



Winter Braids Lecture Notes

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Character varieties and knot symmetries

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Character varieties and knot symmetries

JOAN PORTI

Abstract

Those are notes of the mini-course given in the school Winter Braids VII, held in Caen from February 27th to March 2nd 2017. They overview the variety of representations and characters of a three-manifold in $SL_2\mathbb{C}$, putting emphasis on explicit computations. The notes also discuss the canonical component of a hyperbolic knot, and a recent joint work with Luisa Paoluzzi, on the invariant components of the variety of characters for knot symmetries.

Those are notes of a mini-course in the Winter Braids VII, so they are intended to non-experts and they require only a basic background in geometry and topology of three-manifolds.

Varieties of representations and characters of a finitely generated group Γ in a Lie group G constitute a rich research area; here I focus on representations of compact three-manifold groups (mainly knot exteriors) in $SL_2\mathbb{C}$. In particular the notes do not cover representations of surface nor free groups, nor representations in Lie groups G other than $SL_2\mathbb{C}$, despite those are very active and interesting research topics, that have a lot of interactions with three-manifolds. For the discussion of varieties of representations in a general (algebraic reductive) Lie group G see for instance the paper by Sikora [37].

From the point of view of geometry and topology of three-dimensional manifolds, one of the main motivations to consider varieties of characters is the natural isomorphism between the group of orientation preserving isometries of hyperbolic 3-space and $PSL_2\mathbb{C} = SL_2\mathbb{C}/\{\pm Id\}$. Then studying the variety of characters allows to study deformation of hyperbolic structures. As seminal works, let me mention that Thurston used the variety of representations in [39] to prove the hyperbolic Dehn filling theorem, and Hodgson used it to study degenerations of incomplete or singular structures in [18].

A beautiful relationship between three-manifold topology and the algebraic side of the variety of characters started with the work by Culler and Shalen in [11]. They use the algebraic structure of the variety of characters to find essential surfaces on three-manifolds, corresponding to the ideal points of algebraic curves. This is used for instance in the study of exceptional (non-hyperbolic) Dehn fillings on hyperbolic knots and cusped manifolds, see [10, 2].

There has been an intense research in varieties of characters of three-manifolds in $SL_2\mathbb{C}$. Let me mention a few subjects, like the A -polynomial [8], the dynamics of actions on the variety of characters [6], or the study of the algebraic and arithmetic properties of the variety of characters [31]. There are a lot of remarkable works on varieties of characters that I do not mention, I just make a choice that omits relevant results. However, as those notes insist in the computational side, I want to mention the pioneering work of Riley on varieties of representations of three-manifolds [34, 35, 36], from which I have picked up several ideas, especially when dealing with reduction mod p .

This text starts with the basic definitions of varieties of representations and characters and puts a lot of emphasis in *explicit computations*, specially for groups on two generators. Next the discussion focuses in hyperbolic knots, in particular in the so called canonical or canonical component of the variety of characters. The final section is devoted to a joint work with Luisa Paoluzzi [30], that reflects a different behavior on the variety of characters for different kinds of knot symmetries.

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1. Varieties of representations and characters

Consider a finitely generated group Γ with presentation

$$\Gamma = \langle \gamma_1, \dots, \gamma_n \mid (r_j)_{j \in J} \rangle,$$

and the matrix group

$$\mathrm{SL}_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}.$$

A group morphism $\Gamma \rightarrow \mathrm{SL}_2\mathbb{C}$ is called a *representation* of Γ in $\mathrm{SL}_2\mathbb{C}$.

The set of all representations of Γ in $\mathrm{SL}_2\mathbb{C}$ is called the *variety of representations* and it is denoted by

$$R(\Gamma) = \mathrm{hom}(\Gamma, \mathrm{SL}_2\mathbb{C}).$$

Proposition 1.1. *$R(\Gamma)$ is an affine algebraic set (the zero set of polynomials in \mathbb{C}^m).*

Proof. Map each representation to a n -tuple of matrices, consisting of the image of the generators:

$$\begin{aligned} R(\Gamma) &\rightarrow \mathrm{SL}_2\mathbb{C} \times \overset{(n)}{\cdots} \times \mathrm{SL}_2\mathbb{C} \subset \mathbb{C}^{4n} \\ \rho &\mapsto (\rho(\gamma_1), \dots, \rho(\gamma_n)) \end{aligned}$$

and embed $\mathrm{SL}_2\mathbb{C}$ in \mathbb{C}^4 by using the entries of the matrices. Then $R(\Gamma)$ is in bijection to the zero set in \mathbb{C}^{4n} of the polynomials given by the matrix entries of the group relations r_j and by the determinant equal to one. \square

Let me denote by $I \subset \mathbb{C}[x_1, \dots, x_{4n}]$ the ideal generated by the polynomials in the proof of Proposition 1.1. The function ring is

$$\mathbb{C}[R(\Gamma)] = \mathbb{C}[x_1, \dots, x_{4n}]/I.$$

Remark 1.2. (1) The algebraic structure of Proposition 1.1 is independent of the generating set. More precisely, the structure of $\mathbb{C}[R(\Gamma)]$ as \mathbb{C} -algebra does not depend on the presentation [21].

(2) The defining ideal I may be non-prime or non-radical. Thus the most accurate way to think of $R(\Gamma)$ is not as a (reducible) variety but as an affine scheme [21, 37]. For instance in the polynomial ring $\mathbb{C}[x]$ consider the ideals generated by x^2 and x : they yield the same variety as zero locus (the origin) in the line, but different schemes, the respective spectra of $\mathbb{C}[x]/(x)$ and $\mathbb{C}[x]/(x^2)$.

Action by conjugation. The group $SL_2\mathbb{C}$ acts on $R(\Gamma)$ by conjugation:

$$\begin{aligned} SL_2\mathbb{C} \times R(\Gamma) &\rightarrow R(\Gamma) \\ A, \rho &\mapsto \gamma \mapsto A\rho(\gamma)A^{-1} \end{aligned}$$

The action is algebraic, but as $SL_2\mathbb{C}$ is non-compact, the quotient $R(\Gamma)/SL_2\mathbb{C}$ may be non-Hausdorff. This problem is overcome by taking the algebraic quotient, that we describe next.

Definition 1.3. For $\gamma \in \Gamma$, the *trace function* of γ is defined as:

$$\begin{aligned} \tau_\gamma: R(\Gamma) &\rightarrow \mathbb{C} \\ \rho &\mapsto \text{tr}(\rho(\gamma)) \end{aligned}$$

Notice that $\tau_\gamma \in \mathbb{C}[R(\Gamma)]$, i.e. it is a polynomial function.

Instead of $\gamma \in \Gamma$ one may fix the representation $\rho \in R(M)$. The *character* of ρ is the map

$$\begin{aligned} \chi_\rho: \Gamma &\rightarrow \mathbb{C} \\ \gamma &\mapsto \text{tr}(\rho(\gamma)) \end{aligned}$$

To construct the quotient of the action of $SL_2\mathbb{C}$ on $R(\Gamma)$, consider the algebra of polynomial functions invariant by the action by conjugation $\mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$. The starting point is the following theorem of Procesi, stated in [33] as the invariant of a product of matrices:

Theorem 1.4 (Procesi [33]). *Let F_n be the free group of rank n . The \mathbb{C} -algebra of invariant functions $\mathbb{C}[R(F_n)]^{SL_2\mathbb{C}}$ is finitely generated by trace functions $\tau_{\gamma_1}, \dots, \tau_{\gamma_N}$, for some $\gamma_1, \dots, \gamma_N \in \Gamma$, i.e.*

$$\mathbb{C}[R(F_n)]^{SL_2\mathbb{C}} = \langle \tau_{\gamma_1}, \dots, \tau_{\gamma_N} \rangle.$$

Corollary 1.5. *Let Γ be a finitely generated group. As a \mathbb{C} -algebra, $\mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$ is finitely generated by trace functions $\tau_{\gamma_1}, \dots, \tau_{\gamma_N}$, for some $\gamma_1, \dots, \gamma_N \in \Gamma$, i.e.*

$$\mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}} = \langle \tau_{\gamma_1}, \dots, \tau_{\gamma_N} \rangle.$$

Proof. We have a surjective morphism $F_n \twoheadrightarrow \Gamma$ for some free group F_n . It induces an injection $R(\Gamma) \hookrightarrow R(F_n)$, which in turn induces another surjection $\mathbb{C}[R(F_n)] \twoheadrightarrow \mathbb{C}[R(\Gamma)]$. To prove the corollary, we need to show that the map restricted to invariant functions

$$\mathbb{C}[R(F_n)]^{SL_2\mathbb{C}} \rightarrow \mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$$

is also a surjection. This follows from the fact that $SL_2\mathbb{C}$ is the complexification of a compact real group $SU(2)$ and hence every $SU(2)$ -invariant polynomial is also $SL_2\mathbb{C}$ -invariant. Therefore given any function $f \in \mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$, choose any $\tilde{f} \in \mathbb{C}[R(F_n)]$ that is mapped to f and render it $SU(2)$ -invariant by averaging, as $SU(2)$ is compact. This construction is precisely the proof of *reductivity* of $SL_2\mathbb{C}$ (see for instance [27, 40, 38, 25, 12]). \square

Corollary 1.6. *The set of characters $X(\Gamma) = \{\chi_\rho \mid \rho \in R(\Gamma)\}$ has a natural structure of complex algebraic affine set such that*

$$\mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}.$$

Proof. We construct the affine algebraic set V whose algebra of functions is $\mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$ and show that it is in natural bijection with $X(\Gamma)$. Let $\gamma_1, \dots, \gamma_N \in \Gamma$ be as in Corollary 1.5. The kernel of the map $\mathbb{C}[x_1, \dots, x_N] \rightarrow \mathbb{C}[R(\Gamma)]^{SL_2\mathbb{C}}$ that maps $x_i \mapsto \tau_{\gamma_i}$ is an ideal I , thus the natural variety to consider is $V \subset \mathbb{C}^N$ the zero set of I . This gives a natural map from

$X(\Gamma)$ to V , by mapping each character to the point whose coordinates are the evaluation at $\gamma_1, \dots, \gamma_N$. The map is injective, because by Corollary 1.5, for each $\gamma \in \Gamma$, τ_γ is a polynomial on $\tau_{\gamma_1}, \dots, \tau_{\gamma_N}$, hence the value of a character is determined by those N coordinates. On the other hand, the projection $R(\Gamma) \rightarrow V$ is surjective [40, Theorem 4.6], thus the map $X(\Gamma) \rightarrow V$ is also surjective. \square

Definition 1.7. The *variety of characters* of Γ is the algebraic affine set $X(\Gamma)$.

Remark 1.8. Defining the variety of characters from the algebra of invariant functions $\mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}}$ has several consequences:

1. This is an *affine scheme*, namely the spectrum of the ring $\mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}}$. This is not only a matter of language, as the ring $\mathbb{C}[X(\Gamma)]$ may contain nilpotent elements (the defined ideal may be non-reduced). See [21, 37].
2. This approach is independent of the choice of generators, as $R(\Gamma)$ is independent of the choice of generators, see Remark 1.2.
3. For each $\text{SL}_2\mathbb{C}$ -invariant polynomial function $f: R(\Gamma) \rightarrow \mathbb{C}$ there exists a unique $\tilde{f}: X(\Gamma) \rightarrow \mathbb{C}$ such that the following diagram commutes

$$\begin{array}{ccc} R(\Gamma) & \xrightarrow{f} & \mathbb{C} \\ \downarrow & \nearrow \tilde{f} & \uparrow \\ X(\Gamma) & & \end{array}$$

4. By looking at the non-Hausdorff points, one can prove that $X(\Gamma)$ is the largest Hausdorff quotient. Namely, for every $\text{SL}_2\mathbb{C}$ -invariant continuous map $f: R(\Gamma) \rightarrow Y$, if Y is a Hausdorff topological space then there exists a unique continuous map $\tilde{f}: X(\Gamma) \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} R(\Gamma) & \xrightarrow{f} & Y \\ \downarrow & \nearrow \tilde{f} & \uparrow \\ X(\Gamma) & & \end{array}$$

See Remark 1.14 (3).

5. This construction of an affine invariant quotient is classical for actions of *reductive* groups, see [27, 40, 38, 25].

To better understand the projection $R(\Gamma) \twoheadrightarrow X(\Gamma)$ we need to discuss irreducible representations.

Irreducible representations. This paragraph is devoted to showing that the set of characters of irreducible representations is in bijection to the space of orbits. In fact a more general result establishes a bijection between the variety of characters and the set of conjugacy classes of semi-simple (direct sum of irreducible) representations [21].

Definition 1.9. A representation $\rho \in R(\Gamma)$ is *reducible* if there exists a line in \mathbb{C}^2 invariant by $\rho(\Gamma)$. Otherwise it is called *irreducible*.

Lemma 1.10 ([11]). *For a representation $\rho \in R(\Gamma)$, the following are equivalent:*

- (i) ρ is reducible,

(ii) $\rho(\Gamma)$ is conjugated to a subgroup of the group of upper triangular matrices:

$$\rho(\Gamma) \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

(iii) $\text{tr}(\rho(\gamma)) = 2, \forall \gamma \in [\Gamma, \Gamma]$.

Here $\Gamma' = [\Gamma, \Gamma]$ is the normal subgroup generated by commutators $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$.

Proof. Assertion (i) is a particular case of (ii), as the invariant line is $\mathbb{C} \times 0$, and, by conjugation, any invariant line can be assumed to be this one, so (i) and (ii) are equivalent. Assertion (ii) implies (iii), as the commutator of two upper triangular matrices is conjugated to a matrix

$$(1.1) \quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

This is a particular case of *parabolic* matrix. To see the converse, let $\gamma \in \Gamma'$ such that $\rho(\gamma)$ is nontrivial. As $\text{tr}(\rho(\gamma)) = 2$ and $\det(\rho(\gamma)) = 1$, $\rho(\gamma)$ is parabolic (conjugated to (1.1)), so it has a unique invariant line in \mathbb{C}^2 . We claim that this line is the same for every element in Γ' . By contradiction, if γ_1 and γ_2 are commutators so that $\rho(\gamma_1)$ and $\rho(\gamma_2)$ are nontrivial elements with different invariant lines, after conjugation we may assume that those lines are the coordinate axis and therefore

$$\rho(\gamma_1) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\gamma_2) = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$

with $\alpha, \beta \neq 0$. As $\text{tr}(\rho(\gamma_1\gamma_2)) = 2 + \alpha\beta \neq 2$, we get a contradiction. Thus $\rho(\Gamma')$ has a unique invariant line, and since Γ' is a normal subgroup of Γ , this line must be invariant by the whole $\rho(\Gamma)$. It remains to consider the case $\rho(\Gamma')$ trivial, but then the claim follows from the description of the abelian subgroups of $\text{SL}_2\mathbb{C}$, they are either diagonal or consist of parabolic matrices, conjugated to (1.1), up to sign, with the same invariant line. \square

As a consequence of this lemma, being irreducible or reducible can be read at the character level. Hence we may talk about *reducible* or *irreducible* characters. Let $R^{\text{red}}(\Gamma)$, $X^{\text{red}}(\Gamma)$ denote the respective sets of reducible representations and characters and similarly $R^{\text{irr}}(\Gamma)$, $X^{\text{irr}}(\Gamma)$ for irreducible ones.

Corollary 1.11 ([11]). $R^{\text{red}}(\Gamma)$ and $X^{\text{red}}(\Gamma)$ are Zariski closed subsets.

Lemma 1.12 ([11]). For $\rho \in R^{\text{irr}}(\Gamma)$, there exist $\gamma_1, \gamma_2 \in \Gamma$ such that ρ can be conjugated to ρ' with

$$\rho'(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad \text{and} \quad \rho'(\gamma_2) = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$$

with $\lambda \neq \pm 1$ and $b \neq 0$.

Proof. We must find two elements $\gamma_1, \gamma_2 \in \Gamma$ such that $\text{tr}(\rho(\gamma_1)) \neq \pm 2$ and the group generated by $\rho(\gamma_1)$ and $\rho(\gamma_2)$ is irreducible. Seeking a contradiction, assume that $\text{tr}(\rho(\gamma)) = \pm 2$ for every $\gamma \in \Gamma$. From the relation

$$\text{tr}(\rho(\gamma_1\gamma_2)) + \text{tr}(\rho(\gamma_1\gamma_2^{-1})) = \text{tr}(\rho(\gamma_1))\text{tr}(\rho(\gamma_2)) \quad \forall \gamma_1, \gamma_2 \in \Gamma,$$

(Lemma 2.1 below) it is straightforward that the map $\Gamma \rightarrow \{\pm 1\}$ defined by $\gamma \mapsto \frac{1}{2} \text{tr}(\rho(\gamma))$, $\forall \gamma \in \Gamma$, is a group homomorphism. Notice that any commutator in Γ is mapped to $+1$ by this homomorphism, hence by Lemma 1.10 $\rho(\Gamma)$ is reducible, yielding a contradiction. Once we have $\gamma_1 \in \Gamma$ with $\text{tr}(\rho(\gamma_1)) \neq \pm 2$, we notice that $\rho(\gamma_1)$ has two invariant lines, L_1 and $L_2 \subset \mathbb{C}^2$. If we have an element γ_2 such that $\rho(\gamma_2)$ does not preserve any of the subspaces L_i we are done. Otherwise we have at least $\mu_1, \mu_2 \in \Gamma$ such that $\rho(\mu_i)$ preserves L_i but not L_{3-i} . Then it is easy to see that $\rho(\mu_1\mu_2)$ does not preserve none of the L_i , and we take $\gamma_2 = \mu_1\mu_2$. \square

Proposition 1.13. (i) Let $\rho \in R^{\text{irr}}(\Gamma)$, then ρ' is conjugate to ρ if and only if $\chi_{\rho'} = \chi_{\rho}$.

(ii) The action of $\mathrm{PSL}_2(\mathbb{C})$ on $R^{\mathrm{irr}}(\Gamma)$ identifies orbits with $\mathrm{PSL}_2(\mathbb{C})$.

(iii) There is a principal analytic bundle with fibers orbits by conjugation:

$$\mathrm{PSL}_2(\mathbb{C}) \rightarrow R^{\mathrm{irr}}(\Gamma) \rightarrow X^{\mathrm{irr}}(\Gamma).$$

Proof. Given an irreducible representation, it can be written as in the proof of Lemma 1.12. Let me show that the coefficients λ , a , b , and d , and hence $\rho(\gamma_1)$ and $\rho(\gamma_2)$, are determined by traces (up to conjugation). Firstly $\lambda + 1/\lambda = \mathrm{tr}(\rho(\gamma_1))$ which determines $\lambda^{\pm 1}$. Once λ is chosen instead of $1/\lambda$, then the elements a , d and $b = ad - 1$ are determined from the traces of $\rho(\gamma_2)$ and $\rho(\gamma_1\gamma_2)$:

$$\begin{aligned} a + d &= \mathrm{tr}(\rho(\gamma_2)), \\ \lambda a + \frac{1}{\lambda} d &= \mathrm{tr}(\rho(\gamma_1\gamma_2)). \end{aligned}$$

If we chose $1/\lambda$ instead of λ , then a and d are switched and $b = ad - 1$ is unchanged, this amounts to conjugating $\rho(\gamma_1)$ and $\rho(\gamma_2)$ simultaneously by

$$\pm \begin{pmatrix} 0 & i\sqrt{b} \\ i/\sqrt{b} & 0 \end{pmatrix}.$$

Finally, for any $\gamma \in \Gamma$, $\rho(\gamma)$ is obtained from $\mathrm{tr}(\rho(\gamma))$, $\mathrm{tr}(\rho(\gamma\gamma_1))$, $\mathrm{tr}(\rho(\gamma\gamma_2))$, and $\mathrm{tr}(\rho(\gamma\gamma_1\gamma_2))$. This proves the first assertion. The second assertion is proved by checking that the only matrices that commute with the image of ρ' as in Lemma 1.10 are $\pm \mathrm{Id}$. Finally, the coefficients in the construction are analytic functions on the traces, so we have a locally defined analytic section to the projection $R^{\mathrm{irr}}(\Gamma) \rightarrow X^{\mathrm{irr}}(\Gamma)$. \square

Finally, we mention a few more results on the quotient.

Remark 1.14. (1) For reducible characters, the fibre of the map $R(\Gamma) \rightarrow X(\Gamma)$ has a unique conjugacy class of diagonal representations.

(2) In addition, all orbits in the fibre of a reducible character accumulate to the conjugacy class of diagonal representations (namely, a conjugacy class of upper triangular matrices accumulates to diagonal matrices, without changing the character).

(3) It follows from the previous two items that $X(\Gamma)$ is the largest Hausdorff quotient of $R(\Gamma)$ by the action of $\mathrm{SL}_2\mathbb{C}$, as claimed in Remark 1.8.

The previous remark is elementary to prove, it is a particular case of a general result of invariant theory, about the existence of a unique closed orbit in the fibre.

2. Computing varieties of characters

The goal of this section is to compute some explicit examples. We focus on groups on two generators, when they have a simple presentation the variety of characters is not hard to compute. However general computations must deal with a possible large complexity.

At the end of the section we discuss an explicit method of González-Acuña and Montesinos [15] to compute the variety of characters of any finitely generated group. Their method gives an algorithm for finitely presented groups, though the algorithm is not optimal. Another advantage of this method is that it allows to take reduction mod p for every prime $p \neq 2$.

Computation in $X(F_2)$. Before varieties of characters of groups with two generators, let me discuss the free group F_2 , in particular the Fricke-Klein theorem. The following lemma yields the trace relations on pairs of matrices to compute varieties of characters.

Lemma 2.1 (Trace relations). For $A, B \in \mathrm{SL}_2\mathbb{C}$:

(i) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;

(ii) $\text{tr}(A^{-1}) = \text{tr}(A)$;

(iii) $\text{tr}(AB) + \text{tr}(A^{-1}B) = \text{tr}(A)\text{tr}(B)$.

Proof. (i) is well known for any pair of matrices, (ii) uses the fact that a matrix in $SL_2\mathbb{C}$ has the same eigenvalues as its inverse. To prove (iii), apply Cayley Hamilton's theorem to A :

$$A^2 - \text{tr}(A)A + \text{Id} = 0,$$

multiply the equality by $A^{-1}B$ and take traces. □

Theorem 2.2 (Fricke-Klein). *Let $F_2 = \langle a, b \mid \rangle$, we have an isomorphism*

$$(\tau_a, \tau_b, \tau_{ab}): X(F_2) \xrightarrow{\cong} \mathbb{C}^3.$$

See for instance [14] for a proof. The theorem says that for all $\gamma \in F_2$, τ_γ is a polynomial on the three variables τ_a , τ_b , and τ_{ab} , and that the three variables are algebraically independent. The algebraic independence follows for instance from taking the traces of γ_1 , γ_2 , and $\gamma_1\gamma_2$ for a representation as ρ' in Lemma 1.12. The proof that the variables generate the algebra of polynomials is algorithmic, using Lemma 2.1. We illustrate it with examples.

Example 2.3. Let us compute the traces of a few elements in $F_2 = \langle a, b \mid \rangle$. To simplify, write the coordinates of Fricke-Klein as:

$$(2.1) \quad x = \tau_a, \quad y = \tau_b, \quad z = \tau_{ab}.$$

(i) Trace of powers. Using $B = A$ in (iii) of Lemma 2.1:

$$\tau_{a^2} = \tau_a^2 - \tau_1 = x^2 - 2.$$

and similarly with $B = A^2$:

$$\tau_{a^3} = \tau_a\tau_{a^2} - \tau_a = x^3 - 3x.$$

In general, τ_{a^n} is defined recursively as a Chebyshev polynomial: starting from $\tau_{a^0} = 2$ and $\tau_{a^1} = x$, then

$$\tau_{a^n} = x\tau_{a^{n-1}} - \tau_{a^{n-2}}.$$

(ii) Consider now the commutator $[a, b] = aba^{-1}b^{-1}$. By application of the trace relations (Lemma 2.1):

$$\tau_{aba^{-1}b^{-1}} = \tau_{aba^{-1}}\tau_b - \tau_{ab a^{-1}b} = y^2 - \tau_{aba^{-1}b},$$

$$\tau_{aba^{-1}b} = \tau_{ab}\tau_{a^{-1}b} - \tau_{a^2} = z\tau_{a^{-1}b} - x^2 + 2,$$

$$\tau_{a^{-1}b} = \tau_a\tau_b - \tau_{ab} = xy - z.$$

Putting together the three computations:

$$(2.2) \quad \tau_{[a,b]} = x^2 + y^2 + z^2 - xyz - 2.$$

From the computation of the commutator, notice that with two applications of Lemma 2.1 we can express τ_γ as a polynomial on the trace of elements of shorter word length.

Computing $X(\Gamma)$ for groups of rank two.

Example 2.4 (Free abelian group of rank 2). For $\mathbb{Z} = \langle a, b \mid [a, b] = 1 \rangle$, using coordinates as in (2.1), we claim that

$$X(\mathbb{Z}^2) = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 4 = 0\}.$$

Set $V = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 4 = 0\}$. The inclusion $X(\mathbb{Z}^2) \subset V$ comes from requiring that the trace of the commutator (2.2) is 2, the trace of the identity. To have equality, chose a representation

$$\rho(a) = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad \rho(b) = \begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}.$$

Then, given $(x, y, z) \in V$, it is elementary to prove that there exist $\lambda, \mu \in \mathbb{C} - \{0\}$ such that $x = \lambda + 1/\lambda$, $y = \mu + 1/\mu$ and $z = \lambda\mu + 1/(\lambda\mu)$. Notice that there are two pairs of solutions, (λ, μ) and $(1/\lambda, 1/\mu)$ corresponding to conjugation by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example 2.5 (The trefoil). Set $\Gamma = \langle a, b \mid aba = bab \rangle$ the fundamental group of the trefoil knot exterior. Since a and b are conjugate, we only need two coordinates, i.e. it is a plane curve:

$$x = \tau_a = \tau_b \quad \text{and} \quad y = \tau_{ab^{-1}}.$$

We take traces on the equality $abab^{-1} = ab$ and use the trace relations in Lemma 2.1:

$$\begin{aligned} \tau_{ab} &= \tau_a \tau_b - \tau_{ab^{-1}} = x^2 - y, \\ \tau_{abab^{-1}} &= \tau_{ba} \tau_{ab^{-1}} - \tau_{b^2} = y(x^2 - y) - x^2 + 2. \end{aligned}$$

Hence $\tau_{abab^{-1}} = \tau_{ab}$ becomes:

$$yx^2 - 2x^2 - y^2 + y + 2 = (y - 2)(x^2 - y - 1) = 0.$$

This plane curve has two components, one of them is $y = 2$, and corresponds to abelian representations (the abelianization maps a and b to the same element). On the other hand on the component $x^2 = y + 1$ we have

$$\tau_{ab} = x^2 - y = 1 \quad \text{and} \quad \tau_{aba} = \tau_a \tau_{ab} - \tau_b = x \cdot 1 - x = 0.$$

By looking at the eigenvalues, a representation maps $(ab)^3$ and $(aba)^2$ to $-\text{Id}$. Hence writing $\alpha = ab$ and $\beta = aba$:

$$\Gamma \cong \langle \alpha, \beta \mid \alpha^3 = \beta^2 \rangle.$$

This presentation corresponds to a Seifert fibration, the central element $\alpha^3 = \beta^2$ is represented by the fibre, and α and β are loops around singularities of order 2 and 3. From this presentation, it is easy to check that every point in this curve can be realized as a character, thus

$$X(\Gamma) = \{(x, y) \in \mathbb{C}^2 \mid (y - 2)(x^2 - y - 1) = 0\}.$$

Example 2.6 (The figure eight knot). The fundamental group of the figure eight knot exterior admits a presentation

$$(2.3) \quad \Gamma = \langle a, b \mid wa = bw \rangle,$$

where $w = ab^{-1}a^{-1}b$ and a and b represent meridians of the knot, in particular they are conjugate (by w). As for the trefoil, we chose coordinates

$$x = \tau_a = \tau_b \quad \text{and} \quad y = \tau_{ab^{-1}}.$$

From the relation $w = b^{-1}wa$ and using the trace relations in Lemma 2.1, we can easily compute

$$\tau_w - \tau_{b^{-1}wa} = (y - 2)(y^2 - (x^2 - 1)y + x^2 - 1) = 0.$$

Thus the variety of characters must satisfy this equation. We deduce that it is precisely the defining equation by looking at its components:

- $y - 2 = 0$ is the component consisting of characters of abelian representations.
- $y^2 - (x^2 - 1)y + x^2 - 1 = 0$ is the so called *canonical component*, that contains lifts of the holonomy of the hyperbolic structure. The existence of this component is the content of the next section (Theorem 3.1).

Thus

$$X(\Gamma) \cong \{(x, y) \in \mathbb{C}^2 \mid (y-2)(y^2 - (x^2-1)y + x^2 - 1) = 0\}.$$

Here we have used a theorem in hyperbolic geometry to determine the equations of the variety of characters. In Theorem 2.13 we give a result of González-Acuña and Montesinos [15] on a sufficient set of relations required to compute $X(\Gamma)$.

Example 2.7 (Two-bridge knots). The method for the trefoil and the figure-eight knot is common to two-bridge knots. The fundamental group of a two-bridge knot admits a presentation

$$\Gamma = \langle a, b \mid a w = w b \rangle$$

where w is a certain word on a and b depending on the knot, and a and b are different representatives of a meridian. As a and b are conjugate, two coordinates are sufficient:

$$x = \tau_a = \tau_b \quad \text{and} \quad y = \tau_{ab^{-1}}.$$

There is always a component $y = 2$ consisting of abelian representations. Then the other components are contained in \mathbb{C}^2 . We show in next chapter that each component of $X(\Gamma)$ has dimension at least one (Theorem 3.5). On the other hand, any component cannot be the whole plane \mathbb{C}^2 , because we shall also prove that one of the representations is a smooth point of a curve (Theorem 3.6). Thus the variety of characters of a two bridge knot is always a plane curve, possibly with several components (with arbitrarily many of them, by [28]).

For computations and results on the variety of characters of two-bridge knots, see for instance [22, 28, 36, 5].

Example 2.8 (The figure eight knot II). There is another approach due to Alice Whittlemore [43] to compute the variety of characters of the figure eight knot exterior. Using the presentation in (2.3), set

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix}$$

with $s + s^{-1} = \tau_a = \tau_b = x$ and $\tau_{a^{-1}b} = y$, as this is the generic expression for an irreducible representation (it also may contain reducible nonabelian representations). Then

$$\rho(wa) - \rho(bw) = (y^2 - (x^2 - 1)y + x^2 - 1) \begin{pmatrix} 0 & -1 \\ y-2 & 0 \end{pmatrix}.$$

This proves precisely that the component of $X(\Gamma)$ that contains irreducible characters is defined by $y^2 - (x^2 - 1)y + x^2 - 1 = 0$, without requiring any result in hyperbolic geometry, though the computation is slightly more involved.

Example 2.9 (The figure eight knot III). There is yet a third approach using the presentation of the figure eight-knot exterior as a fibered manifold with fibre a punctured torus. Here we use the presentation corresponding to the fibration:

$$\Gamma = \langle \alpha, \beta, \mu \mid \mu\alpha\mu^{-1} = \alpha\beta, \mu\beta\mu^{-1} = \beta\alpha\beta \rangle.$$

Let $\phi: F_2 = \langle \alpha, \beta \mid \rangle \rightarrow F_2$ be the conjugation by μ , i.e. the monodromy. One can compute the subvariety characters of $X(F_2)$ invariant by ϕ :

$$X(F_2)^\phi = \{\chi \in X(F_2) \mid \chi \circ \phi = \chi\}.$$

Set coordinates

$$x_1 = \tau_\alpha \quad x_2 = \tau_\beta, \quad x_3 = \tau_{\alpha\beta}.$$

Then the condition $\chi \circ \phi = \chi$ yields

$$\begin{aligned} x_1 &= \tau_{\phi(\alpha)} = \tau_{\alpha\beta} = x_3; \\ x_2 &= \tau_{\phi(\beta)} = \tau_{\beta\alpha} = \tau_{\beta\alpha}\tau_\beta - \tau_\alpha = x_2x_1 - x_1; \\ x_3 &= \tau_{\phi(\alpha\beta)} = \tau_{\alpha\beta^2\alpha\beta} = x_1x_2x_3 - x_1x_3 - x_2. \end{aligned}$$

The last equation follows from the previous two, hence, setting $x_3 = x_1$, we get the plane curve:

$$X(F_2)^\phi \cong \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 + x_2 = x_1x_2\}.$$

Next we consider the fibre of the restriction map

$$\text{res}: X(\Gamma) \rightarrow X(F_2)^\phi$$

More precisely, one needs to find how many possibilities there are for $\rho(\mu)$ conjugating ρ and $\rho \circ \phi$, where ρ is a representation with a given character in $X(F_2)^\phi$. The fibre of res depends on whether the character is irreducible or not. It can be easily shown that the map $\text{res}: X(M) \rightarrow X(F_2)^\phi$ is surjective and its fiber consist of

- (a) 2 points at irreducible characters of $X(F_2)^\phi$;
- (b) 1 point at reducible characters of $X(F_2)^\phi$ that are nontrivial on Γ' ;
- (c) a line of abelian characters at characters that are trivial on Γ' .

From this, one can determine:

- (1) The number of components of $X(\Gamma)$: there is a component that is a branched covering of $X(F_2)^\phi$ and the component of abelian characters, that collapse to a point in $X(F_2)^\phi$, the trivial character.
- (2) The variety of $\text{PSL}_2\mathbb{C}$ -characters, that is precisely isomorphic to $X(F_2)^\phi$.

Remark 2.10. This method works for any punctured torus bundle. The monodromy is conjugated to the composition of the maps

$$\begin{array}{ccc} r: F_2 \rightarrow F_2 & l: F_2 \rightarrow F_2 & i: F_2 \rightarrow F_2 \\ a \mapsto a & a \mapsto ab & a \mapsto a^{-1} \\ b \mapsto ba & b \mapsto b & b \mapsto b^{-1} \end{array}$$

Each one induces a map in the variety of characters $X(F_2) \cong \mathbb{C}^3$:

$$(2.4) \quad r^*(x_1, x_2, x_3) = (x_1, x_3, x_1x_3 - x_2),$$

$$(2.5) \quad l^*(x_1, x_2, x_3) = (x_3, x_2, x_2x_3 - x_1),$$

$$(2.6) \quad i^*(x_1, x_2, x_3) = (x_1, x_2, x_3).$$

Hence $X(F_2)^\phi$ can be computed algorithmically [32].

Example 2.11 (Whitehead link). We use the presentation $\Gamma = \langle a, b \mid aw = wa \rangle$ where $w = bab^{-1}a^{-1}b^{-1}ab$. With the coordinates

$$x = \tau_a, \quad y = \tau_b, \quad z = \tau_{ab},$$

we can compute

$$X(\Gamma) = \{(x, y, z) \in \mathbb{C}^3 \mid pq = 0\}$$

with

$$\begin{cases} p = xy - (x^2 + y^2 - 2)z + xyz^2 - z^3, \\ q = x^2 + y^2 + z^2 - xyz - 4. \end{cases}$$

Here $q = 0$ is the component of abelian characters. Notice that in $p = 0$ there are points with $x = \pm 2$ and $y = \pm 2$. Then the values of z are ± 2 and $\pm 1 \pm \sqrt{-1}$. The values $z = \pm 1 \pm \sqrt{-1}$ correspond to different lifts of the holonomy representation.

In general, a two bridge link that is not a knot admits a presentation as above, with a different expression for w . Then the variety of characters is a surface in \mathbb{C}^3 , that may have several components.

General Γ of finite type and reduction mod p . In this paragraph we overview the work of González-Acuña and Montesinos [15], based on works of Vogt [41] and Magnus [24], for the variety of characters of a group of finite type.

We start describing the variety of characters of a free group. For a free group of rank 3, $F_3 = \langle a, b, c \mid \rangle$ we have that

$$(\tau_a, \tau_b, \tau_c, \tau_{ab}, \tau_{bc}, \tau_{ca}): X(F_3) \rightarrow \mathbb{C}^6$$

is a branched covering. In addition there is one further coordinate algebraically dependent: τ_{abc} and τ_{acb} are the solutions of the equation

$$z^2 - Pz + Q = 0$$

with

$$\begin{aligned} P &= \tau_a \tau_{bc} + \tau_b \tau_{ca} + \tau_c \tau_{ab} - \tau_a \tau_b \tau_c, & \text{and} \\ Q &= \tau_a^2 + \tau_b^2 + \tau_c^2 + \tau_{ab}^2 + \tau_{bc}^2 + \tau_{ca}^2 + \tau_{ab} \tau_{bc} \tau_{ca} - \tau_a \tau_b \tau_{ab} - \tau_b \tau_c \tau_{bc} - \tau_c \tau_a \tau_{ca} - 4. \end{aligned}$$

Namely:

$$\tau_{abc} + \tau_{acb} = P \quad \text{and} \quad \tau_{abc} \tau_{acb} = Q.$$

We next discuss the free group of rank 4, $F_4 = \langle a, b, c, d \mid \rangle$. In this case we do not give the explicit equations, but just the following remark:

Remark 2.12 ([15]). τ_{abcd} is a polynomial on the traces of words on a, b, c and d of length ≤ 3 with coefficients in $\frac{1}{2}\mathbb{Z}$.

Thus the construction of [15] provides a natural way to construct $X(F_n)$ as an affine algebraic set defined by polynomials with coefficients in $\frac{1}{2}\mathbb{Z}$.

Theorem 2.13 ([15]). *For a group with presentation $\Gamma = \langle \gamma_1, \dots, \gamma_n \mid \{w_j\}_{j \in J} \rangle$,*

$$X(\Gamma) \cong \{ \chi \in X(F_n) \mid \chi(\gamma_i w_j) = \chi(\gamma_i), \quad i = 1, \dots, n, j \in J \}$$

as varieties.

This theorem and the explicit description of $X(F_n)$ in [15] provide an algorithm to compute the variety of characters of a finitely presented group.

By Remark 2.12, $\forall \gamma \in F_n$, τ_γ is a polynomial in the traces of words of length ≤ 3 on the generators, with coefficients in $\frac{1}{2}\mathbb{Z}$. Thus from [15] we have:

Remark 2.14. One can take reduction mod p , $p \neq 2$ prime, of $X(\Gamma)$ to compute the variety of characters in $SL_2 \mathbb{F}$, where \mathbb{F} is an algebraically closed field of characteristic p .

When computing the reduction mod p , for almost every prime p , the number of components and their dimension does not change, but for finitely many primes there can be ramification phenomena. We discuss an example in Remark 4.7 in Section 4.

3. Hyperbolic knots and the canonical component

Let $K \subset S^3$ be a hyperbolic knot, namely $S^3 - K$ admits a complete hyperbolic structure of finite volume. The hyperbolic space is denoted by \mathbb{H}^3 and we recall that the group of orientation preserving isometries of \mathbb{H}^3 is naturally isomorphic to $PSL_2(\mathbb{C})$. There is a representation, called the *holonomy representation*

$$\text{hol}: \pi_1(S^3 - K) \rightarrow PSL_2(\mathbb{C})$$

that is discrete and faithful, so that the complete hyperbolic structure on $S^3 - K$ is provided by $\mathbb{H}^3/\text{hol}(\pi_1(S^3 - K))$. By Mostow rigidity theorem, the hyperbolic structure is unique. Hence

the holonomy is unique up to conjugation, and up to complex conjugation if the manifold $S^3 - K$ is not oriented. By a theorem of Culler [9], the holonomy representation lifts to $SL_2\mathbb{C}$:

$$\begin{array}{ccc} & & SL_2\mathbb{C} \\ & \nearrow \rho_0 & \downarrow \\ \pi_1(S^3 - K) & \xrightarrow{\text{hol}} & PSL_2(\mathbb{C}) \end{array}$$

This yields 4 characters χ_0 , two lifts of the holonomy representation and two more lifts of its complex conjugated.

To simplify, denote

$$X(K) = X(S^3 - K).$$

The main theorem of this section is:

Theorem 3.1. *The character χ_0 of a lift of the holonomy ρ_0 is a smooth point of $X(K)$ and the component X_0 of $X(K)$ that contains χ_0 is a \mathbb{C} -curve.*

Definition 3.2. The component X_0 is called the *canonical component*.

Remark 3.3. A priori, the canonical component does not need to be unique, as we just have mentioned that there are four characters χ_0 that are lifts of the holonomy of a hyperbolic structure, up to orientation. To my knowledge, for all known examples of hyperbolic knots the canonical component is unique. Other than knots, Casella, Luo, and Tillmann [7], have an example of hyperbolic manifold with at least two canonical components.

Different canonical components are isomorphic, either by complex conjugation or by multiplying the characters by $(-1)^\epsilon$ for some epimorphism $\epsilon: \pi_1(S^3 - K) \rightarrow \mathbb{Z}/2\mathbb{Z}$ (again, this holds for general hyperbolic manifolds, notice that for knots ϵ is unique).

Remark 3.4. (1) For a two bridge knot K , all the components of $X(K)$ are curves, and it is proved in [28] that the number of components is arbitrarily large. One can also show that there is a unique canonical component. The components build by [28] come from pulling back the canonical components of other knots by surjections of the fundamental group. (Notice that in this section we deal only with hyperbolic knots, otherwise there are two-bridge knots that are torus knots, like the trefoil, described before).

(2) For Montesinos knots K , the number of components of any dimension of $X(K)$ can be arbitrarily large [29].

(3) Theorem 3.1 is motivated by Thurston's hyperbolic Dehn filling theorem: some of the representations in a neighborhood of the holonomy of the complete structure are holonomies of non complete structures on the knot exterior whose metric completion is a Dehn filling.

(4) One may prove further that the canonical component is locally parametrized around χ_0 by the trace of any peripheral element, see Theorem 3.10. In particular this proves that the A -polynomial is nontrivial (See [8] for the definition of A -polynomial).

For non-hyperbolic knots, the proof of the non-triviality of the A -polynomial is due to Dunfield and Garoufalidis [13] and Boyer and Zhang [3], based on a theorem of Kronheimer and Mrowka [20].

(5) Theorem 3.1 works for manifolds of finite volume, not only for knot exteriors, and the dimension is the number of cusps.

The proof of Theorem 3.1 is based in the following two results:

Theorem 3.5. [39, 11] *Let $Y \subset X(K)$ be an irreducible component containing an irreducible character. Then $\dim Y \geq 1$.*

Theorem 3.6. *The dimension of the Zariski tangent space at χ_0 is*

$$\dim T_{\chi_0}^{\text{Zar}} X(K) = 1.$$

Theorem 3.5 is a lower bound for the dimension and it is a particular case (for manifolds other than knots) proved in Thurston's notes [39], see also [11, Proposition 3.2.1]. Theorem 3.6 gives an upper bound of the dimension and, due to the properties of the Zariski tangent space, combination of both theorems yields smoothness of χ_0 . (The dimension of the Zariski tangent space is an upper bound for the dimension, with equality precisely at smooth points). This takes care also of the scheme issue, and it is smooth as both variety and as scheme.

In his notes, Thurston only requires Theorem 3.5 to prove the hyperbolic Dehn filling, by using other arguments, like Mostow rigidity and the openness theorem for complex functions.

It is important to mention a completely different approach to deformation spaces by means of ideal triangulations defined by Neumann and Zagier in [26].

Proof of Theorem 3.5. We follow Thurston's notes. Let $M = S^3 - \mathcal{N}(K)$ be the compact manifold obtained by removing a tubular neighborhood of the knot. Let ρ_0 be an irreducible representation of M whose character lies in Y . Since $\pi_1(M^3)$ is normally generated by the meridian, we may assume that $\rho_0(\pi_1(\partial M))$ is not contained in $\{\pm \text{Id}\}$, the center of $\text{SL}_2\mathbb{C}$. We shall choose a generic curve with base point in ∂M , so that

- (1) $\text{tr}(\rho_0(\alpha)) \neq \pm 2$, and
- (2) the restriction $\rho_0|_{(\alpha, \pi_1(\partial M))}$ is irreducible.

We may assume that α is a simple closed curve, so

$$M' = M - \mathcal{N}(\alpha)$$

is a manifold with boundary a surface of genus 2. In particular, $\chi(M') = \frac{1}{2}\chi(\partial M) = -1$, so it has the homotopy type of a 2-CW complex with 1 0-cell, r 1-cells and $(r - 2)$ 2-cells. This gives a presentation of $\pi_1(M')$ with r generators and $r - 2$ relations. Thus the dimensions of $R(M')$ and $X(M')$ have lower bounds

$$\begin{aligned} \dim R(M') &\geq (r - (r - 2)) \dim \text{SL}_2\mathbb{C} = 6, \\ \dim X(M') &\geq \dim R(M') - \dim \text{SL}_2\mathbb{C} \geq 3. \end{aligned}$$

View α as a peripheral element of $\pi_1(M')$ and choose $\beta \in \pi_1(M')$ another peripheral element that represents a meridian around α , so that the commutator $[\alpha, \beta]$ is the boundary of the punctured torus $\partial M \cap \partial M'$.

Claim 3.7. *Let $\rho \in R(M')$ be in a neighborhood of ρ_0 such that*

$$\text{tr}(\rho(\beta)) = \text{tr}(\rho([\alpha, \beta])) = 2.$$

Then $\rho(\beta) = \text{Id}$.

Assume the claim. As $\rho(\beta) = \text{Id}$ implies $\rho \in R(M)$, the claim tells that, in a neighborhood of ρ_0 in $R(M')$, $R(M)$ is determined by 2 equations. Thus, a component of $R(M)$ that contains ρ_0 has dimension ≥ 4 , and the component of $X(M)$ that contains χ_0 has dimension ≥ 1 . This finishes the proof assuming the claim.

To prove the claim we may assume that $\text{tr}(\rho(\alpha)) \neq \pm 2$. Hence, up to conjugation

$$\rho(\alpha) = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad \lambda \neq \pm 1.$$

As $\text{tr}(\rho(\beta)) = \text{tr}(\rho([\alpha, \beta])) = 2$, then

$$\rho(\beta) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad x \neq 0.$$

In particular

$$\rho([\alpha, \beta]) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad y \neq 0.$$

Now $[\alpha, \beta]$ represents a peripheral curve in the punctured torus $\partial M - D$, for a disk $D \subset \partial M$. Hence it is the commutator in $\pi_1(\partial M - D)$, and by the proof of Lemma 1.10, $\rho(\pi_1(\partial M - D))$ is reducible, with an invariant line in common with $\rho(\alpha)$, which contradicts the hypothesis (2). \square

Before talking about proof of Theorem 3.6, let me discuss the *Zariski tangent space*. Given an affine algebraic set

$$V = \{x \in \mathbb{C}^N \mid p_1(x) = \dots = p_r(x) = 0\}$$

where $p_1, \dots, p_r \in \mathbb{C}[x_1, \dots, x_N]$ are polynomials, the Zariski tangent space at $x \in V$ (as scheme) is

$$(3.1) \quad T_x^{\text{Zar}} V = \{v \in \mathbb{C}^N \mid p_1(x + \varepsilon v), \dots, p_r(x + \varepsilon v) \in o(\varepsilon^2)\}.$$

To compute this space we use a construction due to André Weil. Define the space of *crossed morphisms*

$$Z^1(\Gamma, \text{Ad } \rho) = \{\theta: \Gamma \rightarrow \mathfrak{sl}_2\mathbb{C} \mid \theta(\gamma_1\gamma_2) = \theta(\gamma_1) + \rho(\gamma_1)\theta(\gamma_2)\rho(\gamma_1^{-1})\};$$

there is a natural isomorphism

$$(3.2) \quad \begin{array}{ccc} Z^1(\Gamma, \text{Ad } \rho) & \xrightarrow{\cong} & T_\rho^{\text{Zar}} R(\Gamma) \\ \theta & \mapsto & \gamma \mapsto (\text{Id} + \varepsilon\theta(\gamma))\rho(\gamma) = \rho_\varepsilon(\gamma) \end{array}.$$

Notice that $\theta \in Z^1(\Gamma, \text{Ad } \rho)$ if and only if $\rho_\varepsilon(\gamma_1\gamma_2) = \rho_\varepsilon(\gamma_1)\rho_\varepsilon(\gamma_2) + o(\varepsilon^2)$. The isomorphism (3.2) identifies the orbit by conjugation to the space of *inner crossed morphisms*

$$B^1(\Gamma, \text{Ad } \rho) = \{\theta_a \in Z^1(\Gamma, \text{Ad } \rho) \mid \theta_a(\gamma) = \rho(\gamma)a\rho(\gamma^{-1}) - a, \forall \gamma \in \Gamma\}.$$

Namely, for $a \in \mathfrak{sl}_2\mathbb{C}$, θ_a is tangent to the conjugation by $\exp(\varepsilon a)$. Setting

$$H^1(\Gamma, \text{Ad } \rho) = Z^1(\Gamma, \text{Ad } \rho)/B^1(\Gamma, \text{Ad } \rho)$$

by Proposition 1.13 (and using that the analytic and algebraic Zariski tangent space are isomorphic) we have:

Theorem 3.8. *For $\rho \in R(\Gamma)$ irreducible,*

$$H^1(\Gamma, \text{Ad } \rho) \cong T_{\chi_\rho}^{\text{Zar}} X(\Gamma).$$

Using this cohomological interpretation of the Zariski tangent space, Theorem 3.6 is a consequence of the infinitesimal rigidity theorem of Calabi-Weil, that for a knot exterior is stated as follows:

Theorem 3.9 (Calabi [42], Weil [42]). *Let $\rho \in R(K)$ be a lift of the holonomy representation. Then $H^1(\pi_1(S^3 - K), \text{Ad } \rho) \cong \mathbb{C}$.*

This theorem is proved by means of de Rham cohomology. Let

$$E_\rho = \tilde{M} \times \mathfrak{sl}_2(\mathbb{C})/\pi_1 M$$

denote the flat bundle over M with fibre $\mathfrak{sl}_2(\mathbb{C})$, and let $\Omega^p(M, E_\rho)$ denote the space of E_ρ -valued smooth p forms on M . Then the de Rham cohomology of $\Omega^*(M, E_\rho)$ is naturally isomorphic to $H^*(S^3 - K, \text{Ad } \rho)$.

The key result in the proof of Theorem 3.9 is that every closed form in $\Omega^1(M, E_\rho)$ that is L^2 is exact (one can find a Harmonic representative in the cohomology class, that must vanish by a Bochner-type argument). Once we have this vanishing theorem, we deduce that we have an injection by the long exact sequence of the pair:

$$(3.3) \quad 0 \rightarrow H_{dR}^1(S^3 - K, \mathfrak{sl}_2\mathbb{C}) \rightarrow H_{dR}^1(U, \mathfrak{sl}_2\mathbb{C})$$

where U is an end of the manifold, because the kernel is realized by forms with compact support, in particular L^2 . Then the deformation space and the cohomology group of the peripheral torus are easily computed to be two dimensional, and we get $\dim(H^1(S^3 - K, \mathfrak{sl}_2\mathbb{C})) = 1$ by a Poincaré duality argument, cf. [37].

See [1] for an accessible account on Calabi-Weil infinitesimal rigidity or [19] for further applications and another approach to the Calabi-Weil theorem.

One of the consequences of this approach using (3.3) is that all infinitesimal deformations are described by the end of $S^3 - K$. Furthermore, by showing that the non- L^2 deformations must deform the trace of any peripheral element, as done by Bromberg in [4], it can be shown that:

Theorem 3.10. *Let $\rho_0 \in R(K)$ be a lift of the holonomy representation. Then for any non-trivial peripheral element $\gamma \in \pi_1(S^3 - K)$, its trace is a local parameter around χ_{ρ_0} :*

$$\tau_\gamma: X(S^3 - K) \rightarrow \mathbb{C}.$$

4. Knot symmetries

This section is devoted to a joint work with Luisa Paoluzzi [30].

Let K be a knot in S^3 and $\psi: (S^3, K) \rightarrow (S^3, K)$ a diffeomorphism of finite order p , that preserves the orientation of S^3 . By the Smith conjecture, either $\text{Fix}(\psi) = \emptyset$ or $\text{Fix}(\psi) \cong S^1$ is unknotted in S^3 .

Definition 4.1. The finite order diffeomorphism $\psi: (S^3, K) \rightarrow (S^3, K)$ that preserves the orientation is said to be:

- a *free symmetry* if the group $\langle \psi \rangle \cong \mathbb{Z}/p\mathbb{Z}$ acts freely,
- a *periodic symmetry* if $\text{Fix}(\psi)$ is a circle disjoint from K ,
- a *strong inversion* if it has order 2 and $|\text{Fix}(\psi) \cap K| = 2$,
- a *pseudo-periodic symmetry* otherwise.

Remark 4.2. We shall assume that ψ has order a prime $p \neq 2$, hence ψ is either *free* or *periodic*. More precisely, $\text{Fix}(\psi)$ is either empty or an unknotted circle in S^3 , and $\text{Fix}(\psi) = \text{Fix}(\psi^r)$ for every r coprime with p .

The *goal* of this section is to show that periodic and free symmetries have different behavior in the number of components in the fixed subvariety in the variety of characters.

Example 4.3 (Periodic symmetry). Start with the three-strand braid of Figure 4.1, and glue five copies of it as in Figure 4.2, yielding the knot 10_{123} . This is a hyperbolic knot with a periodic symmetry of order five, the fixed point set being the axis perpendicular to the projection plane.

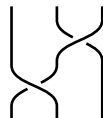


Figure 4.1: The braid with three strands.

The quotient of S^3 by the action of ψ is S^3 with branching locus an unknotted circle. The union of the branching locus and the projection of the knot is the link 6_2^2 , Figure 4.3.

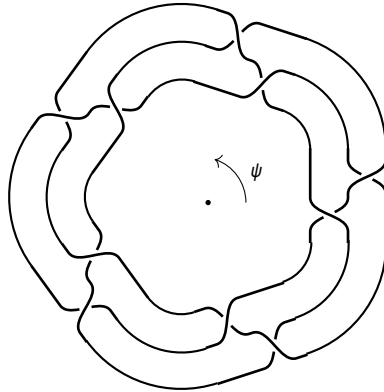


Figure 4.2: The periodic knot 10_{123} with the symmetry ψ of order 5.

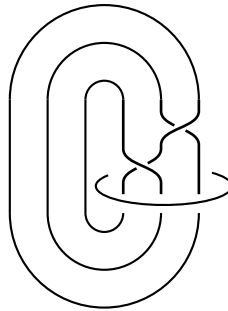


Figure 4.3: The quotient link 6_2^2 , one component is the projection of the fixed point set of ψ , the other one is the projection of the symmetric knot 10_{123} .

This construction is taken from [17], where they consider the knot 8_{18} , that has a period of order 4 instead of 5.

Example 4.4 (Free symmetry). Modify slightly the construction in Example 4.3 by adding a twist of a strip that would contain the three strands of the braid, namely composing with the central element of the braid group, see Figure 4.4.

To visualize the action, view the sphere S^3 as the joint of two circles, arranged as the Hopf link. One of the circles is the axis perpendicular to the projection, the other one is contained in the plane of the projection and is the core of a solid torus containing the braid. The action of ψ is a rotation of order five along each of the circles, because we have added the twist. (In the Example 4.3, the action was trivial along one of the circles).

The quotient is the Lens space $L(5, 1)$, that can be described by $\frac{1}{5}$ -Dehn filling along the trivial knot. Thus the quotient of the knot in S^3 is a knot in a Lens space, that can be again described by a Dehn filling along the link 6_2^2 , see Figure 4.5.

Let $K \subset S^3$ be a hyperbolic knot, the variety of characters is denoted by $X(K) := X(S^3 - K)$. Let $\psi: (S^3, K) \rightarrow (S^3, K)$ be a symmetry of prime order $p \neq 2$, the *variety of invariant characters* is denoted by

$$X(K)^\psi = \{\chi \in X(K) \mid \chi \circ \psi_* = \chi\}.$$

The main result of this section is

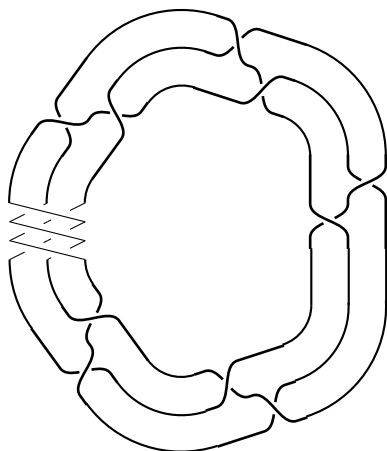


Figure 4.4: The knot with a free symmetry of order 5. In order to visualize the symmetry, the full turn of the strip should be distributed in five times $1/5$ -th of turn along the strip.

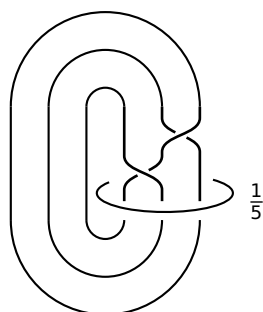


Figure 4.5: Representation of the quotient knot in the lens space $L(5, 1) = S^3/\psi$, by Dehn filling on one component of 6^2_2 .

Theorem 4.5 (Paoluzzi-P. [30]). *If ψ is a periodic symmetry of the knot of order an odd prime p , then $X(K)^\psi$ has at least $\frac{p-1}{2}$ components that are also components of $X(K)$.*

The idea is to contrast this behavior with the case of free symmetry, so we make the following remarks:

Remark 4.6. (a) For each prime $p > 4$ there is a knot K_p with a free symmetry ψ of order p so that $X(K_p)^\psi$ has at most 20 components.

(b) If we look at components of $X(K)$, without ψ -invariance, many further components may appear, for ψ either free or periodic. This can be achieved by considering Montesinos knots.

Another remark is the reduction mod p :

Remark 4.7. When reducing mod p , all the components of the theorem become a single one.

Remark 4.6 (a) and Remark 4.7 are discussed after the proof of the theorem.

Proof of Theorem 4.5. Set $M = S^3 - K$, $\mathcal{O} = M/\psi$. The proof has 3 steps:

- (1) The restriction map $\text{res}: X^{\text{irr}}(\mathcal{O}) \rightarrow X^{\text{irr}}(M)^\psi$ is a bijection.
- (2) Find several components for $X(\mathcal{O})$ using Galois conjugates and the element of finite order.
- (3) Use (1) + (2) to find several components for $X(M)^\psi$

Step 1: Extending ψ -invariant representations. Set $M = S^3 - K$ and $\mathcal{O} = M/\psi$. We have an extension of fundamental groups

$$\begin{array}{c} \pi_1 M \rightarrow \pi_1 \mathcal{O} \rightarrow \mathbb{Z}/p\mathbb{Z} \\ \mu \leftarrow 1 \end{array}$$

that splits because μ has order p . However the extension works without requiring the splitting. Viewing $\pi_1 M$ in $\pi_1 \mathcal{O}$, the induced action of the period ψ on $\pi_1 M$ satisfies

$$\begin{array}{ccc} \psi_*: \pi_1 M & \rightarrow & \pi_1 M \\ \gamma & \mapsto & \mu\gamma\mu^{-1} \end{array}$$

Thus we write

$$\pi_1 \mathcal{O} = \langle \pi_1 M, \mu \mid \mu^p = 1, \mu\gamma\mu^{-1} = \psi_*\gamma, \forall \gamma \in \pi_1 M \rangle.$$

Lemma 4.8. *The restriction map $\text{res}: X^{\text{irr}}(\mathcal{O}) \rightarrow X^{\text{irr}}(M)^\psi$ is a bijection.*

Proof of the lemma. First we show that the restriction of an irreducible character is irreducible. By contradiction, assume that $\rho \in R(\mathcal{O})$ is irreducible but $\text{res}(\rho) \in R(M)$ is reducible. The restriction $\text{res}(\rho)$ is not contained in $\{\pm \text{Id}\}$ because otherwise the image of ρ would be generated by $\pm\rho(\mu)$ and it would be reducible. The restriction $\text{res}(\rho)$ has either one or two invariant lines in \mathbb{C}^2 . If one of these invariant lines was preserved by $\rho(\mu)$, then ρ would be reducible, therefore $\text{res}(\rho)$ has two invariant lines and $\rho(\mu)$ permutes them, but this contradicts that the order of μ is odd.

Once we know that $\text{res}(X^{\text{irr}}(\mathcal{O})) \subset X^{\text{irr}}(M)$, it is clear that $\text{res}(X^{\text{irr}}(\mathcal{O})) \subset X^{\text{irr}}(M)^\psi$ and next we show that every character in $X^{\text{irr}}(M)^\psi$ extends to $\pi_1 \mathcal{O}$ in a unique way. Let $\chi_\rho \in X^{\text{irr}}(M)^\psi$. Since $\chi_{\rho \circ \psi_*} = \chi_\rho$ by Lemma 1.10 $\rho \circ \psi_*$ and ρ are conjugate. There exists a matrix $A \in \text{SL}_2\mathbb{C}$ such that $\rho(\psi_*(\gamma)) = A\rho(\gamma)A^{-1}$, $\forall \gamma \in \pi_1 M$, which is unique up to sign by irreducibility. An extension of ρ must map μ to $\pm A$, and since μ has odd order p , the choice of sign is unique so that $A^p = \text{Id}$. \square

The proof can be easily adapted without requiring that μ has finite order, but just that $\mu^p \in \pi_1 M$, by replacing the identity by an inner automorphism of $\pi_1 M$. On the other hand, the fact that p is odd cannot be skipped.

Step 2: Finding components for $X(\mathcal{O})$. The orbifold \mathcal{O} is hyperbolic and the holonomy representation $\text{hol}: \pi_1 \mathcal{O} \rightarrow \text{PSL}_2\mathbb{C}$ lifts to

$$\rho_0: \pi_1 \mathcal{O} \rightarrow \text{SL}_2\mathbb{C}$$

because p is odd. One way to see that the lift ρ_0 exists is to lift the holonomy of the (smooth incomplete) structure on the complement of the branching locus, and then to consider the induced representation on the orbifold by asking that the meridian is mapped to an element of order p . Another possibility is just to apply Lemma 4.8 to the lift of the holonomy of M .

By a theorem of Vinberg (cf. [23, Theorem 3.1.2]), after conjugation the image $\rho_0(\pi_1 \mathcal{O})$ is contained in $\text{SL}_2(\mathbb{K})$ for \mathbb{K} a number field, i.e. the extension $\mathbb{K}|\mathbb{Q}$ is algebraic. By taking a larger field if necessary, we may assume that the extension $\mathbb{K}|\mathbb{Q}$ is a Galois extension.

Next we use the element of finite order, whose trace lies in the real part of the cyclotomic field, contained in \mathbb{K} . Since $\mu^p = 1$,

$$\text{tr}(\rho_0(\mu)) = -2 \cos \frac{\pi}{p}.$$

Therefore the set of Galois conjugates of this trace gives:

$$\{\tau_\mu(\rho_0^\sigma) \mid \sigma \in \text{Galois}(\mathbb{K})\} = \{-2 \cos \frac{\pi r}{p} \mid r = 1, 3, 5, \dots, p-2\}.$$

Now the equations

$$\tau_\mu = -2 \cos \frac{\pi r}{p},$$

for $r = 1, 3, 5, \dots, p-2$, distinguish $\frac{p-1}{2}$ components

$$Y_1, \dots, Y_{\frac{p-1}{2}}$$

of $X(\mathcal{O})$ that contain different Galois conjugates $\chi_{\rho_0^\sigma}$ of χ_{ρ_0} .

Finally we claim that each Y_r is a curve. By Theorem 3.9, since \mathcal{O} has a cusp, $H^1(\pi_1 \mathcal{O}, \text{Ad } \rho_0) = \mathbb{C}$ hence $H^1(\pi_1 \mathcal{O}, \text{Ad } \rho_0^\sigma) = \mathbb{C}$, $\forall \sigma \in \text{Galois}(\mathbb{K})$, that with Theorem 3.5 proves the claim.

The next step is, using that $\text{res}: X^{\text{irr}}(\mathcal{O}) \rightarrow X^{\text{irr}}(M)^\psi$ is a bijection, the curves $Y_1, \dots, Y_{\frac{p-1}{2}}$ yield different components of $X(M)^\psi$.

Step 3: components for $X(M)^\psi$. We know that $\text{res}: X^{\text{irr}}(\mathcal{O}) \rightarrow X^{\text{irr}}(M)^\psi$ is a bijection. We have $Y_1, \dots, Y_{\frac{p-1}{2}}$ components of $X^{\text{irr}}(\mathcal{O})$ that are curves. We take W_i to be the Zariski closure of $\text{res}(Y_i)$ in $X(M)^\psi$:

$$\begin{array}{ccc} X(\mathcal{O}) & \xrightarrow{\text{res}} & X(M)^\psi \subset X(M) \\ Y_1 & & \\ \vdots & & W_i = \overline{\text{res}(Y_i)} \\ Y_{\frac{p-1}{2}} & & \end{array}$$

Since $Y_i = Y_i^{\text{irr}} \cup (\text{a finite set})$, $W_i = \text{res}(Y_i) \cup (\text{a finite set})$ and $W_1, \dots, W_{\frac{p-1}{2}}$ are different components of $X(M)^\psi$. In addition, as $\dim H^1(\pi_1 M, \text{Ad } \rho_0^\sigma) = \dim H^1(\pi_1 M, \text{Ad } \rho_0) = 1$, $W_1, \dots, W_{\frac{p-1}{2}}$ are also components of $X(M)$. \square

Remark 4.9. We justify now Remark 4.7. Let \mathbb{F} be an algebraically closed field of characteristic p . As an element of $\text{SL}_2 \mathbb{F}$ of order p is conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

all components $Y_1, \dots, Y_{\frac{p-1}{2}}$ in the proof of the theorem become a single one mod p . Therefore all components $W_1, \dots, W_{\frac{p-1}{2}}$ in the theorem become a single one mod p .

Finally, we sketch the examples of Remark 4.6 (a). This is based in the construction of Example 4.4. Denote by A and K_0 the components of the link 6_2^2 (the components of the link can be permuted). See Figure 4.6. Consider a $\frac{q}{p}$ -Dehn filling on A and take the covering

$$(S^3, K_{\frac{q}{p}}) \rightarrow (L(p, q), K_0)$$

where $p > 4$ is prime and p and q are coprime. Notice that when $p = 3$ we do not get a knot but a link. Since p and q are coprime, K_0 lifts to a knot $K_{\frac{q}{p}} \subset S^3$ and since $p > 4$ it is hyperbolic. Then the automorphism of the covering transformation $\psi: (S^3, K_{\frac{q}{p}}) \rightarrow (S^3, K_{\frac{q}{p}})$ is a free symmetry of order p .

The variety of characters $X(6_2^2)$ has precisely two components, the canonical component $X_0(6_2^2)$ and the abelian component $X^{\text{ab}}(6_2^2)$, see [16] or [30]. We use the following two properties:

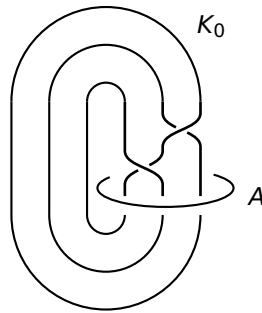


Figure 4.6: The link 6_2^2 , the component A for Dehn filling and the component K_0 that lifts to K .

- (1) The restriction map $X_0(6_2^2) \rightarrow X(\partial\mathcal{N}(A))$ is dominant and its generic fibre is finite. This can be easily checked, see [30].
- (2) $\forall \gamma \in \pi_1 \partial\mathcal{N}(A)$ primitive, $\{\tau_\gamma = 2\}$ is a line in $X(\partial\mathcal{N}(A))$. It suffices to replace $x = 2$ on the equation in Example 2.4, and the equation becomes $(y - z)^2 = 0$.

From these properties, it is straightforward that $X(L(p, q) - K_0)$ has at most C components, for some uniform constant C that does not depend on p or q . By Lemma 4.8 (whose proof does not use that the symmetry has fixed points), $X(S^3 - K_{\frac{q}{p}})^\psi$ has also at most C components, with C uniform, but ψ has order p (that can be any prime > 4). See more details in [30], where it is shown that we can take $C = 20$.

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