^2 = -I$, where $I \in \mathrm{End}(TX)$ is the identity. An almost complex structure J is *compatible* with a symplectic structure ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric. Any symplectic manifold admits a compatible almost complex structure; a Kähler manifold is a symplectic manifold equipped with an integrable compatible almost complex structure.

Affine and projective space have natural *Fubini-Study* Kähler structures as follows. Any Hermitian structure $(\ ) : V \times V \to \mathbb{C}$ defines a symplectic structure on V via its imaginary part,

$$\omega_{V,v}(v_1, v_2) = \mathrm{Im}(v_1, v_2).$$

while its real part gives a Riemannian metric on V. Let K be a Lie group acting on V. If K preserves the Hermitian structure then the action is symplectic and a canonical moment map is given by

$$\langle \Phi_V(v), \xi \rangle = \mathrm{Im}(v, \xi v)/2.$$

Example 3.4.1. Let $K = \mathrm{Sp}(V, \omega)$ be the group of linear symplectomorphisms of V then the map $\xi \mapsto \langle \Phi_V, \xi \rangle$ defines an isomorphism of the Lie algebra $\mathfrak{sp}(V, \omega)$ with $\mathrm{Sym}^2(V^\vee)$, analogous to the isomorphism of the *orthogonal* Lie algebras $\mathfrak{o}(V, g)$ with $\Lambda^2(V)$. The Lie algebra structure induced on $\mathrm{Sym}^2(V^\vee)$ is that induced from the Poisson bracket by the inclusion $\mathrm{Sym}^2(V^\vee) \subset C^\infty(V)$.

Example 3.4.2. Let $K = S^1$ act on $V = \mathbb{C}^n$ with weights $a_1, \ldots, a_n$. If the Hermitian structure on V is the standard one then the moment map on V is Hamiltonian with moment map

$$\Phi(z_1, \ldots, z_n) = \sum_{j=1}^n -a_j |z_j|^2/2$$

In particular, if K acts by scalar multiplication then the moment map is

$$\Phi(z_1, \ldots, z_n) = -\sum_{j=1}^n |z_j|^2/2.$$

The canonical symplectic quotient $V/\!\!/S^1$ is a point. If we shift the moment map by a scalar, $\Phi_c = \Phi + c$, then the symplectic quotient is

$$V/\!\!/S^1 = \left\{ \sum_{j=1}^n |z_j|^2/2 = c \right\} / S^1$$

which identifies with the projective space $\mathbb{P}(V)$ of complex lines in V via $V/\!\!/S^1 \to \mathbb{P}(V), [v] \mapsto \mathrm{span}(v)$.

It follows that projective space $\mathbb{P}(V)$ naturally has a symplectic structure, called the *Fubini-Study* symplectic form $\omega_{\mathbb{P}(V)}$. Explicitly this is given as follows: The tangent space to $\mathbb{P}(V)$ at $[v], v \in V - \{0\}$ naturally identifies with the Hermitian orthogonal to $[v]$. Then

$$\omega_{\mathbb{P}(V),[v]}(v_1, v_2) = \frac{\mathrm{Im}(v_1, v_2)}{(v, v)}.$$

If $z_1, \ldots, z_n$ are coordinates corresponding to a unitary basis then

$$\omega_{\mathbb{P}(V),[z]} = \frac{i \sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j}{2 \sum_{j=1}^n z_j \overline{z}_j}.$$

If K acts on V preserving the Hermitian structure, then it commutes with the action of S^1. The induced action on $\mathbb{P}(V)$ is also symplectic, and has canonical moment map

$$\langle \Phi_{\mathbb{P}(V)}([v]), \xi \rangle = \mathrm{Im}(v, \xi v)/(v, v).$$

Suppose that $K = S^1$, and acts on V with weights $a_1, \ldots, a_n \in \mathbb{Z}$. The action of K on $\mathbb{P}(V)$ is Hamiltonian with moment map

$$\Phi_{\mathbb{P}(V)}([z_1, \ldots, z_n]) = \frac{\sum_{j=1}^n -a_j |z_j|^2/2}{\sum_{j=1}^n |z_j|^2/2}. \tag{3.3}$$

Proposition 3.4.3. *Let K act on V preserving the Hermitian structure. Any smooth invariant subvariety $X \subset \mathbb{P}(V)$ inherits the structure of a Hamiltonian K-manifold from the Fubini-Study Hamiltonian K-manifold structure on $\mathbb{P}(V)$.*

Proof. It suffices to check that the restriction of $\omega_{\mathbb{P}(V)}$ to X is non-degenerate, which holds since $\omega_{\mathbb{P}(V)}(v, Jv) > 0$ for $v \in T_x X, Jv \in T_x X$ since $T_x X$ is J-invariant. □

3.5. **Geometric quantization.** The philosophy of geometric quantization played an important role in the development of equivariant symplectic geometry. Unfortunately good quantization schemes exist only for certain classes of Hamiltonian actions.

Suppose that Q is a manifold and $T^\vee Q$ its cotangent bundle. One thinks of $T^\vee Q$ as the space of classical states for a particle moving on Q, with a vector in $T_q^\vee Q$ representing the momentum. In quantum mechanics the state of the system is given by a *quantum wave-function* $\psi \in L^2(Q)$, whose norm-square $|\psi(q)|^2$ represents the probability of finding the particle at position q, if its position is measured. The construction of $L^2(Q)$ from $T^\vee Q$ can be done in two steps: first cut down the number of directions by half, then pass to functions.

One can try to extend this procedure to arbitrary symplectic manifolds (X, ω) by axiomatizing this two-step process. A *Lagrangian distribution* resp. *complex Lagrangian distribution* is a subbundle $P \subset TX$ resp $TX \otimes_{\mathbb{R}} \mathbb{C}$ such that each fiber P_x is a Lagrangian subspace of $T_x X$ resp. complex Lagrangian subspace of $T_x X \otimes_{\mathbb{R}} \mathbb{C}$. A *polarization* is a Hermitian line bundle L

with connection ∇ such that the curvature of ∇ is $\mathrm{curv}(\nabla) = (2\pi/i)\omega$. A *quantization datum* resp. *complex quantization datum* consists of a Lagrangian distribution resp. complex Lagrangian distribution together with a polarization. The original literature on geometric quantization uses polarization to refer to the Lagrangian distribution. This conflicts with the use of polarization in the geometric invariant theory literature, which we have adopted. The *geometric quantization* of (X, ω) (depending on the choice of (P, L, ∇)) is the vector space of smooth sections of L which are covariant constant with respect to ∇ along P:

$$\mathcal{H}(X, \omega) := \{\sigma \in \Gamma(L), \nabla_v \sigma = 0 \ \forall v \in P\}.$$

We ignore the problem of defining a Hilbert space structure on $\mathcal{H}(X, \omega)$, see [35] for more details.

A case for which a good quantization procedure exists is the case that X is a compact Kähler Hamiltonian K-manifold equipped with polarization $\mathcal{O}_X(1) \to X$. A Lagrangian distribution is provided by the antiholomorphic directions on X, that is, $P = T^{0,1}X \subset TX \otimes_{\mathbb{R}} \mathbb{C}$. Then $\mathcal{H}(X, \omega) = H^0(X, \mathcal{O}_X(1))$. In other words, in the language of geometric quantization *holomorphic sections of the polarizing line bundle are quantum states.*

One can now compare the various operations on symplectic manifolds with those on vector spaces:

Proposition 3.5.1. (a) *(Duals) If J is the complex structure for X then $-J$ is a compatible complex structure for X^-. If $P = T^{0,1}X$ then $\overline{P} = T^{0,1}X^-$. Furthermore, $\overline{L}$ with connection $-\alpha$ is naturally a polarization for X^-. Thus $\mathcal{H}(X^-)$ is the space of complex-conjugates of sections of L, which is naturally identified with the dual $\mathcal{H}(X)^\vee$ of $\mathcal{H}(X)$.*

(b) *(Sums) If X_0, X_1 are Kähler Hamiltonian K-manifolds with polarizations, then $\mathcal{H}(X_0 \cup X_1) = \mathcal{H}(X_0) \oplus \mathcal{H}(X_1)$.*

(c) *(Products) With the same assumptions as in (b), $\mathcal{H}(X_0 \times X_1) = \mathcal{H}(X_0) \otimes \mathcal{H}(X_1)$.*

Example 3.5.2. Let $X = S^2 \cong \mathbb{P}^1$ and ω the standard symplectic form. The moment map for the action of S^1 on $(X, d\omega)$ is has image $[-d, d]$. The d-th tensor product $\mathcal{O}_X(d)$ of the hyperplane bundle $\mathcal{O}_X(1)$ is a polarization of $(X, d\omega)$, so that $\mathcal{H}(X, \omega) = H^0(X, \mathcal{O}_X(d))$ is the space of homogeneous polynomials in two variables of degree d. Note that the weights of $\mathcal{H}(X, \omega)$ are $\{d, d-2, d-4, \ldots, -d\}$, which are the intersections points of the image $\Phi(X)$ with the lattice $d + 2\mathbb{Z} \subset \mathbb{Z}$. The $SU(2)$-action on X induces on $\mathcal{H}(X)$ the structure of an $SU(2)$-module with highest weight d. The product of spheres $S^2_{\lambda_1} \times \ldots \times S^2_{\lambda_n}$ has quantization the tensor product of simple $SU(2)$-modules $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$.

Unfortunately (i) quantizing arbitrary morphisms (i.e. Lagrangian correspondences) is quite difficult, even in this case (ii) there is no good geometric quantization scheme for arbitrary symplectic manifolds. The problem of finding good schemes for say, coadjoint orbits of real Lie groups or moduli spaces of flat connections have vast literatures attached to them.

The reader may notice that we have not said anything yet about the behavior of the quantum state spaces under the symplectic quotient construction. We take this up in Section 5.

4. Geometric invariant theory

In this section we review Mumford's geometric invariant theory [74], see also Brion's review in this volume or the reviews by Newstead [78] or Schmitt [85]. For connections to moduli problems see Newstead [77].

4.1. **Algebraic group actions and quotients.** Let G be a complex linear algebraic group. G is called *reductive* iff every G-module splits into simple G-modules, or equivalently, if G is the complexification of a compact Lie group K. A *Borel subgroup* of a reductive group G is a maximal closed connected solvable subgroup $B \subset G$. The set of Borel subgroups is in bijection with set of right cosets G/B, called the *generalized flag variety* for G, via the map $gB \mapsto gBg^{-1}$. A subgroup $P \subset G$ is *parabolic* iff G/P is complete iff P contains a Borel subgroup. The quotient G/P is called a *generalized partial flag variety.* Let T be a maximal torus of G (for example, the complexification of a maximal torus of the maximal compact subgroup, which was also somewhat confusingly called T.) We denote by $W = N(T)/T$ the Weyl group of T. The action of T on the Lie algebra $\mathfrak{g}$ induces

a *root space decomposition*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{g})} \mathfrak{g}_\alpha$$

where T acts trivially on $\mathfrak{t}$ and on $\mathfrak{g}_\alpha$ by $t\xi = t^\alpha \xi$, and $\mathcal{R}(\mathfrak{g}) \subset \Lambda^\vee$ is the set of roots of $\mathfrak{g}$. Given a choice of positive Weyl chamber let $B^\pm$ be the Borel subgroups whose Lie algebras contain the positive resp. negative root spaces of $\mathfrak{g}$. Each $\lambda \in \mathfrak{t}^\vee$ determines *standard parabolic subgroups* $P_\lambda^\pm$ with Lie algebra $\mathfrak{p}_\lambda^\pm = \mathfrak{b}^\pm + \bigoplus_{\langle h_\alpha, \lambda \rangle = 0} \mathfrak{g}_\alpha$, where $h_\alpha \in \mathfrak{t}$ is the coroot corresponding to $\alpha \in \mathfrak{t}^\vee$. Any parabolic subgroup (in particular, any Borel) is conjugate to a standard parabolic subgroup.

An *action* of G on a variety X is a morphism $G \times X \to X$ such that $g_1(g_2 x) = (g_1 g_2)x$ and $ex = x$, for all $g_1, g_2 \in G, x \in X$. A variety X equipped with a G-action is called a *G-variety*. An *(étale) slice* for the action of G at $x \in X$ is an affine subvariety $V \subset X$ and a G-morphism $G \times_{G_x} V \to X$ that is an isomorphism (étale morphism) onto a neighborhood of X. In contrast with the case of compact group actions, reductive group actions do not in general have slices. Luna's slice theorem [68] asserts that any *closed* orbit of an action of a reductive group on an affine variety has an étale slice. A *categorical quotient* of X by G is a pair (Y, π) where Y is a variety and $\pi : X \to Y$ is a G-invariant morphism that satisfies the universal property for quotients: if $f : X \to Z$ is a G-invariant morphism then f factors uniquely through Y. A *good quotient* of X is a pair (Y, π) where

(a) $\pi : X \to Y$ is G-invariant, affine, surjective,
(b) if $U \subset Y$ is open then $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism
(c) If W_1, W_2 are disjoint closed G-invariant subsets of X then $\pi(W_1), \pi(W_2)$ are disjoint closed subsets of x.

A good quotient is automatically a categorical quotient. A *geometric quotient* is a good quotient that separates orbits.

If G is connected reductive then the generalized flag variety $X = G/B^-$ has a canonical decomposition into *Bruhat cells*

$$X = \bigcup_{w \in W} X_w, \quad X_w := BwB^-/B^- \tag{4.1}$$

and *opposite Bruhat cells*

$$X = \bigcup_{w \in W} Y_w, \quad Y_w := B^- wB^-/B^-. \tag{4.2}$$

The codimension resp. dimensions are given by

$$\operatorname{codim}(X_w) = l(w), \quad \dim(Y_w) = l(w)$$

where $l(w)$ is the minimal number of simple reflections in a decomposition of w. We denote by $x_w = wB^-/B^- = X_w \cap Y_w$ the unique T-fixed point in X_w resp. Y_w. There is a similar decomposition of any generalized flag variety $X = G/P_\lambda^-$ into cells $X_{[w]}$ indexed by $[w] \in W/W_\lambda$.

In the special case $G = GL(r)$, the Weyl group W is naturally identified with the symmetric group and $B^\pm$ are the groups of invertible upper resp. lower triangular matrices. We identify $\mathfrak{k} \to \mathfrak{k}^\vee$; if $\lambda = \operatorname{diag} i(1, \ldots, 1, 0, \ldots, 0)$ has rank s then P_λ is the group of matrices preserving the subspace $\mathbb{C}^s \oplus 0 \subset \mathbb{C}^r$. The quotient $X = G/P_\lambda$ is isomorphic to the Grassmannian $G(s, r)$ of s-dimensional subspaces of $\mathbb{C}^r$. The quotient W/W_λ is natural identified with the set of subsets $I \subset \{1, \ldots, r\}$ of size s via the map $w \mapsto w\{1, \ldots, s\}$. Let $F_1 \subset F_2 \subset \ldots \subset F_r = \mathbb{C}^r$ be the standard flag in $\mathbb{C}^r$. Then the opposite Bruhat cell Y_I has closure the *Schubert variety*

$$\overline{Y}_I = \{E \in G(s, r), \dim(E \cap F_{i_j}) \geq j, j = 1, \ldots, s\}. \tag{4.3}$$

4.2. **Stability conditions.** Let G be a complex reductive group and X a G-variety. A *polarization* of X is an ample G-line bundle $\mathcal{O}_X(1) \to X$. Its d-th tensor power is denoted $\mathcal{O}_X(d)$. Let

$$R(X) = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(d)).$$

The action of X induces an action on $R(X)$ by pull-back. We denote by $R(X)^G \subset R(X)$ the subring of invariants, and by $R(X)^G_{>0}$ the part of $R(X)^G$ of positive degree.

Definition 4.2.1. A point $x \in X$ is

(a) *semistable* if $s(x) \neq 0$ for some $s \in R(X)^G_{>0}$;
(b) *polystable* if x is semistable and $Gx \subset X^{\mathrm{ss}}$ is closed;
(c) *stable* if x is polystable and has finite stabilizer;
(d) *unstable* if x is not semistable.

Example 4.2.2. Suppose that $G = \mathbb{C}^*$ acts on $\mathbb{P}^2$ by $g[z_0, z_1, z_2] = [g^{-1}z_0, z_1, gz_2]$. Then $R(X)_d$ is spanned by $z_0^{d_0} z_1^{d_1} z_2^{d_2}$ with $d_0 + d_1 + d_2 = d$, which has weight $d_0 - d_2$ under $\mathbb{C}^*$. Thus the invariant sections have $d_0 = d_2$. One sees easily that x is

(a) semistable iff $x \neq [1, 0, 0], [0, 0, 1]$
(b) polystable iff $x \in \{[0, 1, 0]\} \cup \{[z_0, z_1, z_2] | z_0 z_2 \neq 0\}$
(c) stable iff $x \in \{[z_0, z_1, z_2] | z_0 z_2 \neq 0\}$

Let X^{ss} resp. X^{ps} resp X^{s} resp. X^{us} denote the semistable resp. polystable resp. stable resp. unstable locus. We will need the following alternative characterizations of poly resp. semistability, see Mumford [74] or Brion's lectures in this volume:

Lemma 4.2.3. *Let $X \subset \mathbb{P}(V)$ be a G-variety. A point $x \in X$ is polystable (resp. semistable) iff the orbit of any lift v in V is closed (resp. does not contain 0).*

Define an equivalence relation on orbits as follows:

Definition 4.2.4. *Orbit-equivalence* is the equivalence relation on X^{ss} defined by $x_0 \sim x_1$ iff $\overline{Gx_0} \cap \overline{Gx_1} \cap X^{\mathrm{ss}} \neq \emptyset$.

Transitivity of this relation follows from:

Proposition 4.2.5. *(see* [74]*) The closure $\overline{Gx}$ of any semistable x contains a unique polystable orbit. Hence two orbits Gx_0, Gx_1 are orbit-equivalent iff their closures contain the same polystable orbit.*

See Theorem 5.4.9 for an analytic proof. The following can be considered the main result of geometric invariant theory [74]:

Theorem 4.2.6 (Mumford)**.** *Let X be a projective G-variety equipped with polarization $\mathcal{O}_X(1)$.*

(a) *There exists a categorical quotient $\pi : X^{\mathrm{ss}} \to X/\!\!/G$.*
(b) *$\pi(X^{\mathrm{s}}) \subset X/\!\!/G$ is open and $\pi|X^{\mathrm{s}} : X^{\mathrm{s}} \to \pi(X^{\mathrm{s}})$ is a geometric quotient.*
(c) *The topological space underlying $X/\!\!/G$ is the space of orbits modulo the orbit-closure relation $X^{\mathrm{ss}}/\sim$.*
(d) *$X/\!\!/G$ is isomorphic to the projective variety with coordinate ring $R(X)^G$.*

Some authors prefer to write $X^{\mathrm{ss}}/\!\!/G$ for the geometric invariant theory quotient, while we drop the superscript from the notation.

4.3. **The Hilbert-Mumford criterion.** Mumford [74], based on previous work of Hilbert for the case of the special linear group acting on projective space, gave a method for explicitly identifying the semistable loci:

Theorem 4.3.1. *(Hilbert-Mumford criterion) Let X be a polarized projective G-variety. $x \in X$ is semistable iff x is semistable for all one-parameter subgroups $\mathbb{C}^* \to G$.*

One direction of the Hilbert-Mumford criterion is trivial: Let X be a polarized G-variety. Suppose that x is G-semistable, so that there exists $s \in R(X)^G_{>0}$ with $s(x) \neq 0$. Then s is also invariant for any one-parameter subgroup, hence x is semistable for any one-parameter subgroup. The other direction is somewhat harder; the proof given in Mumford [74] uses an algebraic theorem of Iwahori. We will give an alternative analytic proof using the Kempf-Ness function in Section 7.2. The following is a fundamental example:

Example 4.3.2. Let $X = (\mathbb{P}^1)^n$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1}(1)^{\boxtimes n}$ the n-fold exterior tensor product. The group $G = SL(2, \mathbb{C})$ acts diagonally on X. We wish to show

(a) $X^{\mathrm{ss}} = \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n,$ at most $n/2$ points equal$\}$.
(b) $X^{\mathrm{s}} = \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n,$ less than $n/2$ points equal$\}$.
(c) $X^{\mathrm{ps}} - X^{\mathrm{s}} = \{(x_1, \ldots, x_n) \in X^{\mathrm{ss}}, \#\{x_1, \ldots, x_n\} = 2\}$. In other words, $n/2$ are equal and the other $n/2$ are also equal.

Indeed, if z_j, w_j are the coordinates on the j-factor then $H^0(\mathcal{O}_X(d))$ is spanned by $z_1^{d_1} w_1^{d-d_1} \dots z_n^{d_n} w_n^{d-d_n}$ where $d_j \in [0,d], j = 1, \dots, n$. If $\mathbb{C}^* \subset G$ is the standard maximal torus given by $g \mapsto \mathrm{diag}(g, g^{-1})$ then $H^0(\mathcal{O}_X(d))^{\mathbb{C}^*}$ is spanned by the polynomials $z_1^{d_1} w_1^{d-d_1} \dots z_n^{d_n} w_n^{d-d_n}$ with $\sum_{j=1}^n d_j = \sum_{j=1}^n d - d_j$, that is, $\sum (d_j/d) = n/2$. Since $d_j/d \in [0,1]$, this means that at least $n/2$ of the d_j's are non-zero. Thus $([z_1, w_1], \dots, [z_n, w_n])$ is $\mathbb{C}^*$-semistable iff at most $n/2$ z_j's and at most $n/2$ w_j's equal zero. Repeating the same for an arbitrary one-parameter subgroup (or equivalently, basis for $\mathbb{C}^2$) proves the claim.

Example 4.3.3. More generally, suppose that $X = (\mathbb{P}^1)^n$ is equipped with the polarization $\mathcal{O}_X(1) := \boxtimes_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(\lambda_i)$ for some positive integers $\lambda_1, \dots, \lambda_n$. Then $x = (x_1, \dots, x_n)$ is semistable iff for all $x \in \mathbb{P}^1$,

$$\sum_{x_j = x} \lambda_j \leq \sum_{x_j \neq x} \lambda_j.$$

For future use we mention the following equivalent form of the Hilbert-Mumford criterion and Lemma 4.2.3:

Corollary 4.3.4. *Let G be a reductive group acting linearly on a finite dimensional vector space V. For any $v \in V$, $\overline{Gv}$ contains 0, if and only if $\overline{\mathbb{C}^* v}$ contains 0 for some one-parameter subgroup $\mathbb{C}^* \subset G$.*

Remark 4.3.5. The statement of the corollary does not hold for arbitrary (that is, not linear) actions resp. arbitrary points. An example I learned from Brion: Let $X = \mathbb{P}(S^3(\mathbb{C}^2) \oplus \mathbb{C})$ with the action induced from the action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ and the trivial action on $\mathbb{C}$. Identifying $S^3(\mathbb{C}^2)$ with homogeneous polynomials in two variables u, v, one sees that the orbit of $[u^2 v, 1]$ contains the orbit of $[u^3, 1]$ in its closure. The stabilizer of $[u^3, 1]$ is a maximal unipotent subgroup of $SL(2,\mathbb{C})$ and so does not contain a copy of $\mathbb{C}^*$. Thus $[u^3, 1]$ cannot be contained in the closure of an orbit of a one-parameter subgroup. On the other hand, the lemma is true for arbitrary actions of abelian groups, as follows from, for example, Atiyah Theorem' 8.2.1 below.

5. The Kempf-Ness theorem

The material in this section is contained in the original paper of Kempf-Ness [52], the book of Mumford-Fogarty-Kirwan [74], and the paper of Guillemin-Sternberg [33]. The notes of Thomas [96] also describe the Kempf-Ness theorem with many examples.

5.1. **Complexification of Lie groups and their actions.** We begin with some basic remarks on the relation between complex and compact group actions. Any compact Lie group K admits a *complexification* G, a complex reductive Lie group G containing K as a maximal compact real subgroup, and whose Lie algebra $\mathfrak{g}$ is equal to $\mathfrak{k} \oplus i\mathfrak{k}$. The complexification G satisfies the universal property that any Lie group homomorphism from K to a complex Lie group H extends to a complex Lie group homomorphism from G. The complexification G admits a *Cartan decomposition*: a diffeomorphism (see Helgason [45, VI.1.1])

$$K \times \mathfrak{k} \to G, \quad (k, \xi) \mapsto k \exp(i\xi). \tag{5.1}$$

If X is a compact complex manifold then the group $\mathrm{Aut}(X)$ of automorphisms is a complex Lie group, with Lie algebra given by the space $H^0(X, TX)$ of holomorphic vector fields on X, see for example Akhiezer [3]. Any action of a compact group K therefore extends to the complexification G.

By a *Kähler Hamiltonian K-manifold* we mean a compact Hamiltonian K-manifold equipped with an integrable K-invariant complex structure. If X is compact then the K-action automatically extends to a G-action preserving the complex structure but not the symplectic structure. By the Kodaira embedding theorem, if the symplectic form is rational then a compact Kähler Hamiltonian K-manifold is isomorphic as a complex G-manifold to a smooth complex algebraic G-variety. However, the symplectic form may not be the pull-back of the Fubini-Study form under any holomorphic embedding of X, see for example Tian [97]. The generating vector fields for $\xi \in \mathfrak{k}$ are the Hamiltonian flows corresponding to the moment map components $\langle \Phi, \xi \rangle$, while the generating vectors fields for $i\xi, \xi \in \mathfrak{k}$ are the *gradient flows* corresponding to $\langle \Phi, \xi \rangle$. In particular, for any $x \in X, \xi \in \mathfrak{k}$, the trajectory $\exp(it\xi)x$ converges to a point $x_\infty \in X$ with $\xi_X(x_\infty) = 0$. Furthermore,

since $\langle \Phi, \xi \rangle$ is a Morse function by Lemma 3.2.9, this convergence is *exponentially fast* in t; the exponential nature of convergence will be used later.

The example of flag varieties will be particularly important later and we briefly describe these actions from the algebraic and symplectic points of view. Let V be a finite dimensional vector space. A *partial flag* in V is a filtration $F = (F_1 \subset F_2 \subset \ldots \subset F_m \subset V)$. The *type* of F is the sequence of dimensions $\dim(F_1) < \dim(F_2) < \ldots < \dim(F_m)$. Given a sequence $t = (t_1 < \ldots < t_m) \in \mathbb{Z}^m$ we let $\mathrm{Fl}(t, V)$ denote the set of partial flags of type t. The general linear group $GL(V)$ acts transitively on $\mathrm{Fl}(t, V)$ with stabilizer the parabolic subgroup of transformations preserving the filtration. A $GL(V)$-equivariant canonical projective embedding of $\mathrm{Fl}(t, V)$ is given by choosing a basis $v_1, \ldots, v_n$ so that $v_1, \ldots, v_{t_j}$ is a basis for F_j for each $j = 1, \ldots, m$, and mapping

$$\mathrm{Fl}(t, V) \to \prod_{j=1}^{m} \mathbb{P}(\Lambda^{t_j} V), \quad F \mapsto \prod_{j=1}^{m} \Lambda_{k=1}^{t_j} v_k.$$

Given a Hermitian metric on V, any partial flag induces a Hermitian splitting

$$V = F_1 \oplus (F_2 \cap F_1^{\perp}) \oplus (F_3 \cap F_2^{\perp}) \ldots \cap (F_m \cap F_{m-1}^{\perp})$$

and such splittings are in one-to-one correspondence with flags. Given real numbers $\lambda_1 > \ldots > \lambda_m$ the flag defines a skew-Hermitian operator acting by $i\lambda_j$ on $F_j \cap F_{j-1}^{\perp}$. Conversely, any such Hermitian operator determines a splitting via its eigenspace decomposition. The unitary group $K = U(V)$ acts transitively on the space of such matrices, which form an orbit of the action of K on the Lie algebra $\mathfrak{k}$. Now $\mathfrak{k}$ may be identified with its dual via any invariant inner product, so one sees that $\mathrm{Fl}(t, V)$ is naturally identified with the coadjoint orbit $K\lambda$ of λ, identified with an element of $\mathfrak{k}^{\vee}$ via the inclusion $\mathfrak{t} \to \mathfrak{k}$ and an identification $\mathfrak{k} \to \mathfrak{k}^{\vee}$. Given a generic $\xi \in \mathfrak{t}_+$, the stable resp. unstable manifolds of the Morse function $\langle \Phi, \xi \rangle$ are the Bruhat resp. opposite Bruhat cells of (4.1) resp. (4.2).

5.2. **Statement and proof.** The Kempf-Ness theorem states the equivalence of the symplectic and geometric invariant theory quotients; the affine case is treated in [52] and the projective case is similar (Theorem 8.3 in [74]).

Theorem 5.2.1. *Let K be a compact group and G its complexification. Let V be a G-module equipped with a K-invariant Hermitian structure. Let $X \subset \mathbb{P}(V)$ be a smooth projective G-variety, and $\Phi : X \to \mathfrak{k}^{\vee}$ the Fubini-Study moment map. Then $\Phi^{-1}(0) \subseteq X^{\mathrm{ps}}$ and the inclusion induces a homeomorphism $X/\!\!/K \to X/\!\!/G$.*

The proof uses the properties of a *Kempf-Ness function* for each $v \in V - \{0\}$:

$$\psi_v : \mathfrak{k} \to \mathbb{R}, \quad \xi \mapsto \log \| \exp(i\xi) v \|^2 / 2.$$

The Kempf-Ness function determines the norm of all vectors in the orbit of v, by the Cartan decomposition (5.1) and K-invariance of the metric. The Kempf-Ness function can be viewed as the integral of the moment map in the following sense:

Lemma 5.2.2. *For all $v \in V$ and $\lambda, \xi \in \mathfrak{k}$ we have $\partial_\lambda \psi_v(\xi) = 2\langle \Phi(\exp(i\xi)[v]), \lambda \rangle$.*

Proof. The proof uses the explicit formula for the Fubini-Study moment map

$$\begin{aligned} \partial_\lambda \psi_v(\xi) &= \frac{d}{dt}|_{t=0} \log \| \exp(i(\xi + t\lambda)) \|^2 / 2 \\ &= \frac{(i\lambda \exp(i\xi)v, \exp(i\xi)v)}{(\exp(i\xi)v, \exp(i\xi)v)} \\ &= 2\langle \Phi(\exp i\xi)[v], \lambda \rangle \end{aligned}$$

□

Corollary 5.2.3. *For any $v \in V$, ψ_v is a convex function with critical points given by the zeros of the map $\xi \mapsto \Phi(\exp(i\xi)[v])$. The second derivatives $\partial_\xi^2 \psi_v$ are strictly positive on $\mathfrak{k} - \mathfrak{k}_x$. For $\xi \in \mathfrak{k}_x$ the function ψ_v is the linear function given by $\psi_v(\xi) = \psi_v(0) + 2\langle \Phi(x), \xi \rangle$.*

Proof. For $\lambda, \nu \in \mathfrak{k}$ we have

$$\begin{aligned}\partial_\nu \partial_\lambda \psi_v(\xi) &= 2\langle L_{J\nu_X}\Phi(\exp(i\xi)[v]), \lambda\rangle \\ &= (\omega(\lambda_X, J\nu_X))(\exp(i\xi)[v]) \\ &= (g(\lambda_X, \nu_X))((\exp i\xi)[v])\end{aligned}$$

which is positive semidefinite since g is a Riemannian metric. By Lemma 5.2.2, the critical points correspond to zeroes of Φ. The formula for $\xi \in \mathfrak{k}_x$ is immediate from the previous lemma. □

If ψ_v is strictly convex (that is, has trivial infinitesimal stabilizer) and has a critical point, then it has a unique global minimum. The following lemma characterizes for which v minima of ψ_v exist:

Lemma 5.2.4. *Let $v \in V - \{0\}$ and $x = [v] \in \mathbb{P}(V)$.*

(a) *ψ_v attains a minimum iff x is polystable.*
(b) *ψ_v is bounded from below iff x is semistable.*

Proof. (a) Recall from 4.2.3 that x is polystable iff Gv is closed. Suppose Gv is closed. Let ξ_j be a minimizing sequence for ψ_v. Then $\exp(i\xi_j)v$ converges to $k\exp(i\xi)v$ for some $k \in K, \xi \in \mathfrak{k}$, since Gv is closed, and ξ must be a global minimum of ψ_v. Conversely, suppose that ψ_v attains a minimum. Necessarily $\langle \Phi(x), \xi\rangle = 0$ for all $\xi \in \mathfrak{k}_x$ since otherwise there exists $\xi_j \in \mathfrak{k}_x$ with $\psi_v(\xi_j) \to -\infty$, using Corollary 5.2.3; this would contradict the existence of a minimum. Let $\xi_j \in \mathfrak{k}, k_j \in K$ be a sequence with the property that $k_j \exp(i\xi_j)v$ converges in V. Using compactness of K, we may assume after passing to a subsequence that k_j converges, so that $\exp(i\xi_j)v$ converges as well. Write $\xi_j = \xi_j^0 + \xi_j^1$ for some sequences $\xi_j^0 \in \mathfrak{k}_x, \xi_j^1 \in \mathfrak{k}_x^\perp$. Then

$$\begin{aligned}\exp(i\xi_j)v &= \exp(i(\xi_j^0 + \xi_j^1))v = (\mathrm{Ad}(\exp(i\xi_j^0))\exp(i\xi_j^1))\exp(i\xi_j^0)v \\ &= \exp(\mathrm{Ad}(\exp(i\xi_j^0))i\xi_j^1)\exp(i\langle\Phi(x), \xi_j^0\rangle)v \\ &= \exp(\mathrm{Ad}(\exp(i\xi_j^0))i\xi_j^1)v.\end{aligned}$$

Since ψ_v is strictly convex on $\mathfrak{k}_x^\perp$, we must have $\|\mathrm{Ad}(\exp(i\xi_j^0))\xi_j^1\|$ bounded and so $\mathrm{Ad}(\exp(i\xi_j^0))\xi_j^1$ converges to some $\xi_\infty \in \mathfrak{k}_x$ with $\exp(i\xi_j)v \to \exp(i\xi_\infty)v$. This proves that Gv is closed. (b) If ψ_v is bounded from below, then any minimizing sequence ξ_j has $\exp(i\xi_j)x$ converging to a critical point of ψ, which is necessarily a zero of Φ. Hence Gx contains a polystable orbit in its closure and is therefore semistable. If ψ_v is not bounded from below, then $\overline{Gv}$ contains 0 and so x is unstable, see Lemma 4.2.3. □

Corollary 5.2.5. $X^{\mathrm{ps}} = G\Phi^{-1}(0)$.

Proof. By Lemmas 5.2.4, 5.2.3, 4.2.3. □

Proof of the Kempf-Ness theorem 5.2.1. Consider the inclusion

$$i/K : \Phi^{-1}(0)/K \to X^{\mathrm{ps}}/G \cong X/\!\!/G.$$

First note that i/K is injective: Suppose that $x_0, x_1 \in \Phi^{-1}(0)$ are such that $Gx_0 = Gx_1$. Since $G = K\exp(\mathfrak{k})$ by (5.1), we have $\exp(i\xi)x_1 = kx_0$ for some $\xi \in \mathfrak{k}, k \in K$. Choose a lift v of x_1. Then both $0, \xi$ are critical points of ψ_v, and since ψ_v is convex this implies $\xi \in \mathfrak{k}_{x_1}$ and so $Kx_0 = Kx_1$. Next note that i/K is surjective by Corollary 5.2.5. Finally i/K is a homeomorphism: Any bijection from a Hausdorff space to a compact space is a homeomorphism. (Alternative, the gradient flow of the norm-square of the moment map discussed in Section 7 defines a continuous inverse to i/K.) □

Remark 5.2.6. Let X be a compact Kähler Hamiltonian K-manifold. An analog of the Kempf-Ness function may be obtained by integrating the one-form given by the moment map: Define $\alpha \in \Omega^1(\mathfrak{k}), \alpha_{x,\lambda}(\xi) = \langle\Phi(\exp(i\lambda)x), \xi\rangle$. Then anti-symmetry of ω implies that α is closed, hence exact by the Poincaré lemma, hence $\alpha_x = \mathrm{d}\psi_x$ for some $\psi_x : \mathfrak{k} \to \mathbb{R}$. Equivariance of Φ implies that $\alpha_{kx} = \alpha_x$, so that $\psi_{kx} = \psi_x$. Say that a point $x \in X$ is *polystable* iff ψ_x attains a minimum, *semistable* iff ψ_x is bounded from below. With these definitions the following Kähler analog of the Kempf-Ness theorem holds, c.f. Mundet [50], Heinzner-Loose [43], Heinzner-Huckleberry [44], Bruasse-Teleman [20], Teleman [94]: Let $X/\!\!/G$ be the quotient of the semistable locus by the

orbit closure equivalence relation. Then the same arguments show that $\Phi^{-1}(0)$ is contained in the semistable locus and the inclusion induces a homeomorphism $X/\!\!/K \to X/\!\!/G$.

We will return to a more complete discussion of the Kempf-Ness function later. We illustrate the theorem with the *Clebsch-Gordan* theory of existence of invariants in tensor products of representations of $G = SL(2,\mathbb{C})$. The weight lattice $\Lambda^\vee$ for G is naturally identified with the set $\mathbb{Z}/2$ of non-negative half-integers and for any $\lambda \in \Lambda^\vee, \lambda \geq 0$ we denote by V_λ the corresponding simple G-module. (The identification with half integers is more natural than the identification with integers since the canonical inner product on the Lie algebra, defined by the trace in the standard representation, assigns length $\sqrt{2}$ to the highest root.) Given $\lambda_1, \ldots, \lambda_n$ we ask whether $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ contains an invariant vector. Now $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong V_{d/2}$ and so $R(\mathbb{P}^1) = \oplus_\lambda V_\lambda$. If we equip $X = (\mathbb{P}^1)^n$ with the ample line bundle $\mathcal{O}_X(1) := \boxtimes_{j=1}^n \mathcal{O}_{\mathbb{P}^1}(\lambda_j)$ then

$$R(X) = \bigoplus_{d \geq 0} \bigotimes_{j=1}^n H^0(\mathcal{O}_{\mathbb{P}^1}(d\lambda_j)) = \bigoplus_{d \geq 0} \bigotimes_{j=1}^n V_{d\lambda_j}.$$

So

$$R(X/\!\!/G) = R(X)^G = (\bigoplus_{d \geq 0} \bigotimes_{j=1}^n V_{d\lambda_j})^G.$$

This is non-zero if and only if $X/\!\!/G$ is empty. The Kempf-Ness Theorem 5.2.1 gives $X/\!\!/G \cong X/\!\!/K \cong (S^2_{\lambda_1} \times \ldots \times S^2_{\lambda_n})/\!\!/SU(2)$ where S^2_λ denotes the two-sphere equipped with re-scaled symplectic form λ and $SU(2)$ acts via the double cover $SU(2) \to SO(3)$. By Proposition 3.3.5,

Corollary 5.2.7. *$(\otimes_{j=1}^n V_{d\lambda_j})^G$ is non-trivial for some d iff*

$$\lambda_j \leq \sum_{i \neq j} \lambda_i, j = 1, \ldots, n. \tag{5.2}$$

This gives a geometric proof of the well-known Clebsch-Gordan rules. A basis for the space of invariants is induced from a choice of parenthesization of the tensor product above, see for example [22]. The relation between the different invariants is also connected to symplectic geometry [84].

5.3. **Quantization commutes with reduction.** The proof of the Kempf-Ness Theorem 5.2.1, which seems otherwise somewhat miraculous, has a conceptual interpretation given by Guillemin-Sternberg [33] in terms of geometric quantization (Section 3.5) as follows. Namely, rather than choosing a lift of $x \in X$ to $V - \{0\}$, which is the total space of $\mathcal{O}_X(-1)$, it is more natural from the viewpoint of geometric quantization to choose a lift l in the positive line bundle $\mathcal{O}_X(1) \to X$. Define the *Guillemin-Sternberg* function

$$\psi_l^\vee : \mathfrak{k} \to \mathbb{R}, \quad \xi \mapsto \log \| \exp(i\xi) l \|^2/2.$$

The same computation as in the Kempf-Ness case, except for a change of sign, implies that for $\lambda, \nu, \xi \in \mathfrak{k}$ we have

$$\partial_\lambda \psi_l^\vee(\xi) = -2\langle \Phi(\exp(i\xi)x), \lambda \rangle, \quad \partial_\nu \partial_\lambda \psi_l^\vee(\xi) = -2\omega_{\exp(i\xi)x}(\lambda_X(x), J\nu_X(x)).$$

In particular, suppose that $s \in H^0(X, \mathcal{O}_X(1))^G$ is an invariant section. Then

$$\psi^\vee_{s(x)}(\xi) = \log \| \exp(i\xi) s(x) \|^2/2 = \log \| s(\exp(i\xi)x) \|^2/2.$$

Now convexity of $\psi^\vee_{s(x)}$ implies that any critical point of $\|s\|^2$ occurs at $\Phi^{-1}(0)$ and is a local maximum, and s is approximately Gaussian. This type of behavior is quite standard for "typical quantum states", which introductory physics lectures often show as concentrating near some submanifold of the corresponding classical state space in Gaussian fashion.

Suppose that K acts freely on the zero level set $\Phi^{-1}(0)$. The complex structure J on X induces an almost complex structure $J/\!\!/K$ on $X/\!\!/K$ by identifying $\pi^* T(X/\!\!/K)$ with the subbundle of $TX|\Phi^{-1}(0)$ perpendicular to the generating vector fields $\xi_X, \xi \in \mathfrak{k}$. This complex structure is integrable since the Nijenhuis tensor vanishes. Similarly the polarization $\mathcal{O}_X(1) \to X$ naturally descends to a polarization $\mathcal{O}_{X/\!\!/K}(1) \to X/\!\!/K$, defined by restricting to $\Phi^{-1}(0)$ and quotienting by the action of K.

Theorem 5.3.1 (Quantization commutes with reduction)**.** *Let X be a compact Hamiltonian K-manifold equipped with moment map $\Phi : X \to \mathfrak{k}^\vee$, polarization $\mathcal{O}_X(1) \to X$ and a compatible K-invariant Kähler structure J, such that K acts freely on the zero level set $\Phi^{-1}(0)$, and let $R(X)_d$ denote the space of sections of $\mathcal{O}_X(d)$ as above. For each $d \geq 0$ there is a canonical isomorphism $\rho : R(X)_d^K \to R(X/\!\!/K)_d$*

Proof. For smooth projective varieties $X \subset \mathbb{P}(V)$ this is a combination of Mumford's Theorem 4.2.6 and the Kempf-Ness Theorem 5.2.1. More generally let X be a compact polarized Kähler Hamiltonian K-manifold. Any section $s \in H^0(X, \mathcal{O}_X(1))^K$ naturally defines a section $\rho(s) \in H^0(X/\!\!/K, \mathcal{O}_{X/\!\!/K}(1))$ by restriction to $\Phi^{-1}(0)$ and descent to the quotient. Then ρ is an injection, since any invariant section has maximum norm on $\Phi^{-1}(0)$. Proving surjectivity required a somewhat complicated argument in the approach of Guillemin-Sternberg, and the following alternative algebraic argument is substantially easier: By Kodaira embedding X is biholomorphic to smooth subvariety of $\mathbb{P}(V)$, and the polarization $\mathcal{O}_X(1)$ is isomorphic as a holomorphic line bundle to the pull-back of the hyperplane bundle on $\mathbb{P}(V)$, although the symplectic structure and moment map may not be pull-backs. By the extension of Kempf-Ness to Kähler varieties discussed in 5.2.6, the semistable locus corresponding to the polarization $\mathcal{O}_X(1)$ has quotient by G diffeomorphic to $X/\!\!/K$. Given a section $s \in H^0(X/\!\!/K, \mathcal{O}_{X/\!\!/K}(1))$, s naturally lifts to an invariant section on the semistable locus X^{ss} with maximum on $\Phi^{-1}(0)$. Since the norm of this section is bounded, it extends over all of X. □

Guillemin-Sternberg also proved "quantization commutes with reduction" for another of class of Hamiltonian actions for which there exists a good quantization scheme, namely cotangent bundles [34]. Quantization commutes with reduction was generalized to arbitrary compact Hamiltonian manifolds using "Spin-c" quantization by Meinrenken [72], and further generalized to "non-abelian localization" by Teleman and Paradan, see the last section of these notes.

5.4. **Polystable points.** By Lemma 5.2.5, the polystable orbits are the orbits of points $x \in \Phi^{-1}(0)$. In this section we investigate these and the orbit-closure equivalence relation in more detail. The following was observed by Kempf-Ness [52] in the linear case and by Slodowy [93] in general, see also Sjamaar [91].

Proposition 5.4.1. *Let X be a Kähler Hamiltonian K-manifold, and $x \in \Phi^{-1}(0)$. Then G_x is the complexification of K_x; in particular, G_x is reductive.*

Proof. Suppose that $x \in \Phi^{-1}(0)$ and $gx = x$. Write $g = k^{-1}\exp(\xi)$ for some $\xi \in i\mathfrak{k}, k \in K$. Let $\psi_x = \psi_{kx}$ be Kempf-Ness functions for x resp. kx, see Remark 5.2.6. Then $\exp(\xi)x = kx$ so $\operatorname{grad}\psi_x(\xi) = \operatorname{grad}\psi_{kx}(0) = \operatorname{grad}\psi_x(0) = 0 = \operatorname{grad}\psi_x(0)$. By convexity, ψ_x is constant along the line $t\xi$, so $\xi \in i\mathfrak{k}_x$. Hence $x = kx$ so $k \in K_x$, which implies $g \in (K_x)_{\mathbb{C}}$. The reverse inclusion $(K_x)_{\mathbb{C}} \subset G_x$ is obvious. □

Remark 5.4.2. Stabilizer groups are not in general reductive. For example let $X = SL(2,\mathbb{C}) \times_B \mathbb{P}^1$. Then every stabilizer is either solvable or unipotent, and so no projective embedding of X has semistable points.

Second we show that polystable points are "seen by one-parameter subgroups." For this we need to review some results on existence of holomorphic slices. Let X be a complex manifold with a holomorphic action of a group G. Let $x \in X$. Recall that a *slice* at x is a G_x-invariant submanifold S of X containing x such that GS is open in X and the natural G-equivariant map from $G \times_{G_x} S \to X$ is an isomorphism onto GS. Sjamaar [91] has proved the following analog of slice theorems of Luna and Snow:

Theorem 5.4.3 (Sjamaar)**.** *Let G be a connected complex reductive group with maximal compact K. Let X be a Kähler Hamiltonian K-manifold such that the action of K extends to a holomorphic action of G. Suppose that $x \in \Phi^{-1}(0)$. Then there exists a slice at x.*

Corollary 5.4.4. *An orbit Gx contains a polystable point y in its closure, iff there exists a one-parameter subgroup $\mathbb{C}^* \subset G$ and a point $z \in Gx$ such that $\mathbb{C}^* z$ contains a polystable point in its closure.*

Proof. Let y be a polystable point. We may assume that $\Phi(y) = 0$. By Theorem 5.4.3, there exists a slice S at y. Now S is biholomorphic to its tangent space T_yS, equivariantly for the action of K_x, in a neighborhood U of y. Furthermore, since this map is holomorphic, the map is equivariant for the *infinitesimal* G-action. By Lemma 4.3.4, there exists a one-parameter subgroup $\mathbb{C}^* \to G$ and a point $v \in T_yS$ such that the closure of $\mathbb{C}^*v$ contains $0 \in T_yS$. By choosing v sufficiently small, we ensure that $\{zv, |z| \leq 1\}$ is in the image of U. Let $s \in S$ be the pre-image of v. Then $\{zs, |z| \leq 1\}$ contains y in its closure, as required. □

Using this corollary we prove a finite-dimensional analog of the *Jordan-Hölder* theory for semistable vector bundles, see for example Seshadri [88].

Definition 5.4.5. For any $\lambda \in \mathfrak{k}$, let $x_\lambda = \lim_{t\to\infty} \exp(-ti\lambda)x$ the *associated graded point* of x with respect to λ.

Remark 5.4.6. The fact that $\exp(-ti\lambda)x$ is the gradient flow of a Morse function (see 3.2.9) implies that the gradient trajectory converges *exponentially fast* to x_λ, that is, $\operatorname{dist}(\exp(-ti\lambda)x, x_\lambda) \leq C_0 e^{-C_1 t}$ for some constants C_0, C_1.

Definition 5.4.7. $\lambda \in \mathfrak{k}$ is *Jordan-Hölder* for $x \in X^{\mathrm{ss}}$ iff x_λ is polystable.

Example 5.4.8. Let $X = \mathbb{C}^2$ and $G = (\mathbb{C}^*)^2$ acting by $(g_1, g_2)(z_1, z_2) = (g_1z_1, g_2z_2)$. Then any (λ_1, λ_2) with $\lambda_1, \lambda_2 > 0$ is Jordan-Hölder.

Theorem 5.4.9. *Let X be a compact Kähler Hamiltonian K-manifold and $x \in X$ a semistable point.*

(a) *The set of Jordan-Hölder vectors for x is a non-empty K_x-invariant cone in $\mathfrak{k}$.*
(b) *The orbit Gx_λ of the associated graded x_λ of a Jordan-Hölder λ is the unique polystable orbit in $\overline{Gx}$.*

Proof. (a) The set of Jordan-Hölder vectors is non-empty: Since x is semistable, Gx contains a polystable y in its closure. By Corollary 5.4.4, any polystable y is in the closure $\mathbb{C}^*z$ for some one-parameter subgroup $\mathbb{C}^* \subset G$ and $z \in Gx$. Suppose that $z = g^{-1}x$ for some $g \in G$. Then $(\mathrm{Ad}(g)\mathbb{C}^*)x = g\mathbb{C}^*z$ contains gy in its closure, and gy is polystable as well. Convexity of the set of Jordan-Hölder vectors follows immediately from convexity of the Kempf-Ness function, since if $\operatorname{grad}(\psi) \to 0$ along any two directions then it also goes to zero in any intermediate direction.

(b) Suppose that y_0, y_1 are polystable points in the closure of Gx, and $y_j = (\exp(i\xi_j)x)_{\lambda_j}$ for some vectors $\xi_j, \lambda_j \in \mathfrak{k}$, $j = 0, 1$. Then $\operatorname{grad}\psi(-t\lambda_j + \xi_j) = \Phi(\exp(-t\lambda_j + \xi_j))x) \to \Phi(y_j) = 0$ as $t \to \infty$. The distance between $\exp(i(\xi_j + t\lambda_j))x$ is given as follows: Let $\delta_t = (\xi_1 + t\lambda_1) - (\xi_0 + t\lambda_0)$, $\xi_{s,t} = (1-s)(\xi_1 + t\lambda_1) + s(\xi_0 + t\lambda_0)$ and $x_{s,t} = \exp(i\xi_{s,t}x)$. Then the square of the distance from $x_{0,t}$ to $x_{1,t}$ is given by

$$\begin{aligned}\left(\int_0^1 \|\frac{d}{ds}x_{s,t}\|\mathrm{d}s\right)^2 &\leq \int_0^1 \|\frac{d}{ds}x_{s,t}\|^2\mathrm{d}s \\ &= \int_0^1 g\left(\frac{d}{ds}x_{s,t}, \frac{d}{ds}x_{s,t}\right)\mathrm{d}s \\ &= \int_0^1 \partial^2_{\delta_t}\psi(\xi_{s,t})\mathrm{d}s = \partial_{\delta_t}\psi(\xi_{s,t})|_{s=0}^{s=1}.\end{aligned}$$

Now $\operatorname{grad}\psi$ converges exponentially to zero along $\xi_{j,t}$ as $t \to \infty$ for $j = 0, 1$, since $\exp(i\xi_{j,t})x$ converges exponentially fast to x_{λ_j}, see Remark 5.4.6. On the other hand, $\|\delta_t\| < C_0 + C_1 t$ for some constants C_0, C_1, by definition of δ_t. Hence $\operatorname{dist}(x_{\lambda_0}, x_{\lambda_1}) = \lim_{t\to\infty} \operatorname{dist}(x_{0,t}, x_{1,t}) = 0$ and the claim follows. □

Remark 5.4.10. We have included (b) to emphasize a somewhat confusing point: distant points in $\mathfrak{k}$ may map to near points in X if the gradient of ψ on the path between them is sufficiently small.

Remark 5.4.11. In fact, the full strength of Sjamaar's (or Luna's) slice theorem is not needed here; it suffices to find a slice for the *infinitesimal* action of G which is substantially easier. Some terminology: If a Lie group with Lie algebra $\mathfrak{g}$ acts on a manifold we say that a submanifold U is $\mathfrak{g}$-*invariant* if the generating vector fields are tangent to U. A *slice* for the infinitesimal

action of $\mathfrak{g}$ at x is a $\mathfrak{g}_x$-invariant holomorphic submanifold S containing x, such that the natural map $\mathfrak{g} \times_{\mathfrak{g}_x} TS \to TX|S$ is an isomorphism. Using the implicit function theorem, one sees that any sequence of points converging to x may be translated by the action of G (which is now only defined in a neighborhood of the identity) into a sequence of points in S. Thus if an orbit Gy in X contains $x \in S$ in its closure, then $Gy \cap S$ also contains x in its closure, and by Lemma 4.3.4 $\mathbb{C}^* y \cap S$ contains x in its closure for some one-parameter subgroup $\mathbb{C}^* \subset G$.

6. Schur-Horn convexity and its generalizations

In this section we discuss the generalization of Clebsch-Gordan theory to arbitrary groups, in particular, the theory of existence of invariants in tensor products of representations of $GL(r)$, the connections (via the Kempf-Ness theorem) with eigenvalue problems, and a combinatorial answer by Knutson, Tao, and the author [60].

6.1. **The Borel-Weil theorem.** Let G be a connected complex reductive group. Let λ be any dominant weight for G and V_λ a simple G-module with highest weight λ. Let P_λ^- be the opposite standard parabolic corresponding to λ, and G/P_λ^- the generalized flag variety corresponding to λ. We denote by $\mathbb{C}_\lambda^\vee$ the one-dimensional representation of P_λ^- corresponding to $-\lambda$, and by $\mathcal{O}_X(\lambda) = G \times_{P_\lambda^-} \mathbb{C}_\lambda^\vee$.

Theorem 6.1.1 (Borel-Weil [87])**.** *Let $X = G/P_\lambda^-$ with λ a weight. Then $H^0(X, \mathcal{O}_X(\lambda)) \cong V_\lambda$ if λ is dominant and vanishes otherwise.*

Proof. First consider the case $G = SL(2, \mathbb{C})$. We identify $\Lambda^\vee$ with $\mathbb{Z}/2$. Then $H^0(\mathcal{O}_X(\lambda))$ is the set of homogeneous polynomials in two variables of degree 2λ, if λ is non-negative, and zero otherwise. In the first case one checks easily that $H^0(\mathcal{O}_X(\lambda))$ is simple with highest weight λ.

Next let G be an arbitrary connected complex reductive group. Let $X = G/B^-$ and $X_1 = BB^-/B^- \cong B/T \cong U$ the open Bruhat cell, (here U is a maximal unipotent) so that $H^0(X_1, \mathcal{O}_X(\lambda)|X_1)^U = H^0(U, \mathbb{C})^U \cong \mathbb{C}$. Thus $H^0(X_1, \mathcal{O}_X(\lambda)|X_1)$ contains a unique highest weight vector, which we denote by s_λ. We wish to determine whether s_λ extends over the complement of X_1 in X. It suffices to check the order of vanishing of s_λ on the divisors X_{s_α}, as α ranges over simple roots. For each root α, we let $h_\alpha \in \mathfrak{t}$ denote the corresponding coroot, so that $\mathfrak{sl}(2, \mathbb{C})_\alpha := \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha$ is the three-parameter Lie algebra corresponding to α. Let $SL(2, \mathbb{C})_\alpha \to G$ denote the homomorphism induced by the inclusion $\mathfrak{sl}(2, \mathbb{C})_\alpha \to \mathfrak{g}$. The orbit $C_\alpha = SL(2, \mathbb{C})_\alpha B^-/B^-$ of $SL(2, \mathbb{C})_\alpha$ on X is isomorphic to $SL(2, \mathbb{C})_\alpha / SL(2, \mathbb{C})_\alpha \cap B^- \cong \mathbb{P}^1$. The curve C_α intersects the Bruhat cell X_{s_α} in the unique point $x_{s_\alpha} = s_\alpha B^-/B^-$. The order of vanishing of s_α along X_{s_α} is necessarily the order of vanishing of $s_\alpha|C_\alpha$ at x_{s_α}. Now $\mathcal{O}_X(\lambda)$ restricts to the line bundle $\mathcal{O}_{\mathbb{P}^1}(\langle \lambda, h_\alpha \rangle)$ on C_α, and the section s_λ restricts to the highest weight section on $C_\alpha - x_{s_\alpha}$. It extends over x_α iff $\langle \lambda, h_\alpha \rangle \geq 0$, by the discussion for the $SL(2, \mathbb{C})$ case.

Now G/B^- fibers over G/P_λ^- with projective fibers and so

$$H^0(G/B^-, \mathcal{O}_{G/B^-}(\lambda)) = H^0(G/P_\lambda, \mathcal{O}_{G/P_\lambda^-}(\lambda)).$$

Since the result is proved for G/B^-, this completes the proof. □

From the point of view of symplectic geometry, the Borel-Weil theorem says that the geometric quantization of a coadjoint orbit equipped with an integral symplectic form (that is, one that is the curvature of some line bundle) is a simple K-module. Indeed, let Φ denote the moment map induced by the action of K on $\mathcal{O}_X(\lambda)$. Since the weight of T on the fiber of $\mathcal{O}_X(\lambda)$ over B^-/B^- is $-\lambda$, Φ maps X onto the coadjoint orbit $K\lambda$ through λ, see Proposition 3.2.12. Thus in the notation introduced in Section 3.5, $\mathcal{H}(K\lambda) = V_\lambda$.

6.2. **The Schur-Horn-Kostant problem.** The Schur-Horn theorem [86], [48] reads:

Theorem 6.2.1. *The set of possible diagonal entries of a Hermitian operator with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the hull of the set of permutations of λ.*

Example 6.2.2. If $K = SO(3)$ then by Proposition 3.2.4 the coadjoint orbit through $\mathrm{diag}(\lambda, -\lambda)$ may be identified with the sphere of radius λ via the isomorphism $\mathfrak{k}^\vee = \mathfrak{so}(3)^\vee \to \mathbb{R}^3$, and the moment map for the maximal torus action is projection onto the z-axis, and so has moment image

$[-\lambda, \lambda]$. The action of the Weyl group $W = \mathbb{Z}_2$ on $\mathfrak{t}$ is identified with the sign representation, and so $[-\lambda, \lambda] = \text{hull}\{-\lambda, \lambda\} = \text{hull}(W\lambda)$ as claimed.

Kostant [63] generalized this result to arbitrary compact connected groups:

Theorem 6.2.3. *Let K be a compact connected group. The projection of a coadjoint orbit $K\lambda$ of an element $\lambda \in \mathfrak{t}^\vee$ is the convex hull of the orbit $W\lambda$ of λ under the Weyl group W.*

Using the Kempf-Ness and Borel-Weil theorems 5.2.1, 6.1.1, the Schur-Horn-Kostant theorem is equivalent to the following well-known fact in representation theory:

Theorem 6.2.4. *With K as above, let λ be a dominant weight. The set of μ/d such that the weight space $V_{d\lambda,(\mu)} \subset V_{d\lambda}$ is non-trivial for some $d \in \mathbb{Z}_+$ is the rational convex hull of $W\lambda$.*

Proof. We identify $X = K\lambda = G/P_\lambda^-$ and $\mathbb{C}_\mu$ the trivial bundle over X with T-weight μ so that $V_{d\lambda,(\mu)} = H^0(X, \mathbb{C}_\mu^* \otimes \mathcal{O}_X(d\lambda))^T$ by Borel-Weil 6.1.1, which is the space of sections over the quotient $(X/\!\!/T)_{\mathbb{C}}$ by Mumford's Theorem 4.2.6. We may use the Hilbert-Mumford criterion to determine whether there are any semistable points: Given a one-parameter subgroup generated by some $\xi \in \mathfrak{t}_+$, a point $x \in X$ flows under $\exp(t\xi)$ to y_w as $t \to -\infty$ where $x \in Y_w := B^- w B^-/B^-$ is the opposite Bruhat cell, see (4.1). The weight of T on the fiber over y_w is $\mu - w\lambda$. Thus $x \in Y_w$ is semistable for ξ iff $\langle w\lambda - \mu, \xi\rangle \leq 0$ iff $\mu \in w\lambda - (\mathfrak{t}_+)^\vee$. In particular Y_1 is contained in the semistable locus for the one-parameter subgroup generated by $-\xi$ with ξ dominant iff $\mu \in \lambda - (\mathfrak{t}_+)^\vee$. The semistable locus for the torus action is non-empty iff a generic point is semistable for all one-parameter subgroups iff

$$\mu \in \bigcap_{w \in W} w(\lambda - (\mathfrak{t}_+)^\vee). \tag{6.1}$$

The dual cone to $\text{hull}(w\lambda, w \in W)$ at $w\lambda$ is generated by $(s_\alpha - 1)w\lambda$ where α ranges over simple roots, which is equal to $w(\mathfrak{t}_+)^\vee$. It follows that (6.1) is equivalent to $\mu \in \text{hull}(w\lambda, w \in W)$ as claimed. □

Proof of Theorem 6.2.3. Let $X = K\lambda$ be as above. The moment map corresponding to the projective embedding $K\lambda \to \mathbb{P}(V_\lambda^\vee)$ is the projection π of X onto $\mathfrak{t}^\vee$ by Proposition 3.2.5 (d). Hence the moment map for the projective embedding $K\lambda \to \mathbb{P}(V_\lambda^\vee \otimes \mathbb{C}_\mu)$ is $\pi - \mu$. By Kempf-Ness $X/\!\!/T_{\mathbb{C}} \cong X/\!\!/T$, where $T_{\mathbb{C}}$ is the complexification of T. Finally $X/\!\!/T$ non-trivial iff 0 is in the image of $\pi - \mu$ iff μ is contained in the image of π. □

6.3. **The Horn-Klyachko problem.** In the previous section we investigated the existence of semistable points for an action of a torus. The Horn problem [49] deals with the following question, which we will rephrase in terms of existence of semistable points for the action of a non-abelian group:

Question 6.3.1. *Given the eigenvalues of Hermitian matrices $H_1, \ldots, H_{n-1}$, what are the possible eigenvalues of $H_1 + \ldots + H_{n-1}$?.*

Since the eigenvalues are real, we may order them in non-increasing order

$$\lambda_1(H_j) \geq \lambda_2(H_j) \ldots \geq \lambda_r(H_j).$$

Then the most famous inequality is the well-known

$$\lambda_1(H_1 + H_2) \leq \lambda_1(H_1) + \lambda_1(H_2).$$

We will give a complete list of such inequalities. Before we give the answer, we note that this question has a symplectic reformulation as follows. Taking $H_n = -H_1 - \ldots - H_{n-1}$, obtain a tuple $(H_1, \ldots, H_n)$ with $H_1 + \ldots + H_n = 0$. Thus the problem is a special case of the *generalized Horn problem*:

Question 6.3.2. *Let K be a compact Lie group. For which $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}_+^\vee$ is the symplectic quotient $(K\lambda_1 \times \ldots \times K\lambda_n)/\!\!/K$ non-empty?*

By the Kempf-Ness and Borel-Weil theorems, this problem is equivalent to the following

Question 6.3.3. *Let K be a compact Lie group. For which dominant weights $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}_+^\vee$ is space of invariants $(V_{d\lambda_1} \otimes \ldots \otimes V_{d\lambda_n})^K$ non-trivial for some $d \geq 0$?*

In the case $K = SU(2)$ this question was answered in Section 5.2. We give a partial answer for the case $K = SU(n)$ using max-min description of eigenvalues; this implies inequalities on the invariant theory problem. Then we give a necessary and sufficient answer using the Hilbert-Mumford criterion, following an argument of Klyachko [55]. Finally we give a brief description of works of Belkale [10], Knutson-Tao [58], and Knutson-Tao-Woodward [60] giving a *minimal set* of inequalities.

We begin with the elementary max-min approach for $K = U(n)$. If H is a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ then

$$\lambda_j = \max_{\substack{V \subset \mathbb{C}^r \\ \dim(V)=j}} \min_{v \in V - \{0\}} \frac{(v, Hv)}{(v, v)}, \quad j \in \{1, \ldots, r\}.$$

This has a generalization to partial sums of eigenvalues as follows: For every subspace $E \subset \mathbb{C}^r$ and Hermitian operator H we denote by H_E the operator on E given by composing H with restriction and projection. Then for any $J = \{j_1 < \ldots < j_s\} \subset \{1, \ldots, r\}$ we have

$$\sum_{j \in J} \lambda_j = \max_{\substack{F_1 \subset \ldots \subset F_s \\ \dim(F_l)=j_l}} \min_{\substack{E \in G(s,n) \\ \dim(E \cap F_l) \geq l}} \mathrm{Tr}(H_E).$$

Suppose that $J_1, \ldots, J_n$ are such that for every set of flags $F_1, \ldots, F_n$, there exists a space $E \in G(s, r)$ such that $\dim(E \cap F_{i,l}) \geq j_{i,l}$ for $i = 1, \ldots, n$ and $l = 1, \ldots, s$. Then

$$\begin{aligned} \sum_{i=1}^n \sum_{j \in J_i} \lambda_{i,j} &= \sum_{i=1}^n \max_{\substack{F_{i,1} \subset \ldots \subset F_{i,s} \\ \dim(F_{i,l})=j_{i,l}}} \min_{\substack{E_i \in G(s,r) \\ \dim(E_i \cap F_{i,l}) \geq l}} \mathrm{Tr}(H_{i,E_i}) \\ &\leq \sum_{i=1}^n \mathrm{Tr}(H_{i,E}) = \mathrm{Tr}(\sum_{i=1}^n H_i | E) = 0. \end{aligned}$$

Example 6.3.4. Suppose that $J_1 = \{1\}, J_2 = \{r\}, J_3 = \{r\}$. Since every subspace of dimension 1 intersects $\mathbb{C}^r$ in a subspace of dimension 1, namely itself, we obtain the inequality $\lambda_{1,1} + \lambda_{2,r} + \lambda_{3,r} \leq 0$. In terms of sums of matrices, this translates to the fact that $\lambda_r(H_1) + \lambda_r(H_2) \leq \lambda_r(H_1 + H_2)$ for any Hermitian matrices H_1, H_2.

The existence of such an E for generic flags is implied by the non-vanishing of the *Schubert coefficient* $\#[\overline{Y}_{J_1}] \cap \ldots \cap [\overline{Y}_{J_n}]$ in the homology $H(Gr(s, r))$ of the Grassmannian $Gr(s, r)$, where $\overline{Y}_{J_i}$ are the *Schubert varieties* of (4.3). (The singular homology has no torsion and with real coefficients is isomorphic to the de Rham cohomology, so there is no conflict with notation.) Thus

Theorem 6.3.5. *If the Horn problem for $\lambda_1, \ldots, \lambda_n$ has a solution, then $\sum_{l=1}^n \sum_{j \in J_i} \lambda_{i,j} \leq 0$ for all $s < r$ and $J_1, \ldots, J_n$ of size s such that $\#[\overline{Y}_{J_1}] \cap \ldots \cap [\overline{Y}_{J_n}] > 0$ in $H(\mathrm{Gr}(s, r))$.*

Unfortunately, from this point of view it is very difficult to see whether the list of all such inequalities is sufficient. Klyachko [55] noticed that this follows from the Hilbert-Mumford criterion. (See Fulton [30] for a more detailed discussion.) Let $O_{\lambda_j} = K\lambda_j \cong G/P^-_{\lambda_j}$ for some *dominant* $\lambda_1, \ldots, \lambda_n$; for simplicity we assume that λ_j are generic. The quotient $(O_{\lambda_1} \times \ldots \times O_{\lambda_n})/\!/G$ is non-empty iff the semistable locus in $O_{\lambda_1} \times \ldots O_{\lambda_n}$ is non-empty, iff a generic point $F = (F_1, \ldots, F_n)$ in $O_{\lambda_1} \times \ldots \times O_{\lambda_n}$ is semistable for all one-parameter subgroups. Let $\xi \in \mathfrak{k}$ generate a one-parameter subgroup. Under the action of $\exp(z\xi), z \to 0$, the point $F_j \in O_{\lambda_j}$ flows to a T-fixed point x_{w_j} where Y_{w_j} contains F_j. Thus F is ξ-semistable iff

$$\sum_{j=1}^n \langle \lambda_j, w_j^{-1} \xi \rangle \leq 0. \tag{6.2}$$

So F is $\mathrm{Ad}(g)\xi$-semistable iff the same inequalities hold for w_j such that $F_j \in gY_{w_j}$. Let $g_j \in G$ be such that $F_j = g_j B/B$. Then F_j lies in gY_{w_j} iff $g^{-1}B/B \in g_j^{-1} Y_{w_j^{-1}}$. Hence the semistable locus for the diagonal action of G is non-empty iff the inequalities (6.2) hold for dominant ξ whenever $(w_1, \ldots, w_n)$ are such that the intersection of the varieties $g_j^{-1} Y_{w_j^{-1}}$ is non-empty for generic $(g_1, \ldots, g_n)$. This gives a necessary and sufficient set of inequalities. From now on we drop

the inverses on the Weyl group elements w_j, since they appear in both the inequalities and the intersection condition.

The next step is to reduce to inequalities for which the intersection number $\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_n}]$ is non-zero. If the intersection is positive dimensional for generic $(g_1, \ldots, g_n)$ then it represents a non-zero homology class of positive degree, and by Poincaré duality there exists an element $w_{n+1} \in W$ such that $\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_{n+1}}] \neq 0$. Then expanding the product of the last two $[\overline{Y}_{w_n}] \cap [\overline{Y}_{w_{n+1}}]$ and choosing w'_n so that $[\overline{Y}_{w'_n}]$ has positive coefficient in $[\overline{Y}_{w_n}] \cap [\overline{Y}_{w_{n+1}}]$ one obtains w'_n such that $\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_{n'}}] \neq 0$. Then $w_n\lambda - w'_n\lambda \in \mathfrak{t}_+$ and so the inequality for $(w_1, \ldots, w'_n)$ implies that for $(w_1, \ldots, w_n)$. The conclusion is that a generic point is semistable iff

$$\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_n}] > 0 \implies \sum_{l=1}^{n} \langle \lambda_l, w_l \xi \rangle \leq 0 \quad \forall \xi \in \mathfrak{t}_+.$$

It suffices to check the inequalities for ξ in a set of generators for $\mathfrak{t}_+$. In particular, for K semisimple it suffices to check them for ξ equal to a fundamental coweight $\omega_j^\vee$, that is, for a generator of $\mathfrak{t}_+$. An argument similar to the one above shows that these inequalities correspond to non-zero intersection numbers in the corresponding generalized partial flag varieties:

Theorem 6.3.6. *Let K be a compact connected semisimple group with complexification G. A necessary and sufficient set of inequalities for the Horn-Klyachko problem is given by*

$$\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_n}] > 0 \implies \sum_{l=1}^{n} \langle \lambda_l, w_l \omega_l^\vee \rangle \leq 0 \quad \forall \xi \in \mathfrak{t}_+.$$

as $\omega_l^\vee$ ranges over fundamental coweights, $[w_1], \ldots, [w_n]$ range over elements of W/W_{ω_j}, $Y_{w_1}, \ldots, Y_{w_n} \subset G/P_{\omega_j}$ are the corresponding opposite Bruhat cells in the partial flag variety G/P_{ω_j}, with the condition that $\#[\overline{Y}_{w_1}] \cap \ldots \cap [\overline{Y}_{w_n}] \neq 0$ in $H(G/P_{\omega_j})$.

For example, suppose that $K = U(r)$ (and Klyachko's argument was restricted to this case) so that $\mathfrak{t}$ is naturally identified with $\mathbb{R}^n$ and the j-th fundamental weight is identified with $\omega_j = e_1 + \ldots + e_j$, where e_j is the j-th standard basis vector. In this case one obtains that $(O_{\lambda_1} \times \ldots \times O_{\lambda_n})/\!\!/G$ is non-empty iff for each $j \in \{1, \ldots, r\}$ and subsets $J_1, \ldots, J_n \subset \{1, \ldots, r\}$ of size k,

$$\#([\overline{Y}_{J_1}] \cap \ldots \cap [\overline{Y}_{J_n}]) > 0 \implies \sum_{l=1}^{n} \sum_{j \in J_l} \lambda_{l,j} \leq 0$$

c.f. Theorem 6.3.5. So the Hilbert-Mumford approach implies the sufficiency as well as the necessity of these inequalities. Generalizations to groups of arbitrary type and other actions are described in Berenstein-Sjamaar [12] and Ressayre [83].

The cohomology of the Grassmannian $G(s, r)$ has a number of combinatorial models, for example, the famous Littlewood-Richardson rule. A recent "puzzles" model introduced by Knutson and Tao, see [60], is simple enough that we give a brief description. The *puzzle board* is the diagram shown in Figure 6.1. There are r little triangles along each big edge in the board. The *puzzle*

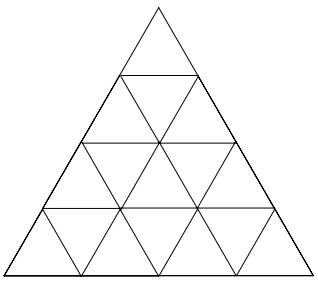

Figure 6.1: Puzzle board

pieces are shown in Figure 6.2. together with their rotations. A *puzzle* is a way of filling in the puzzle board with puzzle pieces so that all of the edges match.

Example 6.3.7. An example of a puzzle is shown in Figure 6.3.

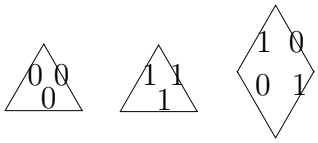

Figure 6.2: Puzzle pieces

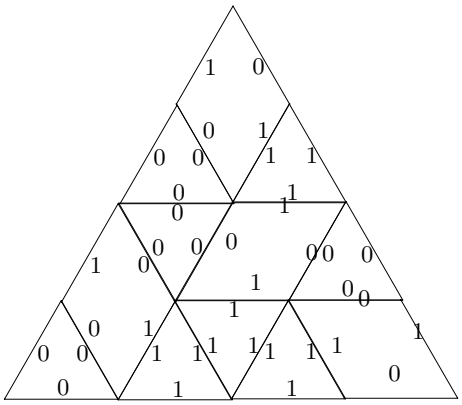

Figure 6.3: An example of a puzzle

For each puzzle, let I denote the positions of the 1's on the northwest boundary, J the positions of the 1's on the northeast boundary, and K the positions of the edge along the southern boundary, reading left to right.

Example 6.3.8. For Figure 6.3, $I = \{2, 4\},\ J = \{2, 4\},\ K = \{2, 3\}$.

Theorem 6.3.9. [60] *The coefficient of* $[\overline{Y}_K]$ *in* $[\overline{Y}_I] \cap [\overline{Y}_J] \in H(G(s, r))$ *is the number of puzzles* n_{IJ}^K *with boundary data* I, J, K.

There are several possible proofs: one given by Knutson and Tao checks the equivalence with the Littlewood-Richardson rule. A second proof [61], joint with the author, proves associativity of the product defined by the puzzle numbers by a simple combinatorial trick, and then checks equality with the Schubert coefficients on generators. The formula generalizes to intersection numbers of arbitrary numbers of Schubert varieties, by considering puzzle boards with arbitrary numbers of "large boundaries". For example, for $n = 4$ one can take a diamond-shaped puzzle board.

Combining this combinatorial description with Klyachko's argument gives the following:

Corollary 6.3.10. *If there is a puzzle whose* 1*'s on the boundary are in positions* I, J, K *then the inequality*

$$\sum_{i \in I} \lambda_i(H_1) + \sum_{j \in J} \lambda_j(H_2) \leq \sum_{k \in K} \lambda_k(H_1 + H_2)$$

holds for any Hermitian matrices A, B*, and these inequalities together with the trace equality*

$$\sum_{i=1}^{n} \lambda_i(H_1) + \sum_{j=1}^{n} \lambda_j(H_2) = \sum_{k=1}^{n} \lambda_k(H_1 + H_2)$$

give sufficient conditions for a triple $(\lambda(H_1), \lambda(H_2), \lambda(H_1 + H_2))$ *to occur.*

Example 6.3.11. The puzzle in Example 6.3.7 gives the inequality $\lambda_2(H_1) + \lambda_4(H_1) + \lambda_2(H_2) + \lambda_4(H_2) \leq \lambda_2(H_1 + H_2) + \lambda_3(H_1 + H_2)$.

The following theorem of Knutson, Tao, and the author [60] (see also the review [59]), extending previous work of Belkale [10], describes a *minimal* set of inequalities:

Theorem 6.3.12. *The inequalities corresponding to* I, J, K *with* $n_{IJ}^K = 1$ *together with the trace equality form a complete and irredundant set of necessary and sufficient conditions for the Horn problem for the sum of two Hermitian matrices.*

Many other problems of this type can be solved in the same way; for example see Agnihotri-Woodward [2] for a discussion of the possible eigenvalues of a product of unitary matrices, and relations with the invariant theory of *quantum* groups. In this case the existence of a good combinatorial model computing the eigenvalue inequalities is still open. Work of Belkale-Kumar [11] and Ressayre [83] gives a minimal set of inequalities for the Klyachko-Horn problem for general type groups.

7. The stratifications of Hesselink, Kirwan, and Ness

According to work of Kirwan [53] and Ness [76], the semistable locus of a G-variety $X \subset \mathbb{P}(V)$ can be considered the open stratum in a Morse-type stratification of X. A theorem of Ness describes the equivalence of this stratification with one introduced by Hesselink [46], which measures the *degree of instability* of a point by its maximal Hilbert-Mumford weight.

7.1. **The Kirwan-Ness stratification.** Let X be a Hamiltonian K-manifold with proper moment map $\Phi : X \to \mathfrak{k}^\vee$. Let $(\ ,\ ) : \mathfrak{k} \to \mathfrak{k} \to \mathbb{R}$ be an invariant metric on $\mathfrak{k}$ inducing an identification $\mathfrak{k} \to \mathfrak{k}^\vee$. Let

$$\phi = \frac{1}{2}(\Phi, \Phi) : X \to \mathbb{R}$$

denote the norm-square of the moment map. The notation $\Phi(x)_X \in \mathrm{Vect}(X)$ denotes the vector field determined by $\Phi(x)$, and $\Phi(x)_X(x) \in T_x X$ its evaluation at x.

Lemma 7.1.1. $\mathrm{crit}(\phi) = \{x \in X, \Phi(x)_X(x) = 0\}$.

Proof. We have $\mathrm{d}\phi(x) = (\Phi(x), \mathrm{d}\Phi(x)) = -\iota_{\Phi(x)_X(x)}\omega_x$. Since ω is non-degenerate, $\mathrm{d}\phi(x)$ vanishes iff $\Phi(x)_X(x) \in T_x X$ does. □

Example 7.1.2. Let $X = \mathbb{P}^2$ and $K = U(1)^2$ acting by $(g_1, g_2)[z_0, z_1, z_2] = [z_0, g_1^{-1} z_1, g_2^{-1} z_2]$. Consider the moment map $\Phi([z_0, z_1, z_2]) \mapsto (|z_1|^2/2, |z_2|^2/2) - (1/4, 1/4)$, which has image the convex hull

$$\Delta(X) = \mathrm{hull}\{(-1/4, -1/4), (-1/4, 3/4), (3/4, -1/4)\}.$$

The critical sets are the level sets of Φ at $(0,0), (-1/4, 0), (0, -1/4), (1/4, 1/4), (-1/4, -1/4), (-1/4, 3/4), (3/4, -1/4)$, see Figure 7.1.

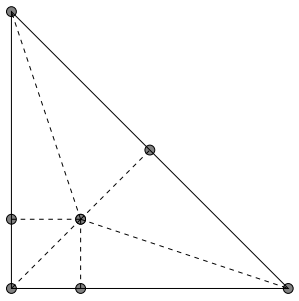

Figure 7.1: Critical values for $X = \mathbb{P}^2$

Lemma 7.1.3. $\Phi(\mathrm{crit}(\phi))$ *is a discrete union of K-orbits in $\mathfrak{k}^\vee$, called the set of* types *for X.*

Proof. Suppose first that K is abelian. Consider the *orbit-type decomposition*

$$X = \bigcup_{H \subset K} X_H, \quad X_H = \{x \in X | K_x = H\}.$$

where the union is over subgroups $H \subset K$. It follows from standard slice theorems that each X_H is a smooth manifold. Let $\mathfrak{h}$ denote the Lie algebra of H. By Lemma 3.3.2, $\Phi(X_H)$ is an open subset of an affine subspace parallel to $\mathrm{ann}(\mathfrak{h})$. Thus $\Phi(X_H \cap \mathrm{crit}(\phi)) = \{\lambda \in \Phi(X_H) | \lambda \in \mathfrak{h}\}$ which is the set containing the unique point in $\Phi(X_H)$ closest to 0, if it exists, and empty, otherwise. Since Φ is proper, the pre-image of any compact set under Φ contains only finitely many orbit-types, which proves the theorem in the abelian case.

Suppose that K is possibly non-abelian with maximal torus T. The action of T on X is also Hamiltonian with moment map Φ_T obtained by composing Φ with the projection of $\mathfrak{k}^\vee$ onto $\mathfrak{t}^\vee$. Let $\phi_T = (\Phi_T, \Phi_T)/2$. Since ϕ is K-invariant, any critical point is conjugate to a point $x \in \mathrm{crit}(\phi)$ with $\Phi(x) \in \mathfrak{t}^\vee$. Then $x \in \mathrm{crit}(\phi)$ iff $x \in \mathrm{crit}(\phi_T)$ iff $\Phi(x)$ is a type for the action of T. Hence the types for K are locally finite. □

Choose a compatible K-invariant metric on X, and let $\mathrm{grad}(\phi) \in \mathrm{Vect}(X)$ denote the gradient of ϕ.

Lemma 7.1.4. *The gradient of ϕ is* $\mathrm{grad}(\phi)(x) = J(x)\Phi(x)_X(x)$.

Proof. Using the proof of Lemma 7.1.1, for $v \in T_x X$

$$g_x(\mathrm{grad}(\phi)(x), v) = D_x\phi(v) = -\omega_x(\Phi(x)_X(x), v) = g_x(J(x)\Phi(x)_X(x), v).$$

The claim follows. □

Let $\varphi_t : X \to X$ be the flow of $-\mathrm{grad}(\phi)$; since Φ is proper, so is ϕ and so φ_t exists for all times $t \in [0, \infty)$.

Proposition 7.1.5 (Duistermaat, see [66], [104])**.** *Any trajectory of φ_t has a limit.*

For the construction of the Kirwan stratification the actual convergence of φ_t is not needed. For each type λ, let $C_\lambda = \Phi^{-1}(K\lambda) \cap \mathrm{crit}(\phi)$ denote the corresponding component of the critical set of ϕ. Since the set of types is discrete, any two limit points are contained in some $C_\lambda \subset \mathrm{crit}(\phi)$, and in fact in the same connected component of $\mathrm{crit}(\phi)$. Let X_λ denote the set of points $x \in X$ flowing to C_λ,

$$X_\lambda := \{\overline{\{\varphi_t(x), t \in [0, \infty)\}} \cap C_\lambda \neq \emptyset.\}.$$

Note that since ϕ is not Morse-Bott in general, there is no guarantee that X_λ is smooth. The *Kirwan-Ness stratification* is the decomposition [53], [76]:

$$X = \bigcup_\lambda X_\lambda.$$

Theorem 7.1.6 (Kirwan)**.** *There exists an invariant metric on X so that each stratum X_λ is smooth. The spectral sequence for the equivariant stratification $X = \cup_\lambda X_\lambda$ collapses at the second page, so that*

$$H_K(X) \cong \bigoplus_\lambda H_K(X_\lambda).$$

In particular the canonical map $H_K(X) \to H_K(\Phi^{-1}(0))$ (which is isomorphic to $H(X/\!\!/K)$ if K acts freely on $\Phi^{-1}(0)$) is a surjection and the equivariant Poincaré polynomial of X

$$p_X^K(t) = \sum t^j \,\mathrm{rank}\, H_K^j(X)$$

is given by

$$p_X^k(t) = \sum_\lambda (-1)^{\mathrm{codim}(X_\lambda)} p_{X_\lambda}^K(t).$$

If X acts freely on $\Phi^{-1}(0)$ this means that the difference $p_X^K(t) - p_{X/\!\!/K}(t)$ is a finite sum of contributions from fixed point sets of one-parameter subgroups. We will see a version of this formula for sheaf cohomology in the last chapter.

In the case that X is a Kähler Hamiltonian K-manifold with proper moment map, the Kirwan-Ness stratification has a more explicit description. For each type λ let $\varphi_{\lambda,t}$ denote the time t flow of $-\mathrm{grad}\langle\Phi, \lambda\rangle$, Z_λ the component of the fixed point set X^λ of the action of λ containing C_λ, Y_λ the subset of X flowing to Z_λ under $\varphi_{\lambda,t}$, K_λ the centralizer of λ, and $U(1)_\lambda$ the one-parameter subgroup generated by λ. Then $K_\lambda/U(1)_\lambda$ acts naturally on Z_λ in Hamiltonian fashion with moment map denoted Φ_λ, obtained by restricting Φ to Z_λ and projecting out the direction generated by λ. We denote by Z_λ^{ss} the set of points flowing to $\Phi_\lambda^{-1}(0)$ under the flow of minus the gradient of the norm-square of Φ_λ. Let Y_λ^{ss} denote the inverse image of Z_λ^{ss} in Y_λ.

Theorem 7.1.7 (Kirwan [53])**.** *Let X be a Kähler Hamiltonian K-manifold with proper moment map $\Phi : X \to \mathfrak{k}^\vee$. For the Kähler metric each X_λ is a G-invariant complex submanifold, each Y_λ is a P_λ-invariant complex submanifold, and $G \times_{P_\lambda} Y_\lambda^{ss} \to X_\lambda$, $[g, y] \mapsto gy$ is an isomorphism of complex G-manifolds.*

We give a proof, and explain the relation with a theorem of Ness [76], in the following section. In the point of view we will present, a key fact is that the gradient flow of the norm-square of the moment map is essentially equivalent to the gradient flow of the Kempf-Ness function, as was pointed out in Donaldson-Kronheimer [26, Section 6]. Let X be a Kähler Hamiltonian K-manifold with proper moment map. For any $x \in X$, let x_t denote the trajectory of the gradient flow of $-\phi$ starting at x. On the other hand, let $\psi : \mathfrak{k} \to \mathbb{R}$ be a Kempf-Ness function for x, $\operatorname{grad}\psi(\xi) = \Phi(\exp(i\xi)x)$. We may also consider the gradient flow of ψ, with respect to the given metric on $\mathfrak{k}$.

Proposition 7.1.8. *Let X, x, ψ be as above. The map $\mathfrak{k} \to X$, $\xi \mapsto \exp(i\xi)x$ maps the gradient trajectories of ψ onto the gradient trajectories of $-\phi$.*

Proof. Follows from $\operatorname{grad}\psi(\xi)_X(\exp(i\xi)(x)) = \Phi(\exp(i\xi)x)(\exp(i\xi)(x))$ and Lemma 7.1.4. □

In particular, since the trajectories of ψ exist for all time by the bound on Φ, any trajectory of $-\operatorname{grad}(\phi)$ is contained in a single G-orbit: $x_t \in Gx, \forall x \in X, t \in \mathbb{R}$.

Corollary 7.1.9. *ψ is bounded from below iff the gradient flow for $-\phi$ converges to $\Phi^{-1}(0)$.*

Proof. In the algebraic case, this is nothing but a reformulation of 4.3.4. For the Kähler case, note that if ψ is bounded from below then $\operatorname{grad}(\psi)$ converges to zero along any gradient trajectory, and by equivalence of gradient flows 7.1.8 it follows that Φ must converge to zero. The converse follows as in the proof of Theorem 5.4.9, using that $\operatorname{grad}(\psi)$ converges to zero exponentially fast along any one-parameter subgroup whose limit corresponds to a polystable point. □

7.2. **The Hesselink stratification.** Let $X \subset \mathbb{P}(V)$ be a projective G-variety, or more generally a compact Kähler Hamiltonian K-manifold. The Hesselink stratification uses the weights appearing in the Hilbert-Mumford criterion to construct a stratification on X: Define for any $\lambda \in \mathfrak{k}$ the *Hilbert-Mumford degree*

$$\deg_\lambda(x) = \langle \Phi(x_\lambda), \lambda \rangle.$$

Definition 7.2.1. A point $x \in X$ is

(a) *degree semistable* iff $\deg_\lambda(x) \leq 0$ for all λ,
(b) *degree stable* iff $\deg_\lambda(x) < 0$ for all λ,
(c) *degree unstable* iff x is not semistable, and
(d) *degree polystable* iff it is degree semistable and $\deg_\lambda(x) = 0$ implies $\deg_{-\lambda}(x) = 0$ for all λ.

Degree semistability might also be called Hilbert-Mumford semistability, but this seems a little unwieldy. We have already seen in the proof of the Kempf-Ness theorem that degree semistability is equivalent to semistability. The equivalence of degree polystability with polystability is proved in Mundet [50]. It follows from Section 5.4 that a point $x \in X$ is polystable but not stable iff its Jordan-Hölder cone contains a line.

The set of destabilizing one-parameter subgroups is studied by Hesselink in the algebraic case [46], [47], see also Ramanan-Ramanathan [82]. For any λ we denote by G_λ the centralizer of λ and by $\mathbb{C}^*_\lambda$ the one-parameter subgroup generated by λ. Obviously $\mathbb{C}^*_\lambda \subset G_\lambda$. Let $x \in X$ and Z_λ denote the component of X^λ containing x_λ. Then the action of G_λ on Z_λ descends to an action of $G_\lambda/\mathbb{C}^*_\lambda$. Furthermore, the inner product on $\mathfrak{k}$ determines a splitting $\mathfrak{g}_\lambda = \mathbb{C}\lambda \oplus \mathfrak{g}_\lambda/\mathbb{C}\lambda$ which defines a lift of $G_\lambda/\mathbb{C}^*_\lambda$ to the polarizing line bundle, at least up to finite cover. So we may consider Z_λ as a polarized $G_\lambda/\mathbb{C}^*_\lambda$-variety, with the caveat that the polarization depends on the choice of inner product on $\mathfrak{k}$.

Theorem 7.2.2. (a) *Any unstable x has a unique (up to scalar multiple)* optimal one-parameter subgroup *generated by $\lambda \in \mathfrak{k}$ with the property that x_λ is a semistable point for the action of $G_\lambda/\mathbb{C}^*_\lambda$ on Z_λ.*

(b) *The optimal one-parameter vector λ has the property that it* maximally destabilizing*: $\deg_\mu(x)/\|\mu\| \leq \deg_\lambda(x)/\|\lambda\|$ for all $\mu \in \mathfrak{k} - \{0\}$ and equality holds iff $\mathbb{R}_+\mu = \mathbb{R}_+\lambda$.*

We prove this theorem in the next section. Let Λ denote the set of equivalence classes of one-parameter subgroups appearing in Hesselink's theorem (with equivalence given by the adjoint action) we call the decomposition $X = \cup_\lambda X_\lambda$ the *Hesselink stratification* of X.

Remark 7.2.3. The Hesselink stratification is the finite-dimensional analog of the Shatz stratification [89] of the moduli stack of vector bundles on a curve by the type of the Harder-Narasimhan filtration.

The following is proved in the algebraic case by Ness [76]:

Theorem 7.2.4. *The Hesselink and Kirwan-Ness stratifications agree.*

This is a generalization of her earlier theorem with Kempf [52], which describes the same result for the open strata only; it includes the Hilbert-Mumford criterion, by definition of degree semistability. We will prove the Hesselink and Ness theorems as well as the Hilbert-Mumford criterion using results on convex functions: Let V be a Euclidean vector space. For any function $f : V \to \mathbb{R}$, we denote by $\mathrm{grad}(f) \in \mathrm{Vect}(V)$ the gradient vector field of f, and for any $v \in V$ let v_t denote the trajectory of $-\mathrm{grad}(f)$. A smooth function $f : V \to \mathbb{R}$ is *strictly convex* iff the Hessian of f is positive definite at every point in v. The following is an easy consequence of strict convexity:

Lemma 7.2.5. *Let V be a Euclidean vector space, $f : V \to \mathbb{R}$ a convex function. If f has a critical point x then it is a global minimum. Furthemore, if f is strictly convex then x is the unique critical point.*

If f has no global minimum, then convexity still implies that f has a unique direction of maximum descent, under modest technical assumptions: We say that f has a *well-behaved gradient* if the gradient of f is bounded and the limit of $\mathrm{grad}(f)$ exists along any gradient trajectory v_t.

Proposition 7.2.6. *Suppose that $f : V \to \mathbb{R}$ is convex and has a well-behaved gradient. Then there exists a unique $\lambda \in V$ so that*

(a) *any gradient trajectory v_t of $-f$ has $\mathrm{grad}(f)(v_t) \to \lambda$ as $t \to -\infty$.*
(b) *Suppose that $\mu \in V$ and $(\mathrm{grad}(f)(-\mu t), \mu)$ approaches a limit c_μ as $t \to \infty$. Then $c_\mu/\|\mu\| \leq \|\lambda\|$, with equality if and only if μ is a positive scalar multiple of λ.*

Proof. (a) The idea is that gradient trajectories "automatically discover the direction of maximal descent". Suppose that $v_{j,t}, j = 0, 1$ are two gradient trajectories and

$$\lim_{t\to\infty} \mathrm{grad}(f)(v_{j,t}) \to \lambda_j$$

for some $\lambda_j \in V, j = 0, 1$. Consider the path

$$\gamma_{t_0,t_1}(s) = sv_{0,t_0} + (1-s)v_{1,t_1}.$$

Let $f_{t_0,t_1}(s) = f(\gamma_{t_0,t_1}(s))$. Case (i): λ_0, λ_1 are both non-zero, so that $v_{j,t} \sim \lambda_j t$ as $t \to \infty$, that is, $\|v_{j,t} + \lambda_j t\|/\|\lambda_j t\| \to 0$ as $t \to \infty$. Choose t_0, t_1 so that $f(v_{0,t_0}) = f(v_{1,t_1})$. By convexity $\frac{d}{ds} f_{t_0,t_1}(s)$ is non-positive at $s = 0$ and non-negative at $s = 1$. On the other hand $\frac{d}{ds} f_{t_0,t_1}(j) = (\mathrm{grad}(f)(j), v_{1,t_1} - v_{0,t_0}) \sim (\lambda_j, -\lambda_1 t + \lambda_0 t)$, so $(\lambda_0, -\lambda_1 + \lambda_0) \leq 0$ and $(\lambda_1, -\lambda_1 + \lambda_0) \geq 0$. But $(\lambda_1 - \lambda_0, \lambda_1 - \lambda_0) > 0$ implies that $(\lambda_1, -\lambda_1 + \lambda_0) < (\lambda_0, -\lambda_1 + \lambda_0)$, which is a contradiction. Case (ii): one of the λ_j, say λ_0 vanishes, then λ_1 is necessarily non-zero. Then $Df_{t_0,t_1}(0) \to 0$ as $t_0, t_1 \to \infty$ and $f_{t_0,t_1}(0)/t_0 \sim 0$ as $t_0, t_1 \to \infty$, but $f_{t_0,t_1}(1) \sim c_0 - t_1(\lambda_1, \lambda_1)$ as $t_1 \to \infty$, which contradicts convexity.

(b) First suppose $\mu = \lambda$. The function $(\mathrm{grad}(f), \lambda)$ has gradient trajectory $t\lambda$, so $(\mathrm{grad}(f), \lambda)$ is non-increasing along $-t\lambda$. Since $\mathrm{grad}(f)$ is bounded and $(\mathrm{grad}(f)(-t\lambda), \lambda)$ is decreasing, the limit

$$c_\lambda = \lim_{t\to\infty} (\mathrm{grad}(f)(-t\lambda), \lambda)$$

exists. Hence $f(-t\lambda) \sim -c_\lambda t$. Suppose by way of contradiction that $c_\lambda \neq (\lambda, \lambda)$. Then $|f(-t\lambda) - f(v_t)| \geq Ct$ for some constant $C > 0$. Since $v_t \sim -t\lambda$, $|f(-t\lambda) - f(v_t)|/\|v_t + t\lambda\| \to \infty$ as $t \to \infty$. Together with the mean value inequality this contradicts the assumption that the gradient of f is bounded.

More generally, let $\mu \notin \mathbb{R}_+\lambda$. Then $f(-\mu t) \sim -c_\mu t$ for some constant c_μ and $f(-\lambda t) \sim -(\lambda, \lambda)t$. Let μ_1, λ_1 be the unit vectors in the direction of μ, λ. If $c_\mu/\|\mu\| < -\|\lambda\|$, then $f(-\mu t)$ goes faster to $-\infty$ than $f(-\lambda_1 t)$. Consider the path $\gamma(s) = (1-s)t\mu_1 + st\lambda_1$. On the one hand, $(\mu_1 - \lambda_1, \lambda_1) \leq 0$

implies that $\frac{d}{ds} f(\gamma(s))|_{s=1} \leq 0$ for $t \gg 0$. On the other hand, $f(\gamma(s))|_{s=0} \ll f(\gamma(s))|_{s=1}$ for $t \gg 0$, which is a contradiction. Hence $c_\mu/\|\mu\| \geq \|\lambda\|$. If equality holds, then the same argument shows we must have $(\mu_1 - \lambda_1, \lambda_1) = 0$, and since both μ_1, λ_1 are unit vectors this implies $\mu_1 = \lambda_1$. □

Proof of Kirwan's theorem 7.1.7. Let $V = \mathfrak{k}$ and $f = \psi$ be a Kempf-Ness function. The gradient $\mathrm{grad}(\psi)$ is well-behaved since X is compact and the gradient flow converges by 7.1.5. Proposition 7.2.6 then implies that for each $x \in X$ there is a unique direction $-\lambda$ of maximal descent for the Kirwan-Ness function. Let X_λ denote the set of points whose directions are *conjugate* to λ and U_λ the set of points whose directions are equal to λ. Equality of the gradient flows Theorem 7.1.8 implies that X_λ is the same as Kirwan's, that is, equals the set of points whose gradient flow converges to $\Phi^{-1}(K\lambda)$. Uniqueness of λ implies that if $x \in U_\lambda$ and $g \in G$ is such that $gx \in U_\lambda$, then $g \in P_\lambda$. Indeed, note $G = KP_\lambda$ and U_λ is P_λ-stable. Hence it suffices to consider the case $g \in K$, and then $g\lambda$ is also a direction of maximal descent. Hence $g\lambda = \lambda$ which implies that $g \in K_\lambda$, hence $g \in P_\lambda$. This implies $X_\lambda = G \times_{P_\lambda} U_\lambda$, which proves the first part of Kirwan's theorem.

To prove the second part, let x_λ denote the associated graded point for some $x \in U_\lambda$. Since $\overline{Gx}$ intersects $\Phi^{-1}(K\lambda)$, $\overline{G_\lambda x_\lambda}$ intersects $\Phi^{-1}(K\lambda)$, so x_λ is semistable for the action of G_λ on Z_λ. Conversely, the pre-image of Z_λ^{ss} is contained in U_λ, since both are G_λ-invariant and contain $\Phi^{-1}(\lambda)$. It follows that $U_\lambda = Y_\lambda^{\mathrm{ss}}$ of Section 7.1. This proves Kirwan's theorem. □

Proof of Hesselink's theorem 7.2.2. Let $x \in X$ and let $-\lambda$ be the direction of maximal descent of the Kempf-Ness function. We must show that λ generates the unique one-parameter subgroup such that x_λ is G_λ-semistable. Suppose that μ generates another one-parameter subgroup. Part (b) of Proposition 7.2.6 gives the inequality $c_\mu/\|\mu\| < \|\lambda\|$ where $c_\mu = (\Phi(x_\mu), \mu)$. Suppose that the G_μ orbit of x_μ is semistable; then its closure intersects $\Phi^{-1}(\mu_1)$ where $\mu_1 \in \mathbb{R}_+\mu$ is such that $c_\mu/\|\mu\| = \|\mu_1\|$. But then the closure of Gx also intersects $\Phi^{-1}(\mu_1)$. By Theorem 7.1.8, $\|\lambda\|$ is the infimum of $\|\Phi\|$ on the orbit Gx. Indeed, $\|\Phi\|$ is decreasing on gradient trajectories of ψ, which all converge to λ. This contradicts $\|\mu_1\| < \|\lambda\|$. □

Remark 7.2.7. Suppose $\omega \in \Omega^2(X)$ is a closed two form that is not symplectic, but satisfies $\omega(\xi_X, J\xi_X) > 0$ for any $\xi \in \mathfrak{k}$. The proof above works equally well for moment maps associated to such two-forms. That is, only non-degeneracy of the two-form on the directions generated by the action is used in the proof.

8. Moment polytopes

According to work of Atiyah, Guillemin-Sternberg, and Kirwan, the quotient of the image of the moment map is convex. (This section could have been placed before that on Schur-Horn convexity.)

8.1. **Convexity theorems for Hamiltonian actions.** Let X be a Hamiltonian K-manifold with moment map Φ. The *moment image* of X is $\Phi(X) \subset \mathfrak{k}$. The quotient

$$\Delta(X) := \Phi(X)/K \subset \mathfrak{k}^\vee/K$$

can be identified with a subset of the convex cone $\mathfrak{t}_+^\vee \cong \mathfrak{k}^\vee/K$.

Example 8.1.1. If $X = \mathbb{P}^{n-1}$ and $G = U(1)^n$ acts by the standard representation, then the moment image is the *standard n-simplex*

$$\Phi(X) = \{(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{\geq 0} \mid \mu_1 + \ldots + \mu_n = 1\},$$

see (3.3). The coordinate hyperplane $\{z_j = 0\} \subset X$ maps to the j-th facet $\{\mu_j = 0\} \subset \Phi(X)$.

Another description of the moment polytope $\Delta(X)$ involves the *shifted* symplectic quotients: for $\lambda \in \mathfrak{k}^\vee$, the quotient

$$X/\!\!/_\lambda K := \Phi^{-1}(K\lambda)/K = (\mathcal{O}_\lambda^- \times X)/\!\!/K$$

is the *symplectic quotient of X at λ*. The shifted symplectic quotient is the classical analog of the multiplicity space $\mathrm{Hom}_K(V_\lambda, V)$ of a representation V in the following sense:

Proposition 8.1.2. *Let X be a polarized projective G-variety and λ a dominant weight. Then $R(X/\!\!/_\lambda G)_d = \mathrm{Hom}_G(V_{d\lambda}, R(X)_d)$ for any $d \geq 0$.*

Proof. Combining the Borel-Weil and Kempf-Ness theorems gives $R(X/\!\!/_\lambda K)_d = R(K\lambda^- \times X)_d^K = (V_{d\lambda}^\vee \otimes R(X)_d)^K = \mathrm{Hom}_K(V_{d\lambda}, R(X)_d)$. □

The following is immediate from the definitions:

Lemma 8.1.3. *$\Delta(X) = \{\lambda \mid X/\!\!/_\lambda K \neq \emptyset\}$ is the set of λ for which the shifted symplectic quotient $X/\!\!/_\lambda K$ is non-empty.*

The set $\Delta(X)$ is the "classical analog" of the set of simple modules appearing in a G-module. Let $\Delta_{\mathbb{Q}}(X) := \Lambda_{\mathbb{Q}}^\vee \cap \Delta(X)$ denote the set of rational points in $\Delta(X)$; furthermore $\Delta_{\mathbb{Q}}(X)$ is dense in $\Delta(X)$, see for example [65].

Theorem 8.1.4. *$\Delta_{\mathbb{Q}}(X) = \Delta(X) \cap \Lambda_{\mathbb{Q}}^\vee$ is equal to the set of points λ/d such that $V_\lambda \subset R(X)_d$.*

Proof. By Lemma 8.1.3 and Proposition 8.1.2. □

Recall that a *convex polyhedron* is the intersection of a finite number of half spaces, while a *convex polytope* is the convex hull of a finite number of points. The fundamental theorem of convex geometry says that any *compact* convex polyhedron is a convex polytope and vice-versa.

Theorem 8.1.5 (Atiyah [5], Guillemin-Sternberg [32] for the abelian case, Kirwan [54] for the non-abelian case)**.** *Let K be a compact, connected Lie group and X a compact connected Hamiltonian K-manifold. Then $\Delta(X)$ is a convex polytope. If K is abelian, then $\Delta(X)$ is the convex hull of the image $\Phi(X^K)$ of the fixed point set X^K of K.*

$\Delta(X)$ is the *moment polytope* of X. The arguments of Atiyah and Guillemin-Sternberg in [5], [32] are Morse-theoretic. The equivariant version of Darboux's theorem implies that the functions $\langle \Phi, \xi \rangle$ have only critical sets of even index, and this implies that the level sets $\langle \Phi, \xi \rangle^{-1}(c)$ are connected. Using an inductive procedure one shows that for any subtorus $K_1 \subset K$, the level sets of the moment map for Φ_1 are connected as well. Taking K_1 of codimension one, this shows that the intersection of $\Phi(X)$ with any rational line is connected and it follows that $\Phi(X)$ is convex. The reader is referred to the original papers for details. Kirwan's non-abelian version uses the Morse theory of the norm-square of the moment map. See Lerman-Meinrenken-Tolman-Woodward [65] for a derivation of non-abelian convexity from the abelian case.

Brion [16], following earlier work of Mumford [76, Appendix], pointed out the following proof of convexity, which in language of geometric quantization would be called a "quantum" proof: Suppose $\lambda_j/d_j \in \Delta_{\mathbb{Q}}(X), j = 0, 1$. Let $v_j \in R(X)_{d_j}$ be the corresponding highest weight vectors. Then for any $n_0, n_1 \in \mathbb{N}$, $v_0^{n_0} v_1^{n_1} \in R(X)_{n_0 d_0 + n_1 d_1}$ is a highest weight vector, so

$$\frac{n_0\lambda_0 + n_1\lambda_1}{n_0 d_0 + n_1 d_1} = \frac{d_0 n_0}{d_0 n_0 + d_1 n_1}(\lambda_0/d_0) + \frac{d_1 n_1}{n_0 d_0 + n_1 d_1}(\lambda_1/d_1) \in \Delta_{\mathbb{Q}}(X).$$

This implies that $\Delta_{\mathbb{Q}}(X)$ is convex.

The inequalities of the previous section (for example, the Horn-Klyachko problem) can now be seen as the inequalities describing the moment polytopes of products of coadjoint orbits.

8.2. **Convexity theorems for orbit-closures.** In the case that X is Kähler, Atiyah [5] also described the images of orbit-closures under the moment map, in the case that K is abelian. Of course if the orbit-closure is smooth, then this falls into the previous convexity theorem, but Atiyah's theorem also includes the case of singular orbit-closures:

Theorem 8.2.1. [7, Theorem 2] *Let K be a torus, G its complexification, and X a Kähler Hamiltonian K-manifold. Let $Y \subset X$ be a G-orbit. Then*

(a) *$\Delta := \Phi(\overline{Y})$ is a convex polytope with vertices $\Phi(\overline{Y} \cap X^G)$;*
(b) *For each open face $F \subset \Delta$, $\Phi^{-1}(F) \cap \overline{Y}$ is a single G-orbit.*
(c) *Φ induces a homeomorphism of $\overline{Y}/G$ onto Δ.*

We will describe Atiyah's arguments since they are brief and are closely related to the one-parameter subgroups of Hesselink as well as the Jordan-Hölder subgroups of Section 5.4.9. The proof depends on the following

Lemma 8.2.2. *Let $Y \subset X$ be a G-orbit and $y \in Y$. Then*

(a) *$y_\lambda = \lim_{t\to\infty}(\exp(it\lambda)y)$ exists and lies in the fixed point set X^λ;*

(b) $\lim_{t\to\infty}\langle\Phi(\exp(it\lambda)y),\lambda\rangle$ *exists and is a constant* d_λ *independent of* y.
(c) $d_\lambda = \sup_{y\in Y}\langle\Phi(y),\lambda\rangle$.

Suppose that λ is generic so that $X^G = X^\lambda$. The Lemma implies

$$\sup_{y\in Y}\langle\Phi(y),\lambda\rangle = \sup_{y\in X^G\cap\overline{Y}}\langle\Phi(y),\lambda\rangle.$$

Hence $\Phi(Y)$ is contained in the convex hull of $\Phi(X^G\cap\overline{Y})$. To see that $\Phi(\overline{Y}) = \Delta$, Atiyah notes that for any $y\in Y$ and direction $\xi\in\mathfrak{k}$ of unit length, there exists a time $t(\xi)$ such that

$$\langle\Phi(\exp(it(\xi)\xi y),\xi\rangle = \frac{1}{2}(\Phi(y)+d(\xi)).$$

The set of points $\exp(i\xi)y$ with $\|\xi\|\le t(\xi/\|\xi\|)$ defines a neighborhood U of y in Y with $\Phi(\overline{U}) = \Phi(y)+\frac{1}{2}(\Delta-\Phi(y))$; this immediately implies that $\Phi(\overline{Y})$ is both open and closed in Δ and hence equal to Δ.

To prove the third part of the Theorem, Atiyah considers for any $\lambda\in\mathfrak{k}$ and fixed point component $Z\subset X^\lambda$, the unstable manifold Z^u consisting of all points that flow to Z under $\exp(it\lambda)$. By the stable manifold theorem Z^u is a smooth manifold and the limit of the flow defines a smooth G-equivariant projection $Z^u\to Z$. In particular, if Z is any component of X^λ containing a limit point of Y then $Y\subset Z^u$ and $\overline{Y}\cap Z$ is a single G-orbit. From this it follows that $\Phi(Z\cap\overline{Y})$ is a face of Δ with fibers the orbits of the compact torus K, see [7, p. 10], and this completes the proof.

Remark 8.2.3. Atiyah's theorem makes the theory of polystable points and Jordan-Hölder vector described in Section 5.4 substantially easier in the abelian case. One sees that the "Jordan-Hölder" cone of Theorem 5.4.9 is the dual cone to the face of the polytope containing 0, in the case that Y is a semistable orbit.

Atiyah's convexity theorem for orbit-closures has been generalized to Borel subgroups by Guillemin and Sjamaar [39].

9. Multiplicity-free actions

In certain cases Hamiltonian or algebraic actions may be classified by combinatorial data related to the moment map. In this section we discuss an example of this, the *multiplicity-free case*, from both the algebraic and symplectic points of view.

9.1. **Toric varieties and Delzant's theorem.** A *toric variety* is a normal G-variety X such that G is an algebraic torus and X contains an open G-orbit. Affine toric varieties are naturally classified by monoids M in the group $\Lambda^\vee$ of weights of G, with the corresponding toric variety given by $\mathrm{Spec}(\mathbb{C}[M])$. Each such monoid spans a rational cone in $\Lambda^\vee_{\mathbb{Q}}$, and defines a dual cone in $\Lambda_{\mathbb{Q}}$. Toric varieties with trivial generic stabilizer are classified by *fans* in $\Lambda_{\mathbb{Q}}$, that is, collections of cones such that any intersection of a cone is again a cone in the fan, see Oda [79] or Fulton [29].

Example 9.1.1. Suppose that $X=\mathbb{P}^2$ with action given by $(w_1,w_2)[z_0,z_1,z_2] = [z_0,w_1^{-1}z_1,w_2^{-1}z_2]$. There are seven orbits, given by non-vanishing of various coordinates, and in particular, three closed orbits $[1,0,0],[0,1,0],[0,0,1]$, whose cones are generated by pairs of vectors $(1,1),(-1,0)$, $(1,1),(0,-1)$, and $(0,-1),(0,-1)$. The fan contains these three cones, and their intersections; this is the *dual fan* to the moment polytope $\mathrm{hull}((0,0),(1,0),(0,1))$.

A Hamiltonian torus action is *multiplicity free* or *completely integrable* if all the symplectic quotients are points, or equivalently, each fiber of the moment map is an orbit of the torus.

Example 9.1.2. The $U(1)^n$ action on $\mathbb{P}^n$ is multiplicity-free, since the fibers of the moment map are given by $[z_0,\dots,z_n]$ with $|z_1|,\dots|z_n|$ fixed, which are orbits of $U(1)^n$.

Multiplicity-free Hamiltonian torus actions are classified by a theorem of Delzant.

Definition 9.1.3. A polytope $\Delta\subset\mathfrak{k}^\vee$ is called *Delzant* if the normal cone at any vertex is generated by a basis of the weight lattice $\Lambda^\vee\subset\mathfrak{k}^\vee$.

Theorem 9.1.4 (Delzant [24]). *There exists a one-to-one correspondence between Delzant polytopes and multiplicity-free torus actions on compact connected manifolds with trivial stabilizer, given by $X \mapsto \Phi(X)$. Any compact connected multiplicity-free Hamiltonian torus action has the structure of a smooth projective toric variety.*

Note that any compatible complex structure is unique up to isomorphism, but not up to Kähler isomorphism. That is, any toric variety has many non-equivalent Kähler structures, see Guillemin [38].

There are "local" and "local-to-global" parts of the proof; the local part follows from the equivariant Darboux theorem, while the "local-to-global" part uses the vanishing of a certain sheaf cohomology group over the polytope.

Existence of a smooth projective toric variety with a given polytope follows from, for example, Lerman's method of *symplectic cutting* [64] which we now describe. We begin with the simplest case, when X is a Hamiltonian S^1-manifold with moment map $\Phi : X \to \mathbb{R}$. The diagonal S^1-action on $X \times \mathbb{C}$ is Hamiltonian with moment map

$$\Phi_{X\times\mathbb{C}} : (x, z) \mapsto \Phi(x) - |z|^2/2.$$

Its symplectic quotient at any value λ

$$X_{\geq\lambda} := (X \times \mathbb{C})/\!\!/_\lambda S^1$$

is called the *symplectic cut* of X at λ admits a decomposition

$$(X \times \mathbb{C})/\!\!/_\lambda S^1 \cong X/\!\!/_\lambda S^1 \cup (X \times \mathbb{C}^*)/\!\!/_\lambda S^1 \cong X/\!\!/_\lambda S^1 \cup \Phi^{-1}((\lambda, \infty)).$$

It follows from the definitions that the inclusion of $\Phi^{-1}((\lambda, \infty))$ in $X_{\geq\lambda}$ is symplectic and so $X_{\geq\lambda}$ is obtained by removing $\Phi^{-1}((-\infty, \lambda))$ and "closing off" the boundary by quotienting it by S^1.

More generally, suppose that K is a torus, $\xi \in \mathfrak{k}$ any rational vector, and $\lambda \in \mathbb{R}$. Let $U(1)_\lambda$ denote the one-parameter subgroup generated by λ, with moment map $\langle \Phi, \lambda \rangle$. Then the symplectic cut $X_{\geq\lambda} = (X \times \mathbb{C})/\!\!/_\lambda U(1)_\lambda \cong X/\!\!/_\lambda U(1)_\lambda \cup \{\langle \Phi, v \rangle > \lambda\}$ admits the structure of a Hamiltonian K-manifold with moment polytope $\Phi(X_{\geq\lambda}) = \Phi(X) \cap \{\langle \mu, v \rangle \geq \lambda\}$.

Example 9.1.5. Let $X = \mathbb{P}^2$ equipped with $U(1)^2$-action given by with weights $(0, 0), (-2, 0), (0, -2)$. The moment polytope is then the convex hull of $(0, 0), (2, 0), (0, 2)$. Let $\lambda = (0, -1)$ so that the one-parameter subgroup generated by λ acts with moment map

$$[z_0, z_1, z_2] \mapsto -2|z_1|^2/(|z_0|^2 + |z_1|^2 + |z_2|^2).$$

The symplectic cut at -1 is then a toric variety with polytope the convex hull of $(0, 0), (0, 2), (1, 0), (1, 1)$, see Figure 9.1.

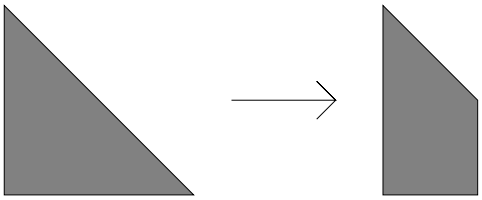

Figure 9.1: Effect of cutting on a moment polytope

Suppose that Δ is a Delzant polytope defined by inequalities

$$\Delta = \{\mu \in \mathfrak{k}^\vee \mid \langle \mu, v_j \rangle \geq \lambda_j, j = 1, \ldots, m\}$$

for some vectors $v_j \in \mathfrak{k}$ and some constants $\lambda_j \in \mathbb{R}, j = 1, \ldots, m$. Let $X = T^\vee K$, with moment image $\mathfrak{k}^\vee$ and the standard Kähler structure. Performing a symplectic cut for each inequality gives a Kähler manifold with Hamiltonian K action and moment polytope Δ.

Alternatively any smooth projective toric variety is a symplectic or geometric invariant theory quotient of affine space $X = \mathbb{C}^m$. There is an explicit description of the semistable locus given by Audin [8] and Cox [23].

9.2. **Multiplicity-free actions and spherical varieties.** Let K be a compact connected Lie group. Recall that a K-module V is *multiplicity-free* iff $\mathrm{Hom}_K(V_\lambda, V)$ is dimension at most one, for any simple K-module V_λ iff $\mathrm{End}_K(V)$ is abelian, using Schur's lemma. The definition in part (a) of the following was introduced in Guillemin-Sternberg [40]:

Theorem 9.2.1. *(see* [102, Appendix]*) The following conditions are equivalent, and if they hold the action is* multiplicity-free*:*

(a) $C^\infty(X)^K$ *is an abelian Poisson algebra.*
(b) *The symplectic quotient* $X/\!\!/_\lambda K := \Phi^{-1}(K\lambda)/K$ *is a point for all* λ*.*

Proof. We denote by $r_\lambda : C^\infty(X)^K \to C^\infty(X/\!\!/_\lambda K)$ the map of Poisson algebras induced by the symplectic quotient construction, if λ is free. In general, we define $C^\infty(X/\!\!/_\lambda K) := C^\infty(X)^K/\{f, f|\Phi^{-1}(\lambda) = 0\}$. A lemma of Arms, Cushman, and Gotay [4], see Sjamaar-Lerman [92], says that this quotient is a non-degenerate Poisson algebra, that is, the bracket vanishes only on constant functions. Suppose (a). Since r_λ is surjective, $C^\infty(X/\!\!/_\lambda K)$ is abelian as well, and so $X/\!\!/_\lambda K$ must be discrete, hence a point by Kirwan's results. Conversely, if all the reduced spaces are points and $f, g \in C^\infty(X)^K$ then $r_\lambda(\{f, g\}) = 0$ for all λ implies that $\{f, g\} = 0$. □

The complex analogs of multiplicity-free Hamiltonian actions are called *spherical varieties*. Let G be a connected complex reductive group. For the following, see Brion-Luna-Vust [18], the review [57], or the second part of Brion's review in this volume.

Theorem 9.2.2. *The following conditions for a normal G-variety X are equivalent; if they hold X is called* spherical*:*

(a) *some (hence any) Borel subgroup B has an open orbit;*
(b) *the space of rational functions $\mathbb{C}(X)$ is a multiplicity-free G-module;*
(c) *some (hence any) Borel subgroup B has finitely many orbits.*

Remark 9.2.3. For an arbitrary group action, existence of a dense orbit does not imply finitely many orbits. For example, consider the action of $SL(n, \mathbb{C})$ on the space of $n \times n$ matrices on the left: any two invertible matrices are related by an element of $SL(n, \mathbb{C})$, but there are infinitely many orbits of degenerate matrices distinguished by their kernels.

The classification of toric varieties is generalized to spherical varieties as a special case of a theorem of Luna-Vust [69] which gives a classification of spherical varieties by their generic isotropy group and a *colored fan*, see the contribution of Pezzini in this volume or Knop [57]. Each colored fan is a collection of *colored cones*, convex cones in the space Λ_X dual to the space $\Lambda_X^\vee$ of characters corresponding to B-semiinvariant functions $\mathbb{C}(X)^{(B)}$, together with a finite set of B-stable divisors, satisfying certain conditions. The classification of generic isotropy groups that appear, which are called *spherical subgroups*, is the subject of an open conjecture of Luna, see the contribution of Bravi in this volume. The relation between multiplicity-free Hamiltonian actions and spherical varieties is given by the following, which is a consequence of the Kempf-Ness theorem:

Proposition 9.2.4. *A smooth G-variety $X \subset \mathbb{P}(V)$ is spherical if and only if it is a multiplicity-free Hamiltonian K-manifold.*

Proof. By Proposition 8.1.2 $X/\!\!/_\lambda K = \mathrm{pt}$ iff $\mathrm{Hom}_G(V_\lambda, H^0(X, \mathcal{O}_X(d)))$ is dimension one or zero for all $d \geq 0$. This holds for all λ and $d \geq 0$ iff $\mathbb{C}(X)_d$ is a multiplicity-free G-module for all $d \geq 0$ iff X is spherical. □

In contrast to the toric case, not every multiplicity-free Hamiltonian action admits the structure of a spherical variety [103].

9.3. **Moment polytopes of spherical varieties.** We have already seen several examples of the following:

Theorem 9.3.1. *Let X be a smooth polarized spherical K-variety with moment polytope Δ and trivial generic stabilizer. Then $H^0(X, \mathcal{O}_X(1))$ is the multiplicity-free K-module whose weights are the integral points $\Delta \cap \mathfrak{k}^\vee$ of Δ.*

Proof. By Proposition 8.1.2 and the fact that the symplectic quotients are points. □

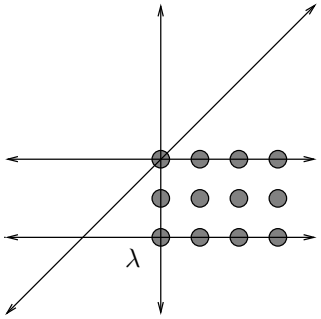

Figure 9.2: Decomposition of $V_\lambda \otimes \mathrm{Sym}(\mathbb{C}^2)$ via Brion's method

The moment polytope of a spherical variety X is described by a finite set of linear inequalities corresponding to the B-stable divisors of X which was described by Brion [17] in the non-abelian case. Let X be a spherical G-variety and $L \to X$ a G-equivariant line bundle. First some notation: Let $\mathbb{C}(X)$ denote the space of rational functions on X, and $\mathbb{C}(X)^{(B)}$ the space of B-semiinvariant vectors. Let $\Lambda_X^\vee \subset \Lambda^\vee$ denote the group of weights appearing in $\mathbb{C}(X)^{(B)}$. Let $\mathcal{D}(X)$ denote the set of prime B-stable divisors of X. Each $D \in \mathcal{D}(X)$ defines a valuation $\mathbb{C}(X)^{(B)} \to \mathbb{Z}$ and so a vector v_D in the dual Λ_X of $\Lambda_X^\vee$. Let $\mathbb{C}(X, L)$ denote the space of rational sections of L, and $s \in \mathbb{C}(X, L)^{(B)}$ with weight $\mu(s)$. Let $n_D(s)$ denote the order of vanishing of s at D. Consider the identification $\mathbb{C}(X)^{(B)} \to \mathbb{C}(X, L)^{(B)}, f \mapsto fs$. The section fs is global iff fs vanishes to at least zeroth order on each $D \in \mathcal{D}(X)$, iff f vanishes at least to order $-n_D$. Thus

Proposition 9.3.2. *Let X be a spherical G-variety, and $L \to X$ a G-line-bundle. The space of weights for elements of $\mathbb{C}(X, L)^{(B)}$ is*

$$\Delta(X, L) = \{\mu \in \Lambda_X^\vee | v_D(\mu) \geq -n_D(s)\} + \mu(s).$$

Example 9.3.3. Here is a typical application which appears in Brion [17] and seems to be due to Macdonald [70]:

Theorem 9.3.4. *Let V_λ be a simple $GL(r)$ module with highest weight $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$. Then $V_\lambda \otimes \mathrm{Sym}(\mathbb{C}^r)$ admits a multiplicity-free decomposition into simple modules V_μ with highest weights $\mu = (\mu_1, \ldots, \mu_r)$ satisfying*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \ldots \mu_r \geq \lambda_r.$$

Proof. We prove only the case $r = 2$; the general case is similar. $V_\lambda \otimes \mathrm{Sym}(\mathbb{C}^2)$ is isomorphic to the space of holomorphic sections of the line bundle $\pi_1^* L_\lambda$ over $X = \mathbb{P}^1 \times \mathbb{C}^2 = \{([w_0, w_1], (z_0, z_1)\}$, where $\pi_1 : \mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{P}^1$ is projection on the first factor. We take B to be the subgroup of upper-triangular invertible matrices. The B-invariant divisors are given by a single G-invariant divisor $D_1 = \{(w, z) | z \in w\}$ and two B-stable divisors $D_2 = \{w = [1, 0]\}$ and $D_3 = \{z \in \mathbb{C} \oplus 0\}$. The space of singular vectors $\mathbb{C}(X)^{(B)}$ is generated by $z_1 - w_1 z_0/w_0$ and z_0 with highest weights $(0, 1)$ resp. $(1, 0)$. The B-stable divisors are defined by $D_1 = \{z_1/z_0 = w_1/w_0\}, D_2 = \{w_1 = 0\}, D_3 = \{z_1 = 0\}$ respectively. Hence $z_1 - w_1 z_0/w_0$ vanishes to order 1 resp. $-1, 0$ on D_1 resp. D_2, D_3; z_1 vanishes to order 0 resp. $0, 1$ on D_1 resp. D_2, D_3. So $v_{D_1} = (0, 1)$, $v_{D_2} = (0, -1)$, $v_{D_3} = (1, 0)$. Taking s to be the section of $\mathbb{P}^1$ with weight (λ_1, λ_2), which vanishes to order 0 on D_1, $\lambda_1 - \lambda_2$ in D_2, and 0 on D_3 one obtains $n_{D_1} = 0, n_{D_2} = \lambda_1 - \lambda_2, n_{D_3} = 0$. This yields the inequalities $\mu_2 \geq \lambda_2$, $-\mu_2 \geq -\lambda_2 - (\lambda_1 - \lambda_2) = -\lambda_1$, $\mu_1 \geq \lambda_1$ as claimed. See Figure 9.2. □

Remark 9.3.5. Not every B-stable divisor defines a facet of the moment polytope. This is already apparent in the case of the Borel-Weil theorem, where for a group of rank r there are r B-stable divisors (the Schubert varieties of codimension one) but the moment polytope is simply a point.

Based on his work on the toric case, Delzant asked the question of whether compact multiplicity-free actions are classified by their moment polytopes and generic stabilizers, and answered the question affirmatively in the rank two case [25]. A result of Knop [56] reduces this to the question of whether affine spherical varieties are classified by their moment polytopes and generic stabilizers

of the compact group actions; this conjecture has recently been proved by Losev [67], see also his review in this volume.

In the torus case we have

Corollary 9.3.6. *With $X, K, \mathcal{O}_X(1)$ as above, if K is a torus then the dimension of $H^0(X, \mathcal{O}_X(1))$ is the number of integral points $\Delta \cap \mathfrak{k}^\vee$ of Δ.*

The dimension of $H^0(X, \mathcal{O}_X(k))$ can be computed by Riemann-Roch for sufficiently large k, since $\mathcal{O}_X(1)$ is by assumption positive. This led to an interesting series of papers on formulas for the number of lattice points in a convex polytope which generalize the Euler-Maclaurin formula and were later proved combinatorially, see [19] for references.

10. Localization via sheaf cohomology

In this section we review various "fixed point methods" for computing moment polytopes, in the context of sheaf cohomology. These include not only the "localization" methods which take as input fixed point data for a one-parameter subgroup, but also the "non-abelian localization" principle which uses the Kirwan-Ness stratification.

10.1. **Grothendieck's local cohomology.** A powerful technique for computing cohomology groups, and therefore for computing moment polytopes, is Grothendieck's local cohomology theory, exposed in [31] and Hartshorne [41]. Let X be a G-variety and $Y \subset X$ a G-subvariety. Let $E \to X$ be a G-equivariant coherent sheaf. Denote by $\Gamma_Y(X, E)$ the group of sections whose support is contained in Y. We denote by H^i_Y the i-th derived functor of Γ_Y, so that the *local cohomology group* $H^i_Y(X, E)$ is a G-module. These modules have the following properties:

Theorem 10.1.1. (a) *(Long Exact Sequence) There is an exact triangle*

$$\dots H_Y(X, E) \to H(X, E) \to H(X - Y, E|X - Y) \to \dots$$

(b) *(Gysin isomorphism) Suppose $Y \subset X$ is smooth. Then*

$$H^j_Y(X, E) \cong H^{j-\mathrm{codim}(Y)}(Y, E|Y \otimes \mathrm{Eul}(N)^{-1})$$

where N is the normal bundle of Y in X and $\mathrm{Eul}(N)^{-1} := \det(N) \otimes \mathrm{Sym}(N)$ (this is an inverse of the K-theory Euler class $\mathrm{Eul}(N) = \Lambda(N^\vee)$ although we do not discuss K-theory here)

(c) *(Spectral sequence associated to a stratification) Let $X_1 \subset X_2 \subset \dots \subset X_m = X$ be a filtration of X. There is a spectral sequence*

$$\bigoplus_{i=1}^{m} H_{X_i - X_{i-1}}(X_i, E|X_i) \implies H(X, E).$$

Let $\chi(X, E) = \bigoplus (-1)^i H^i(X, E)$ be the Euler characteristic, considered as a virtual G-representation, and $\chi_Y(X, E)$ the Euler characteristic of the local cohomology along Y. These will generally not be finite-dimensional, but rather in our cases of interest the multiplicity of each simple module is finite. Thus the formula below holds in the completion of the representation ring, as an immediate consequence of the spectral sequence:

Corollary 10.1.2. *Suppose that $X_1 \subset \dots \subset X_m = X$ is a filtration of X such that the differences $X_i - X_{i-1}$ are smooth with normal bundles $N_i \to X_i - X_{i-1}$. Then*

$$\chi(X, E) = \sum_i (-1)^{\mathrm{codim}(X_i - X_{i-1})} \chi(X_i - X_{i-1}, E|_{X_i - X_{i-1}} \otimes \mathrm{Eul}(N_i)^{-1}) \tag{10.1}$$

if both sides are well-defined in the sense that the multiplicity of any simple module is finite.

This formula applies to various filtrations associated to group actions to give "localization" formulae.

Example 10.1.3. (Weyl character formula and Borel-Weil-Bott, c.f. Atiyah-Bott [6]) Let $X = G/B^-$ and $E = \mathcal{O}_X(\lambda)$ so that if λ is dominant then $H^0(X, E) = V_\lambda$ by Borel-Weil 6.1.1. The Bruhat decomposition $X = \cup_{w \in W} X_w$ gives a filtration $X_i = \cup_{w \in W, l(w) \geq i} X_w$. Each cell X_w fibers

over $x_w = wB/B$ with fiber $X_w \cong M_w := \mathfrak{b} \cap \mathrm{Ad}(w)\mathfrak{b}$. The normal bundle X_w has restriction to x_w given by $N_w = (\mathfrak{b}/\mathfrak{b} \cap \mathrm{Ad}(w)\mathfrak{b})^\vee$. The formula (10.1) gives

$$\begin{aligned}\chi(X, \mathcal{O}_X(\lambda)) &= \bigoplus_{w \in W} (-1)^{l(w)} \chi(X_w, E|X_w \otimes \mathrm{Sym}(N_w) \otimes \det(N_w)) \\ &= \bigoplus_{w \in W} (-1)^{l(w)} \chi(x_w, E \otimes \mathrm{Sym}(N_w) \otimes \det(N_w) \otimes \mathrm{Sym}(M_w^\vee)|x_w) \\ &= \bigoplus_{w \in W} (-1)^{l(w)} \mathbb{C}_{w\lambda} \otimes \mathrm{Sym}(\mathfrak{b}^-) \otimes \mathbb{C}_{w\rho - \rho}\end{aligned}$$

where ρ is the half-sum of positive roots. Thus its character is

$$\sum_{w \in W} (-1)^{l(w)} \frac{t^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1 - t^{-\alpha})}. \tag{10.2}$$

Thus if λ is dominant then

Proposition 10.1.4. *(Weyl character formula) The character of the action of T on V_λ is given by* (10.2).

In general, suppose that w is such that $w(\lambda + \rho) - \rho$ is dominant. From the spectral sequence we see that the only contribution to $\chi(X, \mathcal{O}_X(\lambda))$ comes from $H^{l(w)}(X, \mathcal{O}_X(\lambda))$, since $l(w) = \mathrm{codim}(X_w)$. This is a simple G-module of highest weight $w(\lambda + \rho) - \rho$, since it has the same character as that of $V_{w(\lambda+\rho)-\rho}$ by the Weyl character formula. If no such w exists, then the Fourier expansion of the character vanishes on dominant weights and is W-invariant and so $H(X, \mathcal{O}_X(\lambda))$ is trivial. Thus:

Proposition 10.1.5. *(Borel-Weil-Bott* [15]*) Let $X = G/B^-$. $H^j(X, \mathcal{O}_X(\lambda)) \cong V_{w(\lambda+\rho)-\rho}$ if $w(\lambda + \rho) - \rho$ is dominant for some (unique) $w \in W$ and $j = l(w)$, and is zero otherwise.*

10.2. **One-parameter localization.** The derivation of the Weyl character formula given in the previous section generalizes to varieties with circle actions as follows. Let X be a compact $G \times \mathbb{C}^*$-variety, and $X^{\mathbb{C}^*}$ its $\mathbb{C}^*$-fixed point set. Let $\mathcal{F}$ be the set of components of $X^{\mathbb{C}^*} = \{x \in X | zx = x \ \forall z \in \mathbb{C}^*\}$. For each $F \in \mathcal{F}$, define

$$X_F := \{x \in X | \lim_{z \to 0} zx \in F\}.$$

Let N_F denote the normal bundle of F in X. It admits a decomposition $N_F = N_F^+ \oplus N_F^-$ into positive and negative weight spaces for the $\mathbb{C}^*$-action.

Proposition 10.2.1. *(Bialynicki-Birula decomposition* [13]*) Suppose that X is smooth. Then each X_F is a smooth $G \times \mathbb{C}^*$-stable subvariety, equipped with a morphism $\pi_F : X_F \to F, \quad x \mapsto \lim_{z \to 0} zx$ which induces on X_F the structure of a vector bundle whose fibers are isomorphic to the fibers of the normal bundle $N_F^+ \to F$ of F in X.*

By filtering by the dimension of N_F^+, applying the localization formula (10.1), and pushing forward with π_F one obtains

Theorem 10.2.2 (Localization for one-parameter subgroups)**.** *Let $E \to X$ be any $G \times \mathbb{C}^*$-equivariant coherent sheaf. Then*

$$\chi(X, E) = \sum_{F \subset X^{\mathbb{C}^*}} \chi(F, E|F \otimes \mathrm{Sym}(N_F^{+,\vee}) \otimes \mathrm{Sym}(N_F^-) \otimes \det(N_F^-))$$

in the completion of the representation ring of G.

One could equally well choose the stratification for the inverted $\mathbb{C}^*$-action, which would lead to the same formula with N_F^+, N_F^- inverted. In the equivariant cohomology literature such a choice of direction is called a choice of *action chamber*, see Duistermaat [27].

The spectral sequence contains more information than the localization formula, namely, information about the individual cohomology groups. For example,

Example 10.2.3. Let $X = \mathbb{P}^2$ equipped with the $G = (\mathbb{C}^*)^2$ action by $(g_1, g_2)[z_0, z_1, z_2] = [z_0, g_1^{-1}z_1, g_2^{-1}z_2]$. Then $H^0(X, \mathcal{O}_X(d))$ is spanned by homogeneous polynomials of degree d. Its Euler characteristic has character

$$(\chi(X, \mathcal{O}_X(d)))(g) = \sum_{d_1+d_2 \leq d, d_1, d_2 \geq 0} g_1^{d_1} g_2^{d_2}.$$

One can also see this easily from the localization formula, which gives (for the $\mathbb{C}^*$-action induced by the map $z \mapsto (z, z^2)$) three fixed points with normal weights $(1,0), (0,1)$, resp. $(-1,0), (-1,1)$ resp. $(1,-1), (0,-1)$ and so

$$(\chi(X, \mathcal{O}_X(d)))(g) = (1-g_1)^{-1}(1-g_2)^{-1} - g_1^{d+1}(1-g_1)^{-1}(1-g_1^{-1}g_2)^{-1} + g_2^{d+1}g_1^{-1}(1-g_1^{-1}g_2)^{-1}(1-g_2)^{-1}. \tag{10.3}$$

Now suppose that X' is the blow-up of X at $[1,0,0]$. Let $\pi : X' \to X$ denote the projection, $\mathcal{O}_{X'}(d,e) = \pi^*\mathcal{O}_X(d) \otimes E^e$. The action of $\mathbb{C}^*$ on X' has four fixed points (the point at $[1,0,0]$ is replaced by two fixed points in the exceptional divisor with fiber weights $(e,0), (0,e)$). Hence

$$\begin{aligned}(\chi(X', \mathcal{O}_{X'}(d,e)))(g) &= g_1^e(1-g_1)^{-1}(1-g_1^{-1}g_2)^{-1} \\ &- g_2^{e+1}g_1^{-1}(1-g_1g_2^{-1})^{-1}(1-g_2)^{-1} - g_1^d(1-g_1)^{-1}(1-g_1^{-1}g_2)^{-1} \\ &+ g_2^d(1-g_1^{-1}g_2)^{-1}(1-g_2)^{-1}.\end{aligned} \tag{10.4}$$

Its Fourier transform is shown below in Figure 10.1. The contributions with weights g_1^e contributes

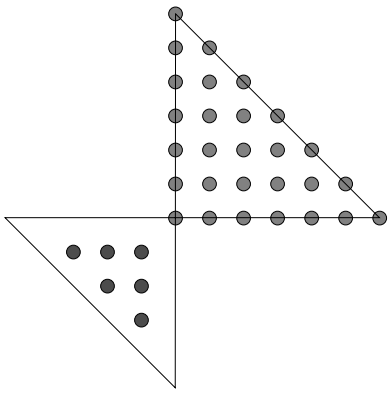

Figure 10.1: Euler characteristic of a line bundle on blow-up of $\mathbb{P}^2$

only to H^0, while the contribution with weight $g_2^{e+1}g_1^{-1}$ contributes only to H^1. The former is the only term whose Fourier transform has support in the larger triangle, while the latter is the only term whose Fourier transform has support in the smaller. Hence the dots in the smaller triangle correspond to vectors in H^1 while those in the larger correspond to H^0. Very similar results are obtained by a deformation method introduced by Witten [101], and studied by a number of other authors since then, see for example [105].

10.3. **Localization via orbit stratification.** Other stratifications lead to interesting but less well-known localization formulae. For example, suppose that G acts on X with only finitely many orbits Y. We then obtain a formula

$$\chi(X, E) = \sum_{Y \subset X} (-1)^{\operatorname{codim}(Y)} \chi(Y, E|Y \otimes \operatorname{Eul}(Y)^{-1})$$

assuming that each simple module appears with finite multiplicity as before. In particular, suppose that X is a toric variety and $E = \mathcal{O}_X(1)$ a polarization. Indexing the orbits Y_F by faces F of the moment polytope Δ we see that

$$\chi(Y, E|Y \otimes \operatorname{Eul}(Y)^{-1}) = \sum_{\mu \in \Lambda^\vee \cap C_F} t^\mu \det(N_F)$$

where the sum is over μ is the outward normal cone C_F to Δ at F, and $\det(N_F)$ is the determinant N_F of the normal bundle to Y_F. This is closely related to the *Brianchon-Gram formula*: for any convex polytope Δ,

$$\chi_\Delta = \sum (-1)^{\operatorname{codim}(F)} \chi_{C_F}$$

where χ_{C_F} is the characteristic function of C_F [90].

10.4. **Non-abelian localization.** Let X be a polarized smooth G-variety and $E \to X$ a G-equivariant coherent sheaf. Combining the Kirwan-Hesselink-Ness stratification with the Euler characteristic formula (10.1) gives

$$\chi(X,E) = \sum_\lambda \chi(X_\lambda, E|_{X_\lambda} \otimes \operatorname{Eul}(N_{X_\lambda})^{-1})$$

where the sum is over types λ or equivalently critical sets for the norm-square of the moment map. This is a sheaf cohomology version of a "non-abelian localization principle" suggested by Witten in the setting of equivariant de Rham cohomology [100]. In fact, this terminology in the sheaf cohomology setting is somewhat confusing: the formula is already quite interesting in the abelian case (non-abelian should read "not necessarily abelian") and the formula is not really a localization formula, since there is a contribution from the (dense) open stratum. Since $X_\lambda = G \times_{P_\lambda} Y_\lambda^{\mathrm{ss}}$, we have

$$\chi(X_\lambda, E|_{X_\lambda} \otimes \operatorname{Eul}(N_{X_\lambda})^{-1}) = \operatorname{Ind}_{G_\lambda}^G \chi(Y_\lambda^{\mathrm{ss}}, E|_{Y_\lambda^{\mathrm{ss}}} \otimes \operatorname{Eul}(N_{X_\lambda}|Y_\lambda^{\mathrm{ss}})^{-1}).$$

(Here Ind denotes holomorphic induction, that is, if V is a G_λ-module then $\operatorname{Ind}_{G_\lambda}^G(V) = \chi(G \times_{P_\lambda^-} V)$.) Since Y_λ^{ss} fibers over Z_λ^{ss} with affine fibers,

$$\begin{aligned}(10.5)\quad &\chi(Y_\lambda^{\mathrm{ss}}, E|_{Y_\lambda^{\mathrm{ss}}} \otimes \operatorname{Eul}(N_{X_\lambda}|Y_\lambda^{\mathrm{ss}})^{-1}) \\ &\qquad = \chi(Z_\lambda^{\mathrm{ss}}, E|_{Y_\lambda^{\mathrm{ss}}} \otimes \operatorname{Sym}(N_{X_\lambda}X|Y_\lambda^{\mathrm{ss}}) \otimes \det(N_{X_\lambda}X|_{Z_\lambda^{\mathrm{ss}}}) \otimes \operatorname{Sym}(N_{Z_\lambda^{\mathrm{ss}}}Y_\lambda^{\mathrm{ss}})^\vee).\end{aligned}$$

This can be put into a more understandable form if we recognize that $N_{X_\lambda}X|_{Z_\lambda^{\mathrm{ss}}}$ resp. $N_{Z_\lambda^{\mathrm{ss}}}Y_\lambda^{\mathrm{ss}}$ is the positive resp. negative part of the normal bundle of Z_λ^{ss} in Y_λ^{ss}. One obtains a formula due to Teleman [95] in the algebraic case and Paradan [81] in the general symplectic setting; the latter proof uses techniques of transversally elliptic operators:

Theorem 10.4.1.

$$\chi(X,E) = \sum_\lambda \operatorname{Ind}_{G_\lambda}^G(\chi(Z_\lambda^{\mathrm{ss}}, E|_{Z_\lambda^{\mathrm{ss}}} \otimes \operatorname{Eul}(N_{Z_\lambda^{\mathrm{ss}}}Y_\lambda^{\mathrm{ss}})_+^{-1}))$$

where the $+$ indicates the particular choice of (formal) inverse to the K-theory Euler class given in the previous formula.

Example 10.4.2. Let $X = \mathbb{P}^1$ and $E = \mathcal{O}(d)$ so $\chi(X,E)$ has character $z^{-d} + z^{-d+2} + \ldots + z^d$. The stratification $\mathbb{P}^1 = \{0\} \cup \mathbb{C}^* \cup \{\infty\}$ leads to the formula

$$z^{-d} + \ldots + z^d = (\sum_{n \in \mathbb{Z}} z^{d+2n}) - z^{d+2}/(1-z^2) - z^{-d-2}/(1-z^{-2}).$$

Example 10.4.3. We describe the non-abelian localization formula for the action of $SL(3,\mathbb{C})$ on a partial flag variety for the exceptional group of type G_2, corresponding to the decomposition of a simple G_2-module into $SL(3,\mathbb{C})$-modules. Let ω_1, ω_2 denote the fundamental weights for $SL(3,\mathbb{C})$. The dual positive Weyl chamber for G_2 is the span of ω_1 and $\omega_1 + \omega_2$. Let $P_{\omega_1+\omega_2}$ denote the maximal parabolic of G_2 corresponding to $\omega_1 + \omega_2$, and $X = G_2/P_{\omega_1+\omega_2}^-$, that is, the coadjoint orbit through $\omega_1 + \omega_2$. The action is spherical and moment polytope the convex hull of $\omega_1, \omega_2, \omega_1 + \omega_2$. We leave the computation of the moment polytope to the reader; it can be computed using one-parameter localization. By Borel-Weil and the computation of the moment polytope,

$$\chi(\mathcal{O}_X(k)) = \sum_{\lambda \in k\Delta \cap Q} \chi_\lambda = \operatorname{Res}_{SL(3,\mathbb{C})}^{G_2}(\chi_{k(\omega_1+\omega_2)})$$

the character of the irreducible G_2-representation with highest weight $k(\omega_1 + \omega_2)$, restricted to $SL(3,\mathbb{C})$; here Q is the lattice generated by the long roots shifted by $k(\omega_1 + \omega_2)$.

We compute the Kirwan-Ness stratification as follows. Let F_1 be the open face connecting $\omega_2, \omega_1+\omega_2$, F_2 the open face connecting $\omega_1, \omega_1+\omega_2$, and F_3 the open face connecting ω_1, ω_2. Let $F_{ij} = F_i \cap F_j$. The inverse image $\Phi^{-1}(F_{12})$ contains a unique point, x_1, which is T-fixed. None of the other T-fixed points map to $\mathfrak{t}^\vee_+$. Therefore, the remaining points in $\Phi^{-1}(\mathrm{int}(\mathfrak{t}^\vee_+))$ (the interior of the positive Weyl chamber) have one-dimensional stabilizers. Since $\Phi^{-1}(\mathrm{int}(\mathfrak{t}^\vee_+))$ has dimension $2\dim(T)$, it is a multiplicity free action, so the inverse image of any face $F \subset \mathrm{int}\,\mathfrak{t}^\vee_+$ has infinitesimal stabilizer the annihilator of the tangent space of F. The stabilizers of the faces F_1, F_2, F_3 are

$$\mathfrak{t}_1 = \mathrm{span}(h_1), \mathfrak{t}_2 = \mathrm{span}(h_2), \mathfrak{t}_3 = \mathrm{span}(h_3)$$

where h_1, h_2, h_3 are the coroots of $SL(3,\mathbb{C})$. The level set $\Phi^{-1}((\omega_1+\omega_2)/2)$ is a critical set of ϕ with type $\lambda = ((\omega_1+\omega_2)/2$. The fixed point component Z_ξ has moment image $\Phi(Z_\xi) =$

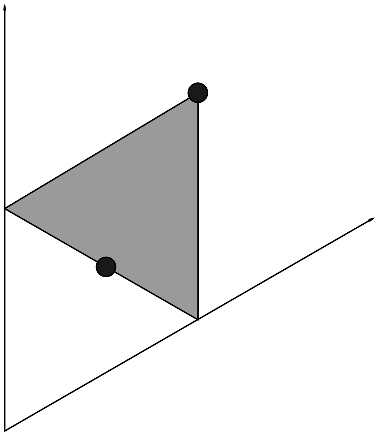

Figure 10.2: Critical values of the norm-square of the moment map for $X = G_2/P_{\omega_1+\omega_2}$

$\mathrm{hull}(2\omega_2-\omega_1, 2\omega_1-\omega_2)$. The unstable manifold Y_ξ has image under the moment map for T (that is, for the maximal torus of the compact group $SU(3)$)

$$\pi^G_T\Phi(\overline{Y_\xi}) = \mathrm{hull}(2\omega_2-\omega_1, 2\omega_1-\omega_2, \omega_1+\omega_2).$$

None of the other facets F_j contain points ξ with $\xi \in \mathfrak{t}_j$. Therefore, there are no other critical points of ϕ in $\Phi^{-1}(\mathrm{int}(\mathfrak{t}^\vee_+))$. Finally consider the inverse image of the vertices F_{13}, F_{23}. Any $x \in \Phi^{-1}(F_{jk})$ has $G_x \neq T$, hence G_x cannot intersect the semisimple part $[G_{\Phi(x)}, G_{\Phi(x)}]$. Therefore, G_x is one-dimensional. let Z_x denote the fixed point component of G_x containing x. Since G_x is one-dimensional, the image $\Phi(Z)$ is codimension one, and so meets $\Phi^{-1}(\mathrm{int}(t^\vee_+))$. But this implies that the $\mathfrak{g}_x$ is conjugate to either $\mathfrak{t}_j$ or $\mathfrak{t}_k$, and so $\mathfrak{g}_x$ cannot equal the span of F_{jk}. Therefore, set of types for the action is $\{\omega_1+\omega_2, \frac{1}{2}(\omega_1+\omega_2)\}$. (In fact the Kirwan-Ness stratification coincides with the orbit stratification for $G_\mathbb{C}$. That is, X is a two-orbit variety, with one open orbit and one of complex codimension two [28].)

We now compute the contributions from the Kirwan-Ness strata. For $\xi = \omega_1+\omega_2$, Z^{ss}_ξ is equal to a point, and the bundle N_ξ is the representation with weights β_5, β_6. Hence

$$\chi_{G_\xi}(Z^{\mathrm{ss}}_\xi, E \otimes \mathrm{Eul}(N_\xi)^{-1}_+) = \sum_{(\lambda,\alpha_1)>k,(\lambda,\alpha_2)>k} z^\lambda.$$

Its induction to G is

$$\mathrm{Ind}^G_{G_\xi}\chi_{G_\xi}(Z^{\mathrm{ss}}_\xi, E \otimes \mathrm{Eul}(N_\xi)^{-1}_+) = \sum_{(\lambda,\alpha_1)>k,(\lambda,\alpha_2)>k} \chi_\lambda.$$

For $\xi = (\omega_1+\omega_2)/2$, we have $Z^{\mathrm{ss}}_\xi \cong \mathbb{C}^*$ and N_ξ trivial. Therefore,

$$\chi_{G_\xi}(Z^{\mathrm{ss}}_\xi, E \otimes \mathrm{Eul}(N_\xi)^{-1}_+) = \sum_{(\lambda,\xi)\geq k(\xi,\xi)} z^\lambda$$

where the sum is over vectors λ such that $\lambda - k(\omega_1 + \omega_2)$ is in some lattice $\Lambda_1^\vee$, and satisfying the inequality above. Hence

$$\mathrm{Ind}_{G_\xi}^G(\chi_{G_\xi}(Z_\xi^{\mathrm{ss}}, E \otimes \mathrm{Eul}(N_\xi)_\xi^{-1})) = \sum_{\lambda \in k\Delta} \chi_\lambda - \sum_{(\lambda,\alpha_1)>k,(\lambda,\alpha_2)>k} \chi_\lambda.$$

Since the contributions from $\xi = (\omega_1 + \omega_2), \frac{1}{2}(\omega_1 + \omega_2)$ must have finite sum, the lattice $\Lambda_1^\vee$ must be the long root lattice. The contribution (for $k = 6$) is shown in Figure 10.3.

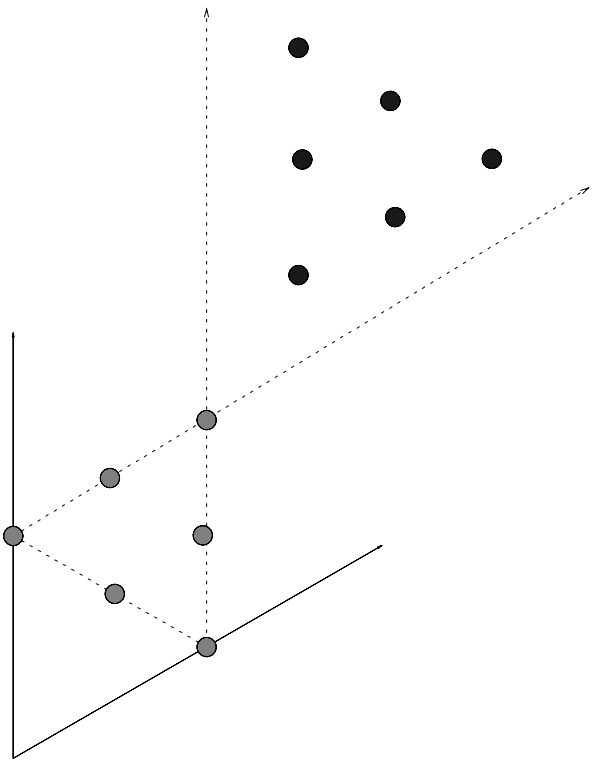

Figure 10.3: $\mathrm{Ind}_T^G \chi_{Z^{\mathrm{ss}}_{(\omega_1+\omega_2)/2},T}(E)$

The positive contribution of the open stratum is finite (6 representations, for $k = 6$) and the negative contribution infinite, that is $\dim(H^{odd}(M_\xi, L^k)) = \infty$, for any k. One can show that the higher cohomology lies in H^1, using the spectral sequence. The sum of the contributions is $\chi(\mathcal{O}_X(k)) = \sum_{\lambda \in k\Delta} \chi_\lambda$ as claimed. This completes the example.

Taking invariants in Theorem 10.4.1 gives a formula expressing the difference between $\chi(X, E)^G$ and $\chi(X/\!\!/G, E/\!\!/G)$:

Theorem 10.4.4.

$$\chi(X, E)^G - \chi(X/\!\!/G, E/\!\!/G) = \sum_{\lambda \neq 0} \chi(Z_\lambda^{\mathrm{ss}}, E|_{Z_\lambda^{\mathrm{ss}}} \otimes \mathrm{Eul}(N_{Z_\lambda^{\mathrm{ss}}} Y_\lambda^{\mathrm{ss}})_+^{-1} \otimes \mathrm{Eul}(\mathfrak{g}/\mathfrak{p}_\lambda^-))^{G_\lambda}$$

The spectral sequence also contains information about the individual cohomology groups. For example, let $\mathbb{C}_\lambda^* \subset G_\lambda$ denote the one-parameter subgroup generated by λ. The weight of $\mathbb{C}_\lambda^*$ on $\det(N_{X_\lambda} X|_{Z_\lambda^{\mathrm{ss}}})$ is positive, if λ is non-trivial. Indeed, $N_{X_\lambda} X|_{Z_\lambda^{\mathrm{ss}}}$ is the negative part of the tangent bundle. Furthermore, $\mathfrak{g}/\mathfrak{p}_\lambda^-$ has positive weights under $\mathbb{C}_\lambda^*$. Thus

Corollary 10.4.5 (Teleman [95])**.** *Suppose that the weights of $\mathbb{C}_\lambda^*$ on $E|Z_\lambda$ are positive for all types λ. (This is automatically the case if $E = \mathcal{O}_X(d)$ is the d-th tensor product of a polarization $\mathcal{O}_X(1)$ of X). Then $H^j(X, E)^G = H^j(X/\!\!/G, E/\!\!/G)$ for all j.*

In particular, if the higher cohomology of E vanishes then so does that of $E/\!\!/G$.

The index maps naturally induce a diagram in K-theory

$$\begin{array}{ccc} K_G(X) & \longrightarrow & K(X/\!\!/G) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z} & \end{array}$$

which fails to commute by the above explicit sum of fixed point contributions for one-parameter subgroups. There are similar results in the equivariant cohomology of X due to Paradan [80] and the author [104], based on earlier work of Witten [100]: a natural diagram of equivariant cohomology groups

$$\begin{array}{ccc} H_G(X) & \longrightarrow & H(X/\!\!/G) \\ & \searrow \quad \swarrow & \\ & \mathbb{R} & \end{array}$$

fails to commute by an explicit sum of fixed point contributions from one-parameter subgroups. The first explicit version of non-abelian localization is due to Jeffrey-Kirwan [51], and expresses the difference as a sum over certain fixed point sets of the maximal torus. The versions of Paradan, myself [104], and Beasley-Witten [9] express the difference as a sum over critical points of the norm-square of the moment map. The left hand arrow in the diagram above takes some work to define: morally speaking it is defined by $\alpha \mapsto \int_{X\times\mathfrak{g}} \alpha$, but this is not well-defined for polynomial equivariant classes. Rather, the left-hand side must be defined by a suitable limit procedure, either by taking the leading term in Riemann-Roch, or (in the context of equivariant de Rham cohomology with smooth coefficients) shifting by equivariant Liouville form and taking the zero limit of the shift, see [104]. From this point of view, the K-theory approach is more natural.

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Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A. • ctw@math.rutgers.edu