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ARNAUD DEBUSSCHE

**Quelques problèmes concernant le comportement pour les grands temps des équations d'évolution dissipatives**

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**Arnaud DEBUSSCHE**

Sujet :

**Quelques problèmes concernant le comportement  
pour les grands temps des équations d'évolution dissipatives**

soutenue le 12 décembre 1989 devant la Commission d'examen

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## **SOME PROBLEMS RELATED TO THE LONG TIME BEHAVIOUR OF DISSIPATIVE EVOLUTION EQUATIONS.**

**Abstract :** In this work, we consider the long time behaviour of dissipative evolution equations. More precisely we study the existence of attracting sets such as attractors and inertial manifolds.

In the first part, we describe a general method to construct inertial manifolds for a nonlinear parabolic equation. We obtain an existence theorem under the same type of assumptions as the methods that already exist. Our method is based on the resolution of a hyperbolic partial differential equation (the Sacker's equation) such that the graph of its solution is a positively invariant manifold.

The second part is devoted to the existence of approximate inertial manifolds. These are substitute to inertial manifolds when their existence is not known. We prove in two cases (the reaction diffusion equation and the Cahn-Hilliard equation) the existence of an infinite family of approximate inertial manifolds with increasing order of approximation. Our method is general and can be applied to other equations.

Finally, in the third part, we study a singular perturbation of the Cahn-Hilliard equation in space dimension one obtained by adding a second order derivative in time whose coefficient  $\varepsilon$  is small. We prove the existence of attractors for the perturbed equation. Moreover, the Hausdorff semi distance from these attractors to the attractor of the unperturbed equation converges to zero when  $\varepsilon$  goes to zero.

**Key words:** Inertial manifold, Sacker's equation, elliptic regularization, approximate inertial manifold, Cahn-Hilliard equation, reaction diffusion equation, attractor, singular perturbation.



## PLAN DE LA THESE :

*Introduction.*

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## INTRODUCTION

Dans cette thèse, nous abordons certains aspects de l'étude du comportement pour les grands temps des solutions des équations d'évolution dissipatives. Ce domaine a connu un essor assez grand depuis quelques temps. Ceci est dû, entre autres, à l'apparition d'ordinateurs de plus en plus puissants capables de travailler sur de tels problèmes.

Une façon d'aborder ces équations est de les considérer comme des systèmes dynamiques de dimension infinie. On peut alors essayer de généraliser les concepts qui existent en dimension finie. C'est ainsi qu'est apparue la notion d'attracteur. Un attracteur associé à un système dynamique est un ensemble invariant qui attire toutes les trajectoires de ce système. Si le semi-groupe associé à l'équation d'évolution possède certaines propriétés de compacité et si on sait qu'il existe un ensemble absorbant borné  $B$  alors on peut montrer que l'ensemble oméga limite de  $B$  est un attracteur. Cette méthode générale s'applique à de nombreuses équations (voir [1],[2]). L'attracteur paraît donc être l'objet idéal pour décrire le comportement asymptotique des solutions, malheureusement c'est en général un ensemble dont la structure peut être très compliquée (voir l'attracteur étrange de Ruelle et Takens [3] ou le "Smale's horseshoe" [4],[5]), ce qui empêche son étude pratique.

Pour cette raison, C.Foias, G.R.Sell et R.Temam ont récemment introduit la notion de variété inertielle qui généralise celle de variété centrale. Une variété inertielle est une variété régulière de dimension finie positivement invariante qui attire les orbites du système à vitesse exponentielle. Quand elle existe, une variété inertielle permet de restreindre l'étude du système de dimension infinie à sa projection sur cette variété: la forme inertielle. Les techniques classiques de construction de variété centrale (méthode de Lyapunov-Perron, méthode de Hadamard...) ont pu être généralisées pour plusieurs équations.

Toutefois, l'existence de variété inertielle n'a pas été établie pour certaines équations très importantes. En général ces équations possèdent des variétés inertielles approchées. Une variété inertielle approchée est une variété régulière de dimension finie telle que toutes les solutions du système entrent dans un voisinage très mince de celle-ci en un temps fini. Outre le fait d'exister sous des hypothèses très larges, les variétés inertielles approchées présentent l'avantage d'être données par des formules explicites.

Cette thèse est composée de trois parties dont nous exposons maintenant le contenu.

## **1<sup>ère</sup> partie: Variété inertielle et équation de Sacker.**

Dans cette partie, nous proposons une nouvelle méthode de construction de variété inertielle. L'idée est de résoudre une équation aux dérivées partielles hyperbolique dont la solution possède un graphe positivement invariant (l'équation de Sacker).

Plus précisément, supposons que l'on étudie l'équation parabolique suivante

$$du/dt + Au + f(u) = 0, \quad (1)$$

que l'on suppose bien posée dans un espace de Hilbert  $H$ , où  $A$  désigne un opérateur du type  $(-\Delta)^r$  et  $f$  une nonlinéarité. On cherche une fonction  $\Phi: H_1 \rightarrow H_2$ , où  $H = H_1 + H_2$  et  $\dim H_1 < \infty$ , telle que son graphe soit invariant pour le système dynamique associé à (1). On note  $P_i$  le projecteur sur  $H_i$  ( $i=1,2$ ). En projetant (1) sur  $H_1$  et  $H_2$ , on voit que si  $p(t) + \Phi(p(t))$  est une solution de (1) sur la variété inertielle, on a

$$d\Phi(p(t))/dt + A\Phi(p(t)) + P_2 f(p(t) + \Phi(p(t))) = 0,$$

$$dp/dt + Ap(t) + P_1 f(p(t) + \Phi(p(t))) = 0,$$

et, en éliminant le temps,  $\Phi$  doit vérifier l'équation introduite par R.J. Sacker [6]

$$-D\Phi(p)(Ap + P_1 f(p + \Phi(p))) + A\Phi(p) + P_2 f(p + \Phi(p)) = 0, \quad (2)$$

( $P_1$  et  $P_2$  commutent avec  $A$ ). On cherche les solutions de (2) dont le support est inclus dans une boule de  $P_1 H$ .

Pour cela, on résout d'abord une équation approchée obtenue en remplaçant  $H$  par un sous espace de dimension finie  $m$  et en ajoutant une viscosité artificielle de coefficient  $\varepsilon$ . Cette équation approchée est une équation elliptique nonlinéaire que l'on résout par une méthode de point fixe dans  $W^{1,\infty}$ . Les estimations sont obtenues à partir de principes du maximum. Une grosse difficulté vient de l'application de ces principes du maximum aux dérivées de  $\Phi$  dont on ne connaît pas le comportement à la frontière de la boule. On obtient des estimations suffisantes pour passer à la limite  $\varepsilon \rightarrow 0$  et  $m \rightarrow \infty$  sous une hypothèse assez forte sur l'opérateur  $A$  (hypothèse d'écart spectral). Cette restriction est commune à toutes les méthodes.

## **2<sup>ème</sup> partie: Construction de familles de variétés inertielles approchées.**

La deuxième partie est consacrée à la construction de familles de variétés inertielles

approchées. Ce travail a été fait en collaboration avec M.Marion et fait suite à deux de ses précédents articles [7,8] dans lesquels étaient construites deux variétés inertielles approchées pour l'équation de réaction diffusion et six pour l'équation de Cahn-Hilliard.

On montre pour ces deux équations que la construction peut être généralisée afin d'obtenir une famille infinie de variétés inertielles approchées dont l'ordre d'approximation est croissant. Le principe de la méthode est de décomposer la solution  $u$  de l'équation en  $u(t) = p(t) + q(t)$  où  $p(t)$  représente les grandes structures et  $q(t)$  représente les petites (on a encore décomposé  $H=H_1+ H_2$  avec  $\dim H_1 < \infty$ ). Pour le cas de la réaction diffusion, on obtient en projetant l'équation

$$dp/dt + Ap = P_1 f(p + q), \quad (3)$$

$$dq/dt + Aq = P_2 f(p + q), \quad (4)$$

où  $A = -\Delta$  muni de conditions aux limites, et  $f$  est une nonlinéarité à croissance polynomiale. On montre qu'à partir d'un certain temps  $q$ , ainsi que toutes ses dérivées, est petit. Ce qui légitime l'approximation de (4) par

$$Aq = P_2 f(p). \quad (5)$$

Si on fixe  $p$ , on note  $\Phi_1(p)$  la solution de (5). Le graphe de  $\Phi_1$  est la première variété inertielle approchée. On peut améliorer l'approximation de (4) en introduisant une approximation de  $dq/dt$  obtenue en dérivant (4) et en négligeant  $d^2q/dt^2$ . Pour obtenir  $\Phi_2$  dont le graphe sera la deuxième variété inertielle approchée, on résout successivement

$$p^1 + Ap = P_1 f(p + \Phi_1(p))$$

$$Aq^1 = P_2 f'(p)(p^1),$$

$$q^1 + Aq = P_2 f(p + \Phi_1(p)),$$

et on pose

$$\Phi_2(p) = q.$$

On peut répéter ce procédé, la  $k^{\text{ième}}$  variété inertielle approchée est construite en introduisant une approximation de  $d^{k-1}q/dt^{k-1}$  obtenue en négligeant  $d^k q/dt^k$ . On montre que les solutions de l'équation de réaction diffusion entrent en un temps fini dans un voisinage de la  $k^{\text{ième}}$  variété inertielle approchée de taille  $K_k \delta^k$ , où  $K_k$  ne dépend pas de  $m = \dim H_1$  et  $\delta$  est proportionnel à la  $(m+1)^{\text{ième}}$  valeur propre de  $A$ . Pour l'équation de Cahn-Hilliard, on procède de la même manière, mais pour des raisons techniques la construction et les démonstrations sont légèrement différentes.

### 3<sup>ème</sup> partie: Une perturbation singulière de l'équation de Cahn-Hilliard.

Dans cette troisième partie, on étudie une perturbation singulière de l'équation de Cahn-Hilliard obtenue en ajoutant une dérivée seconde en temps dont le coefficient  $\varepsilon$  est petit

$$\varepsilon \frac{d^2 u}{dt^2} + \nu \Delta^2 u - \Delta f(u) = 0, \quad (6)$$

où  $f$  est un polynôme de degré impair et de coefficient dominant positif. On s'intéresse plus particulièrement à l'existence d'un attracteur pour cette équation et au comportement de celui-ci lorsque  $\varepsilon$  tend vers zéro.

L'équation de Cahn-Hilliard ( $\varepsilon = 0$ ) a déjà été étudiée dans [9], il y est montré qu'il existe un attracteur pour la topologie forte dans  $\{u \in L^2 : |m(u)| \leq \alpha\}$ . Dans ce travail, on montre en utilisant la méthode classique mentionnée auparavant que (6) possède des attracteurs au sens de la topologie faible dans les espaces  $V^s = \{u \in H^s \times H^{s-2} : |m(u)| + |m(du/dt)| \leq \alpha\}$  pour  $s=1,2$  ou  $3$  ( $m(u)$  désigne la moyenne spatiale de  $u$ ). Pour des raisons techniques, nos résultats ne s'appliquent qu'en dimension d'espace  $n=1$ .

On montre que les attracteurs de (6) dans  $V^3$  que l'on note  $A_\varepsilon$  sont bornés indépendamment de  $\varepsilon$  dans  $V^3$ , ces bornes sont assez longues à obtenir et sont issues d'estimations a priori fines sur les solutions de (6). Ensuite, on définit une injection de l'attracteur de l'équation de Cahn-Hilliard (qui est un sous ensemble de  $L^2$ ) dans  $V^2$  que l'on note  $A$  et on montre que la semi distance de Hausdorff de  $A$  à  $A_\varepsilon$  tend vers zéro lorsque  $\varepsilon$  tend vers zéro

$$\lim_{\varepsilon \rightarrow 0} \sup_{(u_\varepsilon, v_\varepsilon) \in A_\varepsilon} \inf_{(u, v) \in A} \|u - u_\varepsilon\|_{H^2} + \|v - v_\varepsilon\|_{L^2} = 0.$$

En d'autres termes, l'attracteur de (6) est semi continu supérieurement en  $\varepsilon=0$ . On emploie le même genre de techniques que dans [10].

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**1<sup>ère</sup> PARTIE:**

**VARIETE INERTIELLE  
ET EQUATION DE SACKER.**





# INERTIAL MANIFOLDS AND SACKER'S EQUATION

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## **I. INTRODUCTION**

Much progress has been made in the study of dissipative evolution equations because they can be considered as infinite dimensional dynamical systems. For instance, the notion of attractor that existed for finite dimensional systems has been extended and many equations have a global attractor which is compact and connected (see e.g. R. Temam [1] or J.K. Hale [1]).

If  $(S(t))_{t \geq 0}$  is the semigroup associated to a dynamical system in a Banach space  $E$ , the global attractor is a compact subset of  $E$  that attracts all the bounded sets of  $E$  :

$$(1.1) \quad \forall B \subset E, B \text{ bounded} : \\ d(S(t)B, \mathcal{A}) \rightarrow 0 \text{ when } t \rightarrow +\infty,$$

moreover it is invariant :

$$(1.2) \quad \forall t \geq 0, S(t)\mathcal{A} = \mathcal{A}.$$

Such a set is interesting in order to describe the behaviour of the orbits of the system for large time. Indeed, we know that in numerous cases the attractor is finite dimensional and we have very good estimates of its dimension that correspond to the physical situation (see R. Temam [1]).

But attractors are not well adapted to practical purposes (especially numerical utilizations) for two reasons :

- they can attract the orbits very slowly ;

- their geometry can be very complicated (perhaps fractals) and they are not regular object.

For those reasons, the notion of inertial manifold has been introduced by C. Foias, G. Sell and R. Temam [1],[2]). It is defined as follows : let  $(S(t))_{t \geq 0}$  be a dynamical system in a Banach space  $E$ , a set  $\mathcal{M}$  is an inertial manifold for this semigroup if :

- (1.3)     -  $\mathcal{M}$  is a Lipschitzian manifold,  
           -  $\mathcal{M}$  is positively invariant :  
                $\forall t \geq 0 \quad , \quad S(t) \mathcal{M} \subset \mathcal{M} \quad ,$   
           -  $\mathcal{M}$  attracts the orbits of  $(S(t))_{t \geq 0}$  with an exponential speed,  
           -  $\mathcal{M}$  is finite dimensional.

When an inertial manifold exists, the system is very well approximated by the inertial system that we obtain by restricting  $(S(t))_{t \geq 0}$  to  $\mathcal{M}$ .

We are interested in an equation of the form :

$$(1.4) \quad \begin{cases} \frac{du}{dt} + Au + R(u) = 0 , \\ u(0) = u_0 , \end{cases}$$

in a Hilbert space  $H$  ; here  $A$  is an unbounded linear operator (for instance  $A = (-\Delta)^r$ ,  $r > 0$ , with boundary conditions) and  $R$  is a locally Lipschitzian non linear function from  $H$  to  $D(A^{-\gamma})$  ( $0 \leq \gamma \leq \frac{1}{2}$ ).

(In fact, we are not able to prove the existence of an inertial manifold in that case, we will have to modify the function  $R$  to avoid problems for large values of the norm of  $u$ ).

There already exist some methods to construct an inertial manifold. They all look for it as the graph of a Lipschitzian function and the main necessary hypothesis is a spectral gap condition on the operator  $A$  ; for instance, we do not know if there exists an inertial manifold for the Navier-Stokes equation (see C. Foias, G. Sell and R. Temam [1],[2] ; C. Foias, B. Nicolaenko, G. Sell and R. Temam [1],[2] ; P. Constantin, C. Foias, B. Nicolaenko and R. Temam [1],[2],[3] ; J. Mallet-Paret and G. Sell [1] ; S.N. Chow and K. Lu [1]). When that spectral gap condition is not satisfied, there often exist approximative inertial manifolds, that object has been introduced recently by C. Foias, O. Manley and R. Temam for the Navier-Stokes equation and has been extended to others (see C. Foias, O. Manley and R. Temam [1],[2] ; Temam [2] ; M. Marion [1],[2]).

In this paper, we are interested in another construction of inertial manifolds based on an equation of R.J. Sacker [1]. In that paper it was observed that if an invariant manifold is the graph of a function  $\Phi$ , then this function must satisfy an hyperbolic equation. Although the article is devoted to finite dimension, the method applies as well in infinite dimension (with a function  $\Phi$  taking its values in

infinite dimension). Our aim in the present article is to show how one can construct an inertial manifold for a broad class of equations by solving this hyperbolic equation. Classically the hyperbolic equation is resolved by elliptic regularization. We are able to derive inertial manifolds in a constructive manner for the same type of equations as in the references quoted above, with the same type of hypotheses.

In a related work M. Luskin and G. Sell [1], E. Fabes, M. Luskin and G. Sell [1],[2] used this method to construct an inertial manifold, but in their article, the condition on the existence are much stronger than in the others references quoted above as far as the spectral gap condition and as far as the type of equation are concerned. In this article we consider the same class of equations as in the other references and recover the *same* type of gap condition, the method being furthermore constructive.

Let us describe the construction of the hyperbolic equation ; we first modify equation (1.4) by replacing the nonlinear term  $R$  by a truncated form  $R_\theta$  that has a compact support to avoid problem with large values of the norm of  $u$  (this will be done precisely in the next section), we search an inertial manifold for the system :

$$(1.5) \quad \frac{du}{dt} + Au + R_\theta(u) = 0 ,$$

where  $u$  lies in a Hilbert space  $H$ .

Let  $P$  be a finite dimensional orthogonal projector and  $Q = \text{Id}_H - P$  (hence we have  $H = PH \oplus QH$ ). We look for a function  $\Phi$  from  $PH$  to  $QH$  whose graph is positively invariant for the equation (1.5). We project (1.5) on  $PH$  and on  $QH$  :

$$(1.6) \quad \frac{dp}{dt} + Ap + PR_\theta(p+q) = 0 ,$$

$$(1.7) \quad \frac{dq}{dt} + Aq + QR_\theta(p+q) = 0 ,$$

where  $p(t) = Pu(t)$  and  $q(t) = Qu(t)$  (we assumed that  $P$  and  $A$  commute, that will be satisfied [see section II]).

If  $u(0)$  is on the graph of  $\Phi$  and if its graph is positively invariant :

$$\forall t \geq 0 : q(t) = \Phi(p(t)) .$$

Therefore :

$$(1.8) \quad \frac{d\Phi(p)}{dt} + A\Phi(p) + QR_\theta(p+\Phi(p)) = 0 .$$

Using (1.6), we have :

$$\begin{aligned}
\frac{d\Phi(p)}{dt} &= D\Phi(p) \left( \frac{dp}{dt} \right) \\
&= D\Phi(p) \left( -Ap - PR_{\theta}(p+\Phi(p)) \right) \\
&= - \left( Ap + PR_{\theta}(p+\Phi(p)) \right) \cdot \nabla \Phi(p) .
\end{aligned}$$

We infer from (1.8) that  $\Phi$  is a solution of :

$$(1.9) \quad - \left( Ap + PR_{\theta}(p+\Phi(p)) \right) \cdot \nabla \Phi + A\Phi + QR_{\theta}(p+\Phi(p)) = 0 .$$

$R_{\theta}$  has a compact support, let  $\rho$  be given such that the ball centered at  $0$  of radius  $\rho/2$  in  $H$  contains its support, we will search  $\Phi$  with a support in the ball centered at  $0$  of radius  $\rho$  in  $PH$  (this is natural since we will construct  $R_{\theta}$  such that its support contains an absorbing set of the equation), thus we are lead to the Dirichlet boundary condition :

$$(1.10) \quad \Phi = 0 \quad \text{on} \quad \partial B(0, \rho) .$$

This construction of a partial differential equation that characterizes invariant surfaces is due to R.J. Sacker [1]. He was interested in a perturbation of a differential equation that possesses an invariant torus, he used that equation to show that the perturbed equation has an invariant surface that is a graph on the torus ; his result is restricted by an assumption that is similar to the spectral gap condition. The method that we will use to solve (1.9) is similar to his but new difficulties appear :

- $\Phi$  lies in an infinite dimensional space (R.J. Sacker only considered the finite dimensional case).
- The a priori estimates are based on a maximum principle that is very easy to use when the quantity that we want to estimate has its maximum inside the domain, this is always the case on a torus but not in our case where the domain (which is a ball) has a boundary. This difficulty appears in lemma 4.4 here after.

In the following sections, we will first give the notation and the precise assumptions (section II), then we will give the existence theorem and a few remarks (section III) ; finally, in section IV, we will prove the theorem.

## II NOTATION AND HYPOTHESES

Let  $H$  be a Hilbert space with the norm  $\| \cdot \|$  and scalar product  $(\cdot, \cdot)$ .

$A$  denotes a linear closed unbounded positive self-adjoint operator in  $H$  with compact inverse ; under these assumptions, there exists an orthonormal basis of  $H$  consisting of eigenvectors of

$A : (w_1, \dots, w_k, \dots)$ , we denote by  $(\lambda_1, \dots, \lambda_k, \dots)$  the associated eigenvalues :

$$(2.1) \quad \begin{cases} \forall i, j \in \mathbb{N}^* : \\ (w_i, w_j) = \delta_{ij} , \\ A w_i = \lambda_i w_i , \\ \text{and } 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow +\infty . \end{cases}$$

Moreover we can define  $A^s$  for every  $s$  in  $\mathbb{R}$ , the domain  $D(A^s)$  when endowed with the norm  $|A^s \cdot|$  is a Hilbert space.

$P$  denotes the orthogonal projector on the space spanned by  $(w_1, \dots, w_N)$  ( $N$  is to be chosen later).

$P_j$  denotes the  $j^{\text{th}}$  coordinate in  $PH$  :

$$(2.2) \quad \forall u \in H : \quad P_j u = (u, w_j) .$$

$Q$  (resp.  $Q_\ell$ ,  $\ell \geq N+1$ ) denotes the orthogonal projector on the space spanned by  $(w_{N+1}, \dots, w_k, \dots)$  (resp.  $(w_{N+1}, \dots, w_\ell)$ ,  $\ell \geq N+1$ ).  $H_\ell = PH \oplus Q_\ell H$  is the space spanned by  $(w_1, w_2, \dots, w_\ell)$ , it is endowed with the norm induced by the norm in  $H$ .

Let us notice that :

$$(2.3) \quad \begin{cases} \forall s > 0 , \quad \forall p \in PH : \lambda_1^s |p| \leq |A^s p| \leq \lambda_N^s |p| , \\ \forall s > 0 , \quad \forall q \in QH : \lambda_{N+1}^s |q| \leq |A^s q| . \end{cases}$$

$R$  is a nonlinear operator from  $H$  to  $D(A^{-\gamma})$  ( $0 \leq \gamma \leq 1/2$ ), Lipschitzian on the bounded sets of  $H$  and continuously differentiable. As we already said, we will replace  $R$  by a truncated form  $R_\theta$ ; in order to do that, we make some further assumptions on the system (1.4). We assume that for every  $u_0$  in  $H$ , (1.4) possesses a unique solution in  $C(\mathbb{R}^+, H) \cap L^2(0, T; D(A^{1/2}))$  for all  $T > 0$ . Moreover the semigroup  $(S(t))_{t \geq 0}$  associated to (S) possesses a bounded absorbing set  $\mathfrak{B}_0$  in  $H$  which is positively invariant (these assumptions are often satisfied, see e.g R. Temam [1] for numerous examples).

Let us choose  $\rho$  such that the ball of  $H$  centered at  $0$  of radius  $\rho/4$  contains  $\mathfrak{B}_0$ , we choose a  $C^\infty$  function  $\theta$  :

$$(2.4) \quad \begin{cases} \theta : \mathbb{R}^+ \rightarrow [0, 1] , \\ \theta(s) = 1 \quad \text{for } s \in [0, \rho/3] , \\ \theta(s) = 0 \quad \text{for } s \in [\rho/2, +\infty[ , \\ \sup_{s \geq 0} |\theta'(s)| \leq 2 . \end{cases}$$

We write :

$$R_\theta(u) = \theta(|u|) R(u) \quad , \quad \forall u \in H .$$

$R_\theta$  is a  $C^1$  function from  $H$  to  $D(A^{-\gamma})$  with a compact support ; thus it is bounded together with its differential.

From now on, we will always consider the truncated equation in which  $R_\theta$  replaces  $R$ , thus we will omit the index  $\theta$  in order to make notation simpler (except in section III.3).

Let us write :

$$(2.5) \quad K_0 = \sup_{u \in H} |A^{-\gamma} R(u)| \quad ,$$

$$(2.6) \quad K_1 = \sup_{u \in H} \sup_{\substack{v \in H \\ |v|=1}} |A^{-\gamma} DR(u)(v)| \quad .$$

$\Omega$  will be the ball in  $PH$  centered at  $0$  and of radius  $\rho$  :

$$\Omega = B_{PH}(0, \rho) \quad .$$

Now we define some functional spaces that we will need :

$H_0^1(\Omega, Q_\ell H)$  is the space of the functions from  $\Omega$  to  $Q_\ell H$  which are  $L^2$  together with their first partial derivatives and which equal zero on  $\partial\Omega$ , it is a Hilbert space when endowed with the norm :

$$(2.7) \quad |\Phi|_{H_0^1} = \left( |A^{1/2} \Phi|_{L^2}^2 + |\nabla \Phi|_{L^2}^2 \right)^{1/2} \quad ,$$

where

$$|A^{1/2} \Phi|_{L^2}^2 = \int_{\Omega} |A^{1/2} \Phi(p)|^2 dp \quad .$$

(this norm is equivalent to the usual one :  $|\Phi|_{L^2}^2 = \int_{\Omega} |\Phi(p)|^2 dp$  on  $L^2(\Omega, Q_\ell H)$  since  $Q_\ell H$  is a

finite dimensional space ; of course, this is no longer true on  $L^2(\Omega, QH)$ )

and

$$|\nabla \Phi|_{L^2}^2 = \sum_{i=1}^N \int_{\Omega} |D_i \Phi(p)|^2 dp \quad .$$

( $p$  is a variable in  $\Omega$  which is embedded in  $\mathbb{P}H$ ; we write  $p = \sum_{i=1}^{N-1} p_i w_i$ ,  $D_i \Phi(p)$  is the partial derivative of  $\Phi$  with respect to  $p_i$ ).

$L^\infty(\Omega, QH)$  (resp.  $L^\infty(\Omega, Q_\ell H)$ ,  $L^\infty(\Omega, QD(A^{1/2}))$ ) is the space of the essentially bounded functions from  $\Omega$  to  $QH$  (resp.  $Q_\ell H$ ,  $QD(A^{1/2})$ ), it is a Banach space when endowed with the norm :

$$\begin{aligned} |\Phi|_{L^\infty} &= \text{ess sup}_{p \in \Omega} |\Phi(p)| \\ (\text{resp. } |\Phi|_{L^\infty} &= \text{ess sup}_{p \in \Omega} |\Phi(p)| \\ &|A^{1/2}\Phi|_{L^\infty} = \text{ess sup}_{p \in \Omega} |A^{1/2}\Phi(p)|) \end{aligned}$$

$W^{1,\infty}(\Omega, QH)$  (resp.  $W^{1,\infty}(\Omega, Q_\ell H)$ ) is the space of functions in  $L^\infty(\Omega, QH)$  (resp.  $L^\infty(\Omega, Q_\ell H)$ ) whose gradients are in  $L^\infty(\Omega, QH)^N$  (resp.  $L^\infty(\Omega, Q_\ell H)^N$ ), it is a Banach space when endowed with the norm :

$$(2.8) \quad |\Phi|_{W^{1,\infty}} = |\Phi|_{L^\infty} + |D\Phi|_{L^\infty},$$

where :

$$|D\Phi|_{L^\infty} = \text{ess sup}_{p \in \Omega} \sup_{h \in \mathbb{R}^N} \left( \frac{|h \cdot D\Phi(p)|}{|h|} \right),$$

and for  $h = (h_1, \dots, h_N)$  in  $\mathbb{R}^N$

$$h \cdot D\Phi(p) = \sum_{i=1}^N h_i D_i \Phi(p) \quad , \quad |h| = \left( \sum_{i=1}^N h_i^2 \right)^{1/2}$$

We will use the subspaces of  $W^{1,\infty}(\Omega, Q_\ell H)$  and  $W^{1,\infty}(\Omega, QH)$  :

$$\begin{aligned} V_\ell &= \left\{ \Phi \in W^{1,\infty}(\Omega, Q_\ell H) : |\Phi|_{L^\infty} \leq M_0 \text{ and } |D\Phi|_{L^\infty} \leq M_1 \right\}, \\ V &= \left\{ \Phi \in W^{1,\infty}(\Omega, QH) : |\Phi|_{L^\infty} \leq M_0 \text{ and } |D\Phi|_{L^\infty} \leq M_1 \right\}, \end{aligned}$$

( $M_0$  and  $M_1$  will be chosen later).

This section ends with the assumptions on the spectrum of  $A$ .

We choose  $M_0$  such that :



$$(2.9) \quad K_0 \lambda_{N+1}^{\gamma-1} \leq M_0 ,$$

(let us recall that  $K_0 = \sup_{u \in H} |A^{-\gamma} R(u)|$ ) and we assume that there exists  $N$  such that :

$$(2.10) \quad \lambda_{N+1} - \lambda_N \geq K_1 (\lambda_N^{\gamma/2} + \lambda_{N+1}^{\gamma/2})^2 ,$$

(let us recall that  $K_1 = \sup_{u \in H} |A^{-\gamma} DR(u)|_{\mathcal{L}(H)}$ ).

This is a necessary and sufficient condition to the existence of a real  $M_1$  such that :

$$\text{and} \quad \begin{cases} M_1 > 0 , \\ -K_1 \lambda_N^\gamma M_1^2 + M_1 (\lambda_{N+1} - \lambda_N - K_1 (\lambda_N^\gamma + \lambda_{N+1}^\gamma)) - K_1 \lambda_{N+1}^\gamma \geq 0 , \end{cases}$$

that implies :

$$(2.11) \quad \begin{cases} \lambda_{N+1} - \lambda_N - K_1(1+M_1) \lambda_N^\gamma > 0 , \\ \frac{K_1(1+M_1) \lambda_{N+1}^\gamma}{\lambda_{N+1} - \lambda_N - K_1(1+M_1) \lambda_N^\gamma} \leq M_1 . \end{cases}$$

We make the further assumption that we can choose  $M_1$  such that :

$$(2.13) \quad \left( \sum_{i=1}^N \lambda_i \right) \geq K_1(1+M_1) \left( \sum_{i=1}^N \lambda_i^\gamma \right) ,$$

$$(2.14) \quad \lambda_{N+1}^{\gamma-1} K_1(1+M_1) < 1 .$$

### III MAIN RESULT

We state the main result, this is an existence theorem for the inertial manifold (the proof will be given in section IV). This section ends with some comments on the results.

1) **THEOREM 3.1** : *Under the assumptions made in section II, there exists a function  $\Phi$  in  $W^{1,\infty}(\Omega, \text{QH}) \cap L^\infty(\Omega, \text{QD}(A^{1/2}))$  solution of :*

$$(3.1) \quad \begin{cases} - (Ap + PR(p + \Phi(p))) \cdot \nabla \Phi + A\Phi + QR(p + \Phi(p)) = 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, if we extend  $\Phi$  outside  $\Omega$  by setting :  $\Phi|_{\text{PH}\Omega} = 0$  , then its graph is an inertial manifold for the system :

$$\frac{du}{dt} + Au + R(u) = 0 .$$

2) (2.10) is a spectral gap condition, this is the most restricting condition among (2.10)-(2.14) ; for instance if  $\lambda_N \sim c N^\alpha$  (this is the case when  $A = (-\Delta)^\Gamma$  on a n-cube in  $\mathbb{R}^n$  with periodic boundary conditions, then  $\alpha = \frac{2\Gamma}{n}$  (see R. Courant and D. Hilbert [1])), in order to satisfy (2.10) we need  $\alpha > \frac{1}{1-\gamma}$  and (2.13), (2.14) are then easily checked. If we consider the Navier-Stokes equation, we have  $\gamma = \frac{1}{2}$  and  $\alpha = \frac{2}{n}$  , thus our result is not sufficient in this case.

3) Theorem 3.1 gives the existence of an inertial manifold for the truncated system :

$$(3.2) \quad \frac{du}{dt} + Au + R_\theta(u) = 0 .$$

Consider an orbit  $\{u(t)\}_{t \geq 0}$  of the initial system :

$$(3.3) \quad \frac{du}{dt} + Au + R(u) = 0 ,$$

as  $R_\theta$  and  $R$  are equal on the absorbing set of (3.3), there exists a time  $t_0$  such that :

$$\forall t \geq t_0 , R(u(t)) = R_\theta(u(t)) ,$$

thus  $\{u(t)\}_{t \geq t_0}$  is an orbit of (3.2) and if we denote by  $\mathcal{M}$  the inertial manifold of system (3.2) :

$$\begin{aligned} \text{either} \quad & u(t) \in \mathcal{M} \quad \text{for } t \geq t_0 \\ \text{or} \quad & d(u(t), \mathcal{M}) \text{ decays exponentially.} \end{aligned}$$

4) (3.1) is an hyperbolic equation in the unknown  $\Phi$  , it is quite natural to try the method of characteristics to solve it ; we see the term  $-(Ap + PR(p + \Phi)) \cdot \nabla \Phi$  as a directional derivative. We introduce the curves :

$$\begin{cases} \frac{dp}{dt} = - (Ap + PR(p + \Phi)) , \\ p(0) = p_0 , \end{cases}$$

on these curves, we have :

$$\frac{d\Phi(p(t))}{dt} + A\Phi(p(t)) + QR(p(t) + \Phi(p(t))) = 0 ,$$

which can be integrated :

$$\begin{aligned} \Phi(p(t)) = & - \int_{t_0}^t e^{A(\sigma-t)} QR(p(\sigma) + \Phi(p(\sigma))) d\sigma \\ & + e^{A(t_0-t)} \Phi(p(t_0)) . \end{aligned}$$

Letting  $t_0$  go to  $-\infty$  (assuming  $\Phi$  is bounded) and taking  $t$  equals 0 :

$$\Phi(p_0) = - \int_{-\infty}^0 e^{A\sigma} QR(p(\sigma) + \Phi(p(\sigma))) d\sigma .$$

And we can solve this integral equation by a fixed-point method.

This is the principle of the Lyapunov-Perron method developed by C. Foias, G. Sell and R. Temam [1],[2] (see also R. Temam [1] chap. 8 and S.N. Chow and K. Lu [1]).

#### IV PROOF OF THEOREM 3.1

In order to prove theorem 3.1, we will first solve a Galerkin approximation of equation 3.1 (which is an infinite dimensional hyperbolic system, whose unknowns are the coordinates of  $\Phi$  in the base  $(w_{N+1}, \dots, w_k, \dots)$ ).

Let  $\ell \geq N+1$ , we search  $\Phi_\ell$  from  $\Omega$  to  $Q_\ell H$  solution of :

$$\begin{cases} - (Ap + PR(p + \Phi_\ell(p))).\nabla\Phi_\ell + A\Phi_\ell + Q_\ell R(p + \Phi_\ell(p)) = 0 & \text{in } \Omega , \\ \Phi_\ell = 0 & \text{on } \partial\Omega . \end{cases}$$

This is done by regularizing this system with a second order elliptic term that tends to zero. Under the assumptions made in section II (especially (2.10)), we can find a solution of the regularized system through a fixed point method in a subset of  $W^{1,\infty}(\Omega, Q_\ell H)$ , in the same time we obtain a priori estimates that enable us to pass to the limit to obtain a solution of the Galerkin approximation of equation (3.1), then to make  $\ell \rightarrow +\infty$  and thus to have a solution of equation (3.1).

LEMMA 4.1 : Let  $\psi \in V_\ell$ , then there exists a unique  $\Phi$  in  $H_0^1(\Omega, Q_\ell H)$  such that :

$$(4.1) \quad \begin{cases} \varepsilon \Delta\Phi + (Ap + PR(p+\psi)).\nabla\Phi - A\Phi - Q_\ell R(p+\psi) = 0 , \\ \Phi = 0 & \text{on } \partial\Omega . \end{cases}$$

We write  $\Phi = \tau_{\varepsilon, \ell} \psi$ .

(Let us recall that  $\forall \ell = \left\{ \Phi \in W^{1, \infty}(\Omega, Q_\ell H) / \|\Phi\|_{L^\infty} \leq M_0 \text{ and } \|D\Phi\|_{L^\infty} \leq M_1 \right\}$ ).

**Proof**

Let  $a(\Phi_1, \Phi_2)$  be the bilinear continuous form on  $H_0^1(\Omega, Q_\ell H)$  defined by :

$$\begin{aligned} a(\Phi_1, \Phi_2) &= \varepsilon \int_{\Omega} \sum_{i=1}^N (D_i \Phi_1, D_i \Phi_2) \, dp \\ &\quad - \int_{\Omega} ((A p + PR(p+\psi)) \cdot \nabla \Phi_1, \Phi_2) \, dp \\ &\quad + \int_{\Omega} (A^{1/2} \Phi_1, A^{1/2} \Phi_2) \, dp, \end{aligned}$$

and  $L$  the linear continuous form on  $H_0^1(\Omega, Q_\ell H)$  :

$$L(\Phi_2) = - \int_{\Omega} (A^{-\gamma} Q_\ell R(p+\psi), A^\gamma \Phi_2) \, dp.$$

We claim that  $a$  is coercive :

Let  $\Phi \in H_0^1(\Omega, Q_\ell H)$ ,

$$\begin{aligned} a(\Phi, \Phi) &= \varepsilon \int_{\Omega} \sum_{i=1}^N |D_i \Phi|^2 \, dp \\ &\quad - \int_{\Omega} ((A p + PR(p+\psi)) \cdot \nabla \Phi, \Phi) \, dp \\ &\quad + \int_{\Omega} |A^{1/2} \Phi|^2 \, dp, \end{aligned}$$

We observe that :

$$\begin{aligned} &- \int_{\Omega} ((A p + PR(p+\psi)) \cdot \nabla \Phi, \Phi) \, dp \\ &= - \frac{1}{2} \int_{\Omega} (A p + PR(p+\psi)) \cdot \nabla |\Phi|^2 \, dp \\ &= \frac{1}{2} \int_{\Omega} \operatorname{div} (A p + PR(p+\psi)) |\Phi|^2 \, dp, \end{aligned}$$

and this is positive since :

$$\begin{aligned}
\operatorname{div} (A p + P R(p+\psi)) &= \sum_{i=1}^N D_i (\lambda_i p_i + P_i R(p+\psi)) \\
&= \sum_{i=1}^N \lambda_i + P_i D R(p+\psi) (w_i + D_i \psi) \\
&= \sum_{i=1}^N \lambda_i + \sum_{i=1}^N \lambda_i^{-\gamma} P_i D R(p+\psi) (\lambda_i^{\gamma} (w_i + D_i \psi)) \\
&\geq \sum_{i=1}^N \lambda_i - \sum_{i=1}^N |A^{-\gamma} P D R(p+\psi) (\lambda_i^{\gamma} (w_i + D_i \psi))| \\
&\geq \sum_{i=1}^N \lambda_i - K_1 \sum_{i=1}^N |\lambda_i^{\gamma} (w_i + D_i \psi)| \\
&\geq \sum_{i=1}^N \lambda_i - K_1 \left( \sum_{i=1}^N \lambda_i^{\gamma} \right) (1+M_1) .
\end{aligned}$$

(Let us recall that

$$K_1 = \sup_{u \in H} |A^{-\gamma} D R(u)|_{\mathcal{L}(H)}$$

and

$$\begin{aligned}
|D\psi|_{L^\infty} &= \operatorname{ess\,sup}_{p \in \Omega} \sup_{h \in \mathbb{R}^N} \frac{\left| \sum_{i=1}^N h_i D_i \psi \right|}{\left( \sum_{i=1}^N h_i^2 \right)^{1/2}} \\
&\leq M_1 .
\end{aligned}$$

Thus :

$$a(\Phi, \Phi) \geq \inf(\varepsilon, 1) |\Phi|_{H_0^1}^2 .$$

By the Lax-Milgram theorem, there exists a unique  $\Phi$  in  $H_0^1(\Omega, Q_\ell H)$  such that :

$$a(\Phi, \Phi_1) = L(\Phi_1) \text{ for every } \Phi_1 \text{ in } H_0^1(\Omega, Q_\ell H) .$$

Taking  $\Phi_1 \in C^\infty$  with a compact support, we have :

$$\varepsilon \Delta \Phi + (A p + P R(p+\psi)) \cdot \nabla \Phi - A \Phi - Q_\ell R(p+\psi) = 0 ,$$

in the sense of distributions, that implies that  $\Phi$  is in  $H^2(\Omega, Q_\ell H)$  and that (4.1) is satisfied almost everywhere in  $\Omega$ .

**LEMMA 4.2** (Continuity of  $\mathcal{T}_{\varepsilon, \ell}$ ) .

If  $V_\ell$  is endowed with the norm of  $L^\infty(\Omega, Q_\ell H)$ , then

$$\mathcal{T}_{\varepsilon, \ell} : V_\ell \rightarrow H_0^1(\Omega, Q_\ell H)$$

is continuous.

Proof

Let  $(\psi_n)_{n \in \mathbb{N}}$  be a sequence in  $V_\ell$  such that

$$\psi_n \rightarrow \psi \text{ in } L^\infty(\Omega, Q_\ell H).$$

We write  $\Phi_n = \tau_{\varepsilon, \ell} \psi_n$  and  $\Phi = \tau_{\varepsilon, \ell} \psi$  and subtract the equations satisfied by  $\Phi_n$  and  $\Phi$  :

$$\begin{aligned} & \varepsilon \Delta(\Phi_n - \Phi) + (A p + PR(p + \psi_n)) \cdot \nabla(\Phi_n - \Phi) \\ & + (PR(p + \psi_n) - PR(p + \psi)) \cdot \nabla \Phi - A(\Phi_n - \Phi) \\ & - (Q_\ell R(p + \psi_n) - Q_\ell R(p + \psi)) = 0. \end{aligned}$$

We take the scalar product with  $\Phi_n - \Phi$  in  $Q_\ell H$  and integrate over  $\Omega$  (by parts in the two first terms, there are no integral over  $\partial\Omega$  since  $\Phi_n - \Phi$  is in  $H_0^1(\Omega, Q_\ell H)$ ) :

$$\begin{aligned} & -\varepsilon \int_\Omega |\nabla(\Phi_n - \Phi)|^2 dp - \frac{1}{2} \int_\Omega \operatorname{div}(A p + PR(p + \psi_n)) |\Phi_n - \Phi|^2 dp \\ & + \int_\Omega ((PR(p + \psi_n) - PR(p + \psi)) \cdot \nabla \Phi, \Phi_n - \Phi) dp \\ & - \int_\Omega |A^{1/2}(\Phi_n - \Phi)|^2 dp \\ & - \int_\Omega (A^{-\gamma}(Q_\ell R(p + \psi_n) - Q_\ell R(p + \psi)), A^\gamma(\Phi_n - \Phi)) dp \\ & = 0. \end{aligned}$$

The second term is negative since :

$$\operatorname{div}(A p + PR(p + \psi_n)) \geq 0 \quad (\text{see lemma 1}).$$

Thus :

$$\begin{aligned} & -\varepsilon \int_\Omega |\nabla(\Phi_n - \Phi)|^2 dp + \int_\Omega |A^{1/2}(\Phi_n - \Phi)|^2 dp \\ & \leq \int_\Omega ((PR(p + \psi_n) - PR(p + \psi)) \cdot \nabla \Phi, \Phi_n - \Phi) dp \\ & - \int_\Omega (A^{-\gamma}(Q_\ell R(p + \psi_n) - Q_\ell R(p + \psi)), A^\gamma(\Phi_n - \Phi)) dp \\ & \leq \lambda_N^\gamma \sup_{p \in \Omega} |A^{-\gamma}(PR(p + \psi_n) - PR(p + \psi))| \|\nabla \Phi\|_{L^2} \|\Phi_n - \Phi\|_{L^2} \\ & + \lambda_{N+1}^{\gamma-1/2} \sup_{p \in \Omega} |A^{-\gamma}(Q_\ell R(p + \psi_n) - Q_\ell R(p + \psi))| (\operatorname{mes} \Omega)^{1/2} \|A^{1/2}(\Phi_n - \Phi)\|_{L^2} \\ & \leq K_1 \left( \|\nabla \Phi\|_{L^2} + \lambda_{N+1}^{\gamma-1/2} \operatorname{mes}(\Omega)^{1/2} \|\psi_n - \psi\|_{L^\infty} \|A^{1/2}(\Phi_n - \Phi)\|_{L^2} \right). \end{aligned}$$

Therefore if  $|\psi_n - \psi| \rightarrow 0$  then

$$|A^{1/2}(\Phi_n - \Phi)|_{L^2} \rightarrow 0$$

and

$$|\nabla(\Phi_n - \Phi)|_{L^2} \rightarrow 0 .$$

**LEMME 4.3** ( $L^\infty$  estimate)

Let  $\psi$  in  $V_\ell$  and  $\Phi = \mathcal{T}_{\varepsilon, \ell}$  then  $\Phi \in L^\infty(\Omega, Q_\ell H)$  and

$$(4.3) \quad |\Phi|_{L^\infty} \leq M_0 .$$

**Proof**

$\psi$  is a Lipschitzian function (it is in  $W^{1,\infty}(\Omega, Q_\ell H)$ ), hence it is in  $C^{0,s}(\Omega, Q_\ell H)$  for all  $s < 1$  ( $C^{0,s}(\Omega, Q_\ell H)$  is the space of Hölderian continuous function of order  $s$ ).

$R$  is a  $C^1$  function, thus the coefficients of equation (4.1) are in  $C^{0,s}(\Omega, Q_\ell H)$ , and thanks to the classical results on the regularity of the elliptic equations (see D. Gilbarg and N.S. Trudinger [1]) :

$$\Phi \in C^{2,s}(\Omega, Q_\ell H) .$$

We deduce that (4.1) is verified everywhere.

Let us take the scalar product of (4.1) with  $\Phi$ , and use :

$$(4.4) \quad (\Delta \Phi, \Phi) = \frac{1}{2} \Delta |\Phi|^2 - |\nabla \Phi|^2 ,$$

$$(4.5) \quad (\nabla \Phi, \Phi) = \frac{1}{2} \nabla |\Phi|^2 .$$

We obtain :

$$\begin{aligned} & \frac{\varepsilon}{2} \Delta |\Phi|^2 - \varepsilon |\nabla \Phi|^2 + \frac{1}{2} (A p + P R(p + \psi)) \cdot \nabla |\Phi|^2 \\ & - |A^{1/2} \Phi|^2 - (A^{-\gamma} Q_\ell R(p + \psi), A^\gamma \Phi) = 0 . \end{aligned}$$

At the point  $\bar{p}$  where  $|\Phi|^2$  is maximum, we have :

$$\Delta |\Phi(\bar{p})|^2 = 0 \quad \text{and} \quad \nabla |\Phi(\bar{p})|^2 = 0 .$$

( $\bar{p}$  is in the interior of  $\Omega$ , since  $\Phi = 0$  on  $\partial\Omega$ ).

Hence :

$$\begin{aligned} |A^{1/2} \Phi(\bar{p})|^2 & \leq - (A^{-\gamma} Q_\ell R(\bar{p} + \psi(\bar{p})), A^\gamma \Phi(\bar{p})) \\ & \leq |A^{-\gamma} Q_\ell R(\bar{p} + \psi(\bar{p}))| |A^\gamma \Phi(\bar{p})| \\ & \leq K_0 \lambda_{N+1}^{\gamma-1/2} |A^{1/2} \Phi(\bar{p})| , \end{aligned}$$

thanks to (2.5).

We deduce :

$$\begin{aligned} |\Phi|_{L^\infty} &= |\Phi(\bar{p})| \leq \lambda_{N+1}^{-1/2} |A^{1/2} \Phi(\bar{p})| \\ &\leq \lambda_{N+1}^{\gamma-1} K_0 , \\ |\Phi|_{L^\infty} &\leq M_0 , \end{aligned}$$

thanks to (2.9).

**LEMMA 4.4** (*W<sup>1,∞</sup> estimate*)

Let  $\psi$  in  $V_\ell$  and  $\Phi = \tau_{\varepsilon,\ell} \psi$  then  $\Phi \in W^{1,\infty}(\Omega, Q_\ell H)$  and

$$|D\Phi|_{L^\infty} \leq M_1 .$$

Using lemma 4.3, we have :

$$\tau_{\varepsilon,\ell} V_\ell \subset V_\ell .$$

**Proof**

We want to use the same method as in lemma 4.3 to obtain estimates on the derivatives of  $\Phi$ , we will differentiate equation (4.1) and use the same type of arguments. But two difficulties appear :

- we do not know whether the derivatives of  $\Phi$  are sufficiently regular
- the maximum of the derivatives of  $\Phi$  can occur on the boundary.

The first difficulty is easily overcome by approximating  $\psi$  with a sequence of more regular functions.

The second is more difficult to overcome, it is done by looking carefully what happens on the boundary if the maximum occurs there ; the arguments employed are hidden by technical calculus, thus we will first describe the method in a simpler case.

We now begin with the proof.

We first assume that  $\psi$  is in  $C^1(\Omega, Q_\ell H)$  then we have :

$$(4.7) \quad \Phi \in C^3(\Omega, Q_\ell H) .$$

(See D. Gilbarg and N.S. Trudinger [1]).

We differentiate (4.1) with respect to  $p_i$  ( $1 \leq i \leq N$ ) :

$$(4.8) \quad \begin{cases} \varepsilon \Delta D_i \Phi + (A p_i + P R(p+\psi)) \cdot \nabla D_i \Phi + \lambda_i D_i \Phi \\ + \sum_{j=1}^N P_j D R(p+\psi) (w_i + D_i \psi) D_j \Phi \\ - A D_i \Phi - Q_\ell D R(p+\psi) (w_i + D_i \psi) = 0 . \end{cases}$$

Thanks to (4.7), (4.8) is satisfied everywhere in  $\Omega$ .



We choose  $h = (h_1, \dots, h_N)$  in  $\mathbb{R}^N$ , multiply (4.8) by  $h_i$ , make the sum for  $i = 1, \dots, N$  and take the scalar product with  $h.D\Phi = \sum_{i=1}^N h_i D_i \Phi$ , using the analog of (4.4), (4.5) :

$$\begin{aligned} (\Delta h.D\Phi, h.D\Phi) &= \frac{1}{2} \Delta |h.D\Phi|^2 - |\nabla h.D\Phi|^2, \\ (\nabla h.D\Phi, h.D\Phi) &= \frac{1}{2} \nabla |h.D\Phi|^2, \end{aligned}$$

we obtain :

$$\begin{aligned} (4.9) \quad & \frac{\varepsilon}{2} \Delta |h.D\Phi|^2 - \varepsilon |\nabla h.D\Phi|^2 + \frac{1}{2} (A\rho + PR(\rho+\psi)).\nabla |h.D\Phi|^2 \\ & + \left( \sum_{i=1}^N \lambda_i h_i D_i \Phi, h.D\Phi \right) \\ & + \left( \sum_{i,j=1}^N h_i P_j DR(\rho+\psi) (w_i + D_i \psi) D_j \Phi, h.D\Phi \right) \\ & - |A^{1/2} h.D\Phi|^2 - \left( \sum_{i=1}^N h_i Q_\ell DR(\rho+\psi) (w_i + D_i \psi), h.D\Phi \right) \\ & = 0. \end{aligned}$$

We will consider two cases :

• 1st case :  $|h.D\Phi|^2$  attains its maximum at a point  $\bar{p}$  in the interior of  $\Omega$ , then we have :

$$(4.10) \quad \Delta |h.D\Phi(\bar{p})|^2 \leq 0,$$

$$(4.11) \quad \nabla |h.D\Phi(\bar{p})|^2 = 0.$$

We estimate the other terms in (4.9) at the point  $\bar{p}$  :

$$\begin{aligned} (4.12) \quad & \left( \sum_{i,j=1}^N h_i P_j DR(\bar{p}+\psi) (w_i + D_i \psi) D_j \Phi, h.D\Phi \right) \\ & = \left( \sum_{j=1}^N P_j DR(\bar{p}+\psi) \left( \sum_{i=1}^N h_i (w_i + D_i \psi) \right) D_j \Phi, h.D\Phi \right) \\ & \leq \left| \sum_{j=1}^N P_j DR(\bar{p}+\psi) \left( \sum_{i=1}^N h_i (w_i + D_i \psi) \right) D_j \Phi \right| |h.D\Phi| \\ & \leq |P DR(\bar{p}+\psi) \left( \sum_{i=1}^N h_i (w_i + D_i \psi) \right)| |D\Phi|_{L^\infty} |h.D\Phi| \\ & \leq \lambda_N^\gamma K_1 \left| \sum_{i=1}^N h_i (w_i + D_i \psi) \right| |D\Phi|_{L^\infty} |h.D\Phi| \\ & \leq \lambda_N^\gamma K_1 |h| (1 + |D\psi|_{L^\infty}) |D\Phi|_{L^\infty} |h.D\Phi| \\ & \leq \lambda_N^\gamma K_1 |h| (1 + M_1) |D\Phi|_{L^\infty} |h.D\Phi|. \end{aligned}$$

$$\begin{aligned}
(4.13) \quad \left( \sum_{i=1}^N \lambda_i h_i D_i \Phi, h.D\Phi \right) &\leq \left| \sum_{i=1}^N \lambda_i h_i D_i \Phi \right| |h.D\Phi| \\
&\leq \left( \sum_{i=1}^N \lambda_i^2 h_i^2 \right)^{1/2} |D\Phi|_{L^\infty} |h.D\Phi| \\
&\leq \lambda_N |h| |D\Phi|_{L^\infty} |h.D\Phi| .
\end{aligned}$$

$$\begin{aligned}
&|A^{1/2} h.D\Phi|^2 + \left( \sum_{i=1}^N h_i Q_\ell DR(\bar{p}+\psi)(w_i+D_i\psi), h.D\Phi \right) \\
&\geq |A^{1/2} h.D\Phi|^2 - |A^{-\gamma} Q_\ell DR(\bar{p}+\psi) \left( \sum_{i=1}^N h_i (w_i+D_i\psi) \right)| |A^\gamma h.D\Phi| \\
&\geq |A^{1/2} h.D\Phi|^2 - K_1 \left| \sum_{i=1}^N h_i (w_i+D_i\psi) \right| |A^\gamma h.D\Phi| \\
&\geq |A^{1/2} h.D\Phi|^2 - K_1 \lambda_{N+1}^{\gamma-1/2} |h| (1+M_1) |A^{1/2} h.D\Phi| .
\end{aligned}$$

If  $\lambda_{N+1}^{1/2} |h.D\Phi| \geq \frac{K_1}{2} \lambda_{N+1}^{\gamma-1/2} |h| (1+M_1)$ , as the function  $x^2 - K_1 \lambda_{N+1}^{\gamma-1/2} |h| (1+M_1) x$  is increasing when  $x \geq \frac{K_1}{2} \lambda_{N+1}^{\gamma-1/2} |h| (1+M_1)$ , we have :

$$\begin{aligned}
(4.14) \quad &|A^{1/2} h.D\Phi|^2 + \left( \sum_{i=1}^N h_i Q_\ell DR(\bar{p}+\psi)(w_i+D_i\psi), h.D\Phi \right) \\
&\geq \lambda_{N+1} |h.D\Phi|^2 - K_1 \lambda_{N+1}^\gamma |h| (1+M_1) |h.D\Phi| .
\end{aligned}$$

(4.10) to (4.14) with (4.9) give :

$$\begin{aligned}
\lambda_{N+1} |h.D\Phi|^2 &\leq \lambda_N |h| |D\Phi|_{L^\infty} |h.D\Phi| \\
&\quad + \lambda_N^\gamma K_1 |h| (1+M_1) |D\Phi|_{L^\infty} |h.D\Phi| \\
&\quad + \lambda_{N+1}^\gamma K_1 |h| (1+M_1) |h.D\Phi| ,
\end{aligned}$$

$$(4.15) \quad \lambda_{N+1} |h.D\Phi| \leq |h| \left( \lambda_N |D\Phi|_{L^\infty} + K_1 \lambda_N^\gamma (1+M_1) |D\Phi|_{L^\infty} + \lambda_{N+1}^\gamma K_1 (1+M_1) \right) .$$

If  $\lambda_{N+1}^{1/2} |h.D\Phi| \leq \frac{K_1}{2} \lambda_{N+1}^{\gamma-1/2} |h| (1+M_1)$ , then  $\lambda_{N+1} |h.D\Phi| \leq \frac{K_1}{2} \lambda_{N+1}^\gamma |h| (1+M_1)$  and (4.15) is still satisfied. As  $|h.D\Phi|$  attains its maximum at  $\bar{p}$ , we have :

$$(4.16) \quad \left\{ \begin{array}{l} \forall p \in \Omega : \\ \lambda_{N+1} |h \cdot D\Phi| \leq |h| \left( \left( \lambda_N + K_1 \lambda_N^\gamma (1+M_1) \right) \|D\Phi\|_{L^\infty} + \lambda_{N+1}^\gamma K_1 (1+M_1) \right) . \end{array} \right.$$

• 2nd case :  $|h \cdot D\Phi|$  attains its maximum at a point  $\bar{p}$  on the boundary.

In order to describe the arguments we will use, we first give a heuristic proof in a simpler case : we replace  $\Omega$  by the  $N$ -cube  $[-\rho, \rho]^N$  (the following arguments are not valid since we do not have enough regularity in such a domain).

$\bar{p} = (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_N)$  is on the boundary if there exists  $i$  such that  $|\bar{p}_i| = \rho$ , we assume, and this is no restriction, that  $\bar{p}_1 = \rho$ .

$\Phi$  is constant on the boundary thus all tangential derivatives are zero at  $\bar{p}$  :

$$\begin{aligned} D_i \Phi(\bar{p}) &= 0 \quad \text{if } i \neq 1, \\ |h \cdot D\Phi(\bar{p})| &= |h_1 D_1 \Phi(\bar{p})| \\ &\leq |h| |D_1 \Phi(\bar{p})|. \end{aligned}$$

We claim that  $|D_1 \Phi|$  cannot attain its maximum on the boundary, thus we can use the estimate (4.16) of the first case with  $h = (1, 0, \dots, 0)$  :

$$\begin{aligned} \lambda_{N+1} |D_1 \Phi(\bar{p})| &\leq \left( \left( \lambda_N + K_1 \lambda_N^\gamma (1+M_1) \right) \|D\Phi\|_{L^\infty} + \lambda_{N+1}^\gamma K_1 (1+M_1) \right), \\ \lambda_{N+1} |h \cdot D\Phi(\bar{p})| &\leq \lambda_{N+1} |h| |D_1 \Phi(\bar{p})| \\ &\leq |h| \left( \left( \lambda_N + K_1 \lambda_N^\gamma (1+M_1) \right) \|D\Phi\|_{L^\infty} + \lambda_{N+1}^\gamma K_1 (1+M_1) \right). \end{aligned}$$

And (4.16) is valid. We have to prove the claim ; assume that  $|D_1 \Phi|$  attains its maximum at  $p_0$  on the boundary,  $p_0$  must be on the face  $|p_1| = \rho$  (otherwise  $|D_1 \Phi(p_0)| = 0$ ), assume that  $p_{0,1} = \rho$ , then we have :

$$\begin{aligned}
D_i \Phi(p_0) &= 0 & , \text{ if } i \neq 1 , \\
D_{ii} \Phi(p_0) &= 0 & , \text{ if } i \neq 1 , \\
R(p_0 + \psi(p_0)) &= 0 & , \text{ since } |p_0 + \psi(p_0)| \geq |p_0| \geq \rho , \\
\Phi(p_0) &= 0 & , \text{ since } p_0 \text{ is on the boundary .}
\end{aligned}$$

Using equation (4.1) :

$$(4.17) \quad \varepsilon D_{1,1} \Phi(p_0) + \lambda_1 \rho D_1 \Phi(p_0) = 0 ,$$

we take the scalar product of (4.17) with  $D_1 \Phi(p_0)$  :

$$(4.18) \quad \frac{\varepsilon}{2} D_1 |D_1 \Phi(p_0)|^2 + \lambda_1 \rho |D_1 \Phi(p_0)|^2 = 0 .$$

But  $|D_1 \Phi(p_0)|$  is maximum, thus :

$$\begin{aligned}
|D_1 \Phi(p_0)| &> 0 , \\
D_1 |D_1 \Phi(p_0)| &\geq 0 .
\end{aligned}$$

This contradicts (4.18), hence  $|D_1 \Phi|$  cannot obtain its maximum on the boundary.

The proof on that case is complete but it is not valid since  $\Phi$  is not regular enough.

Let us consider the original problem, the calculus are a bit tedious because on the boundary we have to use local coordinates.

$\Phi$  is constant on  $\partial\Omega$ , thus all tangential derivatives are zero on  $\partial\Omega$ .

We note  $\vec{n}(p) = (n_1(p), \dots, n_N(p))$  the unit outward normal at  $p \in \partial\Omega$  and  $\frac{\partial\Phi}{\partial r}$  the derivatives in the direction  $\vec{n}$ , we have :

$$h.D\Phi(p) = \sum_{i=1}^N h_i D_i \Phi(p) = (h \cdot \vec{n}(p)) \frac{\partial\Phi}{\partial r}(p) , \quad \text{when } p \in \partial\Omega .$$

We deduce :

$$(4.19) \quad |h.D\Phi(\bar{p})| \leq |h| \left| \frac{\partial\Phi}{\partial r}(\bar{p}) \right| .$$

We will estimate  $\left| \frac{\partial\Phi}{\partial r}(\bar{p}) \right|$ . First, we claim that  $\left| \frac{\partial\Phi}{\partial r} \right|$  (which is define everywhere in  $\Omega$ ) does not attain its maximum on the boundary. Indeed, if we assume that  $\left| \frac{\partial\Phi}{\partial r} \right|$  attains its maximum at  $p_0$  on the boundary. Let  $(\tau_1, \dots, \tau_{n-1})$  be a local coordinate system on  $\partial\Omega$  near  $p_0$ , then :

$$\begin{aligned}
\Delta\Phi &= \sum_{i,j,k} \frac{\partial^2\Phi}{\partial\tau_j\partial\tau_k} \frac{\partial\tau_k}{\partial p_i} \frac{\partial\tau_j}{\partial p_i} \\
&+ \frac{\partial^2\Phi}{\partial r^2} + \frac{N-1}{r} \frac{\partial\Phi}{\partial r} \\
&+ 2 \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\partial r} \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{r} \\
&+ \sum_{i,j} \frac{\partial\Phi}{\partial\tau_j} \frac{\partial^2\tau_j}{\partial p_i^2} .
\end{aligned}$$

Thus, at  $p_0$ , we have :

$$\begin{aligned}
(4.20) \quad \Delta\Phi(p_0) &= \frac{\partial^2\Phi}{\partial r^2}(p_0) + \frac{N-1}{\rho} \frac{\partial\Phi}{\partial r}(p_0) \\
&+ 2 \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\partial r}(p_0) \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{\rho} .
\end{aligned}$$

(the others terms are zero since they are tangential derivatives).

Moreover :

$$(4.21) \quad A_{p_0} \cdot \nabla\Phi(p_0) = (A_{p_0} \cdot \vec{n}(p_0)) \frac{\partial\Phi}{\partial r}(p_0) ,$$

$$(4.22) \quad R(p_0 + \psi(p_0)) = 0 \quad , \quad (|p_0 + \psi(p_0)| \geq |p_0| \geq \rho) ,$$

$$(4.23) \quad \Phi(p_0) = 0 .$$

Using (4.19) to (4.23) in (4.1) :

$$\begin{aligned}
(4.24) \quad \varepsilon \frac{\partial^2\Phi}{\partial r^2} + 2\varepsilon \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\partial r} \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{\rho} \\
+ \left( \varepsilon \frac{N-1}{\rho} + A_{p_0} \cdot \vec{n} \right) \frac{\partial\Phi}{\partial r} = 0 .
\end{aligned}$$

We take the scalar product of (4.24) with  $\frac{\partial\Phi}{\partial r}(p_0)$  :

$$\begin{aligned}
(4.25) \quad & \frac{\varepsilon}{2} \frac{\partial}{\partial r} \left( \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 \right) + \varepsilon \sum_{i,j} \frac{\partial}{\partial \tau_j} \left( \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 \right) \frac{\partial \tau_j}{\partial p_i} \times \frac{p_i}{\rho} \\
& + \left( \varepsilon \frac{N-1}{\rho} + A p_0 \cdot \vec{n} \right) \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 = 0 .
\end{aligned}$$

(This is the analog of (4.18)).

We assumed that  $\left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2$  is maximum :

$$\begin{aligned}
& \frac{\partial}{\partial r} \left( \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 \right) \geq 0 \quad , \quad \frac{\partial}{\partial \tau_j} \left( \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 \right) = 0 \quad , \\
& \left| \frac{\partial \Phi}{\partial r} (p_0) \right|^2 > 0 \quad ,
\end{aligned}$$

and this contradicts (4.25). Hence  $\left| \frac{\partial \Phi}{\partial r} \right|^2$  cannot have a maximum on the boundary and there exists a point  $p_1$  inside  $\Omega$  such that  $\left| \frac{\partial \Phi}{\partial r} (p_1) \right|^2$  is maximum. We denote by  $\vec{m} = (m_1, \dots, m_N)$  the outward normal at  $p_1$ , then :

$$(4.26) \quad \frac{\partial \Phi}{\partial r} (p_1) = \sum_{i=1}^N m_i D_i \Phi(p_1) .$$

Let  $p$  be a point on the boundary :

$$\begin{aligned}
\left| \sum_{i=1}^N m_i D_i \Phi(p) \right| &= \left| (\vec{m} \cdot \vec{n}(p)) \frac{\partial \Phi}{\partial r} (p) \right| \\
&\leq \left| \frac{\partial \Phi}{\partial r} (p) \right| \\
&< \left| \frac{\partial \Phi}{\partial r} (p_1) \right| ,
\end{aligned}$$

$$(4.27) \quad \left| \sum_{i=1}^N m_i D_i \Phi(p) \right| < \left| \sum_{i=1}^N m_i D_i \Phi(p_1) \right| .$$

We deduce that the maximum of  $\left| \sum_{i=1}^N m_i D_i \Phi(p) \right|$  occurs inside  $\Omega$ , we can use the estimate (4.16) with  $h = (m_1, \dots, m_N)$ :

$$(4.28) \quad \begin{cases} \forall p \in \Omega : \\ \lambda_{N+1} \left| \sum_{i=1}^N m_i D_i \Phi(p) \right| \leq |m| \left( (\lambda_N + K_1(1+M_1) \lambda_N^\gamma) |D\Phi|_{L^\infty} + \lambda_{N+1}^\gamma K_1(1+M_1) \right). \end{cases}$$

Using (4.19), (4.26), (4.28) at  $p_1$  and  $|m| = 1$ , we have for all  $p$  in  $\Omega$  :

$$\begin{aligned} \lambda_{N+1} |h \cdot D\Phi(p)| &\leq \lambda_{N+1} |h \cdot D\Phi(\bar{p})| \\ &\leq \lambda_{N+1} |h| \left| \frac{\partial \Phi}{\partial r}(\bar{p}) \right| \\ &\leq \lambda_{N+1} |h| \left| \frac{\partial \Phi}{\partial r}(p_1) \right| \\ &\leq \lambda_{N+1} |h| \left| \sum_{i=1}^N m_i D_i \Phi(p_1) \right| \\ &\leq |h| \left( (\lambda_N + K_1(1+M_1) \lambda_N^\gamma) |D\Phi|_{L^\infty} + \lambda_{N+1}^\gamma K_1(1+M_1) \right). \end{aligned}$$

And (4.16) is still valid, it is valid for all  $h$  in  $\mathbb{R}^N$ , we deduce :

$$\begin{aligned} \lambda_{N+1} |D\Phi|_{L^\infty} &= \lambda_{N+1} \sup_{p \in B} \sup_{h \in \mathbb{R}^N} \frac{|h \cdot D\Phi|}{|h|} \\ &\leq \left( (\lambda_N + K_1(1+M_1) \lambda_N^\gamma) |D\Phi|_{L^\infty} + \lambda_{N+1}^\gamma K_1(1+M_1) \right), \end{aligned}$$

thanks to (2.11) :

$$|D\Phi|_{L^\infty} \leq \frac{\lambda_{N+1}^\gamma K_1(1+M_1)}{\lambda_{N+1} - \lambda_N - K_1(1+M_1) \lambda_N^\gamma},$$

thanks to (2.12) :

$$|D\Phi|_{L^\infty} \leq M_1.$$

Thus the lemma is proved when  $\psi \in C^1(\Omega, Q_\ell H)$  ; if  $\psi \in W^{1,\infty}(\Omega, Q_\ell H)$  we choose a sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $C^1(\Omega, Q_\ell H) \cap V_\ell$  which converges to  $\psi$ , we write  $\Phi_n = \tau_{\varepsilon, \ell} \psi_n$ , then :

$$\forall n \in \mathbb{N} : |D\Phi_n|_{L^\infty} \leq M_1.$$

$(\Phi_n)$  is bounded in  $W^{1,\infty}(\Omega, Q_\ell H)$ , it admits a subsequence which converges in  $W^{1,\infty}(\Omega, Q_\ell H)$  weak star, thanks to lemma 4.2, the limit must be  $\Phi$ , we deduce :  $\Phi \in W^{1,\infty}(\Omega, Q_\ell H)$ ,

$$|D\Phi|_{L^\infty} \leq \liminf |D\Phi_n|_{L^\infty} \leq M_1.$$

**LEMMA 4.5** (fixed point)

i)  $V_\ell$  is a Banach space when endowed with the  $L^\infty$  norm.

ii)  $\mathcal{T}_{\varepsilon,\ell} : V_\ell \rightarrow V_\ell$  is a strict contraction for that norm.

We deduce that  $\mathcal{T}_{\varepsilon,\ell}$  possesses a fixed point, we denote it by  $\Phi_{\varepsilon,\ell}$ , we have :

$$(4.29) \quad \begin{cases} \varepsilon \Delta \Phi_{\varepsilon,\ell} + (A p + PR(p + \Phi_{\varepsilon,\ell})) \cdot \nabla \Phi_{\varepsilon,\ell} - A \Phi_{\varepsilon,\ell} - Q_\ell R(p + \Phi_{\varepsilon,\ell}) = 0 & \text{in } \Omega, \\ \Phi_{\varepsilon,\ell} = 0 & \text{on } \partial\Omega. \end{cases}$$

**Proof**

i) Left to the reader.

ii) Let  $\psi_i$  in  $V_\ell$  and  $\Phi_i = \mathcal{T}_{\varepsilon,\ell} \psi_i$  ( $i = 1, 2$ ).

We know that  $\Phi_1$  and  $\Phi_2$  are in  $C^{2,s}(\Omega, Q_\ell H)$  ( $s < 1$ ).

We denote by  $\Phi = \Phi_1 - \Phi_2$ , we have :

$$(4.30) \quad \begin{aligned} & \varepsilon \Delta \Phi + (A p + PR(p + \psi_1)) \cdot \nabla \Phi + (PR(p + \psi_1) - PR(p + \psi_2)) \cdot \nabla \Phi_2 \\ & - A \Phi - (Q_\ell R(p + \psi_1) - Q_\ell R(p + \psi_2)) = 0. \end{aligned}$$

We take the scalar product of (4.30) with  $\Phi$  :

$$\begin{aligned} & \frac{\varepsilon}{2} \Delta |\Phi|^2 - \varepsilon |\nabla \Phi|^2 + \frac{1}{2} (A p + PR(p + \psi_1)) \cdot \nabla |\Phi|^2 \\ & + \left( (PR(p + \psi_1) - PR(p + \psi_2)) \cdot \nabla \Phi_2, \Phi \right) \\ & - |A^{1/2} \Phi|^2 - \left( A^{-\gamma} (Q_\ell R(p + \psi_1) - Q_\ell R(p + \psi_2)), A^\gamma \Phi \right) \\ & = 0. \end{aligned}$$

At the point  $\bar{p}$  where  $|\Phi|^2$  is maximum ( $\bar{p}$  is inside  $\Omega$  since  $\Phi = 0$  on  $\partial\Omega$ ) :

$$\begin{aligned} |A^{1/2} \Phi(\bar{p})|^2 & \leq \left( (PR(\bar{p} + \psi_1) - PR(\bar{p} + \psi_2)) \cdot \nabla \Phi_2, \Phi(\bar{p}) \right) \\ & \quad - \left( A^{-\gamma} (Q_\ell R(\bar{p} + \psi_1) - Q_\ell R(\bar{p} + \psi_2)), A^\gamma \Phi(\bar{p}) \right) \\ & \leq \lambda_N^\gamma K_1 |\psi_1 - \psi_2|_{L^\infty} |D\Phi_2|_{L^\infty} |\Phi(\bar{p})| \end{aligned}$$



$$\begin{aligned}
& + K_1 |\psi_1 - \psi_2|_{L^\infty} |A^\gamma \Phi(\bar{p})| \\
& \leq \lambda_N^\gamma \lambda_{N+1}^{-1/2} K_1 M_1 |\psi_1 - \psi_2|_{L^\infty} |A^{1/2} \Phi(\bar{p})| \\
& \quad + \lambda_{N+1}^{\gamma-1/2} K_1 |\psi_1 - \psi_2|_{L^\infty} |A^{1/2} \Phi(\bar{p})| \\
& \leq \lambda_{N+1}^{\gamma-1/2} K_1 (1+M_1) |\psi_1 - \psi_2|_{L^\infty} |A^{1/2} \Phi(\bar{p})| \\
|\Phi(\bar{p})| & \leq \lambda_{N+1}^{-1/2} |A^{1/2} \Phi(\bar{p})| \\
& \leq \lambda_{N+1}^{\gamma-1} K_1 (1+M_1) |\psi_1 - \psi_2|_{L^\infty}, \\
|\Phi_1 - \Phi_2|_{L^\infty} & \leq \lambda_{N+1}^{\gamma-1} K_1 (1+M_1) |\psi_1 - \psi_2|_{L^\infty}.
\end{aligned}$$

Thanks to (2.14),  $\mathcal{T}_{\varepsilon, \ell}$  is a strict contraction.

**LEMMA 4.6 :** (limit  $\varepsilon \rightarrow 0$  and  $\ell \rightarrow +\infty$ )

i) The sequence  $(\Phi_{\varepsilon, \ell})_{\varepsilon > 0}$  possesses a subsequence that converges when  $\varepsilon \rightarrow 0$  in  $W^{1, \infty}(\Omega, Q_\ell H)$  weak star to a function  $\Phi_\ell$  that satisfies :

$$(4.31) \quad \begin{cases} - (A p + PR(p + \Phi_\ell)) \cdot \nabla \Phi_\ell + A \Phi_\ell + Q_\ell R(p + \Phi_\ell) = 0 & \text{in } \Omega, \\ \Phi_\ell = 0 & \text{on } \partial \Omega. \end{cases}$$

ii) Let  $\tilde{\Phi}_\ell$  be the function from  $\Omega$  to  $QH$  defined by :

$$\begin{aligned}
(\tilde{\Phi}_\ell, w_k) &= (\Phi_\ell, w_k) & \text{if } k \leq \ell, \\
(\tilde{\Phi}_\ell, w_k) &= 0 & \text{if } k > \ell,
\end{aligned}$$

then the sequence  $(\tilde{\Phi}_\ell)_{\ell \geq N+1}$  possesses a subsequence that converges when  $\ell \rightarrow +\infty$  in  $W^{1, \infty}(\Omega, QH)$  weak star to a function  $\Phi$ , that satisfies :

$$(4.32) \quad \begin{cases} - (A p + PR(p + \Phi(p))) \cdot \nabla \Phi + A \Phi + QR(p + \Phi) = 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial \Omega. \end{cases}$$

Moreover  $\Phi \in L^\infty(\Omega, QD(A^{1/2})) \cap W^{1, \infty}(\Omega, QH)$ .

**Proof**

i) For all  $\varepsilon > 0$ , we have, since  $\Phi_{\varepsilon, \ell}$  is in  $V_\ell$  :

$$|\Phi_{\varepsilon, \ell}|_{W^{1, \infty}} \leq M_0 + M_1.$$

Hence there is a subsequence  $(\Phi_{\varepsilon'})_{\varepsilon' > 0}$  which converges in  $W^{1, \infty}(\Omega, Q_\ell H)$  weak star to a function  $\Phi_\ell$ , as the imbedding of  $W^{1, \infty}(\Omega, Q_\ell H)$  in  $L^\infty(\Omega, Q_\ell H)$  is compact, we have :

$$\begin{aligned}
\varepsilon \Delta \Phi_{\varepsilon, \ell} &\rightarrow 0 \text{ in } W^{-1, \infty}(\Omega, Q_{\ell} H) , \\
(A_p + PR(p + \Phi_{\varepsilon, \ell})) \cdot \nabla \Phi_{\varepsilon, \ell} &\rightharpoonup (A_p + PR(p + \Phi_{\ell})) \cdot \nabla \Phi_{\ell} \\
&\text{in } L^{\infty}(\Omega, Q_{\ell} H) \text{ weak star} , \\
(A \Phi_{\varepsilon, \ell} + Q_{\ell} R(p + \Phi_{\varepsilon, \ell})) &\rightarrow A \Phi_{\ell} + Q_{\ell} R(p + \Phi_{\ell}) \\
&\text{in } L^{\infty}(\Omega, Q_{\ell} H) \text{ strong} ,
\end{aligned}$$

Using those limits in (4.29), we obtain (4.31).

Moreover :

$$|\Phi_{\ell}|_{W^{1, \infty}(\Omega, Q_{\ell} H)} \leq M_0 + M_1 .$$

We take the scalar product of (4.31) with  $\Phi_{\ell}$  :

$$\begin{aligned}
&- \left( (A_p + PR(p + \Phi_{\ell})) \cdot \nabla \Phi_{\ell}, \Phi_{\ell} \right) + |A^{1/2} \Phi_{\ell}|^2 \\
&\quad + (A^{-\gamma} Q_{\ell} R(p + \Phi_{\ell}), A^{\gamma} \Phi_{\ell}) = 0 , \\
|A^{1/2} \Phi_{\ell}|^2 &\leq \left( \sup_{p \in \Omega} |A_p| + \lambda_N^{\gamma} K_0 \right) M_1 \lambda_{N+1}^{-1/2} |A^{1/2} \Phi_{\ell}| \\
&\quad + \lambda_{N+1}^{\gamma-1/2} K_0 |A^{1/2} \Phi_{\ell}| , \\
|A^{1/2} \Phi_{\ell}|_{L^{\infty}} &\leq \left( \sup_{p \in \Omega} |A_p| + \lambda_N^{\gamma} K_0 \right) M_1 \lambda_{N+1}^{-1/2} + \lambda_{N+1}^{\gamma-1/2} K_0 .
\end{aligned}$$

Hence  $(\Phi_{\ell})_{\ell \geq N+1}$  is bounded in  $W^{1, \infty}(\Omega, Q_{\ell} H) \cap L^{\infty}(\Omega, Q_{\ell} D(A^{1/2}))$ .

ii) We have :

$$|\tilde{\Phi}_{\ell}|_{W^{1, \infty}} = |\Phi_{\ell}|_{W^{1, \infty}} \text{ and } |A^{1/2} \tilde{\Phi}_{\ell}|_{L^{\infty}} = |A^{1/2} \Phi_{\ell}|_{L^{\infty}} .$$

The sequence  $(\tilde{\Phi}_{\ell})_{\ell \geq N+1}$  is bounded in  $W^{1, \infty}(\Omega, QH) \cap L^{\infty}(\Omega, QD(A^{1/2}))$ . Thanks to lemma (4.8) below, there exists a subsequence  $(\tilde{\Phi}_{\ell_k})$  and a function  $\Phi$  in  $W^{1, \infty}(\Omega, QH) \cap L^{\infty}(\Omega, QD(A^{1/2}))$  such that :

$$\begin{aligned}
\tilde{\Phi}_{\ell_k} &\rightharpoonup \Phi \text{ in } W^{1, \infty}(\Omega, QH) \text{ weak star} , \\
\tilde{\Phi}_{\ell_k} &\rightarrow \Phi \text{ in } L^{\infty}(\Omega, QH) \text{ strong} .
\end{aligned}$$

Then, we have :

$$\begin{aligned}
(A_p + PR(p + \tilde{\Phi}_{\ell_k})) \cdot \nabla \tilde{\Phi}_{\ell_k} &\rightharpoonup (A_p + PR(p + \Phi)) \cdot \nabla \Phi \text{ in } L^{\infty}(\Omega, QH) \text{ weak star} , \\
A \tilde{\Phi}_{\ell_k} &\rightarrow A \Phi \text{ in } L^{\infty}(\Omega, QD(A^{-1})) \text{ strong} , \\
QR(p + \tilde{\Phi}_{\ell_k}) &\rightarrow QR(p + \Phi) \text{ in } L^{\infty}(\Omega, QD(A^{-\gamma})) \text{ strong} .
\end{aligned}$$

We deduce that  $\Phi$  satisfies (4.32).

LEMMA 4.7 :

If we extend  $\Phi$ , obtained in lemma 4.6 ii), by zero outside  $\Omega$ , its graph is an inertial manifold for the system :

$$(4.33) \quad \begin{cases} \frac{du}{dt} + Au + R(u) = 0, \\ u(0) = u_0. \end{cases}$$

Proof

We still denote by  $\Phi$  the extended function.

Then we have :

$$-(Ap + PR(p+\Phi)) \cdot \nabla \Phi + A\Phi + QR(p+\Phi) = 0,$$

almost everywhere in PH.

Let  $u_0$  be in H and  $u(t) = S(t) u_0$  be the solution in  $C(\mathbb{R}^+, H) \cap L^2(0, T; D(A^{1/2}))$  for all  $T > 0$  of system (4.33) (see the assumptions on (4.33) on section II), we write :

$$p(t) = Pu(t),$$

$$q(t) = Qu(t),$$

and

$$r(t) = q(t) - \Phi(p(t)).$$

We have :

$$\frac{dq}{dt} + Aq + QR(p+q) = 0,$$

$$\begin{aligned} \frac{d\Phi(p)}{dt} &= \frac{dp}{dt} \cdot \nabla \Phi(p) \\ &= -(Ap + PR(p+q)) \cdot \nabla \Phi(p) \\ &= -(Ap + PR(p+\Phi)) \cdot \nabla \Phi(p) \\ &\quad + (PR(p+\Phi) - PR(p+q)) \cdot \nabla \Phi(p) \\ &= (PR(p+\Phi) - PR(p+q)) \cdot \nabla \Phi(p) \\ &\quad - A\Phi(p) - QR(p+\Phi), \end{aligned}$$

$$(4.34) \quad \begin{aligned} \frac{dr}{dt} &= -(PR(p+\Phi) - PR(p+q)) \cdot \nabla \Phi(p) \\ &\quad - Ar - ((QR(p+q) - QR(p+\Phi))). \end{aligned}$$

We take the scalar product of (4.34) with  $r(t)$  and use :

$$\left( \frac{dr}{dt}, r \right) = \frac{1}{2} \frac{d}{dt} |r|^2,$$

since  $r(t) \in L^2(0, T; D(A^{1/2}))$  and  $\frac{dr}{dt} \in L^2(0, T; D(A^{-1/2}))$ , we obtain :

$$\begin{aligned}
\frac{1}{2} \frac{d|r|^2}{dt} + |A^{1/2} r|^2 &= - \left( (PR(p+\Phi) - PR(p+q)) \cdot \nabla \Phi, r \right) \\
&\quad - (A^{-\gamma} (QR(p+q) - QR(p+\Phi)), A^\gamma r) \\
&\leq |PR(p+\Phi) - PR(p+q)| \|D\Phi\|_{L^\infty} |r| \\
&\quad + |A^{-\gamma} (QR(p+q) - QR(p+\Phi))| |A^\gamma r| \\
&\leq \lambda_N^\gamma K_1 M_1 |r|^2 + K_1 |r| |A^\gamma r| \\
&\leq \lambda_N^\gamma K_1 M_1 \lambda_{N+1}^{-1/2} |r| |A^{1/2} r| \\
&\quad + K_1 \lambda_{N+1}^{\gamma-1/2} |r| |A^{1/2} r| \\
&\leq \lambda_{N+1}^{\gamma-1/2} K_1 (1+M_1) |r| |A^{1/2} r| \\
&\leq \frac{1}{2} \lambda_{N+1}^{2\gamma-1} K_1^2 (1+M_1)^2 |r|^2 + \frac{1}{2} |A^{1/2} r|^2,
\end{aligned}$$

$$\begin{aligned}
\frac{d|r|^2}{dt} + \lambda_{N+1} |r|^2 &\leq \frac{d|r|^2}{dt} + |A^{1/2} r|^2 \\
&\leq \lambda_{N+1}^{2\gamma-1} K_1^2 (1+M_1)^2 |r|^2.
\end{aligned}$$

Thanks to Gronwall's lemma :

$$(4.35) \quad |r(t)|^2 \leq |r(0)|^2 e^{-\lambda_{N+1} \left(1 - \lambda_{N+1}^{2\gamma-1} K_1^2 (1+M_1)^2\right) t}.$$

We deduce that if  $u_0$  is in the graph of  $\Phi$ , then  $r(0) = 0$  and  $r(t) = 0$  for all  $t \geq 0$ . The graph is positively invariant.

If  $u_0$  is not the graph of  $\Phi$ , thanks to (4.35) together with (2.14),  $|r(t)|$  (which is larger than  $d(u(t), \mathcal{M})$ ) decays exponentially.)

Theorem 3.1 is proved, except lemma 4.8, used in lemma 4.6.

**LEMMA 4.8 :**

Let  $(\Phi_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $W^{1,\infty}(\Omega, QH) \cap L^\infty(\Omega, QD(A^{1/2}))$  then there exists a subsequence  $(\Phi_{n_k})$  that converges in  $W^{1,\infty}(\Omega, QH)$  weak star and in  $L^\infty(\Omega, QH)$  strong.

**Proof**

For the convergence in  $W^{1,\infty}(\Omega, QH)$  weak star, we use that  $L^\infty(\Omega, QH)$  is the dual of  $L^1(\Omega, QH)$ ; for the strong convergence in  $L^\infty(\Omega, QH)$ , we use Ascoli-Arzelà theorem :

- $(\Phi_n)$  is bounded in  $W^{1,\infty}(\Omega, QH)$ , thus it is equicontinuous,
- $(\Phi_n)$  is bounded in  $L^\infty(\Omega, QD(A^{1/2}))$ , thus for all  $p$  in  $\Omega$ , the set  $\overline{\{\Phi_n(p) / n \in \mathbb{N}\}}$  is compact in  $QH$  (indeed  $A^{-1/2}$  is a compact operator).

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2ème PARTIE:

CONSTRUCTION DE FAMILLES DE  
VARIETES INERTIELLES APPROCHEES.





## On the construction of families of approximate inertial manifolds

Arnaud DEBUSSCHE<sup>(1)</sup> and Martine MARION<sup>(2)</sup>

### 0. Introduction

The aim of this paper is to present a general method for the construction of approximate inertial manifolds (AIM). The concept of AIM has been recently introduced by Foias-Manley-Teman [7] in relationship with several difficulties arising in the theory of inertial manifolds. For a given partial differential equation, an inertial manifold, when it exists, is a smooth finite dimensional invariant manifold which attracts exponentially all the solutions as time goes to infinity. The long-time dynamics can then be described by the solutions of a finite system of ordinary differential equations. Existence results of such manifolds can be found in Foias-Sell-Temam [8], Constantin et al [4], Mallet-Paret and Sell [12], Fabes-Luskin-Sell [6], Debussche [5] (see also the references therein) but there are still many dissipative partial differential equations for which the existence of inertial manifolds is not known ; there are even in some cases non existence results, see Mallet-Paret and Sell [13], Mora and Sola-Morales [18]. Also, although inertial manifolds are much better suited for computations than attractors, their computations remain a very difficult task. These problems have lead to introduce the weaker concept of AIM : AIM are manifolds which attract the orbit in a small (thin) neighborhood exponentially rapidly. It is shown in [7] that the two dimensional Navier-Stokes equations possess AIM while the existence of exact inertial manifold is not known for these equations. Even for equations possessing inertial manifolds, AIM can be useful, especially for practical purposes. Indeed these manifolds are computable and numerical schemes well adapted for long term integration are based on that concept, Marion-Temam [17] ; see Rosier [22], Rosier-Temam [23], for numerical tests of these methods. We also refer to Sell [24] for another definition of AIM.

The existence of AIM has been proved for several partial differential equations (Foias-Manley-Temam [7], Marion [15, 16], Titi [26], Jolly-Kevrekidis-Titi [9]). But in these references the authors restrict themselves to the construction of a finite number of manifolds. Our aim here is to derive a method for constructing infinitely many AIM providing better and better order approximation to the orbits. We will present our techniques in the case of two model problems that one of the authors previously considered in [15, 16] : a reaction-diffusion equation and a fourth order equation borrowed from

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mathematical physics, namely the Cahn-Hilliard equation. But our methods are general and can be adapted to many dissipative partial differential equations. Related results for the Navier-Stokes equations appear in Temam [25].

The first part of the paper is devoted to the study of a reaction diffusion equation. The precise assumptions are stated in section 1. The problem we investigate can be rewritten as an abstract evolution equation in  $H = L^2(\Omega)$  :

$$(0.1) \quad \frac{du}{dt} + Au + f(u) = 0 ,$$

where  $Au$  is the operator  $-d\Delta u + u$ ,  $d > 0$ , associated to the appropriate boundary condition (Dirichlet, Neumann, periodic). We consider the orthonormal basis of  $H$  consisting of the eigenvectors of  $A$  :

$$\begin{aligned} Aw_j &= \lambda_j w_j, \quad j = 1, 2, \dots \\ 0 < \lambda_1 < \lambda_2 < \dots ; \quad \lambda_j &\rightarrow +\infty \text{ as } j \rightarrow +\infty . \end{aligned}$$

For a fixed  $m$ , let  $P = P_m$  denote the projector in  $H$  onto the space spanned by  $w_1, \dots, w_m$  and let  $Q = Q_m = I - P_m$ . We write  $p = Pu$ ,  $q = Qu$  so that  $u$  is decomposed as the sum

$$u = p + q .$$

We introduce the coupled system of equations for  $p$  and  $q$  :

$$(0.2) \quad \frac{dp}{dt} + Ap + Pf(p+q) = 0$$

$$(0.3) \quad \frac{dq}{dt} + Aq + Qf(p+q) = 0$$

Then, in section 2, we describe our method for constructing a sequence  $\mathfrak{M}_i$ ,  $i \in \mathbb{N}$ , of AIM. The manifold  $\mathfrak{M}_i$  is obtained as the graph of a function  $\phi_i$  mapping  $PH$  into  $QH$ . We show that for large time any solution of (0.1) satisfies

$$\text{dist}(u(t), \mathfrak{M}_i) \leq c_i (\lambda_1/\lambda_{m+1})^{i+1},$$

where  $c_i$  denotes a constant depending on  $i$  but independent of  $m$ . Therefore the orbits enter a neighborhood of  $\mathfrak{M}_i$  that can be made arbitrarily small by choosing  $m$  large enough. And, for  $i_1 < i_2$ , if  $m$  is large enough,  $\mathfrak{M}_{i_2}$  provides a better approximation than  $\mathfrak{M}_{i_1}$ . Our method consists in particular in introducing in (0.3) convenient approximation of  $dq/dt$  and of  $p+q$ . The corresponding technical proofs are given in Section 3. Note that our results hold in space dimension  $n \leq 4$ , while the existence of exact inertial manifolds

for reaction-diffusion equations is generally not known for  $n = 3$  and there are non existence results for  $n = 4$  [13].

We then address in the second part of the paper similar questions for the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u + a \Delta u - b \Delta u^3 = 0, a, b > 0.$$

This equation is a model for the spinodal decomposition that occurs when a binary solution is cooled sufficiently [2, 3, 11, 21]. It contains a fourth order dissipative term and a second order anti-dissipative term and can be rewritten in an abstract form :

$$\frac{du}{dt} + A^2 u + Af(u) = 0, f(u) = au - b u^3,$$

where  $A$  denotes the Laplace operator associated to the considered boundary conditions (Neumann or periodic). The equation and its functional setting are described in section 4. We recall there results borrowed from Nicolaenko-Scheurer [19], Nicolaenko-Scheurer-Temam [20], Marion [16]. We then present in section 5 the principle of the construction of AIM. The algebra and the corresponding proofs are different from the ones for (0.1) but the underlying ideas are similar. In particular we again introduce approximations to time derivatives. We obtain a family of manifolds  $\mathfrak{M}_i$  of dimension  $m$  ( $m \in \mathbb{N}$  fixed) such that for  $t$  sufficiently large,

$$\text{dist}(u(t), \mathfrak{M}_i) \leq c_i (\lambda_2/\lambda_{m+1})^{i+2},$$

where  $c_i$  is independent of  $m$  and  $(\lambda_j)_{j \in \mathbb{N}}$  denotes the family of eigenvalues of  $A$ . Section 6 contains the proofs of technical results concerning time derivatives. Again, our results hold in space dimension  $n = 3$ , where the existence of inertial manifolds is not known.

## **CONTENTS**

### **0. Introduction**

#### **Part I. Reaction-diffusion equations**

1. The equation and some properties of the solution
2. Construction of the family of AIM
3. Proof of the estimates on time derivatives

#### **Part II. The Cahn-Hilliard equation**

4. The equation and some properties of the solution
5. Construction of the family of AIM
6. Proof of the estimates on time derivatives

## Part I. Reaction-diffusion equations

### 1. The equation and some properties of the solution

#### 1.1. The equation and the semi-group

We consider the following problem involving a real valued function  $u(x,t)$  defined on  $\Omega \times \mathbb{R}^+$ , where  $\Omega$  denotes a regular bounded set of  $\mathbb{R}^n$  ( $n \geq 1$ )

$$(1.1) \quad \frac{\partial u}{\partial t} - d \Delta u + f(u) = 0, \text{ in } \Omega \times \mathbb{R}^+.$$

The equation is supplemented with the initial condition

$$(1.2) \quad u(x,0) = u_0(x) \text{ in } \Omega,$$

and one of the three following boundary conditions :

$$(1.3) \quad \begin{array}{ll} \text{Dirichlet} & u = 0 \text{ on } \Gamma = \partial\Omega, \\ \text{Neumann} & \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \\ \text{Periodic} & \Omega = \prod_{i=1}^n ]0, L_i[ \text{ and } u \text{ is } \Omega \text{ periodic.} \end{array}$$

Here,  $d > 0$  is a diffusion coefficient. We assume that the nonlinear term  $f$  is infinitely many differentiable from  $\mathbb{R}$  into  $\mathbb{R}$  and satisfies the growth condition

$$(1.4) \quad c_1 |s|^r - c_3 \leq f(s)s \leq c_2 |s|^r - c_3, \forall s \in \mathbb{R}, \text{ with } r > 2,$$

$$(1.5) \quad f'(s) \geq -c_4, \forall s \in \mathbb{R},$$

where the  $c_i$ 's are positive constants.

For the functional setting of the problem, let us introduce the operator  $Au = -d \Delta u + u$  on  $H = L^2(\Omega)$  equipped with its usual scalar product  $(\cdot, \cdot)$  and norm  $|\cdot|$ . Then  $A$  is a positive linear unbounded operator on  $H$  with domain

$$D(A) = \{ u \in H^2(\Omega), (1.3) \text{ holds} \}.$$

As usual, we denote by  $V$  the space  $D(A^{1/2})$  endowed with the norm  $|A^{1/2} \cdot|$ . We shall denote by  $\|\cdot\|_p$  the norm of  $L^p(\Omega)$ ,  $1 \leq p \leq +\infty$  ( $\|\cdot\|_2 = |\cdot|$ ), while the norm of any other Banach space  $X$  is denoted by  $\|\cdot\|_X$ .

Under assumptions (1.4), (1.5), for  $u_0$  given in  $H$ , the initial-boundary value problem (1.1)-(1.3) possesses a unique solution  $u$  defined for all  $t > 0$  such that

$$u \in C(\mathbb{R}^+; H) \cap L^2(0, T; V) \cap L^r([0, T] \times \Omega), \forall T > 0.$$

Furthermore, if  $u_0 \in V \cap L^r(\Omega)$ , then

$$u \in C(\mathbb{R}^+; V \cap L^r(\Omega)) \cap L^2(0, T; D(A)), \forall T > 0.$$

It is useful here to recall several time uniform estimates satisfied by the solution  $u$  of (1.1) - (1.3) borrowed from Marion [14, 15]. Let  $u_0$  be given in a ball  $B(0, R)$  of  $H$  of center 0 and of radius  $R$ . Then there exists a time  $t_0$  depending only on  $R$  (and of course on the data  $(\Omega, d, f)$ ) such that :

$$(1.6) \quad |Au(t)| \leq \kappa_0, \quad \forall t \geq t_0,$$

$$(1.7) \quad \|u(t)\|_\infty \leq \kappa_0, \quad \forall t \geq t_0,$$

where  $\kappa_0$  denotes a constant depending only on the data. Alternatively (1.6) (resp. (1.7)) expresses that the ball of center 0 and of radius  $\kappa_0$  is an absorbing set in  $D(A)$  (resp.  $L^\infty(\Omega)$ ) for the semi-group associated to (1.1) - (1.3). We recall that the existence of an absorbing set in  $H$  means a dissipative property of the problem. Also the existence of an absorbing set in a space compactly imbedded in  $H$  yields the existence of the universal attractor associated to (1.1) (1.3) (see [1], [10], [14]).

We will also need in the sequel estimates analog to (1.6) (1.7) for the time derivatives of  $u$ . Hereafter, we write :

$$u^{(j)} = \frac{\partial^j u}{\partial t^j}, \quad \text{for } j \in \mathbb{N}.$$

Then, according to [15], there exists a time  $t_j$  depending on  $j$  and  $R$  such that

$$(1.8) \quad |Au^{(j)}(t)| \leq \kappa_j, \quad \forall t \geq t_j,$$

$$(1.9) \quad \|u^{(j)}(t)\|_\infty \leq \kappa_j, \quad \forall t \geq t_j,$$

where  $\kappa_j$  denotes a constant depending on  $j$  and on the data.

It follows from (1.7) that, for  $t \geq t_0$ ,  $u = u(t)$  is solution of an evolution equation with a Lipschitz continuous nonlinear term. Indeed let  $\zeta$  denote a  $C^\infty$  truncation function such that

$\zeta(s) = 1$  if  $0 \leq s \leq 1$  and  $\zeta(s) = 0$  if  $s \geq 2$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(s) = \zeta\left(\frac{s^2}{\kappa_0^2}\right) (f(s) - s).$$

Then, for  $t \geq t_0$ ,  $u(t)$  satisfies

$$\frac{\partial u}{\partial t} - d \Delta u + u + f(u) = 0.$$

This equation rewrites as the following abstract differential equation in  $H$

$$(1.10) \quad \frac{du}{dt} + Au + f(u) = 0.$$

Since we are only interested in long-time behaviours, we will consider from now on (1.10) instead of (1.1).

**Remark 1.1** : All the results presented hereafter can be easily extended to more general reaction-diffusion equations. In particular, they apply to the reaction diffusion systems admitting a positively invariant region considered in [14].

## 1.2. Behaviour of small eddies

Let  $w_j, j \in \mathbb{N}^*$ , denote the orthonormal basis of  $H$  consisting of the eigen vectors of the operator  $A$  :

$$\begin{aligned} Aw_j &= \lambda_j w_j, j = 1, 2, \dots \\ 0 < \lambda_1 \leq \lambda_2 \leq \dots ; \lambda_j &\rightarrow +\infty \text{ as } j \rightarrow +\infty. \end{aligned}$$

We fix an integer  $m \in \mathbb{N}$  and denote by  $P = P_m$  the projector in  $H$  onto the space spanned by  $(w_1, \dots, w_m)$  ; we set  $Q = Q_m = I - P_m$ . In order to simplify the notations, we will write :

$$\lambda = \lambda_m, \Lambda = \lambda_{m+1},$$

and we introduce the number

$$\delta = \lambda_1 / \Lambda_{m+1}.$$

Applying  $P$  and  $Q$  to equation (1.10), we obtain the following coupled system of equations for  $p = Pu$  and  $q = Qu$  :



$$(1.11) \quad \frac{dp}{dt} + Ap + Pf(p+q) = 0 ,$$

$$(1.12) \quad \frac{dq}{dt} + Aq + Qf(p+q) = 0 .$$

Here,  $p$  represents the "large structures" of size larger than  $\lambda_m^{-1/2}$  and  $q$  represents the "small structures" of size smaller than  $\lambda_{m+1}^{-1/2}$ . In the following, we will frequently use that for all  $\alpha > 0$  :

$$(1.13) \quad |A^{\alpha+1/2} p|^2 \leq \lambda |A^\alpha p|^2, \quad \forall p \in PD(A^{\alpha+1/2})$$

$$(1.14) \quad |A^{\alpha+1/2} q|^2 \geq \Lambda |A^\alpha q|^2, \quad \forall q \in QD(A^{\alpha+1/2}).$$

Hereafter, we denote by  $\kappa$  any constant which depends only on the data  $\Omega$ ,  $d$  and  $f$ .

We now give an estimate of the size of  $q$  and of its derivatives for large time.

**Proposition 1.1** : *Assume that (1.4), (1.5) hold. Then, for all  $j$  in  $\mathbb{N}$  there exists a time  $t'_j$  which depends on the data  $(\Omega, d, f)$  and on  $R$  when  $|u_0| \leq R$  such that :*

$$(1.15) \quad |q^{(j)}(t)| \leq \kappa \delta, \quad \forall t \geq t'_j,$$

$$(1.16) \quad |A^{1/2} q^{(j)}(t)| \leq \kappa \delta^{1/2}, \quad \forall t \geq t'_j.$$

**Proof.** Differentiating  $j$  times equation (1.12) with respect to  $t$ , we obtain:

$$(1.17) \quad \frac{d}{dt} q^{(j)} + Aq^{(j)} + Q f_j(u, u^{(1)}, \dots, u^{(j)}) = 0 ,$$

where we have set

$$\frac{d^j}{dt^j} (f(u)) = f_j(u, u^{(1)}, \dots, u^{(j)}),$$

$$q^{(j)} = \frac{d^j}{dt^j} q.$$

By taking the scalar product of (1.17) with  $Aq^{(j)}$ , we have :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |A^{1/2} q^{(j)}|^2 + |Aq^{(j)}|^2 &= - (f_j(u, u^{(1)}, \dots, u^{(j)}), Aq^{(j)}), \\ &\leq |f_j(u, u^{(1)}, \dots, u^{(j)})| |Aq^{(j)}| \\ &\leq \frac{1}{2} |f_j(u, u^{(1)}, \dots, u^{(j)})|^2 + \frac{1}{2} |Aq^{(j)}|^2, \\ \frac{d}{dt} |A^{1/2} q^{(j)}|^2 + |Aq^{(j)}|^2 &\leq |f_j(u, u^{(1)}, \dots, u^{(j)})|^2. \end{aligned}$$

From the construction of  $f$ , we see that  $f$  together with its derivatives are bounded. Therefore, using (1.8) and (1.9), one easily checks that there exists a constant  $K$  such that

$$|f_j(u, u^{(1)}, \dots, u^{(j)})|^2 \leq \kappa^2, \quad \forall t \geq \max_{1 \leq k \leq j} t_k.$$

This gives, thanks to (1.14) :

$$\frac{d}{dt} |A^{1/2} q^{(j)}|^2 + \Lambda |A^{1/2} q^{(j)}|^2 \leq \kappa^2.$$

Integrating this inequality between  $T_j = \max_{1 \leq k \leq j} t_k$  and  $t$ , we find

$$(1.18) \quad |A^{1/2} q^{(j)}(t)|^2 \leq \frac{\kappa^2}{\Lambda} + |A^{1/2} q^{(j)}(T_j)|^2 e^{-\Lambda(t-T_j)}, \quad \forall t \geq T_j.$$

Since

$$|A^{1/2} q^{(j)}(T_j)| \leq |A^{1/2} u^{(j)}(T_j)| \leq \kappa_j$$

we infer from (1.18) that

$$|A^{1/2} q^{(j)}(t)|^2 \leq \frac{2\kappa}{\Lambda}, \quad \forall t \geq t'_j$$

where  $t'_j = \sup_{m \in \mathbb{N}} \max (T_j, T_j + \frac{1}{\lambda_{m+1}} \text{Log} \frac{\lambda_{m+1} \kappa_j^2}{\kappa})$ .

This shows (1.16). Finally, (1.15) follows readily from (1.16) by using (1.14) with  $\alpha = 0$ .

Proposition 1.1 is proved.

## 2. Construction of the family of AIM

The aim of this section is to present the algebra of construction of the sequence  $\mathfrak{M}_i, i \in \mathbb{N}$ , of AIM. These manifolds are closely related to approximations of the equation for  $q$ . These approximations are obtained by using Proposition 1.1, i.e. that  $q$  and its time derivatives are small for large time.

The construction of the first two manifolds  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  follows Marion [15]. The manifold  $\mathfrak{M}_1$  corresponds to the simplest approximation of (1.12). Thanks to Proposition 1.1, we know that  $q$  and  $q^{(1)} = dq/dt$  are small for large time. Therefore, we guess that, in (1.12),  $q^{(1)}$  is small compared to  $Aq$  and  $q$  is small compared to  $p$ . We are led to replace (1.12) by the following approximate equation :

$$(2.1) \quad Aq + Qf(p) = 0 .$$

For  $p$  given in PH, the resolution of (2.1) is easy and we denote by  $q_1$  its solution:

$$(2.2) \quad q_1 = \phi_1(p).$$

The graph of the function  $\phi_1 : PH \rightarrow QD(A)$  defines a smooth ( $C^\infty$ ) manifold  $\mathfrak{M}_1$  in  $H$  of dimension  $m$ . The orbits are attracted by a thin neighborhood of  $\mathfrak{M}_1$  as shown in the following proposition.

**Proposition 2.1.** *Assume that (1.4) (1.5) hold. Then, for  $t$  sufficiently large,  $t \geq t_1^*$ , any orbit of (1.1)-(1.3) remains at a distance in  $H$  of  $\mathfrak{M}_1$  bounded by  $\kappa_1 \delta^2$ ;  $\kappa_1$  is an appropriate constant which depends on the data and  $t_1^*$  depends on the data and on  $R$  when  $|u_0| \leq R$ .*

**Proof.** Let  $u = p+q$  be an orbit of (1.1)- (1.3). For every  $t > 0$ , we define  $q_1(t) = \phi_1(p(t))$ . Then,  $p(t) + q_1(t)$  lies in  $\mathfrak{M}_1$  and

$$\text{dist}(u(t), \mathfrak{M}_1) = \inf_{v \in \mathfrak{M}_1} |u(t) - v| \leq |q(t) - q_1(t)| .$$

Therefore, it suffices to evaluate the norm of

$$\chi_1(t) = q_1(t) - q(t)$$

Subtracting (1.12) from (2.1) with  $q = q_1$  we find

$$A\chi_1 = Q(f(p+q) - f(p)) + q' ,$$

and

$$|A\chi_1| \leq |f(p+q) - f(p)| + |q'|.$$

Since  $f'$  is bounded on  $\mathbb{R}$ , it follows easily that

$$|A\chi_1| \leq \kappa |q| + |q'|,$$

which gives thanks to (1.15) for  $j = 0, 1$ ,

$$(2.3) \quad |A\chi_1| \leq \kappa \delta, \text{ for } t \geq \max(t_0', t_1').$$

Therefore, using (1.14), we obtain :

$$|\chi_1| \leq \kappa \delta^2.$$

This shows Proposition 2.1.

Next, we give as an example the construction of  $\mathfrak{M}_2$ . Making use of  $q_1 = \phi_1(p)$ , we now approximate  $Qf(p+q)$  by  $Qf(p+q_1)$ . Also it follows easily from the proof of Proposition 2.1 that we need now to introduce an approximation of  $q^{(1)}$ . This is obtained by considering the equation for  $q^{(1)}$  :

$$(2.4) \quad q^{(2)} + Aq^{(1)} + Qf'(p+q)(p^{(1)} + q^{(1)}) = 0.$$

In (2.4),  $p^{(1)}$  given by (1.11) is approximated by

$$(2.5) \quad p_1^1 = -Ap - Pf(p)$$

Also  $q^{(2)}$  is neglected and the nonlinear term  $Qf'(p+q)(p^{(1)} + q^{(1)})$  is replaced by  $Qf'(p) P_1^1$ ; the approximate value  $q_1^1$  is given by

$$(2.6) \quad Aq_1^1 + Qf'(p) p_1^1 = 0.$$

Hence, (1.12) is now replaced by the equation :

$$(2.7) \quad q_1^1 + Aq_2 + Qf(p + q_1) = 0$$

The manifold  $\mathfrak{M}_2$  is therefore defined as follows. For  $p \in PH$ , we define  $q_1$  by (2.1). Then we define  $p_1^1$  by (2.5) and the resolution of (2.6) gives  $q_1^1$ .

Finally, by solving (2.7), we obtain

$$q_2 = \phi_2(p)$$

The graph of the function  $\phi_2 : PH \rightarrow QD(A)$  defines a  $C^\infty$  manifold  $\mathfrak{M}_2$  of dimension  $m$  in  $H$ . This manifold provides a better order approximation to the orbits than  $\mathfrak{M}_1$  and this is stated in

**Theorem 2.2** : *Assume that (1.4) (1.5) hold. Then for  $t$  sufficiently large,  $t \geq t_2^*$ , any orbit of (1.1)-(1.3) remains at a distance in  $H$  of  $\mathfrak{M}_2$  bounded by  $\kappa_2 \delta^3$ ;  $\kappa_2$  is an appropriate constant which depends on the data and  $t_2^*$  depends on the data and on  $R$  when  $|u_0| \leq R$ .*

**Proof**. The proof follows the same steps as that of Theorem 2.3 below and is left to the reader.

In order to construct  $\mathfrak{M}_3$ , we will need in particular to consider improved approximations of  $q'$  and of  $p+q$  in (1.12). The approximation of  $p+q$  is provided by  $p+q_2$ . Then, we introduce an approximate value  $q_1^2$  of  $q^{(2)}$  by neglecting  $q^{(3)}$  in the second derivative of (1.12) and that enables us to improve the approximation of (2.3) and to derive a better value for  $q'$ .

More generally, when we construct the manifold  $\mathfrak{M}_k$ , we construct two families  $p_{k-i}^i$ ,  $i = 1, \dots, k-1$  and  $q_{k-i}^i$ ,  $i = 0, \dots, k-1$  such that  $p_{k-1}^1$  provides an approximation of  $p^{(1)}$  and  $q_{k-i}^i$  of  $q^{(i)}$  and that satisfy estimates of the form

$$(2.8) \quad |p_{k-i}^i(t) - p^{(i)}(t)| \leq \kappa_k \delta^{k-i}, \quad i = 1, \dots, k-1,$$

$$(2.9) \quad |q_{k-i}^i(t) - q^{(i)}(t)| \leq \kappa_k \delta^{k-i+1}, \quad i = 0, \dots, k-1,$$

for  $t \geq t_k''$ , where  $\kappa_k$  depends on the data and  $k$ , and  $t_k''$  depends on the data,  $\kappa$  and  $R$  when  $|u_0| \leq R$ . In particular, at step  $\kappa$ , we introduce approximations of the time derivatives of  $p$  and  $q$  up to the order  $(k-1)$ .

The manifolds  $\mathfrak{M}_k$  are defined recursively. For  $k = 1$ ,  $\mathfrak{M}_1$  is given by (2.1). Assume that  $\mathfrak{M}_1, \dots, \mathfrak{M}_{k-1}$  are constructed ( $\kappa \geq 2$ ). We aim now to construct  $\mathfrak{M}_k$ . We start by computing the sequence  $p_{k-i}^i$ ,  $i = 1, \dots, k-1$ . The sequence is defined recursively for increasing values of  $i$  thanks to the following formula :

$$(2.10) \quad p_{k-i}^i = -A p_{k-i+1}^{i-1} - P f_{i-1}(p + q_{k-i-1}, p_{k-i+1}^1 + q_{k-i}^1, \dots, p_{k-i+1}^{i-2} + q_{k-i}^{i-2}, p_{k-i}^{i-1} + q_{k-i-1}^{i-1}).$$

Here, we have set for convenience

$$\begin{aligned} p_1^0 &= p, \quad \forall l \geq 1, \\ q_0 &= 0. \end{aligned}$$

Note that  $p_{k-i}^i$  is defined explicitly and the right hand-side of (2.10) involves either quantities known from the construction of  $\mathfrak{M}_1, \dots, \mathfrak{M}_{k-1}$ , or the term  $p_{k-i+1}^{i-1}$ , defined at step  $(i-1)$  of the recursive scheme. The formula (2.10) is obtained by considering the equation (1.11) differentiated  $(i-1)$  times, that is

$$(2.11) \quad p^{(i)} + A p^{(i-1)} + P f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}) = 0.$$

Then we replace in (2.11) the time derivatives  $p^{(l)}, q^{(l)}$  by  $p_\alpha^l, q_\beta^l$  for convenient values of  $\alpha, \beta$ .

We then compute the family  $q_{k-i}^i, i = k-1, \dots, 1$ . The family is again defined recursively, but now for decreasing values of  $i$ . At step  $i, q_{k-i}^i$ , which is an approximate value of  $q^{(i)}$ , is given by the resolution of

$$(2.12) \quad q_{k-i-1}^{i+1} + A q_{k-i}^i + Q f_i(p + q_{k-i-1}, p_{k-i+1}^1 + q_{k-i}^1, \dots, p_{k-i+1}^{i-1} + q_{k-i}^{i-1}, p_{k-i}^i + q_{k-i-1}^i) = 0,$$

where we agree that

$$q_0^l = 0, \quad \forall l \geq 1.$$

Relation (2.12) is obtained by considering equation (1.12) differentiated  $i$  times and introducing approximations of the different terms. Note that in (2.12) all terms except  $q_{k-i}^i$  are known either from the construction of  $\mathfrak{M}_1, \dots, \mathfrak{M}_k$  or for  $p_{k-i}^i$  from (2.10), or for  $q_{k-i-1}^{i+1}$  from the step  $(i+1)$  of the recursive scheme. For the first step  $i=k-1$ , we have  $q_{k-i-1}^{i+1} = q_0^k = 0$ ; this expresses that the time derivative of order  $k$  of  $q$  is neglected.

Finally,  $q_k = q_k^0$  is defined by the analog of (2.12) for  $k = 0$  that is

$$(2.13) \quad q_{k-1}^1 + A q_k + Q f(p+q_{k-1}) = 0$$

It is clear that  $q_k$  is a function of  $t$  through  $p(t)$  :

$$q_k(t) = \phi_k(p(t))$$

The manifold  $\mathfrak{M}_k$  with equation

$$q = \phi_k(p) , \phi_k : PH \rightarrow QD(A)$$

is a  $\mathfrak{C}^\infty$  manifold of dimension  $m$  in  $H$ . This manifold provides a better order approximation to the orbits than  $\mathfrak{M}_1, \dots, \mathfrak{M}_{k-1}$ . This is stated in the following Theorem which is valid in space dimension  $n \leq 4$ .

**Theorem 2.3.** *Assume that (1.4) (1.5) hold and  $n \leq 4$ . Then, for  $t$  sufficiently large,  $t \geq t_k^*$ , any solution of (1.1) -(1.3) remains at a distance in  $H$  of  $\mathfrak{M}_k$  bounded by  $\kappa_k \delta^{k+1}$ ;  $\kappa_k$  is an appropriate constant which depends on the data and  $k$  and  $t_k^*$  depends on the data,  $k$  and  $R$  when  $|u_o| \leq R$ .*

**Remark 2.4.** As it will appear in the proof of Theorem 2.3, we obtain in fact a stronger result. We show that for any arbit  $u(t) = p(t) + q(t)$  of (1.1)-(1.3), the induced orbit on the manifold  $\mathfrak{M}_k$  :

$$u_k(t) = p(t) + \phi_k(p(t))$$

satisfies for large time

$$|u(t) - u_k(t)| \leq \kappa_k \delta^{k+1} .$$

**Remark 2.5.** The constants  $\kappa_k$  and  $t_k^*$  are independent of  $m$ . Hence, the orbits enter a neighbourhood of  $\mathfrak{M}_k$  that can be made arbitrarily small by increasing the dimension  $m$  of the manifold  $\mathfrak{M}_k$ .

For  $k$  and  $k'$  given,  $k < k'$ , and for  $m$  sufficiently large, the orbits are closer from  $\mathfrak{M}_{k'}$ , than from  $\mathfrak{M}_k$ . However since we are not able to compare  $\kappa_k$  and  $\kappa_{k'}$ , we are not certain that for  $k, k', m$  given,  $k < k'$ , the orbits are closer to  $\mathfrak{M}_k$  than to  $\mathfrak{M}_{k'}$ .

**Remark 2.6.** For a fixed value of  $k$ , the schemes (2.11), (2.13) can be improved and simplified by using the explicit forms for the derivatives of the nonlinearity  $f$ .

**Remark 2.7.** The restriction on the space dimension comes from the use of Sobolev imbedding Theorems in the proof of Theorem 2.3. It is easy to see that, for small values of  $k$ , the conclusions of Theorem 2.3 hold in space dimension  $n$  larger than four. For example, Theorem 2.2 is valid in any space dimension.

**Proof of Theorem 2.3.** Let  $u(t) = p(t) + q(t)$  be a solution of (1.1)-(1.3). We consider the families  $p_{|i}^i, q_{|i}^i, q_i$  related to the manifolds  $\mathfrak{M}_i, 1 \leq i \leq k$ . Our aim is to estimate

$$\chi_k(t) = q_k(t) - q(t)$$

We admit here that (2.8) (2.9), hold (the proof of these estimates is postponed to section 3). Subtracting (1.12) from (2.13), we have :

$$\begin{aligned} A \chi_k &= Q (f(p+q) - f(p+q_{k-1})) + q^{(1)} - q_{k-1}^1, \\ |A \chi_k| &\leq |f(p+q) - f(p+q_{k-1})| + |q^{(1)} - q_{k-1}^1|. \end{aligned}$$

Therefore, since  $f$  is bounded on  $\mathbb{R}$ ,

$$(2.14) \quad |A \chi_k| \leq \kappa |q - q_{k-1}| + |q^{(1)} - q_{k-1}^1|.$$

Now using (2.10) (recall that  $q_{k-1} = q_{k-1}^0$ ), we infer from (2.14)

$$|A \chi_k| \leq \kappa \delta^k, \quad \forall t \geq t_{k-1}''.$$

Hence,

$$|\chi_k| \leq \kappa \delta^{k+1}, \quad \forall t \geq t_{k-1}''.$$

This shows Theorem 2.3.

### **3. Proof of the estimates on time derivatives**

Our aim in this Section is to check the estimates (2.8) and (2.9). We will prove by induction on  $k$  that

$$(3.1)_k \quad |p_{k-i}^i(t) - p^{(i)}(t)| \leq \kappa_k \delta^{k-i}, \quad i = 0, \dots, k-1,$$

$$(3.2)_k \quad |A(q_{k-i}^i(t) - q^{(i)}(t))| \leq \kappa_k \delta^{k-i}, \quad i = 0, \dots, k-1,$$

for  $t \geq t_k''$ , where  $\kappa_k$  depends on the data,  $k$  and  $R$  when  $|u_0| \leq R$ . We recall that  $p_1^0 = p$  and  $q_l^0 = q_l, \forall l \geq 1$ .



i) Initialization of the induction ( $k = 1$ ). The estimate (3.2)<sub>0</sub> restates (2.3) ; while (3.1) is obvious.

ii) The induction argument. We now assume that (3.1)<sub>l</sub> and (3.2)<sub>l</sub> hold for  $0 \leq l \leq k-1$  for  $k \geq 2$  given and prove that (3.1)<sub>k</sub> and (3.2)<sub>k</sub> are satisfied.

Hereafter, we denote by  $\kappa$  any constant which depends on the data and on  $k$  and "for  $t$  large enough" means for  $t \geq T_k$  where  $T_k$  depends on the data,  $k$  and  $R$  when  $|u_0| \leq R$ .

Following the procedure of definition of  $p_{k-i}^i$  and  $q_{k-i}^i$  we will first derive (3.1)<sub>k</sub> by induction on  $i$  with  $i$  increasing and then (3.2)<sub>k</sub> by induction on  $i$  with  $i$  decreasing.

For  $i = 0$  , (3.1)<sub>k</sub> is obvious. Let us assume that the relations (3.1)<sub>k</sub> are proved up to the order  $i - 1$  for  $i \geq 1$ . Subtracting (2.11) to (2.10), we have :

$$p_{k-i}^i - p^{(i)} = -A(p_{k-i+1}^{i-1} - p^{(i-1)}) - P[f_{i-1}(p+q_{k-i-1}, p_{k-i+1}^1 + q_{k-i}^1, \dots, p_{k-i}^{i-1} + q_{k-i-1}^{i-1}) - f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)})].$$

Thanks to the relation (3.1)<sub>k</sub> at the order  $(i - 1)$  and (1.13), we obtain :

$$|A(p_{k-i+1}^{i-1} - p^{(i-1)})| \leq \lambda |p_{k-i+1}^{i-1} - p^{(i-1)}| \leq \kappa \lambda \delta^{k-i+1} \leq \kappa \delta^{k-i}.$$

We infer from (1.8) (1.9) that

$$p+q, p^{(1)} + q^{(1)}, \dots, p^{(i+1)} + q^{(i-1)}$$

are bounded in  $D(A)$  and  $L^\infty(\Omega)$ . Moreover, we easily deduce from (3.1), (3.2) at the preceding steps that  $p+q_{k-i-1}$  is bounded in  $L^2(\Omega)$ ,  $p_{k-i+1}^1 + q_{k-i}^1, \dots, p_{k-i+1}^{i-2} + q_{k-i}^{i-2}$  are bounded in  $D(A)$  and  $p_{k-i}^{i-1} + q_{k-i-1}^{i-1}$  in  $L^2(\Omega)$  for  $t$  large enough. Therefore,

Lemma 3.1 below applies and the second term of (3.5) is majorized by

$$\kappa(|q_{k-i-1} - q| + |A(p_{k-i+1}^1 - p^{(1)})| + |A(q_{k-i}^1 - q^{(1)})| + \dots + |A(p_{k-i+1}^{i-2} - p^{(i-2)})| + |A(q_{k-i}^{i-2} - q^{(i-2)})| + |p_{k-i}^{i-1} - p^{(i-1)}| + |q_{k-i-1}^{i-1} - q^{(i-1)}|),$$

for  $t$  large enough. Now, we use (1.13), (1.14) together with the induction hypotheses to conclude that this is bounded by  $\kappa \delta^{k-i}$  for  $t$  large enough. Therefore (3.1)<sub>k</sub> is proved at the order  $i$ , and this concludes the induction on  $i$ . Thus (3.1)<sub>k</sub> is shown.

Let us now prove that (3.2)<sub>k</sub> is true by induction on  $i$  with  $i$  decreasing. For  $i = k-1$ ,  $q_1^{k-1}$  is defined by

$$Aq_1^{k-1} + Qf_{k-1}(p, p_2^1 + q_1^1, \dots, p_2^{k-2} + q_1^{k-2}, p_1^{k-1}) = 0$$

And,  $q^{(k-1)}$  satisfies

$$q^{(k)} + Aq^{(k-1)} + Qf_{k-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(k-2)} + q^{(k-2)}, p^{(k-1)} + q^{(k-1)}) = 0.$$

Therefore

$$\begin{aligned} |A(q_1^{k-1} - q^{(k-1)})| &\leq |q^{(k)}| + |f_{k-1}(p, p_2^1 + q_1^1, \dots, p_2^{k-2} + q_1^{k-2}, p_1^{k-1}) \\ &\quad - f_{k-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(k-2)} + q^{(k-2)}, p^{(k-1)} + q^{(k-1)})|. \end{aligned}$$

Thanks to proposition 1.1, we have

$$|q^{(k)}| \leq \kappa\delta,$$

for  $t$  large enough. Next we use Lemma 3.1 below to estimate the second term

$$\begin{aligned} |f_{k-1}(p, p_2^1 + q_1^1, \dots, p_2^{k-2} + q_1^{k-2}, p_1^{k-1}) - f_{k-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(k-2)} + \\ q^{(k-2)}, p^{(k-1)} + q^{(k-1)})| &\leq \kappa(|q| + |A(p_2^1 - p^{(1)})| + |A(q_1^1 - q^{(1)})| + \dots + \\ &\quad |A(p_2^{k-2} - p^{(k-2)})| + |A(q_1^{k-2} - q^{(k-2)})| + |p_1^{k-1} - p^{(k-1)}| + |q^{(k-1)}|). \end{aligned}$$

and we infer from Proposition 1.1, (1.13), (3.1)<sub>k</sub> and the induction hypotheses that this is majorized by  $\kappa\delta$  for  $t$  large enough. Therefore

$$|A(q_1^{k-1} - q^{(k-1)})| \leq \kappa\delta,$$

for  $t$  large enough. And (3.2)<sub>k</sub> is proved for  $i = k-1$ . Let us assume that (3.2)<sub>k</sub> is proved at the orders  $k-1, k-2, \dots, i+1$  for  $i \geq 0$ . By subtracting (2.12) to the  $i$ th derivative of (1.12), we obtain

$$\begin{aligned} |A(q_{k-i}^i - q^{(i)})| &\leq |q_{k-i-1}^{i+1} - q^{(i+1)}| + |f_i(p+q_{k-i-1}, p_{k-i+1}^1 + q_{k-i}^1, \dots, p_{k-i+1}^{i-1} + \\ &\quad q_{k-i}^{i-1}, p_{k-i}^i + q_{k-i-1}^i) - f_i(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}, p^{(i)} + q^{(i)})|. \end{aligned}$$

The induction hypothesis together with (1.14) gives

$$|q_{k-i-1}^{i+1} - q^{(i+1)}| \leq \kappa \delta^{k-i},$$

for  $t$  large enough. Similar arguments as above yield also that the second term is bounded by  $K \delta^{k-i}$  for  $t$  large enough. Therefore

$$|A(q_{k-i}^i - q^{(i)})| \leq \kappa \delta^{k-i},$$

and  $(3.2)_k$  is proved for  $i$  and, thanks to the induction argument on  $i$ ,  $(3.2)_k$  is true for all  $i$ . In the same time, the induction on  $k$  is finished and  $(3.1)_k$ ,  $(3.2)_k$  hold for all integer  $k$ . It remains to state and prove Lemma 3.1.

**Lemma 3.1.** *Assume that  $n \leq 4$ . Then, for  $(u, u_1, \dots, u_{l-1}, u_l)$  in  $L^2(\Omega) \times (D(A) \cap L^\infty(\Omega))^{l-1} \times (L^2(\Omega) \cap L^\infty(\Omega))$  and  $(v, v_1, \dots, v_{l-1}, v_l)$  in  $L^2(\Omega) \times D(A)^{l-1} \times L^2(\Omega)$ , we have :*

$$|f_l(u, u_1, \dots, u_{l-1}, u_l) - f_l(v, v_1, \dots, v_{l-1}, v_l)| \leq C(R) (|u-v| + |A(u_1 - v_1)| + \dots + |A(u_{l-1} - v_{l-1})| + |u_l - v_l|),$$

when  $|u| \leq R$ ,  $|v| \leq R$ ,  $|u_j|_\infty \leq R$ ,  $|Au_j| \leq R$ ,  $|Av_j| \leq R$ , for  $j = 1, \dots, l-1$ , and  $|u_l|_\infty \leq R$ ,  $|u_l| \leq R$ ,  $|v_l| \leq R$  where  $C(R)$  depends on the data,  $l$  and  $R$ .

**Proof.** The proof is lengthy and we omit the details.

We first decompose  $f_l$  in two terms :

$$f_l(u, u_1, \dots, u_{l-1}, u_l) = g_l(u, u_1, \dots, u_{l-1}) + f'(u) u_l,$$

where  $g_l$  is a polynomial in  $u_1, \dots, u_{l-1}$  whose coefficient depends on  $f^{(k)}(u)$  for  $k \leq l$ . Then, we write :

$$f_l(u, u_1, \dots, u_{l-1}, u_l) - f_l(v, v_1, \dots, v_{l-1}, v_l) = g_l(u, u_1, \dots, u_{l-1}) - g_l(v, u_1, \dots, u_{l-1}) + g_l(v, u_1, \dots, u_{l-1}) - g_l(v, v_1, u_2, \dots, u_{l-1}) + \dots + g_l(v, v_1, \dots, v_{l-2}, u_{l-1}) - g_l(v, v_1, \dots, v_{l-1}) + f'(u) u_l - f'(v) v_l.$$

The first term is a sum of monomials in  $u_1, \dots, u_{l-1}$  whose coefficients are a constant times  $f^{(k)}(u) - f^{(k)}(v)$ . Therefore since all the derivatives of  $f$  are Lipschitz and  $u_1, \dots, u_{l-1}$  are bounded in  $L^\infty(\Omega)$ , we have :

$$|g_l(u, u_1, \dots, u_{l-1}) - g_l(v, u_1, \dots, u_{l-1})| \leq \kappa(R) |u-v|.$$

The second term is of the type :

$$\sum_k a_{kg}^{(k)}(v) u_2^{\alpha_{2;k}} \dots u_{l-1}^{\alpha_{l-1;k}} (u_1^{\alpha_{1;k}} - v_1^{\alpha_{1;k}}).$$

Using Hölder inequality and the boundedness of the derivatives of  $g$ , for  $\beta_{i,k} = \alpha_{i,k} p_i$  and  $\frac{1}{p_1} + \dots + \frac{1}{p_{l-1}} = 1$ , we obtain

$$\begin{aligned}
& |g(v, u_1, \dots, u_{l-1}) - g(v, v_1, u_2, \dots, u_{l-1})| \\
& \leq \sum_k |a_k| (\sup |g^{(k)}|) |u_1 - v_1|_{L^{\beta_{1,k}}}^{\alpha_{1,k}} |u_2|_{L^{\beta_{2,k}}}^{\alpha_{2,k}} \dots |u_{l-1}|_{L^{\beta_{l-1,k}}}^{\alpha_{l-1,k}}, \\
& \leq \sum_k |a_k| (\sup |g^{(k)}|) (|u_1|_{L^{\beta_{1,k}}} + |v_1|_{L^{\beta_{1,k}}})^{\alpha_{1,k}-1} |u_2|_{L^{\beta_{2,k}}}^{\alpha_{2,k}} \dots \\
& \qquad \qquad \qquad |u_{l-1}|_{L^{\beta_{l-1,k}}}^{\alpha_{l-1,k}} |u_1 - v_1|_{L^{\beta_{1,k}}}, \\
& \leq \kappa(R) |A(u_1 - v_1)|,
\end{aligned}$$

since, for all  $p$ ,  $L^p(\Omega) \subset H^2(\Omega)$  and the  $H^2$  norm is equivalent to the norm  $|A|$ . We treat the other terms in a similar way except for the last one.

$$\begin{aligned}
|f(u) u_l - f(v) v_l| & \leq |(f(u) - f(v)) u_l| + |f(v) (u_l - v_l)|, \\
& \leq \kappa |u - v| |u_l|_{\infty} + \kappa |u_l - v_l|, \\
& \leq \kappa(R) (|u - v| + |u_l - v_l|).
\end{aligned}$$

All these estimates prove Lemma 3.1.

**Remark 3.1.** We used  $n \leq 4$  only in Lemma 3.1. If we look carefully at the proof we can see that Lemma 3.1 is true for every  $n$  if  $l = 0$  or  $1$ , for  $n \leq 8$  if  $l = 2$ ,  $n \leq 6$  if  $l = 3$  and  $n \leq 5$  for  $l = 4$ . Thus theorem 2.3 is true for larger values of  $n$  when  $k = 1, 2, 3, 4, 5$ .

## Part II : The Cahn-Hilliard equation

### 4. The equation and some properties of the solution

#### 4.1. The equation and the semi-group

We now consider the Cahn Hilliard equation whose unknown is a real valued function  $u(x,t)$  defined on  $\Omega \times \mathbb{R}^+$  where  $\Omega$  denotes a regular bounded set of  $\mathbb{R}^n (n \leq 3)$  :

$$(4.1) \quad \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0, \text{ in } \Omega \times \mathbb{R}^+.$$

Here,  $f$  is a polynomial of odd degree with positive leading coefficient

$$f(u) = \sum_{j=1}^{2r-1} a_j u^j, \quad a_{2r-1} > 0,$$

and we assume that

$$r \geq 2 \text{ if } n = 1,2 \text{ and } r = 2 \text{ if } n = 3.$$

Strictly speaking, the Cahn-Hilliard equation corresponds to  $f(u) = au^3 - bu$ ,  $a, b > 0$ , see [23].

This equation is supplemented with the initial condition :

$$(4.2) \quad u(x,0) = u_0(x), \text{ in } \Omega,$$

and with one of the two following boundary conditions :

$$(4.3) \quad \text{Neumann : } \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma = \partial\Omega,$$

$$\text{Periodic : } \Omega = \prod_{i=1}^n ]0;L_i[ \text{ and } u \text{ is } \Omega \text{ periodic}$$

For the mathematical setting of the equation, it is convenient to introduce the operator  $A = -\Delta$  on  $L^2(\Omega)$  equipped with its usual scalar product  $(\cdot, \cdot)$  and norm  $|\cdot|$ . Then  $A$  is a positive linear unbounded operator on  $H$  with domain

$$D(A) = \{u \in H^2(\Omega), (4.3) \text{ holds}\}.$$

Using this abstract setting, (4.1) (4.2) (4.3) rewrites :

$$(4.4) \quad \begin{aligned} \frac{du}{dt} + A^2u + Af(u) &= 0, \\ u(0) &= u_0. \end{aligned}$$

As shown in [19], for  $u_0$  given in  $L^2(\Omega)$ , the initial boundary problem (4.1) (4.3) possesses a unique solution  $u$  defined for all  $t > 0$  such that

$$u \in C(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; D(A)), \quad \forall T > 0$$

Furthermore, if  $u_0 \in D(A) \cap L^{2r}(\Omega)$ , then

$$u \in C(\mathbb{R}^+; D(A) \cap L^{2r}(\Omega)) \cap L^2(0, T; D(A^2)), \quad \forall T > 0.$$

A particular feature of equation (1.1) is that the average of the solution is conserved, for all  $t > 0$  :

$$m(u(t)) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx = m(u_0).$$

Thus, there can not exist bounded absorbing sets in the whole space  $L^2(\Omega)$ . Therefore, following [19,20], we let the semigroup operate on the subset of  $L^2(\Omega)$

$$H_{\alpha} = \{u \in L^2(\Omega) / |m(u)| \leq \alpha\},$$

for some fixed  $\alpha > 0$ .

We now recall some time uniform estimates borrowed from [19,20]. Let  $u_0$  be given in a ball  $B(0, R)$  in  $H_{\alpha}$ . Then there exists a time  $t_0$  depending on the data, on  $\alpha$  and on  $R$  such that.

$$(4.5) \quad |Au(t)| \leq \kappa_0, \quad \forall t \geq t_0$$

where  $\kappa_0$  is a constant depending on the data and  $\alpha$ . This estimate provides the existence of a bounded absorbing set in  $D(A) \cap H_{\alpha}$ . Since  $D(A) \cap H_{\alpha}$  is compactly imbedded in  $H_{\alpha}$ , this yields also the existence of a universal attractor in  $H_{\alpha}$  (see [19,20]).

The estimate (4.5) is extended in [16] to the time derivatives of  $u$  :

$$u^{(j)} = \frac{\partial^j u}{\partial t^j}, \quad j \in \mathbb{N}.$$

For every  $j \geq 1$ , there exists a time  $t_j$  depending on the data,  $\alpha$ ,  $j$  and  $R$  (when  $|u_0| \leq R$ ) such that :

$$(4.6) \quad |Au^{(j)}(t)| \leq \kappa_j, \quad \forall t \geq t_j,$$

where  $\kappa_j$  is a constant depending on the data and  $\alpha$ .

#### 4.2. Behaviour of small eddies

Let  $(w_j)_{j \in \mathbb{N}}$  denote the orthonormal basis of  $H$  consisting of the eigenvectors of  $A$  and  $(\lambda_j)_{j \in \mathbb{N}}$  the associated eigenvalues

$$\begin{aligned} A w_j &= \lambda_j w_j, \quad j = 1, 2, \dots \\ 0 &= \lambda_1 < \lambda_2 \leq \lambda_3 \dots; \lambda_j \rightarrow +\infty, \text{ as } j \rightarrow \infty. \end{aligned}$$

As in Section 1.2, we fix an integer  $m \in \mathbb{N}$  and denote by  $P = P_m$  the projector in  $H$  onto the space spanned by  $w_1, \dots, w_m$ ; we set  $Q = Q_m = I - P_m$  and for the sake of simplicity

$$\lambda = \lambda_m, \quad \Lambda = \lambda_{m+1}.$$

We also introduce :

$$\delta = \lambda_2 / \lambda_{m+1}$$

Note that :

$$(4.7) \quad |A^{\beta+1/2} p|^2 \leq \lambda |A^\beta p|^2, \quad \forall p \in PD(A^{\beta+1/2}),$$

$$(4.8) \quad |A^{\beta+1/2} q|^2 \geq \Lambda |A^\beta q|^2, \quad \forall q \in QD(A^{\beta+1/2}).$$

By projection of (4.1) on  $PH$  and  $QH$ , it comes that  $p = Pu$ ,  $q = Qu$  satisfy the following coupled system

$$(4.9) \quad \frac{dp}{dt} + A^2 p + PA f(p+q) = 0,$$

$$(4.10) \quad \frac{dq}{dt} + A^2 q + QA f(p+q) = 0.$$

The next proposition states that  $q$  and all its time derivatives remain small for large time.

**Proposition 4.1.** *For all  $j$  in  $\mathbb{N}$ , there exists a time  $t'_j$  which depends on the data,  $\alpha$ ,  $j$  and  $R$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$  such that, for  $t \geq t'_j$*

$$(4.11) \quad |q^{(j)}(t)| \leq \kappa_j \delta^2,$$

$$(4.12) \quad |Aq^{(j)}(t)| \leq \kappa_j \delta,$$

where  $\kappa_j$  is a constant depending on the data,  $\alpha$  and  $j$ .

We omit to give the proof of that proposition that can be found in [16]. It is very similar to the proof of Proposition 1.1.

## 5. Construction of the family of AIM

We present in this Section the method of construction of a sequence  $\mathfrak{M}_i, i \in \mathbb{N}$ , of AIM. As in Part I, the manifolds correspond to approximations of equation (4.10) for  $q$ . Also, these approximations are improved by considering approximations of an increasing number of time derivatives of  $q$ . However since the nonlinear term in (4.1) contains derivatives, the algebra of construction differs from the one in Part I.

We start with the construction of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  which is borrowed from [16]. Proposition 4.1 indicates that  $q$  and  $\frac{dq}{dt}$  are small for large time. Therefore we are lead to replace in (4.10)  $p+q$  by  $p$  while  $\frac{dq}{dt}$  is neglected. This gives the following approximate equation :

$$(5.1) \quad A^2 q + Q A f(p) = 0 .$$

For  $p$  given in  $PH$  since  $A$  is injective on  $QH$ , (5.1) possesses a unique solution  $q_1 \in Q D(A^2)$ . The manifold  $\mathfrak{M}_1$  with equation  $q_1 = \Phi_1(p)$  is an analytic manifold in  $H$  of dimension  $m$ . The orbits are attracted by a thin neighborhood of  $\mathfrak{M}_1$  as shown in the following Theorem.

**Theorem 5.1.** *There exists a time  $t_1^*$  depending on the data,  $\alpha$  and  $R$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$  such that the orbits of (4.1)-(4.3) remain when  $t \geq t_1^*$  at a distance in  $H$  of  $\mathfrak{M}_1$  bounded by  $\kappa_1 \delta^3$ ;  $\kappa_1$  is an appropriate constant which depends on the data and  $\alpha$ .*

**Proof.** Let  $u = p+q$  be an orbit of (4.1)-(4.3). For  $t > 0$ , we define

$$q_1(t) = \Phi_1(p(t)).$$

Then,  $p(t) + q_1(t)$  lies in  $\mathfrak{M}_1$  and

$$\text{dist}(u(t), \mathfrak{M}_1) \leq |q(t) - q_1(t)| .$$

Subtracting (4.10) from (5.1), we obtain

$$(5.2) \quad A^2(q_1 - q) = QA(f(p+q) - f(p)) + q^{(1)} .$$



Since, for  $n \leq 3$ ,  $H^2(\Omega)$  is an algebra, it is easy to see that the mapping  $u \rightarrow f(u)$  from  $H^2(\Omega)$  into  $H^2(\Omega)$  is Lipschitzian on the bounded sets of  $H^2(\Omega)$ . Therefore, by (4.5), we have :

$$\|A f(p+q) - A f(p)\| \leq \kappa \|q\|_{H^2(\Omega)}.$$

Then on the subspace

$$\{u \in H^2(\Omega), \int_{\Omega} u dx = 0\},$$

the norm of  $H^2(\Omega)$  is equivalent to the norm  $\|Au\|$ . Thus, using also (4.12), we obtain

$$\|A f(p+q) - A f(p)\| \leq \kappa \delta,$$

for  $t$  large enough. The second term in the right hand side of (5.2) is bounded by using (4.11)

$$\|q^{(1)}\| \leq \kappa \delta^2, \text{ for large } t.$$

Finally, we have obtained that

$$(5.3) \quad \|A^2(q_1 - q)\| \leq \kappa \delta + \kappa \delta^2,$$

and, thanks to (4.8), this gives

$$\|q_1 - q\| \leq \kappa \delta^3.$$

Theorem 5.1 is proved .

It comes from the proof of Theorem 5.1 (especially (5.3)) that the approximation of the time derivative by 0 is of better order than the one of the nonlinearity. Therefore, we can improve the approximation of (4.10) by replacing (5.1) by

$$(5.4) \quad A^2q + QA f(p+\Phi_1(p)) = 0.$$

For  $p \in P H$ , we define

$$\Phi_2(p) = q_2, \text{ the solution of (5.4).}$$

Let  $\mathfrak{M}_2$  be the analytic manifold of dimension  $m$  in  $H$  with equation  $q = \Phi_2(p)$ . Then

**Theorem 5.2.** *There exists a time  $t_2^*$  depending on the data,  $\alpha$  and  $R$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$  such that the orbits of (4.1)-(4.3) remain when  $t \geq t_2^*$  at a distance in  $H$  of  $\mathfrak{M}_2$ , bounded by  $\kappa_2 \delta^4$ ;  $\kappa_2$  is an appropriate constant depends on the data and  $\alpha$ .*

The proof of this Theorem can be found in [16].

Next, in order to construct  $\mathfrak{M}_3$ , we introduce an approximation of  $q^{(1)}$ . This is done by differentiating (4.10) with respect to  $t$  and neglecting  $q^{(2)}$ . The method is analog to the one in Section 2, but again the approximations of the different terms in (4.10) will be too crude and we will be able to improve them in a fourth step with  $q^{(2)}$  still neglected (the details of the computations for the first six manifolds can be found in [16]). Thus, we see the main difference between the two schemes. For the Cahn-Hilliard, when we neglect  $q^{(k)}$  we consider two successive approximations of (4.10) (and therefore two manifolds).

Now, we describe the algebra. The manifolds are defined recursively. Assume that we have obtained  $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_{2k-3}, \mathfrak{M}_{2k-2}$  for some  $k \geq 2$  (this has been done for  $k=2$ ). We aim to construct  $\mathfrak{M}_{2k-1}$  and  $\mathfrak{M}_{2k}$ . We start by  $\mathfrak{M}_{2k-1}$ ;  $\mathfrak{M}_{2k-1}$  is defined thanks to two sequences  $p_{2k-2i-1}^i, i = 1, \dots, k-1$  and  $q_{2k-2i-1}^i, i = 0, \dots, k-1$ , where  $p_{2k-2i-1}^i$  provides an approximation of  $p^{(i)}$  and  $q_{2k-2i-1}^i$  of  $q^{(i)}$ . These families satisfy estimates of the form

$$(5.5) \quad |p_{2k-2i-1}^i - p^{(i)}| \leq \kappa_k \delta^{2k-2i},$$

$$(5.6) \quad |q_{2k-2i-1}^i - q^{(i)}| \leq \kappa_k \delta^{2k-2i+1},$$

for  $t \geq t_k''$  where  $\kappa_k$  depends on the data,  $\alpha$  and  $k$ ;  $t_k''$  depends on the data,  $\alpha$ ,  $R$  and  $k$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$  (this will be shown in Section 6 below).

We start by defining  $p_{2k-2i-1}$  by induction on  $i$ ,  $i$  increasing. We denote :

$$F_l(u, u^{(1)}, \dots, u^{(l)}) = \frac{d^l}{dt^l} F(u).$$

Then,  $p_{2k-2i-1}^i$ , is given by

$$(5.7) \quad p_{2k-2i-1}^i = -A^2 p_{2k-2i+1}^{i-1} - PA f_{i-1} (p + q_{2k-2i-1}, p_{2k-2i}^1 + q_{2k-2i-1}^1, \dots, p_{2k-2i}^{i-1} + q_{2k-2i-1}^{i-1}),$$

where we have set

$$p_l^0 = p, \forall l.$$

Note that (5.7) is an explicit formula and all the terms in the right-hand side of (5.7) are known either from the construction of the previous manifolds or from step (i-1) of the recursive scheme. Also (5.7) is obtained as an approximation of equation (4.9) for  $p$  differentiated (i-1) times.

Then, the sequence  $q_{2k-2i-1}^i$ ,  $i = k-1, \dots, 1$ , is defined recursively on  $i$  for  $i$  decreasing thanks to the formula

$$(5.8) \quad q_{2k-2i-3}^{i+1} + A^2 q_{2k-2i-1}^i + QA f_i(p+q_{2k-2i-2}, p_{2k-2i-1}^1 + q_{2k-2i-2}^1, \dots, \\ p_{2k-2i-1}^i + q_{2k-2i-2}^i) = 0.$$

We agreed that

$$q_0^1 = q_{-1}^1 = 0, \forall l \geq 1.$$

It is easy to check that all necessary terms are known in (5.8). This formula is obtained by considering equation (4.19) for  $q$  differentiated  $i$  times. Note that for  $i = k-1$ , we have  $q_{2k-2i-3}^{i+1} = 0$ ; this means that the derivative  $q^{(k)}$  is neglected in the construction of  $\mathfrak{M}_{2k-1}$ .

Finally,  $q_{2k-1} = q_{2k-1}^0 = \Phi_{2k-1}(p)$  is given by the analog of (5.8) for  $i=0$ :

$$q_{2k-3}^1 + A^2 q_{2k-1} + QA f(p+q_{2k-2}) = 0.$$

We now proceed to the construction of  $\mathfrak{M}_{2k}$ . For that purpose, we introduce two families  $p_{2k-2i}^i$ ,  $i=1, \dots, k-1$  and  $q_{2k-2i}^i$ ,  $i=0, \dots, k-1$  which satisfy the estimates

$$(5.9) \quad |p_{2k-2i}^i - p^{(i)}| \leq \kappa_k \delta^{2k-2i+1},$$

$$(5.10) \quad |q_{2k-2i}^i - q^{(i)}| \leq \kappa_k \delta^{2k-2i+2},$$

$\forall t \geq t''_k$  where  $\kappa_k$  and  $t''_k$  are as in (5.5) (5.6).

The construction is analogous to the one of  $\mathfrak{M}_{2k-1}$ , so that we only give the formulas. We first define  $p_{2k-2i}^i$  by

$$p_{2k-2i}^i = -A^2 p_{2k-2i+2}^{i-1} - PA f_{i-1}(p+q_{2k-2i}, p_{2k-2i+1}^1 + q_{2k-2i}^1, \dots, p_{2k-2i+1}^{i-1} + q_{2k-2i}^{i-1}),$$

for  $i = 1, \dots, k-1$ . Then  $q_{2k-i}^i, i = k-1, \dots, 1$  is given by

$$(5.11) q_{2k-2i-2}^{i+1} + A^2 q_{2k-2i}^i + QA f_i(p+q_{2k-2i-1}, p_{2k-2i-1}^1 + q_{2k-2i-1}^1, \dots, p_{2k-2i-1}^i + q_{2k-2i-1}^i) = 0$$

Again it follows from (5.11) for  $i = k-1$  that  $q^{(k)}$  is neglected in the construction of  $\mathfrak{M}_{2k}$ .

Lastly for  $p$  given in PH,  $\Phi_{2k}(p) = q_{2k}$  is the solution of

$$q_{2k-2}^1 + A^2 q_{2k} + QA f(p + q_{2k-1}) = 0.$$

To conclude, we have defined the mapping  $\Phi_k : PH \rightarrow QD(A^2)$  for all  $k \in \mathbb{N}^*$ . The graph of this mapping is an analytic manifold  $\mathfrak{M}_k$  of dimension  $m$  in  $H$ . The  $\mathfrak{M}_k$ 's provide a better and better order approximation to the orbits as  $k$  increases.

**Theorem 4.3.** *There exists a time  $t_k^*$  depending on the data,  $k$ ,  $\alpha$  and  $R$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$  such the orbits of (4.1)-(4.3) remain when  $t \geq t_k^*$  at a distance in  $H$  of  $\mathfrak{M}_k$  bounded by  $\kappa_k \delta^{k+2}$ ;  $\kappa_k$  is an appropriate constant depending on the data,  $k$  and  $\alpha$ .*

**Remark 4.4.** Remarks similar to Remark 2.4 and 2.5 can be made here. The details are left to the reader.

**Remark 5.4.** On the contrary of the case of reaction-diffusion equations, the schemes that we have just defined can not be improved.

**Proof of Theorem 4.3.** We postpone to Section 6 the proof of estimates (5.5) (5.6) (5.9) (5.10). Let us show here that these estimates yield Theorem 4.3.

Let  $u(t) = p(t) + q(t)$  be a solution of (4.1)-(4.3) and let  $p(t) + q_k(t)$  the induced trajectory on  $\mathfrak{M}_k$ . We aim to estimate

$$x_k(t) = q_k(t) - q(t).$$

It follows from (5.9) and (5.13) that  $q_k$  satisfies

$$q_{k-2}^1 + A^2 q_k + QA f(p + q_{k-1}) = 0$$

By subtracting (4.10) to this equation, we obtain

$$(5.14) \quad A^2\chi_k = QA (f(p+q) - f(p+q_{k-1})) + q^{(1)} - q_{k-2}^1 .$$

It is clear from (4.6) and (5.6) (5.10) that  $p$ ,  $q$  and  $q_{k-1}$  are bounded for large  $t$  by constants depending only on the data,  $k$  and  $\alpha$ . Therefore, since  $f(u)$  is lipschitzian on the bounded sets of  $H^2(\Omega)$ , (5.14) yields

$$(5.15) \quad |A^2\chi_k| \leq \kappa |A(q - q_{k-1})| + |q^{(1)} - q_{k-2}^1| ,$$

and using (5.6) (5.10),

$$|A^2\chi_k| \leq \kappa \delta^k ,$$

$$|\chi_k| \leq \kappa \delta^{k+2} .$$

Theorem 4.3 is proved.

## **6. Proof of the estimates on time derivatives**

The aim of this Section is to derive estimates (5.5) (5.6) (5.9) (5.10). We will prove by induction on  $k$  that

$$(6.1)_k \quad |p_{2k-2i-1}^i - p^{(i)}| \leq \kappa_k \delta^{2k-2i} , \quad i = 0, \dots, k-1 ,$$

$$(6.2)_k \quad |A(q_{2k-2i-1}^i - q^{(i)})| \leq \kappa_k \delta^{2k-2i} , \quad i = 0, \dots, k-1 ,$$

$$(6.3)_k \quad |p_{2k-2i}^i - p^{(i)}| \leq \kappa_k \delta^{2k-2i+1} , \quad i = 0, \dots, k-1 ,$$

$$(6.4)_k \quad |A(q_{2k-2i}^i - q^{(i)})| \leq \kappa_k \delta^{2k-2i+1} , \quad i = 0, \dots, k-1 ,$$

for  $t \geq t_k$  where  $\kappa_k$  depends on the data,  $k$ ,  $\alpha$  and  $t_k$  depends on the data,  $k$ ,  $\alpha$  and  $R$  when  $|u_0| \leq R$  and  $u_0 \in H_\alpha$ . We recall that  $p^0_l = p$ ,  $\forall l$ .

For  $k = 1$ , (6.1)<sub>1</sub> and (6.3)<sub>1</sub> are obvious. Also (5.3) along with (4.8) yield (6.2)<sub>1</sub>. Finally, (6.4)<sub>1</sub> states the result of approximation for the manifold  $\mathfrak{M}_2$  that can be found in [16] and is derived with techniques similar to the ones used here.

Next assume that (6.1)<sub>k</sub>-(6.4)<sub>k</sub> are true up to order  $k-1$  for  $k > 1$ . We aim to prove them at order  $k$ . Following the schemes of construction of the manifolds, the relations (6.1)<sub>k</sub> (6.2)<sub>k</sub> (6.3)<sub>k</sub> (6.4)<sub>k</sub> will be proved successively.

We start by  $(6.1)_k$  and argue by induction on  $i$ , for  $i$  increasing. For  $i = 0$ , relation  $(6.1)_k$  is obvious.

Assume it is true for  $1, 2, \dots, i-1$ . By subtracting (5.7) from the  $(i-1)$ th derivative of (4.9), we obtain

$$(6.5) \quad |p_{2k-2i-1}^i - p^{(i)}| \leq |A^2(p_{2k-2i+1}^{i-1} - p^{(i-1)})| + |A(f_{i-1}(p+q_{2k-2i-1}, p_{2k-2i}^1 + q_{2k-2i-1}^1, \dots, p_{2k-2i}^{i-1} + q_{2k-2i-1}^{i-1}) - f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}))|.$$

Thanks to the induction assumption for  $i-1$  and (4.7), we have for large  $t$

$$(6.6) \quad |A^2(p_{2k-2i+1}^{i-1} - p^{(i-1)})| \leq \lambda^2 |p_{2k-2i+1}^{i-1} - p^{(i-1)}| \leq \kappa \lambda^2 \delta^{2k-2i+2} \leq \kappa \delta^{2k-2i}.$$

Next, for majorizing the second term in the right handside of (6.5), we use Lemma 6.1 below. Indeed, thanks to (4.6) and the induction assumption, it is easy to check that

$$p+q_{2k-2i-1}, p_{2k-2i}^\alpha + q_{2k-2i-1}^\alpha, p+q, p^{(\alpha)} + q^{(\alpha)}, 1 \leq \alpha \leq i-1$$

are bounded in  $H^2(\Omega)$  by constants depending only on the data,  $k$  and  $\alpha$  for large time. Therefore Lemma 6.1 applies and yields

$$(6.7) \quad |A(f_{i-1}(p+q_{2k-2i-1}, p_{2k-2i}^1 + q_{2k-2i-1}^1, \dots, p_{2k-2i}^{i-1} + q_{2k-2i-1}^{i-1}) - f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}))| \leq \kappa \{ \|p_{2k-2i}^1 - p^{(1)}\|_{H^2(\Omega)} + \dots + \|p_{2k-2i}^{i-1} - p^{(i-1)}\|_{H^2(\Omega)} + \|q_{2k-2i-1} - q\|_{H^2(\Omega)} + \|q_{2k-2i}^1 - q^{(1)}\|_{H^2(\Omega)} + \dots + \|q_{2k-2i-1}^{i-1} - q^{(i-1)}\|_{H^2(\Omega)} \}.$$

For  $1 \leq \alpha \leq i-1$ , we have :

$$\|p_{2k-2i}^\alpha - p^{(\alpha)}\|_{H^2(\Omega)} \leq c_1 \{ |p_{2k-2i}^\alpha - p^{(\alpha)}| + |A(p_{2k-2i}^\alpha - p^{(\alpha)})| \},$$

with  $c_1$  a universal constant,

$$\begin{aligned} &\leq (\text{thanks to (4.7)}), \\ &\leq c_1(1+\lambda) |p_{2k-2i}^\alpha - p^{(\alpha)}|. \end{aligned}$$

Therefore using  $(6.3)_{k-i+\alpha}$

$$(6.8) \quad \|p_{2k-2i}^\alpha - p^{(\alpha)}\|_{H^2(\Omega)} \leq c_1(1+\lambda) \kappa \delta^{2k-2i+1}, \\ \leq \kappa \delta^{2k-2i}.$$

Also, for  $0 \leq \alpha \leq i-1$ , we see that

$$\|q_{2k-2i-1}^\alpha - q^{(\alpha)}\|_{H^2(\Omega)} \leq c_2 |A(q_{2k-2i-1}^\alpha - q^{(\alpha)})|,$$

where  $c_2$  is a universal constant and thanks to (6.2)<sub>k-i+α</sub>

$$(6.9) \quad \|q_{2k-2i-1}^\alpha - q^{(\alpha)}\|_{H^2(\Omega)} \leq c_2 \kappa \delta^{2k-2i}$$

To conclude, we infer from (6.5)-(6.9) that

$$|p_{2k-2i-1}^i - p^{(i)}| \leq \kappa \delta^{2k-2i}, \text{ for large } t,$$

which is (6.1)<sub>k</sub>.

The estimates (6.2)<sub>k</sub>, (6.3)<sub>k</sub>, (6.4)<sub>k</sub> are proved thanks to similar arguments, except that the induction is for  $i$  decreasing for (6.2)<sub>k</sub> and (6.4)<sub>k</sub>. The details of these very technical estimates are left to the reader.

To conclude, there remains to check the following Lemma.

**Lemma 6.1. The mapping**

$$F_k : H^2(\Omega)^{k+1} \rightarrow H^2(\Omega),$$

$$(u_0, \dots, u_k) \rightarrow F_k(u_0, u_1, \dots, u_k),$$

is lipschitzian on the bounded sets of  $H^2(\Omega)^{k+1}$ .

The proof is easy since  $f_k$  is a polynomial and  $H^2(\Omega)$  is an algebra (recall that  $\Omega \subset \mathbb{R}^n$ ,  $n \leq 3$ ). The details are left to the reader.

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**3<sup>ème</sup> PARTIE:**

**UNE PERTURBATION SINGULIERE  
DE L'EQUATION DE CAHN-HILLIARD.**



# A SINGULAR PERTURBATION OF THE CAHN-HILLIARD EQUATION

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## INTRODUCTION.

Let us consider the Cahn-Hilliard equation :

$$\frac{\partial u}{\partial t} + \nu \Delta^2 u - \Delta f(u) = 0, \text{ in } \Omega \times \mathbb{R}^+, \quad (0.1)$$

where  $u = u(x, t)$  is a real-valued function,  $\Omega$  is an open set in  $\mathbb{R}^n$  and the nonlinear function  $f$  is a polynomial of even order whose leading coefficient is positive.

We know that (0.1) (together with Neumann or periodic boundary conditions) possesses solutions defined for all time and those solutions converge to a global attractor  $\mathcal{A}$  (see R. Temam [1] and the references quoted there).

We are interested in a singular perturbation of that equation that we obtain by adding a second order time derivative with a small coefficient :

$$\epsilon \frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial u_\epsilon}{\partial t} + \nu \Delta^2 u_\epsilon - \Delta f(u_\epsilon) = 0, \quad (0.2)$$

we want to show the same type of results for (0.2) as for (0.1) : existence of solutions and of a global attractor  $\mathcal{A}_\epsilon$ . Moreover, we want to study the behaviour of  $\mathcal{A}_\epsilon$  when  $\epsilon$  goes to zero and to know whether  $\mathcal{A}_\epsilon$  "converges" to  $\mathcal{A}$ . More precisely, since  $\mathcal{A}$  lies in  $L^2(\Omega)$  and  $\mathcal{A}_\epsilon$  in  $H^2(\Omega) \times L^2(\Omega)$  (indeed, (0.2) is a second order equation thus the solution is  $(u, \frac{\partial u}{\partial t})$  and lies in a product space), we will first define a convenient embedding  $\mathcal{A}^*$  of  $\mathcal{A}$  in  $H^2(\Omega) \times L^2(\Omega)$  (an element  $u$  in  $\mathcal{A}$  will be identified to the pair  $\{u, v\}$  where  $v$  is the derivative at time 0 of a solution of (0.1) passing through  $u$  at time 0, then we will show that the Hausdorff semidistance  $\delta(\mathcal{A}_\epsilon, \mathcal{A}^*)$  for the topology of  $H^2(\Omega) \times L^2(\Omega)$  converges to zero ; in other words the attractor  $\mathcal{A}_\epsilon$  is upper semicontinuous at  $\epsilon = 0$ . For that purpose, we will need a priori estimates independent of  $\epsilon$  for the solution of (0.2). Such a priori

estimates are not easy to get and that will be the object of Section II. Unfortunately, for technical reasons which will be mentioned below, our results are only valid in space dimension one. In space dimension two or three, we do not know whether there exists an attractor  $\mathcal{A}_\epsilon$  for (0.2) while in the unperturbed case the existence of the attractor is known in space dimension one, two or three.

In a related situation, J.K. Hale and G. Raugel [1] proved the upper semicontinuity of the attractor for a singular perturbation of a reaction diffusion equation.

The lower semicontinuity of  $\mathcal{A}_\epsilon$  is a problem of a totally different nature (see J.K. Hale and G. Raugel [2]) ; it needs very precise information on the attractor and we intend to adress it eventually.

This work is organized as follows : in Section I, we set the notation and state the theorems of existence of solutions for (0.1) and (0.2) ; thus we can define the corresponding semigroups. Concerning (0.1), we do not give the proof that the reader can find in R. Temam [1] ; concerning (0.2), the proof are classical and are sketched in appendix A. We end Section I with a few remarks on Lyapunov functions for (0.1) and (0.2). In Section II, we derive a priori estimates on the solutions of (0.2). We will prove the existence of bounded absorbing sets in different spaces. Moreover, these absorbing sets are independent of  $\epsilon$ . In Section III, we use the results of Section II to prove the existence of attractors for (0.2), these are attractors in the weak topology (whereas the attractor of (0.1) is in the strong topology). In Section IV, we prove the upper semicontinuity of the attractor at  $\epsilon = 0$ . We use the same type of methods as J.K. Hale and G. Raugel [1].

## I. Preliminaries

### a) Notations.

$\Omega$  is the interval  $[0, L]$  in  $\mathbb{R}$ . We denote by  $\| \cdot \|$  and  $(\cdot, \cdot)$  the norm and scalar product in  $L^2(\Omega)$ . For each  $u$  in  $L^2(\Omega)$ , we denote by  $m(u)$  its average :

$$m(u) = \frac{1}{L} \int_{\Omega} u(x) dx. \quad (1.1)$$

$\dot{L}^2(\Omega)$  is the subspace of  $L^2(\Omega)$  consisting of functions whose average is zero :

$$\dot{L}^2(\Omega) = \{u \in L^2(\Omega) / m(u) = 0\}.$$

For each  $u$  in  $L^2(\Omega)$ ,  $\tilde{u}$  is its projection on  $\dot{L}^2(\Omega)$  :

$$\tilde{u} = u - m(u). \quad (1.2)$$

We introduce the unbounded linear operator given by

$$A = \Delta^2,$$

$$D(A) = \{u \in H^4(\Omega) / \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \quad (1.3)_N$$

if we consider the Neumann boundary conditions for (0.1), (0.2), or

$$D(A) = H_{per}^4(\Omega), \quad (1.3)_P$$

if we consider periodic boundary conditions ( $H_{per}^4(\Omega)$  is the space of functions of  $H^4(\Omega)$  whose derivatives of order less than three are equal at 0 and  $L$ ).  $A$  is a positive self-adjoint operator, it possesses an orthonormal basis of eigenvectors  $(w_j)_{j \in \mathbb{N}}$ . We denote by  $(\lambda_j)_{j \in \mathbb{N}}$  the associated eigenvalues, we then have :

$$\begin{cases} \forall i, j \in \mathbb{N} : (w_i, w_j) = \delta_{i,j} \\ Aw_i = \lambda_i w_i \\ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow +\infty. \end{cases}$$

Moreover,  $w_0$  is the constant function equal to  $\frac{1}{L}$  and for all  $u$  in  $L^2(\Omega)$  :

$$m(u) = (u, w_0). \quad (1.4)$$

For all  $s$  in  $\mathbb{R}$ , we recall that  $A^s$  is defined by

$$A^s u = \sum_{j=1}^{+\infty} \lambda_j^s u_j w_j, \quad \forall u = \sum_{j=0}^{+\infty} u_j w_j.$$

We denote its domain by

$$V_s = D(A^{s/4}). \quad (1.5)$$

$V_s(1)$  is endowed with the seminorm and semiscalar product :

$$|u|_s = |A^{s/4} u|, \quad (1.6)$$

$$(u, v)_s = (A^{s/4} u, A^{s/4} v), \quad (1.7)$$

and with the norm :

$$\|u\|_s = (|u|_s^2 + m(u)^2)^{1/2}. \quad (1.8)$$

---

(<sup>1</sup>) Usually,  $V_s$  denotes the domain of  $A^{s/2}$ . Here, we prefer to use the domain of  $A^{s/4}$ , so that  $V_s$  is embedded in the Sobolev space  $H^s(\Omega)$  (when  $s$  is positive).

Moreover, for  $u = \sum_{j=0}^{+\infty} u_j w_j$ , we have

$$u \in V_s \text{ iff } \sum_{j=1}^{+\infty} \lambda_j^{s/2} u_j^2 < +\infty,$$

and then :

$$\|u\|_s = \left( \sum_{j=1}^{+\infty} \lambda_j^{s/2} u_j^2 \right)^{1/2}, \quad m(u) = u_0. \quad (1.9)$$

When  $s$  is positive,  $V_s$  is a subspace of  $H^s(0, L)$  and  $\|\cdot\|_s$  is a norm equivalent to the usual norm of  $H^s(0, L)$ .

We easily deduce from (1.9) the interpolation inequality :

$$\|u\|_{\lambda s_1 + (1-\lambda)s_2} \leq \|u\|_{s_1}^\lambda \|u\|_{s_2}^{1-\lambda}, \text{ for all } u \text{ in } V_{s_2}, \\ s_1 < s_2 \text{ and } \lambda \in [0, 1].$$

Let  $f$  denote a polynomial of even order whose leading coefficient is positive and vanishing at 0 :

$$f(x) = \sum_{i=1}^{2p-1} a_i x^i, \quad a_{2p-1} > 0. \quad (1.10)$$

We denote by  $g$  the primitive of  $f$  vanishing at 0 :

$$g(x) = \sum_{i=2}^{2p} \frac{a_{i-1}}{i} x^i.$$

## b) The Cahn Hilliard equation.

The unknown function is a scalar  $u(x, t)$ ,  $x \in [0, L]$ ,  $t \in \mathbb{R}^+$ , and the equation reads :

$$\frac{\partial u}{\partial t} + \nu \Delta^2 u - \Delta f(u) = 0 \text{ in } [0, L] \times \mathbb{R}^+, \quad (1.11)$$

where  $\nu$  is a strictly positive real parameter. We supplement (1.11) with an initial condition:

$$u(x, 0) = u_0(x) \text{ for } x \text{ in } [0, L], \quad (1.12)$$

and with boundary conditions that can be one of two types. Either the Neumann boundary conditions :

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ at } 0 \text{ and } L, \quad (1.13)_N$$

or the periodic boundary conditions :

$$\frac{\partial^\alpha u}{\partial x^\alpha}(0, \cdot) = \frac{\partial^\alpha u}{\partial x^\alpha}(L, \cdot) \text{ for } \alpha \leq 3. \quad (1.13)_P$$

(We will write (1.13) for  $(1.13)_N$  or  $(1.13)_P$ ; there will be no confusion since the two cases are treated in the same way. The domain of  $A$  will be  $(1.3)_N$  if we consider  $(1.13)_N$  and  $(1.3)_P$  if we consider  $(1.13)_P$ ).

We then have an existence and uniqueness theorem for solutions whose proof can be found in R. Temam [1] :

**Theorem 1.1.** *For all  $u_0$  in  $V_0$ , the equation (1.11)(1.12)(1.13) has a unique solution  $u$  which belongs to*

$$C([0, T]; V_0) \cap L^2(0, T; V_2) \cap L^{2p}(0, T; L^{2p}(0, L))$$

for all  $T > 0$ .

The mapping  $S(t) : u_0 \rightarrow u(t)$  is continuous in  $V_0$  and the family  $(S(t))_{t \geq 0}$  is a semigroup.

The function  $J(u) = \frac{\nu}{2} \|u\|_1^2 + \int_\Omega g(u) dx$  decays along the orbits.

Moreover, if  $u_0$  is in  $V_2$ , then :

$$u \in C([0, T]; V_2) \cap L^2(0, T; V_4)$$

for all  $T > 0$ .

**Remark.** Theorem 1.1 is still true in space dimension  $n = 2$  and, if we assume  $p = 2$ , in space dimension  $n = 3$ .

c) The perturbed Cahn-Hilliard equation.

The unknown function is a scalar  $u_\epsilon(x, t)$ ,  $x \in [0, L]$ ,  $t \in \mathbb{R}^+$ , and the equation reads:

$$\epsilon \frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial u_\epsilon}{\partial t} + \nu \Delta^2 u_\epsilon - \Delta f(u_\epsilon) = 0 \text{ in } [0, L] \times \mathbb{R}^+, \quad (1.14)$$

where  $\epsilon$  is a strictly positive real parameter which is supposed to be small.  $f$  and  $\nu$  are as in section b). We supplement (1.14) with the initial conditions

$$\begin{cases} u_\epsilon(x, 0) = u_0(x), \\ \frac{\partial u_\epsilon(x, 0)}{\partial t} = u_1(x), \text{ for all } x \in \Omega, \end{cases} \quad (1.15)$$



and with the same boundary conditions (1.13)<sub>N</sub> or (1.13)<sub>p</sub>.

For the existence and uniqueness of solutions, we have the following theorem whose proof can be found in Appendix A.

**Theorem 1.2.** *For  $s = 1, 2$  or  $3$  and for all  $(u_0, u_1)$  in  $V_s \times V_{s-2}$ , equation (1.14)(1.15)(1.13) possesses a unique solution  $u_\epsilon$  and*

$$\left( u_\epsilon, \frac{\partial u_\epsilon}{\partial t} \right) \in C([0, T], V_s \times V_{s-2}),$$

for all  $T > 0$ . The mapping :

$$S_\epsilon^s(t) : (u_0, u_1) \rightarrow \left( u_\epsilon(t), \frac{\partial u_\epsilon}{\partial t}(t) \right)$$

is continuous from  $V_s \times V_{s-2}$  into itself and the family  $(S_\epsilon^s(t))_{t \geq 0}$  forms a semigroup.

**Remark.** We can easily prove that  $(S_\epsilon^s(t))$  is in fact an homeomorphism from  $V_s \times V_{s-2}$  into itself and is defined for all  $t$  in  $\mathbb{R}$ . Therefore  $(S_\epsilon^s(t))_{t \geq 0}$  is in fact a group. Thus there is a difference with the semigroup introduced in the preceding section for the Cahn-Hilliard equation that has a regularizing effect and thus cannot be invertible.

#### d) Evolution of the average.

In order to study the long time behaviour of the solutions of equation (1.14), it is useful to rewrite the equation as a system for its space average  $m(u_\epsilon)$  and  $\tilde{u}_\epsilon = u_\epsilon - m(u_\epsilon)$ . We take the scalar product of (1.19) with  $w_0 = \frac{1}{L}$  (this amounts to taking the average of (1.14)); we obtain

$$\epsilon \frac{d^2}{dt^2} m(u_\epsilon) + \frac{d}{dt} m(u_\epsilon) = 0. \quad (1.17)$$

We subtract (1.17) from (1.14), and obtain

$$\epsilon \frac{d^2}{dt^2} \tilde{u}_\epsilon + \frac{d}{dt} \tilde{u}_\epsilon + \nu \Delta^2 \tilde{u}_\epsilon - \Delta f(\tilde{u}_\epsilon + m(u_\epsilon)) = 0. \quad (1.18)$$

We supplement (1.17)(1.18) with the obvious initial conditions :

$$m(u_\epsilon)(0) = m(u_0), \quad m \left( \frac{du_\epsilon}{dt} \right) (0) = m(u_1), \quad (1.19)$$

$$\tilde{u}_\epsilon(0) = \tilde{u}_0, \quad \frac{\partial \tilde{u}_\epsilon}{\partial t}(0) = \tilde{u}_1. \quad (1.20)$$

Then (1.14)(1.15)(1.13) is equivalent to the system (1.17)(1.20) (supplemented by boundary conditions for  $\tilde{u}_\epsilon$ ).

Moreover, (1.17)(1.18) can be solved explicitly for  $m(u_\epsilon)$ , and we obtain

$$m(u_\epsilon) = m(u_0) + \epsilon m(u_1)(1 - e^{-t/\epsilon}).$$

We easily deduce :

**Proposition 1.1.** Let  $u$  be the solution of (1.14)(1.15)(1.13) with  $(u_0, u_1) \in V_s \times V_{s-2}$  ( $s = 1, 2$  or  $3$ ), then  $m(u_\epsilon) + \epsilon m\left(\frac{\partial u_\epsilon}{\partial t}\right)$  is independent of  $t$  and

$$\begin{aligned} m(u_\epsilon) &\rightarrow m(u_0) + \epsilon m(u_1) \text{ when } t \rightarrow +\infty, \\ m\left(\frac{\partial u_\epsilon}{\partial t}\right) &\rightarrow 0 \text{ when } t \rightarrow +\infty. \end{aligned}$$

For the Cahn Hilliard equation, the average of the solution is constant and the equivalent system formally reduces to the preceding case with  $\epsilon = 0$ .

e) Lyapunov functions.

Thanks to Theorem 1.1, the function  $J(u) = \frac{\nu}{2} |u|_1^2 + \int_\Omega g(u)dx$  is a Lyapunov function on  $V_0$  for the Cahn-Hilliard equation. For the perturbed equation (1.14), we do not have a Lyapunov function on the whole space  $V_s \times V_{s-2}$ . We take the scalar product of (1.14) with  $A^{-1/2} \frac{du_\epsilon}{dt}$ , we obtain :

$$\frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{du'_\epsilon}{dt} \right|_{-1}^2 + \nu |u_\epsilon|_1^2 \right) + \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 + \left( f(u_\epsilon), \frac{du_\epsilon}{dt} \right)_0 = 0.$$

But,

$$\begin{aligned} \left( f(u_\epsilon), \frac{du_\epsilon}{dt} \right)_0 &= \left( f(u_\epsilon), \frac{du_\epsilon}{dt} \right) - \left( f(u_\epsilon), m \left( \frac{du_\epsilon}{dt} \right) \right) \\ &= \frac{d}{dt} \int_\Omega g(u_\epsilon)dx - \left( f(u_\epsilon), m \left( \frac{du_\epsilon}{dt} \right) \right). \end{aligned}$$

Thus :

$$\frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{du'_\epsilon}{dt} \right|_{-1}^2 + \nu |u_\epsilon|_1^2 + 2 \int_\Omega g(u_\epsilon)dx \right) + \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 - \left( f(u_\epsilon), m \left( \frac{du_\epsilon}{dt} \right) \right) = 0.$$

We cannot conclude in general that the function  $J_\epsilon(u, v) = \epsilon |v|_{-1}^2 + \nu |u|_1^2 + 2 \int_\Omega g(u)dx$  decays along the orbits of (1.14)(1.15)(1.13) (we do not know the sign of  $(f(u_\epsilon), m(\frac{du_\epsilon}{dt}))$ ).

If we restrict the semigroup  $(S_\epsilon^s(t))_{t \geq 0}$  to the subspace :

$$\chi_0 = \{(u, v) \in V_s \times V_{s-2} / m(v) = 0\}$$

(this space is stable for the semigroup thanks to Proposition 1.1), then  $J_\epsilon(u, v)$  decays along the orbits in  $\chi_0$ . Thus, we have a Lyapunov function on the subspace  $\chi_0$ .

## II. Time uniform estimate for the perturbed Cahn-Hilliard equation.

a) In this section, we prove the existence of bounded absorbing sets for the equation (1.14)(1.15)(1.13) in the spaces  $X_s^\alpha$  that we define hereafter. As we will study in Section IV the behaviour when  $\epsilon$  converges to zero, we will always make precise the dependence of our estimate on  $\epsilon$ , and the constants that appear are independent of  $\epsilon$ .  $\epsilon$  is supposed to be in a fixed interval  $]0, \epsilon_0[$ .

Due to Proposition 1.1 (in particular to the fact that  $m(u_\epsilon) + \epsilon m\left(\frac{du_\epsilon}{dt}\right)$  is independent of  $t$ ), it is impossible to find bounded absorbing sets in the whole space  $V_s \times V_{s-2}$ , thus we define for  $s = 1, 2$  or  $3$  and for all  $\alpha$  in  $\mathbb{R}^+$  :

$$X_s^\alpha = \{(u, v) \in V_s \times V_{s-2} : |m(u)| + \epsilon_0 |m(v)| \leq \alpha\}.$$

Thanks to Proposition 1.1,  $X_s^\alpha$  is stable for the semigroup  $(S_\epsilon^s(t))_{t \geq 0}$ , we will prove the existence of bounded absorbing sets for the restriction of  $S_\epsilon^s(t)$  to  $X_s^\alpha$ . We first give a few lemmas that we will need.

Lemma 2.1. For  $s = 1, 2$  or  $3$ , let  $u \in L^2(0, T; V_s)$  satisfy :

$$\frac{du}{dt} \in L^2(0, T; V_{s-2})$$

and

$$\epsilon \frac{d^2 u}{dt^2} + \nu Au \in L^2(0, T; V_{s-2}),$$

then

$$\left( \epsilon \frac{\partial^2 u}{\partial t^2} + \nu \Delta u, A^{s/2-1} \frac{\partial u}{\partial t} \right) = \frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{\partial u}{\partial t} \right|_{s-2}^2 + |u|_s^2 \right),$$

in the sense of distribution on  $]0, T[$ .

This lemma is classical, the reader can find the proof in J.M. Ghidaglia and R. Temam [1].

**Lemma 2.2.** Let  $0 \leq \lambda \leq \min\left(\frac{\nu}{2}\lambda_1, \frac{1}{2\epsilon_0}\right)$ . Then for all  $(u, v)$  in  $V_s \times V_{s-2}$  :

$$(1 - \lambda\epsilon) |v|_{s-2}^2 + \lambda(\lambda\epsilon - 1)(u, v)_{s-2} + \lambda\nu |u|_s^2 \geq \frac{1}{4} |v|_{s-2}^2 + \frac{\lambda\nu}{2} |u|_s^2.$$

The proof of this lemma is easy, we use the fact that for all  $u$  in  $V_{s-2}$  :

$$|u|_{s-2} \leq \frac{1}{\sqrt{\lambda_1}} |u|_s.$$

We now define the three following functionnals :

$$\begin{cases} \text{For all } \varphi, \psi \text{ in } V_1 \times V_{-1} : \\ E_1(\varphi, \psi) = \epsilon |\psi + \lambda\varphi|_1^2 + \nu |\varphi|_{-1}^2 + 2 \int_{\Omega} g(\varphi) dx \\ \\ \text{For all } \varphi, \psi \text{ in } V_2 \times V_0 : \\ E_2(\varphi, \psi) = \epsilon |\psi + \lambda\varphi|_0^2 + \nu |\varphi|_1^2 + 2 \int_{\Omega} f'(\varphi) |\nabla\varphi|^2 dx \\ \\ \text{For all } \varphi, \psi \text{ in } V_1 \times V_{-1}, u \text{ in } V_2 : \\ E_3(\varphi, \psi) = \epsilon |\psi + \lambda\varphi|_{-1}^2 + \nu |\varphi|_1^2 + 2 \int_{\Omega} f'(u) \tilde{\varphi}^2 dx. \end{cases}$$

( $\lambda$  is the constant appearing in Lemma 2.2).

Then we have :

**Lemma 2.3.** There exist some constants  $c_1, \dots, c_6, c'_1, \dots, c'_6$  independent of  $\epsilon$  such that :

$$\begin{aligned} C_1(\epsilon |\psi|_{-1}^2 + |\varphi|_1^2) - C_2 &\leq E_1(\varphi, \psi) \leq C'_1(\epsilon |\psi|_{-1}^2 + |\varphi|_1^2) + C'_2 \|\varphi\|_1^{2p}, \\ C_3(\epsilon |\psi|_0^2 + |\varphi|_2^2) - C_3 &\leq E_2(\varphi, \psi) \leq C'_3(\epsilon |\psi|_0^2 + |\varphi|_2^2) + C'_4 \|\varphi\|_1^{2p}, \\ C_5(\epsilon |\psi|_{-1}^2 + |\varphi|_1^2) - C_6 &\leq E_3(\varphi, \psi) \leq C'_5(\epsilon |\psi|_{-1}^2 + |\varphi|_1^2) + C'_6 \|u\|_1^{2p-2} |\varphi|_0^2. \end{aligned}$$

**Proof of Lemma 2.3 :** Let  $\varphi, \psi$  in  $V_1 \times V_{-1}$ , then

$$\begin{aligned} \epsilon |\psi + \lambda\varphi|_{-1}^2 &\leq 2(\epsilon |\psi|_{-1}^2 + \epsilon\lambda^2 |\varphi|_{-1}^2) \\ &\leq 2\left(\epsilon |\psi|_{-1}^2 + \frac{\epsilon}{4\epsilon_0^2} |\varphi|_{-1}^2\right) \\ &\leq 2\left(\epsilon |\psi|_{-1}^2 + \frac{1}{2\epsilon_0\lambda_1} |\varphi|_1^2\right) \end{aligned}$$

$g$  is a polynomial of even order with a positive leading coefficient, therefore we can find two positive constants  $C_2$  and  $k$  such that

$$k |x|^{2p} \geq g(x) \geq -\frac{C_2}{2|\Omega|}, \text{ for all real } x.$$

We deduce :

$$2 \int_{\Omega} g(\varphi) dx \geq -C_2,$$

and the left hand side of the first inequalities follows.

Thanks to Sobolev inequalities, we know that  $L^\infty(\Omega)$  is embedded in  $H^1(\Omega)$  and we can find a positive constant  $C$  such that :

$$\|\varphi\|_{L^\infty} \leq C \|\varphi\|_1.$$

We deduce :

$$\begin{aligned} 2 \int_{\Omega} g(\varphi) dx &\leq 2k \|\varphi\|_{L^\infty}^{2p} |\Omega| \\ &\leq 2kC \|\varphi\|_1^{2p}, \end{aligned}$$

and the right hand side of the first inequality follows.

For the two other inequalities, we use the same type of arguments.

b) Absorbing set in  $X_1^\alpha$  :

**Proposition 2.1.** *There exist two constants  $R_1, R_2$  independent of  $\epsilon$  such that for all  $u_\epsilon$  solution of (1.14)(1.15)(1.13) with  $(u_0, u_1)$  in  $X_1^\alpha$ , we have*

$$\epsilon \left| \frac{du_\epsilon}{dt}(t) \right|_{-1}^2 + \|u_\epsilon(t)\|_1^2 \leq R_1 + R_2(\epsilon \|u_1\|_{-1}^2 + \|u_0\|_1^2 + \|u_0\|_1^{2p}) e^{-\frac{\lambda}{2}t}.$$

We easily deduce, thanks to Proposition 1.1 :

**Corollary 2.1.** *The semigroup  $(S_\epsilon^1(t))_{t \geq 0}$  possesses a bounded absorbing set in  $X_1^\alpha$  for all  $\alpha$ .*

**Proof of Proposition 2.1** : Let  $\lambda$  be a positive real number smaller than  $\inf\left(\frac{\nu}{2}\lambda_1, \frac{1}{2\epsilon_0}\right)$  (see Lemma 2.2) and  $u_\epsilon$  be the solution of (1.14)(1.15)(1.13) where  $(u_0, u_1)$  is in  $X_1^\alpha$ . We set

$$v = \frac{du_\epsilon}{dt} + \lambda u_\epsilon,$$

and rewrite (1.14) as follows :

$$\epsilon \frac{dv}{dt} + (1 - \lambda\epsilon)v + \lambda(\lambda\epsilon - 1)u_\epsilon + \nu\Delta^2 u_\epsilon - \Delta f(u_\epsilon) = 0. \quad (2.1)$$

We take the scalar product of (2.1) with  $A^{-1/2}v$  in  $L^2$  :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\epsilon \|v\|_{-1}^2 + \nu \|u_\epsilon\|_1^2 + 2 \int_{\Omega} g(u_\epsilon) dx) + (1 - \lambda\epsilon) \|v\|_{-1}^2 \\ & + \lambda(\lambda\epsilon - 1)(u_\epsilon, v)_{-1} + \lambda\nu \|u_\epsilon\|_1^2 + \lambda(f(u_\epsilon), u_\epsilon) + (f(u_\epsilon), m(v)) = 0. \end{aligned} \quad (2.2)$$

The leading term of  $g(u)$  is  $\frac{a_{2p-1}}{2p}u^{2p}$ , the one of  $f(u)u$  is  $a_{2p-1}u^{2p}$ , therefore :

$$\exists C_1 > 0 : \forall u \in \mathbb{R}, \quad f(u)u \geq g(u) - \frac{C_1}{\lambda |\Omega|}$$

(recall that  $a_{2p-1}$  is positive). We deduce

$$\lambda(f(u_\epsilon), u_\epsilon) \geq \lambda \int_{\Omega} g(u_\epsilon) dx - C_1. \quad (2.3)$$

Moreover, since  $(u_\epsilon, \frac{du_\epsilon}{dt})$  belongs to  $X_\alpha^1$  :

$$\begin{aligned} m(v) &= m\left(\frac{du_\epsilon}{dt}\right) + \lambda m(u_\epsilon) \\ &\leq \alpha \left(\lambda + \frac{1}{\epsilon_0}\right). \end{aligned}$$

There exists a constant  $C_2$  such that :

$$\forall u \in \mathbb{R} : |f(u)| \leq \frac{1}{\alpha \left(\lambda + \frac{1}{\epsilon_0}\right)} \left(\frac{\lambda}{2}g(u) + \frac{C_2}{|\Omega|}\right),$$

which gives :

$$|(f(u_\epsilon), m(v))| \leq \frac{\lambda}{2} \int_{\Omega} g(u_\epsilon) dx + C_2. \quad (2.4)$$

We use (2.3), (2.4) in (2.2), we denote  $C_3 = 2(C_1 + C_2)$  and we use Lemma 2.2, we find that for all time  $t$  :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \epsilon \|v\|_{-1}^2 + \nu \|u_\epsilon\|_1^2 + 2 \int_{\Omega} g(u_\epsilon) dx \right) \\ & + \frac{1}{4} \|v\|_{-1}^2 + \frac{\lambda\nu}{2} \|u_\epsilon\|_1^2 + \frac{\lambda}{2} \int_{\Omega} g(u_\epsilon) dx \leq \frac{C_3}{2}. \end{aligned}$$

Thus :

$$\frac{d}{dt} E_1 \left( u_\epsilon, \frac{du_\epsilon}{dt} \right) + \frac{\lambda}{2} E_1 \left( u_\epsilon, \frac{du_\epsilon}{dt} \right) \leq C_3.$$

Thanks to Gronwall's Lemma :

$$E_1 \left( u_\epsilon, \frac{du_\epsilon}{dt} \right) \leq \frac{2C_3}{\lambda} (1 - e^{-\frac{\lambda}{2}t}) + E_1(u_0, u_1) e^{-\frac{\lambda}{2}t}.$$

We deduce Proposition 2.1, thanks to Lemma 2.3.

c) Absorbing set in  $X_2^\alpha$  :

Proposition 2.2 : *There exist two constants  $R_3, R_4$  independent of  $\epsilon$  such that for all bounded set  $B$  in  $X_2^\alpha$ , there exists a time  $t_0(B)$  (independent of  $\epsilon$ ) such that for all solutions of (1.14), (1.15), (1.13) with  $(u_0, u_1)$  in  $B$  and for all time  $t \geq t_0(B)$  :*

$$\epsilon \left| \frac{du_\epsilon}{dt}(t) \right|_0^2 + \|u_\epsilon(t)\|_2^2 \leq R_3 + R_4 \left( \epsilon \left| \frac{du_\epsilon}{dt}(t_0) \right|_0^2 + \|u_\epsilon(t_0)\|_2^2 \right) e^{-\frac{\lambda}{2}(t-t_0)}.$$

We easily deduce, thanks to Proposition 1.1 :

Corollary 2.2 : *The semigroup  $(S_\epsilon^2(t))_{t \geq 0}$  possesses a bounded absorbing set in  $X_2^\alpha$  for all  $\alpha$ .*

Proof of Proposition 2.2.

We take the scalar product in  $L^2(\Omega)$  of (2.2) with  $\tilde{v}$ , we get :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\epsilon \|v\|_0^2 + \nu \|u_\epsilon\|_2^2) + (1 - \lambda\epsilon) \|v\|_0^2 + \lambda(\lambda\epsilon - 1)(u_\epsilon, v)_0 \\ + \lambda\nu \|u_\epsilon\|_2^2 - (\Delta f(u_\epsilon), v) = 0. \end{aligned} \tag{2.5}$$

We rewrite the last term as follows :

$$\begin{aligned} -(\Delta f(u_\epsilon), v) &= (f'(u_\epsilon) \nabla u_\epsilon, \nabla \frac{du_\epsilon}{dt}) \\ &\quad + \lambda (f'(u_\epsilon) \nabla u_\epsilon, \nabla u_\epsilon), \\ -(\Delta f(u_\epsilon), v) &= \frac{1}{2} \int_\Omega f'(u_\epsilon) \frac{d}{dt} |\nabla u_\epsilon|^2 dx \\ &\quad + \lambda \int_\Omega f'(u_\epsilon) |\nabla u_\epsilon|^2 dx, \end{aligned}$$

$$\begin{aligned}
-(\Delta f(u_\epsilon), v) &= \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} f'(u_\epsilon) |\nabla u_\epsilon|^2 dx \right) \\
&\quad + \lambda \int_{\Omega} f'(u_\epsilon) |\nabla u_\epsilon|^2 dx \\
&\quad - \frac{1}{2} \int_{\Omega} f''(u_\epsilon) \frac{du_\epsilon}{dt} |\nabla u_\epsilon|^2 dx.
\end{aligned} \tag{2.6}$$

We estimate the last term of (2.6) :

$$\begin{aligned}
&\left| \frac{1}{2} \int_{\Omega} f''(u_\epsilon) \frac{du_\epsilon}{dt} |\nabla u_\epsilon|^2 dx \right| \\
&\leq \frac{1}{2} \|f''(u_\epsilon)\|_{L^\infty} \left\| \frac{du_\epsilon}{dt} \right\|_{L^2} \|\nabla u_\epsilon\|_{L^4}^2;
\end{aligned}$$

we use Sobolev inequalities and interpolation inequalities :

$$\begin{aligned}
\exists C_1 > 0 : \|f''(u_\epsilon)\|_{L^\infty} &\leq C_1 (\|u_\epsilon\|_1^{2p-3} + 1), \\
\exists C_2 > 0 : \|\nabla u_\epsilon\|_{L^4}^2 &\leq C_2 \|u_\epsilon\|_1^{3/2} \|u_\epsilon\|_2^{1/2}.
\end{aligned}$$

(Recall that the space dimension is 1 ; therefore  $H^{1/4}(\Omega)$  and a fortiori  $H^{1/2}(\Omega)$  is embedded in  $L^4(\Omega)$ ). We deduce :

$$\begin{aligned}
&\left| \frac{1}{2} \int_{\Omega} f''(u_\epsilon) \frac{du_\epsilon}{dt} |\nabla u_\epsilon|^2 dx \right| \\
&\leq C_1 C_2 (\|u_\epsilon\|_1^{2p-3} + 1) \|u_\epsilon\|_1^{3/2} \|u_\epsilon\|_1^{1/2} \left\| \frac{du_\epsilon}{dt} \right\|_{L^2} \\
&\leq C_5 ((\|u_\epsilon\|_1^{2p-3} + 1) \|u_\epsilon\|_1^{3/2})^4 + \frac{1}{8} \left\| \frac{du_\epsilon}{dt} \right\|_{L^2}^2 + \frac{\lambda\nu}{4} \|u_\epsilon\|_2^2 \\
&\leq C_5 ((\|u_\epsilon\|_1^{2p-3} + 1) \|u_\epsilon\|_1^{3/2})^4 + \frac{\lambda}{8} \|u_\epsilon\|_0^2 + \frac{\alpha^2}{8\epsilon_0} \\
&\quad + \frac{1}{8} \|v\|_0^2 + \frac{\lambda\nu}{4} \|u_\epsilon\|_2^2.
\end{aligned} \tag{2.7}$$

We use (2.6) and (2.7) in (2.5), we obtain thanks to Lemma 2.2:

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\epsilon \|v\|_0^2 + \nu \|u_\epsilon\|_2^2 + \int_{\Omega} f'(u_\epsilon) |\nabla u_\epsilon|^2 dx) \\
&\quad + \frac{1}{8} \|v\|_0^2 + \frac{\lambda\nu}{4} \|u_\epsilon\|_2^2 + \lambda \int_{\Omega} f'(u_\epsilon) |\nabla u_\epsilon|^2 dx \\
&\leq C_5 ((\|u_\epsilon\|_1^{2p-3} + 1) \|u_\epsilon\|_1^{3/2})^4 + \frac{\lambda}{8} \|u_\epsilon\|_0^2 + \frac{\alpha^2}{8\epsilon_0}
\end{aligned} \tag{2.8}$$

There exists a constant  $C_6$  such that :

$$\forall u \in \mathbb{R} \quad \lambda f'(u) \geq \frac{\lambda}{4} f'(u) - C_6 ;$$



therefore :

$$\lambda \int_{\Omega} f'(u_{\epsilon}) |\nabla u_{\epsilon}|^2 dx \geq \frac{\lambda}{4} \int_{\Omega} f'(u_{\epsilon}) |\nabla u_{\epsilon}|^2 dx - C_6 |u_{\epsilon}|_1^2. \quad (2.9)$$

Let  $B$  be a bounded set in  $X_2^{\alpha}$ , then  $B$  is bounded in  $X_1^{\alpha}$ . Thanks to Corollary 2.1, there exists a constant  $C_7$  (independent of  $B$ ) and a time  $t_0(B)$  such that for all time  $t \geq t_0(B)$  :

$$2C_5((\|u_{\epsilon}\|_1^{2p-3} + 1) |u_{\epsilon}|_1^{3/2})^4 + \frac{\lambda}{4} |u_{\epsilon}|_0^2 + \frac{\alpha^2}{4\epsilon_0^2} + 2C_6 |u_{\epsilon}|_1^2 \leq C_7. \quad (2.10)$$

(2.9), (2.10) in (2.8) give :

$$\frac{d}{dt} E_2 \left( u_{\epsilon}, \frac{du_{\epsilon}}{dt} \right) + \frac{\lambda}{2} E_2 \left( u_{\epsilon}, \frac{du_{\epsilon}}{dt} \right) \leq C_7,$$

for all time  $t \geq t_0(B)$ . Thanks to Gronnwall's lemma :

$$E_2 \left( u_{\epsilon}(t), \frac{du_{\epsilon}}{dt}(t) \right) \leq \frac{2C_7}{\lambda} (1 - e^{-\frac{\lambda}{2}(t-t_0)}) + E_2 \left( u_{\epsilon}(t_0), \frac{du_{\epsilon}}{dt}(t_0) \right) e^{-\frac{\lambda}{2}(t-t_0)}.$$

We deduce the Proposition 2.2, thanks to Lemma 2.3.

d) Absorbing set in  $X_3^{\alpha}$  :

**Proposition 2.3** : *There exists two constants  $R_5, R_6$  independent of  $\epsilon$  such that for all bounded set  $B$  in  $X_3^{\alpha}$ , there exists a time  $t_1(B)$  (independent of  $\epsilon$ ) such that for all solutions of (1.14),(1.15), (1.13) with  $(u_0, u_1)$  in  $B$  and for all time  $t \geq t_1(B)$  :*

$$\begin{aligned} \epsilon \left| \frac{d^2 u_{\epsilon}}{dt^2}(t) \right|_{-1}^2 + \left| \frac{du_{\epsilon}}{dt}(t) \right|_1^2 + |u_{\epsilon}(t)|_3^2 \\ \leq R_5 + \frac{R_6}{\epsilon} \left( \left| \frac{du_{\epsilon}}{dt}(t_1) \right|_1^2 + |u_{\epsilon}(t_1)|_3^2 + 1 \right) e^{-\frac{\lambda}{2}(t-t_1)}. \end{aligned}$$

We easily deduce, thanks to Proposition 1.1 :

**Corollary 2.3** : *The semigroup  $(S_{\epsilon}^3(t))_{t \geq 0}$  possesses a bounded absorbing set in  $X_3^{\alpha}$  for all  $x$ .*

In order to prove the Proposition 2.3, we need two lemmas :

**Lemma 2.4.** Let  $u_\epsilon$  be the solution of (1.14)(1.15)(1.13) with  $(u_0, u_1)$  in  $X_1^\alpha$ , then there exist a constant  $K_1$  (independent of  $(u_0, u_1)$  and  $\epsilon$ ) and a time  $t_0(|u_1|_{-1}, |u_0|_1, m(u_1))$  such that for all time  $t \geq t_0$  :

$$\int_{t_0}^t \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 ds \leq K_1.$$

**Lemma 2.5.** Let  $f, g$  be two functions defined on  $\mathbb{R}^+$  such that :

(i)  $g$  is positive,

(ii) there exists  $k_1, k_2, k_3, k_4$  in  $\mathbb{R}^+$  and a time  $t_0$  such that for all time  $t \geq t_0$  :

$$\frac{df}{dt} + k_1 f \leq k_2 + k_3 g f + k_4 g,$$

(iii) there exists  $k_5$  in  $\mathbb{R}^+$  such that for all time  $t \geq t_0$  :  $\int_{t_0}^t g(s) ds \leq k_5$ , then, for all time  $t \geq t_0$ ,

$$\begin{aligned} f(t) &\leq \left( \frac{k_2}{k_1} + k_4 k_5 \right) e^{k_3 k_5} \\ &\quad + \left( f(t_0) - \frac{k_2}{k_1} \right) e^{k_3 k_5 - k_1(t-t_0)}. \end{aligned}$$

**Proof of Lemma 2.4.** We take the scalar product of (1.14) with  $A^{-1/2} \frac{du_\epsilon}{dt}$  :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 + \nu \|u_\epsilon\|_1^2 + \int_\Omega g(u_\epsilon) dx \right) \\ &\quad + \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 - \int_\Omega f(u_\epsilon) m \left( \frac{du_\epsilon}{dt} \right) dx = 0. \end{aligned} \tag{2.11}$$

Thanks to Proposition 2.1, we can find a constant  $C'_1$  and a time  $t_0(|u_0|_{-1}, |u_1|_1)$  such that, for all  $t \geq t_0$  :

$$\epsilon \left| \frac{du_\epsilon}{dt}(t) \right|_{-1}^2 + \nu \|u_\epsilon(t)\|_1^2 + \int_\Omega g(u_\epsilon(t)) dx \leq C'_1. \tag{2.12}$$

Thanks to Sobolev inequalities (the space dimension is one) :

$$\begin{aligned} \left| \int_\Omega f(u_\epsilon) dx \right| &\leq C'_2 (\|u_\epsilon\|_1^{2p-1} + 1) \\ &\leq C'_3, \end{aligned} \tag{2.13}$$

for all  $t \geq t_0$ . We use (2.13) in (2.11), we obtain since  $m \left( \frac{du_\epsilon}{dt} \right) = m(u_1) e^{-t/\epsilon}$  :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 + \nu \|u_\epsilon\|_1^2 + \int_\Omega g(u_\epsilon) dx \right) \\ &\quad + \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 \leq C'_3 \frac{\alpha}{\epsilon_0} e^{-t/\epsilon}. \end{aligned} \tag{2.14}$$

We integrate (2.14) from  $t_0$  to  $t$  and use (2.12) :

$$\begin{aligned} \int_{t_0}^t \left| \frac{du_\epsilon}{dt} \right|_{-1}^2 ds &\leq \frac{1}{2} C'_1 + C'_3 \alpha (e^{-t_0/\epsilon} - e^{-t/\epsilon}) \\ &\leq \frac{1}{2} C'_1 + C'_3 \alpha. \end{aligned}$$

Proof of Lemma 2.5 :

We have for all  $t \geq t_0$  :

$$\frac{df}{dt} + k_1 f \leq k_2 + k_3 g f + k_4 g, \quad (2.15)$$

We multiply (2.15) by  $e^{k_1 t - k_3 \int_{t_0}^t g(s) ds}$  :

$$\begin{aligned} &\frac{d}{dt} \left( e^{k_1 t - k_3 \int_{t_0}^t g(s) ds} f(t) \right) \\ &\leq (k_2 + k_4 g) e^{k_1 t - \int_{t_0}^t g(s) ds} \\ &\leq (k_2 + k_4 g + k_1 k_4 \int_{t_0}^t g(s) ds) e^{k_1 t} \\ &\leq \frac{d}{dt} \left( \frac{k_2}{k_1} + k_4 \int_{t_0}^t g(s) ds \right) e^{k_1 t}. \end{aligned} \quad (2.16)$$

We integrate (2.16) from  $t_0$  to  $t$  :

$$\begin{aligned} &e^{k_1 t - k_3 \int_{t_0}^t g(s) ds} f(t) - e^{k_1 t_0} f(t_0) \leq \\ &\left( \frac{k_2}{k_1} + k_4 \int_{t_0}^t g(s) ds \right) e^{k_1 t} - \frac{k_2}{k_1} e^{k_1 t_0}, \\ f(t) &\leq \left( \frac{k_2}{k_1} + k_4 \int_{t_0}^t g(s) ds \right) e^{k_3 \int_{t_0}^t g(s) ds} \\ &\quad + \left( f(t_0) - \frac{k_2}{k_1} \right) e^{-k_1(t-t_0) - k_3 \int_{t_0}^t g(s) ds} \\ &\leq \left( \frac{k_2}{k_1} + k_4 k_5 \right) e^{k_3 k_5} + \left( f(t_0) - \frac{k_2}{k_1} \right) e^{k_3 k_5} e^{-k_1(t-t_0)}. \end{aligned}$$

Proof of Proposition 2.3 :

We differentiate (2.2) with respect to time and we denote by

$$\begin{aligned} W &= \frac{dv}{dt} = \frac{d^2 u_\epsilon}{dt^2} + \lambda \frac{du_\epsilon}{dt}, \\ w &= \frac{du_\epsilon}{dt}, \end{aligned}$$

we obtain :

$$\epsilon \frac{dW}{dt} + (1 - \lambda\epsilon)W + \lambda(\lambda\epsilon - 1)w + \nu \Delta^2 w - \Delta f'(u_\epsilon)w = 0. \quad (2.17)$$

We take the scalar product of (2.17) with  $A^{-1/2}W$  :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\epsilon \|W\|_{-1}^2 + \nu \|w\|_1^2) + (1 - \lambda\epsilon) \|W\|_{-1}^2 + \lambda(\lambda\epsilon - 1)(w, W)_{-1} \\ + \lambda\nu \|w\|_1^2 + (f'(u_\epsilon)w, W)_0 = 0 \end{aligned} \quad (2.18)$$

We rewrite the last term as follows :

$$\begin{aligned} (f'(u_\epsilon)w, W)_0 &= (f'(u_\epsilon)\tilde{w}, \tilde{w}') \\ &\quad + \lambda(f'(u_\epsilon)\tilde{w}, \tilde{w}) \\ &\quad + (f'(u_\epsilon)m(w), \tilde{W}), \\ (f'(u_\epsilon)w, W)_0 &= \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} f'(u_\epsilon)\tilde{w}^2 dx \right) \\ &\quad + \lambda \int_{\Omega} f'(u_\epsilon)\tilde{w}^2 dx \\ &\quad + m(w)(f'(u_\epsilon), \tilde{W}) \\ &\quad - \frac{1}{2} \int_{\Omega} f''(u_\epsilon)\tilde{w}^3 dx \\ &\quad - \frac{1}{2} m(w) \int_{\Omega} f''(u_\epsilon)\tilde{w}^2 dx. \end{aligned} \quad (2.19)$$

We now estimate the last three terms of (2.19). Since  $H^1(\Omega)$  is an algebra ( $\Omega \subset \mathbb{R}$ ) and  $f''$  is a polynomial of order  $2p - 3$  there exists a constant  $c_1$  such that :

$$\|f''(u_\epsilon)\|_1 \leq C_1 (\|u_\epsilon\|_1^{2p-3} + 1).$$

Therefore :

$$\begin{aligned} |m(w)(f''(u_\epsilon), \tilde{W})| &\leq |m(w)| \|f''(u_\epsilon)\|_1 \|W\|_{-1} \\ &\leq C_1 |m(w)| (\|u_\epsilon\|_1^{2p-3} + 1) \|W\|_{-1} \\ &\leq C_2 m(w)^2 (\|u_\epsilon\|_1^{2p-3} + 1)^2 + \frac{1}{8} \|W\|_{-1}^2. \end{aligned} \quad (2.20)$$

Thanks to the embedding of  $H^1(\Omega)$  into  $L^\infty(\Omega)$  there exist two constants  $C_3, C_4$  such that:

$$\begin{aligned} \|f''(u_\epsilon)\|_{L^\infty} &\leq C_3 (\|u_\epsilon\|_{L^\infty}^{2p-3} + 1) \\ &\leq C_4 (\|u_3\|_1^{2p-3} + 1). \end{aligned}$$

Now we use the embedding  $L^3(\Omega) \subset L^6(\Omega) \subset V_{1/3}$  and an interpolation inequality :

$$\begin{aligned} |\tilde{w}|_{L^3}^3 &\leq C_5 |w|_{H^{1/3}}^3 \\ &\leq C_5 |w|_{-1} |w|_1^2. \end{aligned}$$

Thus :

$$\begin{aligned} \left| \int_{\Omega} f''(u_{\epsilon}) \tilde{w}^3 dx \right| &\leq C_4 C_5 (\|u_{\epsilon}\|_1^{2p-3} + 1) |w|_{-1} |w|_1^2 \\ &\leq C_6 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 |w|_1^2 + \frac{\lambda\nu}{8} |w|_1^2. \end{aligned} \quad (2.21)$$

Similarly, there exists a constant  $C_7$  such that :

$$\left| m(w) \int_{\Omega} f''(u_{\epsilon}) \tilde{w}^2 dx \right| \leq C_7 m(w)^2 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 + \frac{\lambda\nu}{8} |w|_1^2. \quad (2.22)$$

We use (2.20)-(2.22) in (2.19) :

$$\begin{aligned} (f'(u_{\epsilon}), W)_0 &\geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx \\ &\quad + \lambda \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx \\ &\quad - C_2 m(w)^2 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 - \frac{1}{8} |W|_{-1}^2 \\ &\quad - C_6 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 |w|_1^2 - \frac{\lambda\nu}{8} |w|_1^2 \\ &\quad - C_7 m(w)^2 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 - \frac{\lambda\nu}{8} |w|_1^2. \end{aligned} \quad (2.23)$$

Now, we use (2.23) and the Lemma 2.2 to obtain from (2.18) :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\epsilon |W|_{-1}^2 + \nu |w|_1^2 + \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx) \\ &\quad + \frac{1}{2} |W|_{-1}^2 + \frac{\lambda\nu}{2} |w|_1^2 + \lambda \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx \\ &\quad \leq C_2 m(w)^2 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 \\ &\quad + C_6 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 |w|_1^2 \\ &\quad + C_7 m(w)^2 (\|u_{\epsilon}\|_1^{2p-3} + 1)^2 |w|_{-1}^2 \\ &\quad + \frac{1}{8} |W|_{-1}^2 + \frac{\lambda\nu}{4} |w|_1^2. \end{aligned}$$

Finally, since there exists a constant  $C_8$  such that :

$$\lambda \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx = \frac{\lambda}{4} \int_{\Omega} f'(u_{\epsilon}) \tilde{w}^2 dx - C_8,$$

we have:

$$\begin{aligned}
& \frac{d}{dt} \left( E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) \right) + \frac{\lambda}{2} E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) \\
& \leq 2C_2 m(w)^2 (\|u_\epsilon\|_1^{2p-3} + 1)^2 + 2C_8 \\
& + \frac{2C_6}{\nu} (\|u_\epsilon\|_1^{2p-3} + 1)^2 |w|_{-1}^2 E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) \\
& + 2C_7 m(w)^2 (\|u_\epsilon\|_1^{2p-3} + 1)^2 |w|_{-1}^2.
\end{aligned}$$

Let  $B$  be a bounded set in  $X_3^\alpha$ , then  $B$  is bounded in  $X_1^\alpha$  and thanks to Corollary 2.1, there exist constants independent of  $B$  and of  $\epsilon$ ,  $k_2, k_3, k_4$ , and a time  $t_1(B)$  such that for all  $(u_0, u_1)$  in  $B$  and all time  $t \geq t_1(B)$ , we have :

$$\begin{aligned}
\frac{d}{dt} E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) + \frac{\lambda}{2} E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) \\
\leq k_2 + k_3 |w|_{-1}^2 E_3 \left( \frac{du_\epsilon}{dt}, \frac{d^2u_\epsilon}{dt^2} \right) + k_4 |w|_{-1}^2.
\end{aligned}$$

Thanks to Lemma 2.4, there exists a time  $t_2(B)$  such that for all  $t \geq t_2(B)$  :

$$\int_{t_1}^t |w|_{-1}^2 ds \leq K_1.$$

Thanks to Lemma 2.5, for all  $t \geq t_0(B) = \sup(t_1(B), t_2(B))$  :

$$\begin{aligned}
E_3 \left( \frac{du_\epsilon}{dt}(t), \frac{d^2u_\epsilon}{dt^2}(t) \right) & \leq \left( 2\frac{k_2}{\lambda} + k_4 K_1 \right) e^{k_3 K_1} \\
& + \left( E_3 \left( \frac{du_\epsilon}{dt}(t_0), \frac{d^2u_\epsilon}{dt^2}(t_0) \right) - 2\frac{k_2}{\lambda} \right) e^{k_3 K_1 - \frac{\lambda}{2}(t-t_0)}.
\end{aligned}$$

Thanks to Lemma 2.3, there exists  $k'_1, k'_2$  constants such that for all  $t \geq t_0$  :

$$\begin{aligned}
\epsilon \left| \frac{d^2u_\epsilon}{dt^2}(t) \right|_{-1}^2 + \left| \frac{du_\epsilon}{dt}(t) \right|_1^2 & \leq k'_1 + k'_2 \left( \epsilon \left| \frac{d^2u_\epsilon}{dt^2}(t_0) \right|_{-1}^2 + \left| \frac{du_\epsilon}{dt}(t_0) \right|_1^2 \right. \\
& \left. + \|u_\epsilon(t_0)\|_1^{2p-2} \left| \frac{du_\epsilon}{dt}(t_0) \right|_0^2 \right) e^{-\frac{\lambda}{2}(t-t_0)}.
\end{aligned} \tag{2.24}$$

We assume that  $t_0$  is sufficiently large so that :

$$\|u_\epsilon(t_0)\|_1^{2p-2} \left| \frac{du_\epsilon}{dt}(t_0) \right|_0^2 \leq k'_3, \tag{2.25}$$

where  $k'_3$  is a constant independent of  $B$  (we use Corollary 2.2). Moreover, thanks to (1.14):

$$\begin{aligned} \epsilon \left| \frac{d^2 u_\epsilon}{dt^2}(t_0) \right|_{-1} &\leq \left| \frac{du_\epsilon}{dt}(t_0) \right|_{-1} + |u_\epsilon(t_0)|_3 + |f(u_\epsilon(t_0))|_1, \\ \epsilon \left| \frac{d^2 u_\epsilon}{dt^2}(t_0) \right|_{-1}^2 &\leq \frac{k'_4}{\epsilon} \left( \left| \frac{du_\epsilon}{dt}(t_0) \right|_{-1}^2 + |u_\epsilon(t_0)|_3^2 + 1 \right), \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} |u_\epsilon(t)|_3 &\leq \epsilon \left| \frac{d^2 u_\epsilon}{dt^2}(t) \right|_{-1} + \left| \frac{du_\epsilon}{dt}(t) \right|_{-1} + |f(u_\epsilon(t))|_1, \\ |u_\epsilon(t)|_3^2 &\leq k'_5 \left( \epsilon^2 \left| \frac{d^2 u_\epsilon}{dt^2}(t) \right|_{-1}^2 + \left| \frac{du_\epsilon}{dt}(t) \right|_1^2 + 1 \right). \end{aligned} \quad (2.27)$$

From (2.24)-(2.27), we deduce the Proposition 2.3.

e) Propositions 1.1 and 2.1 are still true if we consider (1.14) on a bounded set in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , but we do not know whether Propositions (2.2) and (2.3) are still true. If we try to obtain estimates in  $V_2 \times V_0$  by the same techniques as in Proposition 2.2, we have to estimate the last term of (2.6) :

$$\frac{1}{2} \int_{\Omega} f''(u_\epsilon) \frac{du_\epsilon}{dt} |\nabla u_\epsilon|^2 dx.$$

Even if  $n = 2$ , this term is stronger than  $E_2(u_\epsilon, \frac{du_\epsilon}{dt})$ . For instance, we can majorize it by:

$$|f''(u_\epsilon)|_{L^\infty} \left| \frac{du_\epsilon}{dt} \right|_{L^2} |\nabla u_\epsilon|_{L^4}^2. \quad (2.28)$$

If  $n = 2$ ,  $L^4(\Omega)$  is embedded in  $H^{1/2}(\Omega)$  and  $L^\infty(\Omega)$  in  $H^{1+\nu}(\Omega)$  for all  $\eta > 0$ ; therefore using interpolation inequalities, (2.28) is majorized by

$$C(\|u_\epsilon\|_1^{1-\eta} \|u_\epsilon\|_2^\eta + 1) \|u_\epsilon\|_0 \|u_\epsilon\|_1 \|u_\epsilon\|_2$$

and this is too strong (we would need  $\eta \leq 0$  and this is not possible). If  $n = 3$ , (2.28) is stronger. For the Proposition 2.3, we have the same type of problems.

### III. EXISTENCE OF ATTRACTORS.

a) In this section, we first give a result of B. Nicolaenko, B. Scheurer and R. Temam [1] concerning the existence of a maximal attractor for the Cahn-Hilliard equation (1.11)(1.12)-(1.13). Then using a general theorem, we prove the existence of attractors for the perturbed Cahn-Hilliard equation (1.14), (1.15), (1.13) in the spaces  $X_s^\alpha$  endowed with their weak

topology. The results are valid for the Cahn-Hilliard equation in space dimension  $n = 1, 2$  or  $3$ , but for the perturbed equation, since we do not have any absorbing set in space dimension higher than two, the results are restricted to the space dimension  $n = 1$ .

It is known that the attractor for the Cahn-Hilliard equation has finite Hausdorff and fractal dimension. We think that the method developed in J.M. Ghidaglia [1] (which is a slight modification of the usual method presented in R. Temam [1] for instance) can be adapted successfully to the perturbed equation in order to show that the attractors constructed in section b) hereafter have finite fractal and Hausdorff dimension.

b) The Cahn-Hilliard equation.

As we already mentioned in section Id), the average of a solution of (1.11) is constant. Therefore, there does not exist any absorbing set in the whole space  $V_0$  and we have to introduce the subsets of  $V_0$  :

$$H_\alpha = \{u \in V_0 : |m(u)| \leq \alpha\}.$$

One can show (see B. Nicolaenko, B. Scheurer and R. Temam [1] or R. Temam [1]) the following result :

**Theorem 3.1.** *The restriction to  $H_\alpha$  of the semigroup  $(S(t))_{t \geq 0}$  defined in Theorem 1.1 possesses a maximal attractor  $A_\alpha$  in  $H_\alpha$  endowed with the strong topology of  $V_0$ . Furthermore,  $A_\alpha$  is compact and connected (for the strong topology of  $V_0$ ).*

Thanks to Theorem 1.1, we know that if  $u_0$  is in  $V_0$ , then for  $t > 0$ ,  $u(t)$  is in  $V_2$ . As the attractor  $A_2$  is in invariant set, we deduce :

$$A_2 \subset V_2.$$

Using the second assertion of Theorem 1.1 and the same argument as above :

$$A_2 \subset V_4.$$

c) The perturbed Cahn-Hilliard equation.

To show the existence of an attractor for the equation (1.14), (1.15), (1.13), we use a general existence theorem whose proof can be found in R. Temam [1] or Hale [1] :



**Theorem 3.2.** Let  $(E, d)$  be a metric space and  $(S(t))_{t \geq 0}$  a semigroup on  $E$  such that for each  $t$ ,  $S(t)$  is continuous.

Suppose that there exists a bounded absorbing set  $B_a$  such that  $\bigcup_{t \geq t_0} S(t)B_a$  is relatively compact in  $E$  for some  $t_0 > 0$ . Then

$$\mathcal{A} = \omega(B_a) = \bigcap_{s \geq 0} \text{cl} \left( \bigcup_{t \geq s} S(t)B_a \right)$$

is a maximal attractor for the semigroup  $(S(t))_{t \geq 0}$  :

- $\forall t \in \mathbb{R} : S(t)\mathcal{A} = \mathcal{A}$
- for all bounded set in  $E$  :

$$d(S(t)B, \mathcal{A}) \xrightarrow{t \rightarrow +\infty} 0.$$

Moreover  $\mathcal{A}$  is compact and connected.

Our aim is to apply Theorem 3.2 to the semigroups  $(S_\epsilon^s(t))_{t \geq 0}$  on  $X_\alpha^s$  endowed with the weak topology. In section II, we obtained bounded absorbing sets in  $X_\alpha^s$  endowed with the strong topology, we denote them by  $B_{\alpha, \epsilon}^s$ . Let  $t_0$  be such that

$$S_\epsilon^s(t)B_{\alpha, \epsilon}^s \subset B_{\alpha, \epsilon}^s \text{ for } t \geq t_0;$$

then  $\bigcup_{t \geq t_0} S_\epsilon^s(t)B_{\alpha, \epsilon}^s \subset B_{\alpha, \epsilon}^s$  and since  $B_{\alpha, \epsilon}^s$  is compact in  $X_\alpha^s$  weak,  $\bigcup_{t \geq t_0} S_\epsilon^s(t)B_{\alpha, \epsilon}^s$  is relatively compact in  $X_\alpha^s$  weak.

In order to apply Theorem 3.2, it remains to check the weak continuity for the semigroups  $(S_\epsilon^s(t))_{t \geq 0}$ . We have

**Proposition 3.1 :** For  $\alpha$  in  $\mathbb{R}^+$ ,  $s = 1, 2$  or  $3$  and  $t \geq 0$ , the mapping  $S_\epsilon^s(t)$  is weakly continuous on  $X_\alpha^s$ .

**Proof :** We first consider the case  $s = 2$  or  $3$ . Let  $(u_n, v_n)$  be a sequence in  $X_s^\alpha$  weakly convergent to  $(u, v)$  in  $X_s^\alpha$ . Then  $(u_n, v_n)$  converges strongly in  $X_{s-1}^\alpha$  and since  $S_\epsilon^{s-1}(t)$  is strongly continuous  $S_\epsilon^s(t)(u_n, v_n)$  converges to  $S_\epsilon^s(t)(u, v)$  in  $X_{s-1}^\alpha$  strong. On the other hand,  $(u_n, v_n)$  is bounded in  $X_s^\alpha$  and consequently  $S_\epsilon^s(t)(u_n, v_n)$  too, therefore there exists a subsequence  $S_\epsilon^s(t)(u_{n_h}, v_{n_h})$  that converges weakly in  $X_s^\alpha$  to a limit  $(U, V)$ . This subsequence is strongly convergent in  $X_{s-1}^\alpha$ , thus the limit  $(U, V)$  must be equal to  $S_\epsilon^s(t)(u, v)$ . This shows that the whole sequence weakly converges to  $S_\epsilon^s(t)(u, v)$ .

We now consider the case  $s = 1$ . Let  $(u_n, v_n)$  be a sequence in  $X_s^\alpha$  weakly convergent to  $(u, v)$  in  $X_1^\alpha$ . We denote by  $u_\epsilon^n$  (resp.  $u_\epsilon$ ) the solution of (1.19), (1.15), (1.13) with  $u_0 = u_n, u_1 = v_n$  (resp.  $u_0 = u, u_1 = v$ ). We subtract the equations satisfied by  $u_\epsilon^n$  and  $u_\epsilon$ , we obtain :

$$\epsilon \frac{d^2 w_\epsilon^n}{dt^2} + \frac{dw_\epsilon^n}{dt} + \nu \Delta^2 w_\epsilon^n - \Delta f(u_\epsilon^n) + \Delta f(u_\epsilon) = 0, \quad (3.1)$$

where  $w_\epsilon^n = u_\epsilon^n - u_\epsilon$ .

We take the scalar product in  $L^2$  of (3.1) with  $A^{-1} \frac{dw_\epsilon^n}{dt}$  :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 + \nu |w_\epsilon^n|_0^2 \right) + \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 \\ & + \left( (f(u_\epsilon^n) - f(u_\epsilon)), A^{-1/2} \frac{dw_\epsilon^n}{dt} \right) = 0, \\ & \frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 + \nu |w_\epsilon^n|_0^2 \right) + \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 \\ & \leq \frac{1}{2} |f(u_\epsilon^n) - f(u_\epsilon)|_{L^2}^2 + \frac{1}{2} \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2, \\ & \frac{d}{dt} \left( \epsilon \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 + \nu |w_\epsilon^n|_0^2 \right) \\ & \leq |f(u_\epsilon^n) - f(u_\epsilon)|_{L^2}^2. \end{aligned} \quad (3.2)$$

Since  $(u_\epsilon^n)$  is bounded in  $H^1(\Omega)$  and consequently in  $L^\infty(\Omega)$  and since  $f$  is Lipschitzian on the bounded sets of  $\mathbb{R}$ , we have

$$\begin{aligned} & \exists C(u_\epsilon^n, u_\epsilon) \text{ such that} \\ & |f(u_\epsilon^n) - f(u)|_{L^2}^2 \leq C |u_\epsilon^n - u|_{L^2}^2 \\ & \leq C |w_\epsilon^n|_{L^2}^2 \\ & \leq C (|w_\epsilon^n|_0^2 + m(w_\epsilon^n)^2). \end{aligned} \quad (3.3)$$

We take the average of (3.1) :

$$\epsilon \frac{d^2}{dt^2} m(w_\epsilon^n) + \frac{d}{dt} m(w_\epsilon^n) = 0,$$

and that implies :

$$m(w_\epsilon^n) = m(u_n - u) + \epsilon m(v_n - v)(1 - e^{-t/\epsilon}), \quad (3.4)$$

$$m \left( \frac{dw_\epsilon^n}{dt} \right) = m(v_n - v) e^{-t/\epsilon}. \quad (3.5)$$

We deduce from (3.2)-(3.5) :

$$\begin{aligned} \frac{d}{dt} \left( \epsilon \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 + \nu |w_\epsilon^n|_0^2 \right) &\leq |w_\epsilon^n|_0^2 \\ &+ C(m(u_n - u) + \epsilon m(v_n - v)(1 - e^{-t/\epsilon})). \end{aligned}$$

Thanks to Gronwall's Lemma, we have :

$$\begin{aligned} \epsilon \left| \frac{dw_\epsilon^n}{dt} \right|_{-2}^2 + \nu |w_\epsilon^n|_0^2 &\leq e^{Ct} \left( \epsilon |v_n - v|_{-2}^2 + \nu |u_n - u|_0^2 \right) \\ &+ C \int_0^t e^{C(t-s)} (m(u_n - u) + \epsilon m(v_n - v)(1 - e^{-s/\epsilon})) dt. \end{aligned} \quad (3.6)$$

As  $\{v_n - v, u_n - u\}$  converges weakly to zero in  $X_1^\alpha$  and the injection of  $X_1^\alpha$  into  $X_0^\alpha$  is compact, (3.6) shows that  $\left\{ \frac{dw_\epsilon^n}{dt}(t), w_\epsilon^n(t) \right\}$  strongly converges to zero in  $X_0^\alpha$ . In the same manner as in the case  $s = 2$  or  $3$ , this implies that  $S_\epsilon^1(t)\{u_n, v_n\}$  weakly converges to  $S_\epsilon^1(t)\{u, v\}$  in  $X_1^\alpha$ .

Now, Propositions 2.1, 2.2, 2.3 and 3.1 allow us to apply Theorem 3.2 in the three cases  $s = 1, 2$  and  $3$ . Thus, we have

**Theorem 3.3 :** For  $s = 1, 2$  or  $3$  and  $\alpha$  in  $\mathbb{R}^+$ , the restriction of  $S_\epsilon^s(t)$  to  $X_\alpha^s$  possesses a maximal attractor  $\mathcal{A}_{\epsilon, \alpha}^s$  compact and connected in the space  $X_\alpha^s$  endowed with the weak topology.

**Remark.** 1)  $\mathcal{A}_{\epsilon, \alpha}^3$  is a bounded set in  $X_\alpha^3$  and consequently in  $X_\alpha^2$ . As it is an invariant set, we have :

$$\mathcal{A}_{\epsilon, \alpha}^3 \subset \mathcal{A}_{\epsilon, \alpha}^2.$$

In the same way :

$$\mathcal{A}_{\epsilon, \alpha}^2 \subset \mathcal{A}_{\epsilon, \alpha}^1.$$

Unfortunately, we do not know whether  $\mathcal{A}_{\epsilon, \alpha}^2$  and  $\mathcal{A}_{\epsilon, \alpha}^1$  are bounded in  $X_\alpha^3$  and  $X_\alpha^2$  and we do not know whether the other inclusions hold.

2) Using the compactness of the injection of  $X_\alpha^s$  into  $X_\alpha^{s'}$  for all  $s' < s$ , one can see that the bounded sets of  $X_\alpha^s$  are attracted by  $\mathcal{A}_{\epsilon, \alpha}^s$  in the strong topology of  $X_\alpha^{s'}$ .

3) Let  $(u, v)$  be in  $\mathcal{A}_{\epsilon, \alpha}^s$ , then there is a complete orbits  $(u_\epsilon(t), \frac{du_\epsilon(t)}{dt})_{t \in \mathbb{R}}$  in  $\mathcal{A}_{\epsilon, \alpha}$  such that

$$u_\epsilon(0) = u, \quad v_\epsilon(0) = v.$$

For all time  $t$ , we have :

$$m\left(\frac{du_\epsilon}{dt}(t)\right) = m(v)e^{-t/\epsilon};$$

$$m(v) = m\left(\frac{du_\epsilon}{dt}(t)\right) e^{t/\epsilon}.$$

Since  $m\left(\frac{du_\epsilon}{dt}(t)\right)$  is bounded, by letting  $t \rightarrow -\infty$  we have

$$m(v) = 0.$$

Using section Ia), for the perturbed Cahn-Hilliard equation there exists a Lyapunov function defined on a set that contains the attractors. We deduce that the attractors consist of the unstable set of the set of the fixed points.

4) Using the technics developped in J.M. Ghidaglia and R. Temam [2], one can generalize Theorem 3.3 to the case where the right hand side of (1.14) contains a time periodic forcing term.

#### IV. Convergence of the attractor in $X_3^\alpha$ when $\epsilon \rightarrow 0$ .

a) In this section, we prove that when  $\epsilon$  is small,  $\mathcal{A}_{\epsilon,\alpha}^3$  is close to  $\mathcal{A}_\alpha$ . More precisely, we first define a convenient embedding of  $\mathcal{A}_\alpha$  (which is included in  $V_4$ ) into  $X_2^\alpha$  :

$$\mathcal{A}_\alpha^* = \{(u, -\nu\Delta^2 u + \Delta f(u)) / u \in \mathcal{A}_\alpha\}.$$

If  $(u, v) \in \mathcal{A}_\alpha^*$ , and if  $(u(t))_{t \in \mathbb{R}}$  is a complete orbit in  $\mathcal{A}_\alpha$  such that  $u(0) = u$ , then  $v$  is the time derivative of  $u(t)$  at the time 0 :  $v = \frac{du}{dt}(0)$ . We will show that the Hausdorff semidistance  $\delta(\mathcal{A}_{\epsilon,\alpha}^3, \mathcal{A}_\alpha^*)$  converges to zero.

b) Theorem 4.1. *The Hausdorff semidistance  $\delta(\mathcal{A}_{\epsilon,\alpha}^3, \mathcal{A}_\alpha^*)$  converges to zero when  $\epsilon \rightarrow 0$  :*

$$\lim_{\epsilon \rightarrow 0} \sup_{(u_\epsilon, v_\epsilon) \in \mathcal{A}_{\epsilon,\alpha}^3} \inf_{(u, v) \in \mathcal{A}_\alpha^*} (\|u_\epsilon - u\|_2^2 + \|v_\epsilon - v\|_0^2)^{1/2} = 0.$$

Remark. As we already mentioned in the introduction, Theorem 4.1 gives the upper-semicontinuity of the attractor at zero. The lower continuity is much more difficult and we need to have sharp informations on the structure of the attractors (consisting of the stationary solutions and their unstable manifolds).

Proof of Theorem 4.1.

We know that  $\mathcal{A}_{\epsilon, \alpha}^3$  is an invariant set. Let  $(u_\epsilon(t))_{t \in \mathbb{R}}$  be an orbit in  $\mathcal{A}_{\epsilon, \alpha}^3$ , then using Proposition 2.4 we have :

$$\epsilon \left| \frac{d^2 u_\epsilon(t)}{dt^2} \right|_{-1}^2 + \left| \frac{du_\epsilon(t)}{dt} \right|_1^2 + |u_\epsilon(t)|_3^2 \leq R_5, \quad (4.1)$$

for all time  $t$ , with  $R_5$  a constant independent of  $\epsilon$ .

Moreover we saw in the third remark of the preceding section that:

$$m \left( \frac{du_\epsilon}{dt} \right) = 0 \quad (4.2)$$

and

$$m(u_\epsilon) \text{ is independent of } t. \quad (4.3)$$

Let  $\eta$  be a real positive number. For each  $\epsilon$  positive, we choose  $(u_\epsilon, v_\epsilon)$  in  $X_3^\alpha$  such that :

$$\inf_{u, v \in \mathcal{A}_\alpha^*} (\|u_\epsilon - u\|_2^2 + \|v_\epsilon - v\|_0^2) \geq \delta(\mathcal{A}_{\epsilon, \alpha}^3, \mathcal{A}_\alpha^*) - \eta.$$

We denote by  $(u_\epsilon(t), \frac{du_\epsilon}{dt}(t))_{t \in \mathbb{R}}$  a complete orbit in  $\mathcal{A}_{\epsilon, \alpha}^3$  such that

$$u_\epsilon(0) = u_\epsilon, \quad \frac{du_\epsilon}{dt}(0) = v_\epsilon.$$

Then thanks to (4.1), (4.2), (4.3) we have

$$(u_\epsilon)_{\epsilon > 0} \text{ is bounded in } L^\infty(\mathbb{R}, V_3), \quad (4.4)$$

$$\left( \frac{du_\epsilon}{dt} \right)_{\epsilon > 0} \text{ is bounded in } L^\infty(\mathbb{R}, V_1), \quad (4.5)$$

$$\left( \sqrt{\epsilon} \frac{d^2 u_\epsilon}{dt^2} \right)_{\epsilon > 0} \text{ is bounded in } L^\infty(\mathbb{R}, V_{-1}), \quad (4.6)$$

Thus, using classical compactness theorem, (4.4) and (4.5) imply that there exists a subsequence  $(u_{\epsilon'})_{\epsilon' > 0}$  and a function  $u$  in  $C^0(\mathbb{R}, V_2)$  such that for all  $T > 0$ ,  $u_{\epsilon'}$  converges to  $u$  in  $C^0([-T, T], V_2)$ . Moreover (4.4) implies that  $u$  belongs to  $C_b^0(\mathbb{R}, V_2)$ .

We know that  $\frac{du_\epsilon}{dt}$  is in  $C^0(\mathbb{R}, V_0)$ . And (4.5) implies that there is a subsequence (that we still denote by  $(u_{\epsilon'})_{\epsilon' > 0}$ ) such that

$$\frac{du_{\epsilon'}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^\infty(\mathbb{R}, V_1) \text{ weak star.} \quad (4.7)$$

Thanks to (4.6), we have :

$$\epsilon \frac{d^2 u_\epsilon}{dt^2} \rightarrow 0 \text{ in } L^\infty(\mathbb{R}, V_{-1}) \text{ strong} \quad (4.8)$$

We rewrite (1.14) as follows :

$$\frac{du_{\epsilon'}}{dt} = -\epsilon \frac{d^2 u_{\epsilon'}}{dt^2} - \nu \Delta^2 u_{\epsilon'} + \Delta f(u_{\epsilon'}). \quad (4.9)$$

Since the mapping  $u \rightarrow f(u)$  is continuous from  $V_2$  to  $V_2$  we deduce from (4.8) and the convergence of  $u_{\epsilon'}$  to  $u$  in  $C^0([-T, T], V_2)$  for all  $T > 0$  that  $\frac{du_{\epsilon'}}{dt}$  converges to  $-\nu \Delta^2 u + \Delta f(u)$  in  $C^0([-T, T], V_{-2})$  for all  $T > 0$ .

(4.7) implies that :

$$\frac{du}{dt} = -\nu \Delta^2 u + \Delta f(u), \quad (4.10)$$

and (4.5) implies that  $\frac{du_{\epsilon'}}{dt}$  converges to  $\frac{du}{dt}$  in  $C^0([-T, T], V_0)$  for all  $T > 0$  and that  $\frac{du}{dt}$  belongs to  $C_b^0(\mathbb{R}, V_0)$ .

Thus,  $u$  is solution of the equation (1.11) that belongs to  $C_b^0(\mathbb{R}, V_2)$ . From the definition of  $\mathcal{A}_\alpha$ , for all  $t$   $u(t)$  belongs to  $\mathcal{A}_\alpha$ . And from (4.10) :

$$\left( u(t), \frac{du}{dt}(t) \right) \in \mathcal{A}_\alpha^*.$$

Since  $\left( u_{\epsilon'}, \frac{du_{\epsilon'}}{dt} \right)$  converges to  $\left( u, \frac{du}{dt} \right)$  in  $C^0([-T, T], V_2 \times V_0)$  for all  $T > 0$ , we have :

$$\left( u_{\epsilon'}, v_{\epsilon'} \right) = \left( u_{\epsilon'}(0), \frac{du_{\epsilon'}}{dt}(0) \right) \text{ converges to } \left( u(0), \frac{du}{dt}(0) \right) \text{ in } (V_2 \times V_0).$$

And that implies :

$$\lim_{\epsilon \rightarrow 0} \inf_{(u, v) \in \mathcal{A}_\alpha^*} (\|u_\epsilon - u\|_2^2 + \|v_\epsilon - v\|_0^2) = 0,$$

$$0 \leq \limsup_{\epsilon \rightarrow 0} \delta(\mathcal{A}_{\epsilon, \alpha}^3, \mathcal{A}_\alpha^*) \leq \eta.$$

$\eta$  is arbitrary small, therefore :

$$\lim_{\epsilon \rightarrow 0} \delta(\mathcal{A}_{\epsilon, \alpha}^3, \mathcal{A}_\alpha^*) = 0.$$

**APPENDIX**  
**Proof of Theorem (1.2)**

The proof of this theorem is very standard, we use a Faedo-Galerkin method. Let  $(w_j)_{j \in \mathbb{N}}$  be the orthonormal basis in  $L^2(0, L)$  of eigenvectors of  $A$ . We first look for  $u_m$  in  $Sp(w_1, \dots, w_m)$  such that

$$\epsilon \frac{d^2 u_m}{dt^2} + \frac{du_m}{dt} + \nu \Delta^2 u_m - \Delta P_m f(u_m) = 0, \quad (\text{A.1})$$

$$u_m(0) = P_m u_0, \quad (\text{A.2})$$

$$\frac{du_m}{dt}(0) = P_m u_1. \quad (\text{A.3})$$

( $P_m$  is the orthogonal projector on  $Sp(w_1, \dots, w_m)$ ).

The existence and uniqueness for  $u_m$  on an interval  $[0, T_m[$  follow from classical theorem on ordinary differential equations. Using the same arguments as in the proof of proposition 1.1, 2.1, 2.2, 2.3 one easily prove that, for  $s = 1, 2$  or  $3$ , if  $(u_0, u_1)$  belongs to  $V_s \times V_{s-2}$  then, for all  $T \geq 0$ :

$$(u_m)_{m \in \mathbb{N}} \text{ is bounded in } L^\infty(0, T; V_s), \quad (\text{A.4})$$

$$\left( \frac{du_m}{dt} \right)_{m \in \mathbb{N}} \text{ is bounded in } L^\infty(0, T; V_{s-2}). \quad (\text{A.5})$$

(in fact, we can prove (A.4), (A.5) with simpler calculus since we are not interested in the dependence on  $\epsilon$  and we do not need time uniform estimates). ((A.4) and (A.5) implies that  $T_m = +\infty$ ).

Now, using classical compactness theorems, we can take the limit in (A.1), (A.2), (A.3). There exists a function  $u$  in  $L^\infty(0, T; V_s)$  with  $\frac{du}{dt}$  in  $L^\infty(0, T; V_{s-2})$  for all  $T > 0$  such that:

$$\epsilon \frac{d^2 u}{dt^2} + \frac{du}{dt} + \nu \Delta^2 u - \Delta f(u) = 0, \quad (\text{A.6})$$

$$u(0) = u_0, \quad (\text{A.7})$$

$$\frac{du}{dt}(0) = u_1. \quad (\text{A.8})$$

Now, we prove the uniqueness in  $V_1 \times V_{-1}$  together with the continuity of  $S_\epsilon^1(t)$ .

Let  $(u_0, u_1), (v_0, v_1)$  in  $V_1 \times V_{-1}$  and  $u, v$  the corresponding solution of (A.6), (A.7), (A.8). We denote by

$$w = u - v.$$

We have

$$\epsilon \frac{d^2 w}{dt^2} + \frac{dw}{dt} + \nu \Delta^2 w - \Delta(f(u) - f(v)) = 0. \quad (\text{A.9})$$

We take the scalar product of (A.9) with  $A^{-1/2} \frac{dw}{dt}$ , we obtain :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{dw}{dt} \right|_{-1}^2 + \nu |w|_1^2 \right) + \left| \frac{dw}{dt} \right|_{-1}^2 \\ + \left( f(u) - f(v), \frac{dw}{dt} \right)_0 = 0, \\ \frac{1}{2} \frac{d}{dt} \left( \epsilon \left| \frac{dw}{dt} \right|_{-1}^2 + \nu |w|_1^2 \right) + \left| \frac{dw}{dt} \right|_{-1}^2 \\ \leq |f(u) - f(v)|_1 \left| \frac{dw}{dt} \right|_{-1}. \end{aligned}$$

We use a Young inequality :

$$\frac{d}{dt} \left( \epsilon \left| \frac{dw}{dt} \right|_{-1}^2 + \nu |w|_1^2 \right) \leq |f(u) - f(v)|_1^2.$$

Since  $f$  is a polynomial,  $V_1$  is an algebra and  $u, v$  are in  $L^\infty(0, T; V_1)$ , we know that there exists a constant  $C_1$  (depending on  $\sup_{[0, T]} |u|_1$  and  $\sup_{[0, T]} |v|_1$ ) such that

$$|f(u) - f(v)|_1 \leq C_1 |u - v|_1.$$

Therefore :

$$\frac{d}{dt} \left( \epsilon \left| \frac{dw}{dt} \right|_{-1}^2 + \nu |w|_1^2 \right) \leq C_1^2 |w|_1^2.$$

An application of Gronwall's lemma leads to :

$$\epsilon \left| \frac{dw}{dt}(t) \right|_{-1}^2 + \nu |w(t)|_1^2 \leq e^{C_1 t} \left( \epsilon |u_1|_{-1}^2 + \nu |u_0|_1^2 \right). \quad (\text{A.10})$$

We now take the average of (A.9) :

$$\epsilon \frac{d^2}{dt^2} m(w) + \frac{d}{dt} m(w) = 0.$$



Therefore :

$$m(w) = (m(u_0) - m(v_0)) + \epsilon(m(u_1) - m(v_1))(1 - e^{-t/\epsilon}), \quad (\text{A.11})$$

$$m\left(\frac{dw}{dt}\right) = (m(u_1) - m(v_1))e^{-t/\epsilon}. \quad (\text{A.12})$$

(A.10), (A.11) and (A.12) imply the unicity of the solution in  $V_1 \times V_{-1}$  and the continuity of  $S_\epsilon^1(t)$ .

The continuity of  $S_\epsilon^2(t)$  and  $S_\epsilon^1(t)$  are left to the reader. We use the same type of arguments as above. It remains to prove the regularity of the solution, that is :

$$\left(u, \frac{du}{dt}\right) \in C([0, T], V_s \times V_{s-2}).$$

We notice that  $u$  is the solution of the nonhomogeneous linear problem :

$$\epsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + \nu \Delta^2 v = g,$$

supplemented with the same initial and boundary conditions as (A.6) and with :

$$g = \Delta f(u) \in L^\infty(0, T; V_{s-2}).$$

One can find the proof of the regularity for that type of linear problem in Lions-Magenes [1] for instance.

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